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Kurzweil-Stieltjes Integral and its Applications in Generalized Ordinary Differential Equations Theory

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Prohlášení

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Poděkování

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Contents

1 Preface

This thesis is devoted to differential equations with solutions which need not be absolutely continuous. It consists of two parts.

The first part is concerned with generalized linear ordinary differential equation (GLODE) of the form

$$x(t) = \widetilde{x}_0 + \int_a^t \mathrm{d}[A]\, x + f(t) - f(a)\,, \quad t \in [a,b]\,. \tag{1.1}$$

which contains as special case linear differential equations with impulses. Sections 3 and 4 are based on papers [8] and [9] and we present here some results on continuous dependence of solutions of GLODE on a parameter.

Section 5 presents results from [10], [11]. It is concerned with approximations of solutions of (1.1) by solutions of ordinary linear differential equations. We continue research initiated by M. Pelant in his Ph.D. thesis [25].

The second part is based on the paper [12]. The main result (see Theorem 8.1) deals with periodic impulse problems for nonlinear second order impulsive differential equations of the form

$$u'' + c\,u' = g(u) + e(t), \qquad u(0) = u(T), \quad u'(0) = u'(T),$$

$$\begin{cases} u(t_i+) = u(t_i) + J_i(u,u'), \\ u'(t_i+) = u'(t_i) + M_i(u,u'), \quad i = 1,2,\dots,m, \end{cases}$$

where $g \in C(0,\infty)$ can have a strong singularity at the origin and $0 < t_1 < \ldots < t_m < T$, $e \in L_1[0,T]$, $c \in \mathbb{R}$ and J_i, M_i, $i = 1,2,\dots,m$ are continuous mappings of $\mathbf{G}[0,T] \times \mathbf{G}[0,T]$ into $\mathbb{R}$, where $\mathbf{G}[0,T]$ denotes the space of functions regulated on $[0,T]$. In particular, we prove continuation type existence principle which is an analogy of the result of Manásevich and Mawhin (see [22]) valid for the classical case. Unlike most of the previously published results, our existence principle concerns also problems, in which monotonicity of impulse functions is not required.

Recent state summary

Linear systems of ordinary differential equations have been described in details in many textbooks devoted to ordinary differential equations. On the elementary level, solutions are understood in a classical sense. A function $x\colon [a,b] \to \mathbb{R}^n$ is said to be a (classical) solution to the initial value problem

$$x' - P(t)\,x = q(t)\,, \quad x(t_0) = \widetilde{x}_0 \tag{1.2}$$

on the interval $[a,b]$, if it possesses a continuous derivative on $[a,b]$ and it is such that $t_0 \in [a,b]$, $x(t_0) = \widetilde{x}_0$ and the equality

$$x'(t) - P(t)\,x(t) = q(t) \tag{1.3}$$

is satisfied for all $t \in [a,b]$. One can see that in such a case it is natural to assume that the coefficients P, q are continuous on $[a,b]$. The initial value problem (1.2) can be equivalently reformulated as the integral equation

$$x(t) - \widetilde{x}_0 - \int_{t_0}^{t} P(\tau)\,x(\tau)\mathrm{d}\tau = \int_{t_0}^{t} q(\tau)\mathrm{d}\tau \tag{1.4}$$

where the integrals are the Riemann ones.

In a more advanced setting the Lebesgue integral is used, P and q can be Lebesgue integrable functions and x is said to be a (Carathéodory) solution to (1.2) on the interval $[a,b]$, if it is absolutely continuous on $[a,b]$, $t_0 \in [a,b]$, $x(t_0) = \widetilde{x}_0$ and the equality (1.3) is satisfied for almost all $t \in [a,b]$. Equivalently, x is a solution to (1.2), if it is a solution to the integral equation (1.4), where the integrals are the Lebesgue ones.

Let us notice that if we put

$$A(t) = \int_{a}^{t} P(\tau)\,\mathrm{d}\tau \quad \text{and} \quad f(t) = \int_{a}^{t} q(\tau)\,\mathrm{d}\tau, \tag{1.5}$$

the equation (1.4) can be rewritten as

$$x(t) - \widetilde{x}_0 - \int_{t_0}^{t} \mathrm{d}[A(\tau)]\,x(\tau) = f(t) - f(t_0), \quad t \in [a,b]. \tag{1.6}$$

It is very natural to ask how the solutions depend on the change of parameters of the system, i.e. to find conditions ensuring that if $P_k \to P$ and $q_k \to q$ holds, then also $x_k \to x$ is true for solutions of systems (1.2) and

$$x_k' - P_k(t)\,x_k = q_k(t)\,, \quad x_k(t_0) = \widetilde{x}_0, \quad k \in \mathbb{N}. \tag{1.7}$$

It is well known that if $P_k \to P$ and $q_k \to q$ in L_1, then $x_k \rightrightarrows x$ holds for the Carathéodory solutions of (1.7). Moreover, due to Kurzweil and Vorel (c.f. [16, Theorem 1]), the same holds if

there is $m \in L_1[a,b]$ such that $|P_k(t)| \le m(t)$ a.e. on $[a,b]$ for $k \in \mathbb{N}$, (1.8)

$$\int_a^t P_k(s)\,\mathrm{d}s \rightrightarrows \int_a^t P(s)\,\mathrm{d}s \quad \text{and} \quad \int_a^t q_k(s)\,\mathrm{d}s \rightrightarrows \int_a^t q(s)\,\mathrm{d}s \quad \text{on } [a,b] \tag{1.9}$$

If we define $A_k(t) = \int_a^t P_k(s)\,\mathrm{d}s$ and $f_k(t) = \int_a^t q_k(s)\,\mathrm{d}s$, conditions (1.8) and (1.9) respectively reduce to

$$\sup\left\{\operatorname{var}_a^b A_k\colon\ k \in \mathbb{N}\right\} < \infty \tag{1.10}$$

and

$$A_k \rightrightarrows A, \quad f_k \rightrightarrows f \quad \text{on } [a,b]. \tag{1.11}$$

Now, let us give two easy observations.

- Consider very simple ODE $x' = f_n'(t)$, $x(0) = 0$, where

  $$f_n(t) = \begin{cases} 0 & \text{for } 0 \le t \le \frac{1}{n}, \\ t\,\sin(\frac{\pi}{t}) & \text{for } \frac{1}{n} < t \le 1. \end{cases}$$

  We have $f_n \in \mathbf{AC}[0,1]$ for $n \in \mathbb{N}$ and

  $$x_n(t) = f_n(t) \rightrightarrows x(t) = \begin{cases} 0 & \text{for } t = 0, \\ t\,\sin(\frac{\pi}{t}) & \text{for } t > 0. \end{cases}$$

  However, $x \notin \mathbf{AC}[0,1]$, i.e. x is not a solution of any ODE.

- Consider $x_k' = B_k(t)\,x_k$, $x_k(0) = \widetilde{x}_0 \in \mathbb{R}^n$, where

  $$\begin{cases} B_k(t) = P(t) + k\,\chi_{(\tau,\tau+\frac{1}{k})}(t)\ \mathrm{I}, \quad 0 < \tau < 1, \\ P\colon\ [0,T] \to \mathbb{R}^{n\times n} \ \text{ is continuous}, \quad P(t)\,P(s) = P(s)\,P(t) \ \text{ for } t,s \in [0,T]. \end{cases}$$

  We have

  $$x_k(t) \to x(t) = \begin{cases} \exp\left(\int_0^t P(s)\,\mathrm{d}s\right)\widetilde{x}_0 & \text{if } t \le \tau, \\ \exp\left(\int_\tau^t P(s)\,\mathrm{d}s\right)\,\exp(\mathrm{I})\,\exp\left(\int_0^\tau P(s)\,\mathrm{d}s\right)\widetilde{x}_0 & \text{if } t > \tau. \end{cases}$$

  Since $x(\tau+) - x(\tau) = (\exp(\mathrm{I}) - \mathrm{I})\exp\left(\int_0^\tau P(s)\,\mathrm{d}s\right)\widetilde{x}_0 \neq 0$, we can see that x has a jump at $t = \tau$. In other words, x is subjected to a impulse at $t = \tau$.

Motivated by the above observations, let us consider linear equation (1.6) in a more general framework. It is natural to assume, that A is of bounded variation and f is regulated. In this way, equation (1.6) covers both the above observations, i.e. the case that the coefficients of the system are rapidly oscillating and the case that they contain impulses.

The equation (1.6) is a special case of generalized differential equations introduced by J. Kurzweil 1957 in his celebrated paper [17]. From the results by I. I. Gichman, M. A. Krasnosel'skij and S. G. Krein and from the paper [16] it turned out that to get the conditions for continuous dependence of solutions on a parameter, only the knowledge of the indefinite integral is sufficient. This together with certain convergence effects which could not be explained by the known results was the main motivation for [17]. Moreover, in this paper new notion of nonlinear K-integral $\int_{\tau_1}^{\tau_2} \mathrm{D}\, U(\tau,t)$ was introduced. In the special case $U(\tau,t) = f(\tau)\, g(t)$, this integral reduces to the Perron–Stieltjes integral $\int_a^b f\,\mathrm{d}g$. Solutions of generalized differential equation were defined as solutions of the related integral equation where the nonlinear K-integral was involved. In this paper he presented basic existence results and results on continuous dependence of solutions on a parameter. In a series of papers published in the years 1957–1959 he continued his investigations and, in particular, he introduced the notion of the emphatic convergence.

Independently of Kurzweil, in 1959, T. H. Hildebrandt (c.f. [13]) investigated linear integral equations of the form (1.6) with the σ-Young integral and with right-hand side of bounded variation. In particular, he gave conditions ensuring the existence and uniqueness of solutions and constructed the fundamental matrix, essentially in the form of the product integral. In seventies in some sense a complete theory of linear boundary value problems for ordinary and generalized differential and integro-differential equations was established in a series of papers by Š. Schwabik, M. Tvrdý and O. Vejvoda (c.f. [43], [41], [51]). It appeared that the Kurzweil integration theory is an extremely powerful tool for investigating integral and differential equations from the general viewpoint of functional analysis. This stadium of the research was summarized in the monograph [42]. The monograph by Š. Schwabik [39] gave a representative account of the theory of generalized ODE's. In particular, various types of continuous dependence on parameters and variational stability were the topics of this monograph. Furthermore, differential equations with impulses turned out to be special cases of generalized differential equations.

In nineties the research by Š. Schwabik and M. Tvrdý was focused on extension of the results known for the setting on the space **BV** of functions of bounded variation to the space **G** of regulated functions. At the beginning this research was inspired by the work done by Ch. S. Hönig [14] who treated similar questions but on the basis of the interior Dushnik integral which in general differs somewhat from the Perron-Stieltjes integral. Properties of the Perron-Stieltjes integration with respect to regulated functions and analytic representation of the dual spaces to subspaces of **G** were disclosed in [45], a general form of linear bounded and compact operators was given in [40]. Results for the corresponding linear boundary value problems were delivered in [46] and fundamental theory of general linear integral equations in $\mathbb{R}^n$ was established in [47]. This stage of the investigation of differential and integral equations with regulated solutions was summarized in the monograph [49] by M. Tvrdý.

Problems of continuous dependence on a parameter from a new viewpoint have been investigated by M. Ashordia [1] and later by M. Tvrdý [48]. A closely related problem is that of approximating solutions of GLDE by solutions of properly chosen sequence of ODE's which was treated by M. Pelant in his Ph.D. thesis [25]. Sections 3 – 5 of our thesis are devoted to similar topics. In particular, we generalize and extend the result by the above mentioned authors.

In general, known ODE theories can be compared by the following table:

Theory	Integral	Solution
Classical	$\mathcal{Riemann}$, $\mathcal{Newton}$	$\mathbf{C}^1$
Carathéodory	$\mathcal{Lebesgue}$	**AC**
GODE	$\mathcal{Kurzweil}$	**BV**, **G**

Many applications from physics, biology, medicine and economy are modeled by systems with impulses. Let us mention e.g. the following: models of clock mechanism, oscillation of a pendulum with shock impulses, (Andronov, Witt and Khaikin, 1937; Kalitin, 1969); radiation of electric of magnetic waves in a medium with rapidly changing parameters (Friedman, 1956); motion of a particle in a field generated by a potential concentrated in a single point (Gottfried (1966)); control and optimal control, in particular cosmonautics (Lasota and Szafraniec, 1968; Tingh, 1969; Utkin, 1981); hysteresis and generalized variational inequalities (Krejčí, 1996; Krejčí and Laurençot, 2006); dosage schedule and pharmacokinetics in chemotherapy, drug distribution in the human body (Krüger-Thiemer, 1966); mass measles vaccination across age cohorts (Agur, Cojocaru, Mazor, Anderson and Danon, 1993); population growth models with impulsive effects (Ballinger and Liu, 1997); market models with discontinuous current prices (Zavalishchin, 1994). These models are mostly nonlinear. The next observation shows how to utilize the basic properties of GLODE's for solving nonlinear impulse problems.

Consider the impulse boundary value problem

$$\begin{cases} x' - P(t)\,x = q(t), \quad M\,x(0) + N\,x(T) = r, \\ \Delta^+ x(\tau) = \Delta^{[k]} \ \text{ if } \tau = t_k \in D = \{t_1, \dots, t_m\}, \end{cases} \tag{1.12}$$

where $0 < t_1 < \dots < t_m < T$; $P,\ q \in L_1[0,T]$, $M,\ N \in \mathbb{R}^{n\times n}$, $\Delta^{[k]}$, $r \in \mathbb{R}^n$.
Let X be the Cauchy matrix for $x' - P(t)\,x = 0$, $X(0) = \mathrm{I}$
and let $B = M\,X(0) + N\,X(T)$, $\det B \neq 0$.

Put $f(t) = \displaystyle\int_0^t h(s)\,\mathrm{d}s + \sum_{k=1}^m \Delta^{[k]}\,\chi_{(t_k,T]}(t)$ for $t \in [0,T]$.

Then problem (1.12) is equivalent to the system

$$\begin{cases} x(t) - x(0) - \displaystyle\int_0^t P(s)\,x(s)\,\mathrm{d}s = f(t) - f(0) \quad \text{for} \ \ t \in [0,T], \\ M\,x(0) + N\,x(T) = r \end{cases}$$

It possesses an unique solution x for each f and r

$$
\begin{aligned}
x(t) &= X(t)\,B^{-1}\,r + \int_0^T \Gamma(t,s)\,\mathrm{d}\,[f(s)] \\
&= X(t)\,B^{-1}\,r + \int_0^T \Gamma(t,s)\,h(s)\,\mathrm{d}s + \sum_{k=1}^{m} \Gamma(t,t_k)\,\Delta^{[k]} \quad \text{for} \quad t \in [0,T],
\end{aligned}
$$

where Γ is Green's function for

$$x' - P(t)\,x = 0, \qquad M\,x(0) + N\,x(T) = 0.$$

Using a quasilinearization method, we can conclude that x is a solution to the nonlinear problem

$$x' - P(t)\,x = F(t,x), \quad M\,x(0) + N\,x(T) = R(x), \quad \Delta^{+}x(t) = S_k(x) \text{ if } t = t_k \in D$$

if and only if it is a solution to the integral equation

$$x(t) = X(t)\,B^{-1}\,R(x) + \int_0^T \Gamma(t,s)\,F(s,x)\,\mathrm{d}s + \sum_{k=1}^{m} \Gamma(t,t_k)\,S_k(x) \quad \text{on} \quad [0,T].$$

Starting with Hu and Lakshmikantham [15], periodic boundary value problems

$$u'' = f(t,u,u'), \quad u(0) = u(T), \;\; u'(0) = u'(T)$$

$$\begin{cases} u(t_i+) = u(t_i) + J_i(u(t_i)), \\ u'(t_i+) = u'(t_i) + M_i(u'(t_i)), \quad i = 1,2,\dots,m, \end{cases}$$

have been studied by many authors. Usually it is assumed that the functions J_i, $M_i : \mathbb{R} \to \mathbb{R}$, $i = 1,2,\dots,m$, are continuous functions and fulfil some monotonicity type conditions. A rather representative (however not complete) list of related papers is given in references. In particular, in [4], [5], [7], [20], [21] existence results in terms of lower/upper functions obtained by the monotone iterative method can be found. All of these results impose monotonicity of the impulse functions and existence of an associated pair of well-ordered lower/upper functions. The papers [6] and [52] are based on the method of bound sets, however the effective criteria contained therein correspond to the situation when there is a well-ordered pair of constant lower and upper functions. Existence results which apply also to the case when a pair of lower and upper functions which need not be well-ordered is assumed were provided only by Rachůnková and Tvrdý, see [30], [32]–[34]. Analogous results for impulsive problems with quasilinear differential operator were delivered by Rachůnková and Tvrdý in [35]–[37]. When no impulses are acting, periodic problems with singularities have been treated by many authors. For rather representative overview and references, see e.g. [27] or [28]. To our knowledge, up to now singular periodic impulsive problems have not been treated. For singular Dirichlet impulsive problems we refer to the papers by Rachůnková [26], Rachůnková and Tomeček [29] and Lee and Liu [19].

Part I
Generalized Linear ODE

2 Preliminaries

2.1 Basic notation

The following notation and definitions will be used throughout this text: $\mathbb{N} = \{1, 2, 3, \dots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. $\mathbb{R}$ is the set of real numbers; $\mathbb{R}^{m\times n}$ is the space of real $m \times n$ matrices $B = (b_{ij})_{\substack{i=1,\dots,m\\ j=1,\dots,n}}$ equipped with the norm

$$|B| = \max_{j=1,\dots,n} \sum_{i=1}^{m} |b_{ij}|\,;$$

$\mathbb{R}^n = \mathbb{R}^{n\times 1}$ stands for the set of real column n-vectors.

For a matrix $B \in \mathbb{R}^{n\times n}$, $\det B$ denotes the determinant of B. If $\det B \neq 0$, then the matrix inverse to B is denoted by B^{-1}. B^T is the matrix transposed to B. The symbol I stands for the identity matrix and 0 for the zero matrix.

If $a, b \in \mathbb{R}$ are such that $-\infty < a < b < +\infty$, then $[a, b]$ stands for the closed interval $\{x \in \mathbb{R};\, a \le x \le b\}$, (a, b) is its interior and $(a, b]$, $[a, b)$ are the corresponding half-closed intervals.

The sets $D = \{t_0, t_1, t_2, \dots, t_m\}$ of points in the closed interval $[a, b]$ such that

$$a = t_0 < t_1 < t_2 < \cdots < t_m = b$$

are called *divisions* of $[a, b]$. The set of all divisions of the interval $[a, b]$ is denoted by $\mathcal{D}[a, b]$.

For a matrix valued function $B : [a, b] \to \mathbb{R}^{m\times n}$, its variation $\operatorname{var}_a^b B$ on the interval $[a, b]$ is defined by

$$\operatorname{var}_a^b B = \sup_{D\in\mathcal{D}[a,b]} \sum_{i=1}^{m} |B(t_i) - B(t_{i-1})|\,.$$

If $\operatorname{var}_a^b B < +\infty$, we say that the function B is of *bounded variation* on the interval $[a, b]$. $\mathbf{BV}^{m\times n}[a, b]$ denotes the set of all $m \times n$ matrix valued functions of bounded variation on $[a, b]$. We will write $\mathbf{BV}^n[a, b]$ instead of $\mathbf{BV}^{n\times 1}[a, b]$ and $\mathbf{BV}[a, b]$ instead of $\mathbf{BV}^{1\times 1}[a, b]$. The set $\mathbf{BV}^{m\times n}$ equipped with the norm $\|B\|_{BV} = |B(a)| + \operatorname{var}_a^b B$ is Banach space. For further details concerning the space $\mathbf{BV}^{m\times n}[a, b]$, see e.g. [50].

If a sequence of $m \times n$ matrix valued functions $\{B_k\}_{k=1}^{\infty}$ *converges uniformly* to a matrix valued function B_0 on $[c, d] \subset [a, b]$, i.e.

$$\lim_{k\to\infty} \sup_{t\in[c,d]} |B_k(t) - B_0(t)| = 0\,,$$

we write

$$B_k \rightrightarrows B_0 \quad \text{on } [c,d]\,.$$

We say that $\{B_k\}_{k=1}^{\infty}$ converges *locally uniformly* to B_0 on a set $M \subset [a,b]$, if $B_k \rightrightarrows B_0$ on each closed subinterval $J \subset M$.

$\mathbf{C}^{m\times n}[a,b]$ stands for the set of functions $B\colon [a,b] \to \mathbb{R}^{m\times n}$ that are *continuous* on $[a,b]$. $\mathbf{C}^{m\times n}$ equipped with the supreme norm

$$\|B\| = \sup_{t\in[a,b]} |B(t)|$$

is Banach space.

We will write briefly $B(t+) = \lim_{\tau\to t+} B(\tau)$, $B(s-) = \lim_{\tau\to s-} B(\tau)$. Furthermore, we denote

$$\begin{aligned} \Delta^+ B(t) &= B(t+) - B(t) \quad \text{for } t \in [a,b), \quad \Delta^+ B(b) = 0,\\ \Delta^- B(s) &= B(s) - B(s-) \quad \text{for } s \in (a,b], \quad \Delta^- B(a) = 0 \end{aligned}$$

and

$$\Delta B(r) = \Delta^+ B(r) - \Delta^- B(r) = B(r+) - B(r-) \quad \text{for } r \in (a,b).$$

We say that the matrix valued function $B\colon [a,b] \to \mathbb{R}^{m\times n}$ is *regulated* on $[a,b]$, if it possesses finite[1] limits $B(t+)$ and $B(s-)$ for each $t \in [a,b)$, $s \in (a,b]$. The set of all $m \times n$ matrix valued functions regulated on the interval $[a,b]$ is denoted by $\mathbf{G}^{m\times n}[a,b]$. Furthermore, for each regulated function B, we denote

$$\begin{cases} \mathfrak{S}^+(B;[a,b]) = \{t \in [a,b]\colon\ \Delta^+ B(t) \neq 0\}\,,\\ \mathfrak{S}^-(B;[a,b]) = \{t \in [a,b]\colon\ \Delta^- B(t) \neq 0\}\\ \text{and}\\ \mathfrak{S}(B;[a,b]) = \mathfrak{S}^+(B;[a,b]) \cup \mathfrak{S}^-(B;[a,b]). \end{cases} \tag{2.1}$$

Thus, $\mathfrak{S}(B;[a,b])$ is the set of all points of the discontinuity of the function B on the interval $[a,b]$. It is known that for each $B \in \mathbf{G}^{m\times n}[a,b]$ this set is at most countable and for each $\varepsilon > 0$ there are at most finitely many points $t \in [a,b)$ such that $|\Delta^+ B(t)| \geq \varepsilon$ and at most finitely many points $s \in [a,b]$ such that $|\Delta^- B(s)| \geq \varepsilon$. Clearly, each function regulated on $[a,b]$ is bounded on $[a,b]$, i.e. $\|B\| < \infty$ for all $B \in \mathbf{G}^{m\times n}[a,b]$.

By $\mathbf{AC}^{m\times n}[a,b]$ we denote the set of all functions $B\colon [a,b] \to \mathbb{R}^{m\times n}$ such that each component b_{ij}, $i = 1,\dots,m$, $j = 1,\dots,n$ of B is *absolutely continuous* on the interval $[a,b]$.

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[1] i.e. in $\mathbb{R}^{m\times n}$

Analogously to the spaces of functions of bounded variation, $\mathbf{AC}^n[a,b] = \mathbf{AC}^{n\times 1}[a,b]$, $\mathbf{G}^n[a,b] = \mathbf{G}^{n\times 1}[a,b]$, $\mathbf{C}^n[a,b] = \mathbf{C}^{n\times 1}[a,b]$ and $\mathbf{AC}[a,b] = \mathbf{AC}^1[a,b]$, $\mathbf{G}[a,b] = \mathbf{G}^1[a,b]$, $\mathbf{C}[a,b] = \mathbf{C}^1[a,b]$. Obviously,

$$\mathbf{AC}^{m\times n}[a,b] \subset \mathbf{BV}^{m\times n}[a,b] \subset \mathbf{G}^{m\times n}[a,b] \quad \text{and} \quad \mathbf{C}^{m\times n}[a,b] \subset \mathbf{G}^{m\times n}[a,b].$$

Finally, a function $f\colon [a,b] \to \mathbb{R}$ is called a *finite step function* on $[a,b]$ if there is a division $\{\alpha_0, \alpha_1, \dots, \alpha_m\} \in \mathcal{D}[a,b]$ of $[a,b]$ such that f is constant on every open interval (α_{j-1}, α_j), $j = 1, 2, \dots, m$. The set of all finite step functions on $[a,b]$ is denoted by $\mathbf{S}[a,b]$, $\mathbf{S}^{m\times n}[a,b]$ is the set of all $m \times n$ matrix valued functions whose arguments are finite step functions and $\mathbf{S}^{n\times 1}[a,b] = \mathbf{S}^n[a,b]$. It is known that the set $\mathbf{S}^{m\times n}[a,b]$ is dense in $\mathbf{G}^{m\times n}[a,b]$ with respect to the supremal norm, i.e.

$$\left\{ \begin{array}{l} \text{for each } \ \varepsilon > 0 \text{ and each } \ B \in \mathbf{G}^{m\times n}[a,b] \\ \text{there is an } \ \widetilde{B} \in \mathbf{S}^{m\times n}[a,b] \ \text{ such that } \ \|B - \widetilde{B}\| < \varepsilon. \end{array} \right. \tag{2.2}$$

We say that a proposition $P(n)$ holds for almost all (briefly a.a.) $n \in \mathbb{N}$ if it is true for all $n \in \mathbb{N} \setminus K$ where K is a finite set.

2.2 Kurzweil-Stieltjes integral

In this subsection we will recall the definition of the Kurzweil-Stieltjes integral (in what follows KS-integral).

Let $-\infty < a < b < +\infty$. For given $m \in \mathbb{N}$, a division $D = \{t_0, t_1, \dots, t_m\} \in \mathcal{D}[a,b]$ and $\xi = (\xi_1, \xi_2, \dots, \xi_m) \in \mathbb{R}^m$, the couple $P = (D, \xi)$ is called a *partition* of $[a,b]$ if

$$t_{j-1} \le \xi_j \le t_j \quad \text{for all } j = 1, 2, \dots, m\,.$$

The set of all partitions of the interval $[a,b]$ is denoted by $\mathcal{P}[a,b]$.

An arbitrary positive valued function $\delta\colon [a,b] \to (0, +\infty)$ is called a *gauge* on $[a,b]$. Given a gauge δ on $[a,b]$, the partition

$$P = (D, \xi) = \big(\{t_0, t_1, \dots, t_m\},\, (\xi_1, \xi_2, \dots, \xi_m)\big) \in \mathcal{P}[a,b]$$

is said to be δ-fine, if

$$[t_{j-1}, t_j] \subset \big(\xi_j - \delta(\xi_j), \xi_j + \delta(\xi_j)\big) \quad \text{for all } j = 1, 2, \dots, m\,.$$

The set of all δ-fine partitions of the interval $[a,b]$ is denoted by $\mathcal{A}(\delta; [a,b])$.

For functions $f,\, g\colon [a,b] \to \mathbb{R}$ and a partition $P \in \mathcal{P}[a,b]$,

$$P = \big(\{t_0, t_1, \dots, t_m\},\, (\xi_1, \xi_2, \dots, \xi_m)\big)$$

we define

$$\Sigma_P(f\, \Delta g) = \sum_{i=1}^{m} f(\xi_i)[g(t_i) - g(t_{i-1})]\,.$$

We say that $I \in \mathbb{R}$ is the *KS-integral* of f with respect to g from a to b if

$$\forall \varepsilon > 0\ \exists \delta\colon [a,b] \to (0,+\infty)\ \forall P \in \mathcal{A}(\delta;[a,b]) :\ |I - \Sigma_P(f\,\Delta g)| < \varepsilon\,.$$

In such a case we write

$$I = \int_a^b f\,\mathrm{d}g \quad \text{or} \quad I = \int_a^b f(t)\mathrm{d}g(t)\,.$$

It is well known that the KS-integral $\int_a^b f\,\mathrm{d}g$ exists provided $f \in \mathbf{BV}[a,b]$ and $g \in \mathbf{BV}[a,b]$. For the properties of the KS-integral with respect to functions of bounded variation, see [42]. The KS-integral with respect to scalar regulated functions is described in [45], [49], [50].

If $F\colon [a,b] \to \mathbb{R}^{m\times n}$, $G\colon [a,b] \to \mathbb{R}^{n\times p}$ and $H\colon [a,b] \to \mathbb{R}^{p\times m}$ are matrix valued functions, then the symbols

$$\int_a^b F\,\mathrm{d}[G] \quad \text{and} \quad \int_a^b \mathrm{d}[H]\,F$$

stand respectively for the matrices

$$\Big(\sum_{j=1}^{n} \int_a^b f_{i,j}\,\mathrm{d}g_{j,k}\Big)_{\substack{i=1,\dots,m\\ k=1,\dots,p}} \quad \text{and} \quad \Big(\sum_{i=1}^{m} \int_a^b f_{k,i}\,\mathrm{d}h_{i,j}\Big)_{\substack{k=1,\dots,p\\ j=1,\dots,n}},$$

whenever all integrals appearing in the sums exist. Since the integral of a matrix valued function with respect to a matrix valued function is a matrix whose elements are sums of KS-integrals of real functions with respect to real functions, it is easy to reformulate all the statements from Section 5 in [50] for matrix valued functions (cf. [42, I.4]).

The extension of the results known for scalar real valued functions to real vector or matrix valued functions is obvious and hence for the basic facts concerning integrals with respect to regulated functions we shall refer to the corresponding assertions from [45] or [49].

The next theorem summarizes the fundamental properties of KS-integration with respect to regulated functions.

2.1 Theorem. ([49, Theorem 2.3.8 and Theorem 2.3.15]) *If f, $g \in \mathbf{G}[a,b]$ and at least one of the functions f, g has a bounded variation on $[a,b]$, then the integral $\int_a^b f\,\mathrm{d}g$ exists. Furthermore,*

$$\Big|\int_a^b f\,\mathrm{d}g\Big| \le 2\,\big(|f(a)| + \mathrm{var}_a^b f\big)\,\|g\| \quad \textit{if}\ \ f \in \mathbf{BV}[a,b] \ \textit{and}\ g \in \mathbf{G}[a,b], \tag{2.3}$$

and

$$\Big|\int_a^b f\,\mathrm{d}g\Big| \le \|f\|\,\mathrm{var}_a^b\, g \quad \textit{if}\ \ f \in \mathbf{G}[a,b] \ \textit{and}\ g \in \mathbf{BV}[a,b]. \tag{2.4}$$

2.2 Remark. The inequality (2.3) is in [49, Theorem 2.3.8] in the form

$$\Big|\int_a^b f\,\mathrm{d}g\Big| \le \big(|f(a)| + |f(b)| + \operatorname{var}_a^b f\big)\,\|g\|\,,$$

from which (2.3) immediately follows using

$$|f(b)| \le |f(a)| + |f(b) - f(a)| \le |f(a)| + \operatorname{var}_a^b f\,.$$

The inequalities (2.3) and (2.4) imply the following simple convergence assertions.

2.3 Corollary. *If $f \in \mathbf{BV}[a,b]$ and $g_k \in \mathbf{G}[a,b]$ for all $k \in \mathbb{N}$ and $g_k \rightrightarrows g$ on $[a,b]$ then*

$$\int_a^t f\,\mathrm{d}g_k \rightrightarrows \int_a^t f\,\mathrm{d}g \quad \text{and} \quad \int_a^t g_k\,\mathrm{d}f \rightrightarrows \int_a^t g\,\mathrm{d}f \quad \text{on } [a,b].$$

The next natural assertion is very useful for consideration of GLODE was proved in our paper [11]. It provides more sofisticated convergence criteria which will be useful later.

2.4 Theorem. *Let $x,\ x_k \in \mathbf{G}^n[a,b]$, $A,\ A_k \in \mathbf{BV}^{n\times n}[a,b]$ for $k \in \mathbb{N}$. Furthermore, let*

$$x_k \rightrightarrows x \quad \text{on } [a,b], \tag{2.5}$$

$$\alpha^* := \sup\{\operatorname{var}_a^b A_k\colon\ k \in \mathbb{N}\} < +\infty, \tag{2.6}$$

and

$$A_k \rightrightarrows A \quad \text{on } [a,b]. \tag{2.7}$$

Then

$$\int_a^t \mathrm{d}[A_k]\,x_k \rightrightarrows \int_a^t \mathrm{d}[A]\,x \quad \text{on } [a,b]. \tag{2.8}$$

Proof. Let $\varepsilon > 0$ be given. By (2.2) and (2.5), we can find $u \in \mathbf{S}^n[a,b]$ and $k_0 \in \mathbb{N}$ such that

$$\|x - u\| < \varepsilon, \quad \|x_k - u\| < \varepsilon \quad \text{and} \quad \|A_k - A\| < \varepsilon \quad \text{for } \ k \ge k_0.$$

Furthermore, since $\operatorname{var}_a^b u < \infty$, using (2.3) we can see that for $t \in [a,b]$ and $k \ge k_0$ the relations

$$\Big|\int_a^t \mathrm{d}[A_k]\,x_k - \int_a^t \mathrm{d}[A]\,x\Big| = \Big|\int_a^t \mathrm{d}[A_k]\,(x_k - u) + \int_a^t \mathrm{d}[A_k - A]\,u + \int_a^t \mathrm{d}[A]\,(u - x)\Big|$$
$$\le \alpha^*\varepsilon + 2\,(\operatorname{var}_a^b u)\,\varepsilon + \alpha^*\varepsilon = 2\,(\alpha^* + \operatorname{var}_a^b u)\,\varepsilon$$

hold, wherefrom our assertion immediately follows. □

2.5 Corollary. *The assertion of Theorem* 2.4 *remains unchanged, if we replace condition* (2.7) *by*

$$\big(A_k - A_k(a)\big) \rightrightarrows \big(A - A(a)\big) \quad \text{on } [a,b]\,.$$

Proof. By Theorem 2.4, we get

$$\int_a^t \mathrm{d}[A_k]\,x_k = \int_a^t \mathrm{d}\big[A_k - A_k(a)\big]\,x_k \rightrightarrows \int_a^t \mathrm{d}\big[A - A(a)\big]\,x = \int_a^t \mathrm{d}[A]\,x \quad \text{on } [a,b].$$

□

2.6 Remark. Corollary 2.5 is slightly generalized version of result due to Ashordia [1, Lemma 1].

2.3 Generalized linear ordinary differential equations

Here we describe some fundamental properties of generalized linear differential equations. Throughout the whole text we work on a bounded interval $[a,b]$.

2.3.1 Definition and basic properties

Assume that

$$A \in \mathbf{BV}^{n\times n}[a,b],\ \ f \in \mathbf{G}^n[a,b] \tag{2.9}$$

and consider the equation

$$x(t) = x(s) + \int_s^t \mathrm{d}[A]\,x + f(t) - f(s)\,. \tag{2.10}$$

Let $[c,d] \subset [a,b]$. We say that a function $x\colon [c,d] \to \mathbb{R}^n$ is a solution of (2.10) on $[c,d]$ if

$$\int_c^d \mathrm{d}[A]\,x \in \mathbb{R}^n$$

and (2.10) holds for all $t,s \in [c,d]$.

Moreover, if $t_0 \in [c,d]$ and $\widetilde{x}_0 \in \mathbb{R}^n$ are given, we say that $x\colon [c,d] \to \mathbb{R}^n$ is a solution of the initial value problem (2.10), $x(t_0) = \widetilde{x}_0$ on $[c,d]$ if it is a solution of (2.10) on $[c,d]$ and $x(t_0) = \widetilde{x}_0$, i.e. if

$$x(t) = \widetilde{x}_0 + \int_{t_0}^t \mathrm{d}[A]\,x + f(t) - f(t_0) \tag{2.11}$$

for all $t \in [c,d]$.

Notice that under assumption (2.9) each solution of equation (2.10) on $[a,b]$ is regulated on $[a,b]$ (see Theorem 2.10). Equation (2.10) is usually called a *generalized linear differential equation.* Such equations with solutions having values in the space $\mathbb{R}^n$ of real n-vectors have been thoroughly investigated e.g. in the monographs [42] or [39].

2.7 Theorem. ([42, III.1.4]) *Let $A \in \mathbf{BV}^{n\times n}[a,b]$. If $t_0 \in [a,b]$, then the initial value problem (2.11) possesses for any $f \in \mathbf{G}^n[a,b]$, $\widetilde{x}_0 \in \mathbb{R}^n$ a unique solution $x(t)$ defined on $[a,b]$ if and only if*

$$\det\big[\mathrm{I} - \Delta^- A(t)\big] \neq 0 \quad \text{on } (t_0, b] \quad \text{and} \quad \det\big[\mathrm{I} + \Delta^+ A(t)\big] \neq 0 \quad \text{on } [a, t_0).$$

If $t_0 = a$, then the initial value problem (2.11) possesses for any $f \in \mathbf{G}^n[a,b]$, $\widetilde{x}_0 \in \mathbb{R}^n$ a unique solution $x(t)$ defined on $[a,b]$ if and only if

$$\det\big[\mathrm{I} - \Delta^- A(t)\big] \neq 0 \quad \text{for each } t \in (a,b]. \tag{2.12}$$

2.3.2 Fundamental matrix

2.8 Lemma. ([42, III.2.10, III.2.11]) *For a given $A \in \mathbf{BV}^{n\times n}[a,b]$ such that (2.12) and*

$$\det\big[\mathrm{I} + \Delta^+ A(t)\big] \neq 0 \quad \text{on } [a, b) \tag{2.13}$$

there exists a unique $U\colon [a,b] \times [a,b] \to \mathbb{R}^{n\times n}$ such that

$$U(t,s) = \mathrm{I} + \int_s^t \mathrm{d}[A(r)]\, U(r,s) \tag{2.14}$$

for all $t, s \in [a,b]$.

Moreover, if we denote $X(t) = U(t,a)$ for $t \in [a,b]$, we get $\det X(t) \neq 0$ for $t \in [a,b]$,

$$U(t,s) = X(t)\, X^{-1}(s) \quad \text{for all } s,t \in [a,b] \tag{2.15}$$

and

$$X(t) = \mathrm{I} + \int_a^t \mathrm{d}[A]\, X\,, \quad t \in [a,b]. \tag{2.16}$$

Furthermore, the inverse matrix $X^{-1}(t)$ is of bounded variation on $[a,b]$ and it satisfies the relation

$$X^{-1}(t) = X^{-1}(s) - X^{-1}(t)\, A(t) + X^{-1}(s)\, A(s) + \int_s^t \mathrm{d}[X^{-1}]\, A \tag{2.17}$$

for $t, s \in [a,b]$.

By Theorem 2.7, for a given $t_0 \in [a,b]$, the unique solution $x(t)$ of

$$x(t) = \widetilde{x}_0 + \int_{t_0}^t \mathrm{d}[A]\, x \tag{2.18}$$

on $[t_0, 1]$ is given by

$$x(t) = X(t)\, X^{-1}(t_0)\, \widetilde{x}_0\,. \tag{2.19}$$

2.9 Definition. The matrix valued function $X\colon [a,b] \to \mathbb{R}^{n\times n}$ given by Lemma 2.8 is called *the fundamental matrix* of the homogeneous generalized linear differential equation

$$x(t) = x(s) + \int_s^t \mathrm{d}[A]\, x\,, \qquad t,s \in [a,b] \tag{2.20}$$

on the interval $[a,b]$ or, briefly, the fundamental matrix corresponding to the given matrix function A.

2.3.3 Variation-of-constants formula

For our purposes the property (2.12) is crucial. Its importance is well illustrated by the next assertion which is a fundamental existence result for the equation (1.1).

2.10 Theorem. *Let $A \in \mathbf{BV}^{n\times n}[a,b]$ satisfy* (2.12)*. Then, for each $\widetilde{x}_0 \in \mathbb{R}^n$ and each $f \in \mathbf{G}^n[a,b]$, the initial value problem* (1.1) *has a unique solution x on $[a,b]$ and $x \in \mathbf{G}^n[a,b]$. Moreover, $x - f \in \mathbf{BV}^n[a,b]$.*

Proof follows from [46, Proposition 2.5].

2.11 Lemma. ([42, III.1.6]) *Assume that $A \in \mathbf{BV}^{n\times n}[a,b]$, $f \in \mathbf{G}^n[a,b]$. Let $x(t)$ be a solution of the equation* (2.10) *on $[a,b]$. Then the one-sided limits $x(a+)$, $x(t+)$, $x(t-)$, $x(b-)$, $t \in (a,b)$ exist and the relations*

$$\begin{cases} x(t+) = [\mathrm{I} + \Delta^+ A(t)]\, x(t) + \Delta^+ f(t) & \text{for all } t \in [a,b)\,, \\ x(t-) = [\mathrm{I} - \Delta^- A(t)]\, x(t) - \Delta^- f(t) & \text{for all } t \in (a,b] \end{cases} \tag{2.21}$$

hold.

2.12 Theorem. ([42, III.2.13]) *(variation-of-constants formula) Assume that $A \in \mathbf{BV}^{n\times n}[a,b]$ satisfy* (2.12) *and* (2.13)*. Then for any $t_0 \in [a,b]$, $\tilde{x} \in \mathbb{R}^n$, $f \in \mathbf{G}^n[a,b]$ there is an unique solution of the nonhomogenous initial value problem*

$$x(t) = \tilde{x} + \int_{t_0}^t \mathrm{d}[A]\, x + f(t) - f(t_0)$$

which can be expressed in the form

$$x(t) = X(t)X^{-1}(t_0)\tilde{x} + f(t) - f(t_0) - X(t)\int_{t_0}^t \mathrm{d}[X^{-1}(r)]\,\big(f(r) - f(t_0)\big)$$

for $t \in [a,b]$, where $X\colon [a,b] \to \mathbb{R}^{n\times n}$ is the fundamental matrix corresponding to A.

2.3.4 Further properties

2.13 Lemma. ([42, I.4.30]) *(generalized Gronwall's inequality) Assume that $h\colon [a,b] \to \mathbb{R}$ is a nonnegative nondecreasing function, $\varphi\colon [a,b] \to \mathbb{R}$ nonnegative and bounded, i.e. there exist $K \in \mathbb{R}$ such that $\varphi(t) \le K$ for all $t \in [a,b]$.*

a) If h is left-continuous on $(a,b]$ and if there exist nonnegative constants K_1, K_2 such that

$$\varphi(t) \le K_1 + K_2 \int_a^t \varphi \,\mathrm{d}h \qquad \text{for all } t \in [a,b],$$

then

$$\varphi(t) \le K_1 \exp\big[K_2\big(h(t) - h(a)\big)\big] \quad \text{for any } t \in [a,b].$$

b) If h is right-continuous on $[a,b)$ and if there exist nonnegative constants K_1, K_2 such that

$$\varphi(t) \le K_1 + K_2 \int_t^b \varphi \,\mathrm{d}h \qquad \text{for all } t \in [a,b],$$

then

$$\varphi(t) \le K_1 \exp\big[K_2\big(h(b) - h(t)\big)\big] \quad \text{for any } t \in [a,b].$$

Analogously to [42, Theorem III.1.7] where $f \in \mathbf{BV}^n[a,b]$, we have

2.14 Lemma. *Let $A \in \mathbf{BV}^{n\times n}[a,b]$ satisfy* (2.12). *Then*

$$c_A := \sup\{\big|[\mathrm{I} - \Delta^- A(t)]^{-1}\big|\colon\ t \in [a,b]\} < \infty \tag{2.22}$$

and

$$|x(t)| \le c_A\,(|\widetilde{x}_0| + 2\,\|f\|)\,\exp(c_A \operatorname{var}_a^t A) \quad \text{for } t \in [a,b] \tag{2.23}$$

holds for each $\widetilde{x}_0 \in \mathbb{R}^n$, $f \in \mathbf{G}^n[a,b]$ and each solution x of (1.1) *on $[a,b]$.*

Proof. First, notice that for $t \in [a,b]$ such that $|\Delta^- A(t)| < \frac{1}{2}$ we have

$$\big|[\mathrm{I} - \Delta^- A(t)]^{-1}\big| = \left|\sum_{k=1}^{\infty} (\Delta^- A(t))^k\right| \le \sum_{k=1}^{\infty} |\Delta^- A(t)|^k = \frac{1}{1 - |\Delta^- A(t)|} < 2.$$

Therefore, (2.22) follows easily from the fact that the set

$$\{t \in [a,b]\colon\ |\Delta^- A(t)| \ge \frac{1}{2}\}$$

has at most finitely many elements.

Now, let x be a solution of (1.1). Put $B(a) = A(a)$ and $B(t) = A(t-)$ for $t \in (a,b]$. Then, as in the proof of [42, Theorem III.1.7], we get $A - B \in \mathbf{BV}^{n\times n}[a,b]$,

$$A(t) - B(t) = \Delta^- A(t) \quad \text{and} \quad \int_a^t \mathrm{d}[A-B]\,x = \Delta^- A(t) \quad \text{for } t \in [a,b].$$

Consequently

$$x(t) = [\mathrm{I} - \Delta^- A(t)]^{-1} \left(\widetilde{x}_0 + f(t) - f(a) + \int_a^t \mathrm{d}[B]\,x \right)$$

and

$$|x(t)| \le K_1 + K_2 \int_a^t |x| \mathrm{d}h \quad \text{for } t \in [a,b],$$

where

$$K_1 = c_A \left(|\widetilde{x}_0| + 2\,\|f\| \right), \quad K_2 = c_A \quad \text{and} \quad h(t) = \operatorname{var}_a^t B \quad \text{for } t \in [a,b].$$

The function h is nondecreasing and, since B is left-continuous on $(a,b]$, h is also left-continuous on $(a,b]$. Therefore we can use Lemma 2.13 (the generalized Gronwall inequality) to get the estimate (2.23). $\square$

2.15 Corollary. *Let $A \in \mathbf{BV}^{n\times n}[a,b]$ satisfy* (2.12). *Then for each $\widetilde{x}_0 \in \mathbb{R}^n$, $f \in \mathbf{G}^n[a,b]$ and each solution x of* (1.1) *on $[a,b]$, the estimate*

$$\operatorname{var}_a^b (x - f) \le c_A \left(\operatorname{var}_a^b A\right) \left(|\widetilde{x}_0| + 2\,\|f\|\right) \exp(c_A \operatorname{var}_a^b A).$$

is true, where c_A is defined by (2.22).

Proof. By (2.23), we have

$$\|x\| \le c_A \left(|\widetilde{x}_0| + 2\,\|f\|\right) \exp(c_A \operatorname{var}_a^b A).$$

Therefore

$$\begin{aligned} \operatorname{var}_a^b (x - f) &\le \left(\operatorname{var}_a^b A\right) \|x\| \\ &\le c_A \left(\operatorname{var}_a^b A\right) \left(|\widetilde{x}_0| + 2\,\|f\|\right) \exp(c_A \operatorname{var}_a^b A). \end{aligned}$$

$\square$

2.16 Lemma. *Let $A \in \mathbf{BV}^{n\times n}[a,b]$ satisfy* (2.12) *and let c_A be defined by* (2.22). *Then*

$$c_A = \Big(\inf \big\{ \big| \big[\mathrm{I} - \Delta^- A(t)\big]\; x \big| : \; t \in [a,b],\; x \in \mathbb{R}^n,\; |x| = 1 \big\} \Big)^{-1}. \tag{2.24}$$

Proof. We have

$$\begin{aligned}
c_A &= \sup\left\{|[\mathrm{I}-\Delta^- A(t)]^{-1}| \colon\, t\in[a,b]\right\}\\
&= \sup\left\{\frac{|[\mathrm{I}-\Delta^- A(t)]^{-1}|\,|[\mathrm{I}-\Delta^- A(t)]\,x|}{|[\mathrm{I}-\Delta^- A(t)]\,x|} \colon\, t\in[a,b],\ x\in\mathbb{R}^n,\ |x|=1\right\}\\
&\geq \sup\left\{\frac{|x|}{|[\mathrm{I}-\Delta^- A(t)]\,x|} \colon\, t\in[a,b],\ x\in\mathbb{R}^n,\ |x|=1\right\}\\
&= \sup\left\{\frac{1}{|[\mathrm{I}-\Delta^- A(t)]\,x|} \colon\, t\in[a,b],\ x\in\mathbb{R}^n,\ |x|=1\right\}\\
&= \left(\inf\left\{|[\mathrm{I}-\Delta^- A(t)]\,x| \colon\, t\in[a,b],\ x\in\mathbb{R}^n,\ |x|=1\right\}\right)^{-1}.
\end{aligned}$$

Thus, it remains to prove that the inequality

$$c_A \leq \left(\inf\left\{\left|\left[\mathrm{I}-\Delta^- A(t)\right]\ x\right| \colon\, t\in[a,b],\ x\in\mathbb{R}^n,\ |x|=1\right\}\right)^{-1} \tag{2.25}$$

is true, as well. To this aim, first let us notice that for each $t\in[a,b]$ there is a $z\in\mathbb{R}^n$ such that $|z|=1$ and

$$\left|[\mathrm{I}-\Delta^- A(t)]^{-1}\right| = \left|[\mathrm{I}-\Delta^- A(t)]^{-1}\,z\right|. \tag{2.26}$$

Indeed, let $t\in[a,b]$ and let $B=[\mathrm{I}-\Delta^- A(t)]^{-1}$. Let $i_0\in\{1,2,\dots,n\}$ be such that $|B|=\sum_{j=1}^n |b_{i_0,j}|$ and let

$$z=\begin{pmatrix}\operatorname{sgn}(b_{i_0,1})\\ \operatorname{sgn}(b_{i_0,2})\\ .\\ .\\ \operatorname{sgn}(b_{i_0,n})\end{pmatrix}.$$

Then $|z|=1$. Furthermore,

$$\begin{aligned}
|B\,z| &= \max_{i=1,2,\dots,n}\sum_{j=1}^n |b_{i,j}\,z_j| = \max_{i=1,2,\dots,n}\sum_{j=1}^n |b_{i,j}\operatorname{sgn}(b_{i_0,j})|\\
&\leq \max_{i=1,2,\dots,n}\sum_{j=1}^n |b_{i,j}| = |B|.
\end{aligned}$$

On the other hand, we have

$$|B| = \sum_{j=1}^n |b_{i_0,j}| = \left|\sum_{j=1}^n \operatorname{sgn}(b_{i_0,j})\,b_{i_0,j}\right| \leq |B\,z|.$$

Therefore, we can conclude that (2.26) is true.

Now, due to (2.12), there is $w \in \mathbb{R}^n$ such that $z = [\mathrm{I} - \Delta^- A(t)]\, w$. Inserting this instead of z into (2.26), we get

$$\begin{aligned}\big\|[\mathrm{I} - \Delta^- A(t)]^{-1}\big\| &= \frac{|[\mathrm{I} - \Delta^- A(t)]^{-1}\, [\mathrm{I} - \Delta^- A(t)]\, w|}{|[\mathrm{I} - \Delta^- A(t)]\, w|} \\ &= \frac{|w|}{|[\mathrm{I} - \Delta^- A(t)]\, w|} = \frac{1}{\Big|[\mathrm{I} - \Delta^- A(t)]\, \left(\frac{w}{|w|}\right)\Big|} \\ &\le \sup\left\{\frac{1}{|[\mathrm{I} - \Delta^- A(t)]\, x|} : \, x \in \mathbb{R}^n,\, |x| = 1\right\}.\end{aligned}$$

It follows that

$$\begin{aligned}c_A &\le \sup\left\{\frac{1}{|[\mathrm{I} - \Delta^- A(t)]\, x|} : \, t \in [a,b],\, x \in \mathbb{R}^n,\, |x| = 1\right\} \\ &= \big(\inf\{|[\mathrm{I} - \Delta^- A(t)]\, x| : \, t \in [a,b],\, x \in \mathbb{R}^n,\, |x| = 1\big)^{-1},\end{aligned}$$

i.e. (2.25) is true. This completes the proof. □

2.4 Continuous dependence of solutions on a parameter

Together with problem (1.1) let us consider the sequence of initial value problems

$$x(t) = \widetilde{x}_k + \int_a^t \mathrm{d}[A_k]\, x + f_k(t) - f_k(a)\,, \quad t \in [a,b]\,, \tag{2.27}$$

where $k \in \mathbb{N}$. We are interested in finding conditions ensuring the convergence of solutions of (2.27) to a solution of (1.1). First, let us recall the result of Ashordia concerning the case of uniform convergence of $A_k(t)$ to $A(t)$ on a compact interval. This lemma was stated in Theorem 1 in [1]. We present it in the form which we will need for the proof of our main convergence result (see Theorem 3.1).

2.17 Lemma. ([1, Theorem 1]) *Let $A,\, A_k \in \mathbf{BV}^{n\times n}[a,b]$ for $k \in \mathbb{N}$ and let (2.12) hold. Assume that the sequence $\{A_k\}_{k=1}^{\infty}$ satisfies (2.6) and*

$$\big(A_k - A_k(a)\big) \rightrightarrows \big(A - A(a)\big) \quad \text{on } [a,b]. \tag{2.28}$$

Then there exists a fundamental matrix X_k corresponding to A_k for a.a. $k \in \mathbb{N}$ and X_0 corresponding to A on $[a,b]$ and $X_k \rightrightarrows X_0$ on $[a,b]$.

Moreover, let (2.13), $\widetilde{x}_k \to \widetilde{x}_0$, f, $f_k \in \mathbf{BV}^n[a,b]$ for $k \in \mathbb{N}$ and let the sequence $\{f_k\}_{k=1}^{\infty}$ satisfy the condition

$$\big(f_k - f_k(a)\big) \rightrightarrows \big(f - f(a)\big) \quad \text{on } [a,b].$$

Then there exists a unique solution x_k of (2.27) for a.a. $k \in \mathbb{N}$ and there exists a unique solution x_0 of (1.1) on $[a,b]$ and

$$x_k \rightrightarrows x_0 \quad \text{on } [a,b].$$

Sketch of proof. STEP 1. We can show that

$$\det[\mathrm{I}+\Delta^+ A_k(t)] \neq 0 \quad \text{and} \quad \det[\mathrm{I}-\Delta^- A_k(t)] \neq 0 \quad \text{on } [a,b].$$

By (2.28),

$$\Delta^+ A_k \rightrightarrows \Delta^+ A \quad \text{and} \quad \Delta^- A_k \rightrightarrows \Delta^- A \quad \text{on } [a,b].$$

Since $\operatorname{var}_a^b A < +\infty$, the series $\sum_{t\in[a,b]} |\Delta^+ A(t)|$ and $\sum_{t\in[a,b]} |\Delta^- A(t)|$ converge. Thus

$$|\Delta^+ A(t)| < \frac{1}{2} \quad \text{and} \quad |\Delta^- A(t)| < \frac{1}{2} \quad \text{for a.a. } k \in \mathbb{N}.$$

STEP 2. According to the first part of the proof, the fundamental matrices X_0, X_k, $k \in \mathbb{N}$ are defined on $[a,b]$. By Gronwall's inequality applied on $|X_k(t)-X_0(t)|$ we obtain that $X_k \rightrightarrows X_0$ on $[a,b]$.

STEP 3. Using the variation-of-constants formula (Theorem 2.12) and Corollary 2.5 we get $x_k \rightrightarrows x_0$ on $[a,b]$. □

The next fundamental result on the continuous dependence of solutions of generalized linear differential equations on a parameter generalizes the result due to M. Ashordia [1, Theorem 1]. Unlike [1] and [2], we do not utilize the variation-of-constants formula and therefore we need not assume that, in addition to (2.12), also the condition (2.13) is satisfied. Furthermore, both the nonhomogeneous part of the equation and the solution may be only regulated functions, not necessarily of bounded variation.

2.18 Theorem. *Let* A, $A_k \in \mathbf{BV}^{n\times n}[a,b]$, f, $f_k \in \mathbf{G}^n[a,b]$, $\widetilde{x}_0$, $\widetilde{x}_k \in \mathbb{R}^n$ *for* $k \in \mathbb{N}$. *Assume* (2.6), (2.7), (2.12),

$$f_k \rightrightarrows f \quad \textit{on } [a,b] \tag{2.29}$$

and

$$\lim_{k\to\infty} \widetilde{x}_k = \widetilde{x}_0. \tag{2.30}$$

Then the equation (1.1) *has a unique solution* x *on* $[a,b]$. *Furthermore, for each* $k \in \mathbb{N}$ *sufficiently large there exists a unique solution* x_k *on* $[a,b]$ *to the equation* (2.27) *and*

$$x_k \rightrightarrows x \quad \textit{on } [a,b]\,. \tag{2.31}$$

Proof. STEP 1. As in the first part of the proof of Lemma 2.17, we can show that there is a $k_1 \in \mathbb{N}$ such that

$$\det[\mathrm{I}-\Delta^- A_k(t)] \neq 0 \quad \text{on } [a,b]$$

holds for all $k \geq k_1$. In particular, (2.27) has a unique solution x_k for $k \geq k_1$.

Step 2. For $k \ge k_1$, put

$$c_{A_k} := \sup\{\big|[\mathrm{I} - \Delta^- A_k(t)]^{-1}\big| \colon t \in (a,b]\} < \infty$$

Then, by Lemma 2.16, we have

$$\begin{aligned}(c_{A_k})^{-1} &= \inf\left\{\big|\big[\mathrm{I} - \Delta^- A_k(t)\big]\ x\big| \colon t \in [a,b],\ x \in \mathbb{R}^n,\ |x| = 1\right\}\\ &\ge \inf\left\{\big|\big[\mathrm{I} - \Delta^- A(t)\big]\ x\big| \colon t \in [a,b],\ x \in \mathbb{R}^n,\ |x| = 1\right\}\\ &\quad - \sup\left\{\big|\big[\Delta^- (A_k(t) - A(t))\big]\ x\big| \colon t \in [a,b],\ x \in \mathbb{R}^n,\ |x| = 1\right\}.\end{aligned}$$

Since, due to the assumption (2.7),

$$\Delta^- A_k \rightrightarrows \Delta^- A \quad \text{on } [a,b]\,,$$

we conclude that there is a $k_0 \ge k_1$ such that

$$(c_{A_k})^{-1} \ge (c_A)^{-1} - (2\,c_A)^{-1} = (2\,c_A)^{-1} \quad \text{for } k \ge k_0.$$

To summarize,

$$c_{A_k} \le 2\,c_A < \infty \quad \text{for } k \ge k_0. \tag{2.32}$$

Step 3. Set $w_k = (x_k - f_k) - (x - f)$. Then, for $k \ge k_0$,

$$w_k(t) = \widetilde{w}_k + \int_a^t \mathrm{d}[A_k]\, w_k + h_k(t) - h_k(a) \quad \text{on } [a,b],$$

where

$$h_k(t) = \int_a^t \mathrm{d}[A_k - A]\,(x - f) + \left(\int_a^t \mathrm{d}[A_k]\, f_k - \int_a^t \mathrm{d}[A]\, f\right) \quad \text{for } t \in [a,b]$$

and

$$\widetilde{w}_k = (\widetilde{x}_k - f_k(a)) - (\widetilde{x}_0 - f(a))\,.$$

By (2.29) and (2.30) we can see that

$$\lim_{k\to\infty} \widetilde{w}_k = 0. \tag{2.33}$$

Furthermore, since $x - f \in \mathbf{BV}^n[a,b]$ and $A_k \rightrightarrows A$ on $[a,b]$, by Theorem 2.1 we have

$$\lim_{k\to\infty} \left\|\int_a^t \mathrm{d}[A_k - A]\,(x - f)\right\| = 0$$

and, by Lemma 2.4,

$$\lim_{k\to\infty} \int_a^t \mathrm{d}[A_k]\, f_k = \int_a^t \mathrm{d}[A]\, f.$$

To summarize,

$$\lim_{k\to\infty} \|h_k\| = 0. \tag{2.34}$$

On the other hand, applying Theorem 2.14 and taking into account the relation (2.32), we get

$$\|w_k\| \le 2\, c_A\, (|\widetilde{w}_k| + 2\, \|h_k\|)\, \exp(2\, c_A\, \alpha^*) \text{ for } t \in [a,b] \quad \text{and} \quad k \ge k_0,$$

wherefrom, by virtue of (2.33) and (2.34), the relation

$$\lim_{k\to\infty} \|w_k\| = 0$$

follows. Finally, having in mind the assumptions (2.29) and (2.30), we conclude that the relation

$$x_k \rightrightarrows x \quad \text{on } [a,b]$$

is true, as well. This completes the proof. □

It is easy to see that the generalized differential equation (1.1) is equivalent with the equation

$$x(t) = \widetilde{x}_0 + \int_a^t \mathrm{d}[B]\, x + g(t) - g(a)$$

whenever $B - A$ and $g - f$ are constant on $[a,b]$. Therefore Theorem 2.18 can be also reformulated as follows.

2.19 Corollary. *The assertion of Theorem* 2.18 *remains unchanged, if we replace assumptions* (2.7) *and* (2.29) *by*

$$\big(A_k - A_k(a)\big) \rightrightarrows \big(A - A(a)\big) \quad \textit{on } [a,b]\,,$$

and

$$\big(f_k - f_k(a)\big) \rightrightarrows \big(f - f(a)\big) \quad \textit{on } [a,b]\,.$$

Below we will formulate a result concerning the case when the assumption (2.28) is not satisfied. To this aim, let us introduce the following notation.

2.20 Notation. For $F\colon [a,b] \to \mathbb{R}^{m\times n}$ and $J = [\alpha,\beta] \subset [a,b]$, we define

$$F^J(t) = F(t) - F(\alpha) \quad \text{for } t \in J\,.$$

2.21 Theorem. ([49, Theorem 3.3.2]) *Let* $A,\ A_k \in \mathbf{BV}^{n\times n}[a,b]$ *for* $k \in \mathbb{N}$, (2.6) *and* (2.12) *hold. Furthermore, assume that there is a finite set* $D \subset [a,b]$ *such that*

$$A_k^J \rightrightarrows A^J \textit{ on any closed interval } J \subset [a,b] \setminus D\,, \tag{2.35}$$

$$\det[\mathrm{I} - \Delta^- A_k(t)] \ne 0 \textit{ for all } t \in D \textit{ and for a.a. } k \in \mathbb{N}, \tag{2.36}$$

and

$$
\begin{cases}
\text{if } \tau \in D, \text{ then } \forall\, \xi \in \mathbb{R}^n \text{ and } \forall\, \varepsilon > 0\ \exists\, \delta > 0 \text{ such that} \\
\forall\, \delta' \in (0,\delta)\ \exists\, k_0 \in \mathbb{N} \text{ such that the relations} \\
\qquad |u_k(\tau) - u_k(\tau-\delta') - \Delta^- A(\tau)[\mathrm{I} - \Delta^- A(\tau)]^{-1}\xi| < \varepsilon\,, \\
\qquad |v_k(\tau+\delta') - v_k(\tau) - \Delta^+ A(\tau)\xi| < \varepsilon \\
\text{are satisfied } \forall\, k \geq k_0 \text{ and } \forall\, u_k,\ v_k \text{ such that} \\
|\xi - u_k(\tau-\delta')| \leq \delta,\ |\xi - v_k(\tau)| \leq \delta \text{ and} \\
\qquad u_k(t) = u_k(\tau-\delta') + \displaystyle\int_{\tau-\delta'}^{t} \mathrm{d}[A_k]\, u_k(s) \quad \text{on } [\tau-\delta',\tau]\,, \\
\qquad v_k(t) = v_k(\tau) + \displaystyle\int_{\tau}^{t} \mathrm{d}[A_k]\, v_k(s) \quad \text{on } [\tau,\tau+\delta']\,.
\end{cases}
\tag{2.37}
$$

Then for a.a. $k \in \mathbb{N}$ *the fundamental matrix* X_k *corresponding to* A_k *is defined on* $[a,b]$ *and*

$$\lim_{k\to\infty} X_k(t) = X_0(t) \quad \text{on } [a,b], \tag{2.38}$$

where X_0 *is the fundamental matrix corresponding to* A*.*

2.22 Remark. Theorem 2.21 is a slightly modified version of [49, Theorem 3.3.2]. Notation is simplified and, in particular, from the proof given in [49, Theorem 3.3.2] it follows that the assumption $\det[\mathrm{I} - \Delta^- A_k(t)] \neq 0$ on $(a,b]$ for all $k \in \mathbb{N}$ used in [49] is not necessary and can be replaced by a weaker one, i.e. $\det[\mathrm{I} - \Delta^- A_k(t)] \neq 0$ for all $t \in D$ and for a.a. $k \in \mathbb{N}$.

Conditions (2.35) – (2.37) characterize the concept of emphatic convergence introduced by J. Kurzweil (c.f. [18, Definition 4.1]). For more details see [49, Definition 3.2.8] or [39].

In the proof of Theorem 4.2 the following two lemmas are needed. The former one is from [1, Lemma 2]. The latter one is based on [49, Theorem 3.2.5] and on [1, Lemma 2].

2.23 Lemma. ([1, Lemma 2]) *Assume that* $A,\ A_k \in \mathbf{BV}^{n\times n}[a,b]$ *for* $k \in \mathbb{N}$ *and* (2.12), (2.13) *hold. Let* X_k *be fundamental matrix corresponding to* A_k*,* $k \in \mathbb{N}$*, let* X_0 *be fundamental matrix corresponding to* A *and let* $X_k \rightrightarrows X_0$ *on* $[a,b]$*. Then*

$$X_k^{-1} \rightrightarrows X_0^{-1} \quad \text{on } [a,b].$$

2.24 Lemma. *Let* $A,\ A_k \in \mathbf{BV}^{n\times n}[a,b]$ *for* $k \in \mathbb{N}$ *and let* (2.6), (2.12), (2.13) *and* (2.28) *hold.*

Then there exists the fundamental matrix X_k corresponding to A_k for a.a. $k \in \mathbb{N}$ and X_0 corresponding to A on $[a,b]$ and

$$X_k^{-1} \rightrightarrows X_0^{-1} \quad \text{on } [a,b]\,.$$

3 Nonhomogeneous equations

This section deals with the problem of continuous dependence on a parameter of solutions to nonhomogeneous GLODEs.

Together with problem (1.1) let us consider the sequence

$$x(t) = \widetilde{x}_k + \int_a^t \mathrm{d}[A_k]\, x + f_k(t) - f_k(a)\,, \quad t \in [a,b], \tag{3.1}$$

where $k \in \mathbb{N}$. Throughout the section we assume:

$$A,\ A_k \in \mathbf{BV}^{n\times n}[a,b],\ \ f,\ f_k \in \mathbf{G}^n[a,b] \ \text{ and } \ \widetilde{x}_0,\ \widetilde{x}_k \in \mathbb{R}^n \ \text{ for } \ k \in \mathbb{N}. \tag{3.2}$$

Our main result is the following assertion which extends Theorem 2.21 and provides conditions ensuring the continuous dependence of solutions of (3.1) on a parameter k. In comparison with Theorem 2.21, condition (2.37) has to be somewhat modified. We keep the notation introduced in Notation 2.20.

3.1 Theorem. *Assume* (3.2), (2.6), (2.12) *and* (2.36). *Let $\widetilde{x}_k \to \widetilde{x}_0$ and let $x_0(t)$ denote a solution of* (1.1). *Moreover, assume that*

$$\begin{cases} \text{there is a finite set } D \subset [a,b] \text{ such that } A_k^J \rightrightarrows A^J \text{ and} \\ f_k^J \rightrightarrows f^J \ \text{ on any closed interval } J \subset [a,b] \setminus D \end{cases} \tag{3.3}$$

and

$$\begin{cases} \text{if } \tau \in D, \text{ then } \forall\, \xi \in \mathbb{R}^n \text{ and } \forall\, \varepsilon > 0\ \exists\, \delta > 0 \text{ such that} \\ \forall\, \delta' \in (0,\delta)\ \exists\, k_0 \in \mathbb{N} \text{ such that the relations} \\ |u_k(\tau) - u_k(\tau-\delta') - \Delta^- A(\tau)[\mathrm{I} - \Delta^- A(\tau)]^{-1}\xi \\ \qquad -[\mathrm{I} - \Delta^- A(\tau)]^{-1}\Delta^- f(\tau)| < \varepsilon\,, \\ |v_k(\tau+\delta') - v_k(\tau) - \Delta^+ A(\tau)\xi - \Delta^+ f(\tau)| < \varepsilon \\ \text{are satisfied } \forall\, k \ge k_0 \text{ and } \forall\, u_k,\ v_k \text{ fulfilling } (3.5),\ (3.6) \\ \text{and such that } |\xi - u_k(\tau-\delta')| \le \delta,\ |\xi - v_k(\tau)| \le \delta\,, \end{cases} \tag{3.4}$$

where

$$u_k(t) = u_k(\tau-\delta') + \int_{\tau-\delta'}^{t} \mathrm{d}[A_k]\, u_k + f_k(t) - f_k(\tau-\delta') \quad \text{on } [\tau-\delta',\tau]\,, \tag{3.5}$$

$$v_k(t) = v_k(\tau) + \int_\tau^t \mathrm{d}[A_k]\, v_k + f_k(t) - f_k(\tau) \quad \textit{on } [\tau, \tau + \delta'] \,. \tag{3.6}$$

Then for a.a. $k \in \mathbb{N}$ *the solution* x_k *of* (3.1) *exists on* $[a, b]$ *and*

$$\lim_{k\to\infty} x_k(t) = x_0(t) \tag{3.7}$$

for any $t \in [a, b]$, *where* $x_0(t)$ *is the solution of* (1.1). *Moreover,* (3.7) *holds locally uniformly on* $[a, b] \setminus D$.

Proof. First, notice that Corollary 2.19 implies that (3.7) holds locally uniformly on $[a, b] \setminus D$.

Assume that $D = \{\tau\}$, where $\tau \in (a, b)$.

Due to Theorems 2.7 and 2.21, we can see that the problem (3.1) has a solution x_k on $[a, b]$ for a.a. $k \in \mathbb{N}$. Indeed, by Theorem 2.21 there exists a fundamental matrix X_k corresponding to A_k for a.a. $k \in \mathbb{N}$. Moreover, $\det[\mathrm{I} - \Delta^- A_k(t)] \neq 0$ on $[a, b]$ for a.a. $k \in \mathbb{N}$ wherefrom, by Theorem 2.7, our claim follows.

The rest of the proof is divided into three steps. First, we prove that (3.7) is true for $t \in [a, \tau)$, then for $t = \tau$ and finally for $t \in (\tau, b]$.

STEP 1. Let $\alpha \in (a, \tau)$ be given. Then by Corollary 2.19 the relation (3.7) holds uniformly on $[a, \alpha]$. Therefore (3.7) is true for any $t \in [a, \tau)$.

STEP 2. Now we will prove that (3.7) is true also for $t = \tau$. Note that according to (2.21) we have

$$x_0(\tau-) = x_0(\tau) - \Delta^- A(\tau)\, x_0(\tau) - \Delta^- f(\tau) \,.$$

Given an arbitrary $\delta' \in (0, \tau)$ and $k \in \mathbb{N}$, we obtain

$$\begin{aligned} |x_0(\tau) - x_k(\tau)| &\le |x_0(\tau) - \Delta^- A(\tau)\, x_0(\tau) - \Delta^- f(\tau) - x_0(\tau - \delta')| \\ &\quad + |x_0(\tau - \delta') - x_k(\tau - \delta')| + |x_k(\tau - \delta') + \Delta^- A(\tau)\, x_0(\tau) + \Delta^- f(\tau) - x_k(\tau)| \\ &= |x_0(\tau-) - x_0(\tau - \delta')| + |x_0(\tau - \delta') - x_k(\tau - \delta')| \\ &\quad + |x_k(\tau) - x_k(\tau - \delta') - \Delta^- A(\tau)\, [\mathrm{I} - \Delta^- A(\tau)]^{-1}\, x_0(\tau-) \\ &\qquad - [\mathrm{I} - \Delta^- A(\tau)]^{-1}\, \Delta^- f(\tau)| \,, \end{aligned}$$

where we made use of the fact that the relation

$$\mathrm{I} + B\, [\mathrm{I} - B]^{-1} = [\mathrm{I} - B]^{-1}$$

is true whenever the matrix $[\mathrm{I} - B]$ is regular.

Choose $\varepsilon > 0$. According to (3.4) we can choose $\delta \in (0, \varepsilon)$ in such a way that for each $\delta' \in (0, \delta)$ there exists $k_1 = k_1(\delta') \in \mathbb{N}$ such that

$$\begin{aligned} &|u_k(\tau) - u_k(\tau - \delta') - \Delta^- A(\tau)\, [\mathrm{I} - \Delta^- A(\tau)]^{-1}\, x_0(\tau-) \\ &\qquad - [\mathrm{I} - \Delta^- A(\tau)]^{-1}\, \Delta^- f(\tau)| < \varepsilon \end{aligned}$$

holds for any $k \geq k_1$ and for each solution $u_k(t)$ of equation (3.5) satisfying $|x_0(\tau-) - u_k(\tau-\delta^{'})| \leq \delta$.

Set $u_k(t) = x_k(t)$ for $t \in [\tau-\delta^{'}, \tau]$. Choose $\delta^{'} \in (0,\delta)$ such that

$$|x_0(\tau-) - x_0(\tau-\delta^{'})| < \frac{\delta}{2}.$$

Taking into account that $x_k(t) \to x_0(t)$ on $[0,\tau)$ as $k \to \infty$ we get the existence of a $k_0 \in \mathbb{N}$, $k_0 \geq k_1$, such that

$$|x_0(\tau-\delta^{'}) - x_k(\tau-\delta^{'})| < \frac{\delta}{2} \qquad \text{for all } k \geq k_0.$$

Therefore the estimate

$$|x_0(\tau-) - x_k(\tau-\delta^{'})| \leq |x_0(\tau-) - x_0(\tau-\delta^{'})| + |x_0(\tau-\delta^{'}) - x_k(\tau-\delta^{'})| < \delta$$

is true for $k \geq k_0$. Moreover, we have

$$\begin{aligned}|x_k(\tau) - x_k(\tau-\delta^{'}) - \Delta^{-}A(\tau)\,[\mathrm{I} - \Delta^{-}A(\tau)]^{-1}\,x_0(\tau-)\\ -[\mathrm{I} - \Delta^{-}A(\tau)]^{-1}\,\Delta^{-}f(\tau)| < \varepsilon\,.\end{aligned}$$

To summarize, we have

$$|x_0(\tau) - x_k(\tau)| < \frac{\delta}{2} + \frac{\delta}{2} + \varepsilon < 2\,\varepsilon \quad \text{for all } k \geq k_0\,,$$

i.e. $x_k(\tau) \to x_0(\tau)$ for $k \to \infty$.

Step 3. Proof of the convergence on $(\tau,b]$ consists of two parts. First we show that there is a $\delta > 0$ such that $x_k(t) \to x_0(t)$ converges on $(\tau,\tau+\delta)$ as $k \to \infty$. Then this pointwise convergence is extended to $(\tau,b]$.

Let $\varepsilon > 0$ be given and let $\delta_0 \in (0,\varepsilon)$ be such that

$$|x_0(s) - x_0(\tau+)| < \varepsilon \quad \text{for all } s \in (\tau,\tau+\delta_0)\,.$$

By the assumption (3.4), there is a $\delta \in (0,\delta_0)$ such that

$$\forall\,\delta^{'} \in (0,\delta)\ \exists\,k_1 = k_1(\delta^{'}) \in \mathbb{N}$$

and such that

$$|v_k(\tau+\delta^{'}) - v_k(\tau) - \Delta^{+}A(\tau)x_0(\tau) - \Delta^{+}f(\tau)| < \varepsilon$$

is true for each $k \geq k_1$ and for each solution $v_k(t)$ of equation (3.6) satisfying $|x_0(\tau) - v_k(\tau)| \leq \delta$.

Now, for each $\delta' \in (0,\delta)$ the distance between $x_0(\tau+\delta')$ and $x_k(\tau+\delta')$ can be estimated. In view of the fact that

$$x_0(\tau+) - x_0(\tau) = \Delta^+ A(\tau)\, x_0(\tau) + \Delta^+ f(\tau)$$

we have

$$\begin{aligned} |x_0(\tau+\delta') - x_k(\tau+\delta')| &\le |x_0(\tau+\delta') - x_0(\tau+)| \\ + |x_0(\tau+) - x_0(\tau) + x_k(\tau) - x_k(\tau+\delta')| &+ |x_0(\tau) - x_k(\tau)| = |x_0(\tau+\delta') - x_0(\tau+)| \\ + |\Delta^+ A(\tau)\, x_0(\tau) + \Delta^+ f(\tau) &+ x_k(\tau) - x_k(\tau+\delta')| + |x_0(\tau) - x_k(\tau)|\,. \end{aligned}$$

Since $x_k(\tau) \to x_0(\tau)$ for $k \to \infty$, we get an existence of $k_0 \in \mathbb{N}$, $k_0 \ge k_1$ such that $|x_0(\tau) - x_k(\tau)| < \delta$ for all $k \ge k_0$. Since $\tau + \delta' \in (\tau, \tau+\delta_0)$, we have $|x_0(\tau+\delta') - x_0(\tau+)| < \varepsilon$. Setting $v_k(t) = x_k(t)$ on $[\tau, \tau+\delta']$, we get

$$|x_k(\tau+\delta') - x_k(\tau) - \Delta^+ A(\tau)\, x_0(\tau) - \Delta^+ f(\tau)| < \varepsilon \quad \text{for all } k \ge k_0\,.$$

Altogether, for any $k \ge k_0$ the estimate

$$|x_0(\tau+\delta') - x_k(\tau+\delta')| \le \varepsilon + \delta + \varepsilon < 3\,\varepsilon$$

is valid. Consequently, we have $x_k(t) \to x_0(t)$ for $k \to \infty$ on $(\tau, \tau+\delta)$.

Now, choose an arbitrary σ in $(\tau, \tau+\delta)$. Making use of Corollary 2.19 with σ in place of a the proof of the validity of (3.7) for any $t \in [a,b]$ can be completed.

The extension to the case $D = \{\tau_1, \tau_2, \ldots, \tau_m\}$ with $m > 1$ is obvious. □

3.2 Corollary. *Let A, f, A_k, f_k be left-continuous on $(a,b]$ for all $k \in \mathbb{N}$. Assume that* (3.2) *and* (2.6) *holds. Let $\widetilde{x}_k \to \widetilde{x}_0$ and let $x_0(t)$ denote a solution of* (1.1). *Moreover, assume* (3.3) *and*

$$\left\{ \begin{array}{l} \textit{if } \tau \in D, \textit{ then } \forall\, \xi \in \mathbb{R}^n \textit{ and } \forall\, \varepsilon > 0\ \exists\, \delta > 0 \textit{ such that} \\ \forall\, \delta' \in (0,\delta)\ \exists\, k_0 \in \mathbb{N} \textit{ such that the relation} \\ |v_k(\tau+\delta') - v_k(\tau) - \Delta^+ A(\tau)\xi - \Delta^+ f(\tau)| < \varepsilon \\ \textit{is satisfied } \forall\, k \ge k_0 \textit{ and } \forall\, v_k \textit{ fulfilling } (3.6) \\ \textit{and such that } |\xi - v_k(\tau)| \le \delta. \end{array} \right. \tag{3.8}$$

Then for a.a. $k \in \mathbb{N}$ the solution x_k of (3.1) *exists on $[a,b]$ and* (3.7) *holds for any $t \in [a,b]$, where $x_0(t)$ is the solution of* (1.1). *Moreover,* (3.7) *holds locally uniformly on $[a,b] \setminus D$.*

Proof. Since we have $\Delta^- A(t) = 0$ and $\Delta^- A_k(t) = 0$ for all $k \in \mathbb{N}$ and $t \in [a,b]$, the condition (3.4) reduces to (3.8). □

3.3 Corollary. *Let A, f, A_k, f_k be right-continuous on $[a,b)$ for all $k \in \mathbb{N}$. Assume* (3.2), (2.6), (2.36) *and* (2.12). *Let $\widetilde{x}_k \to \widetilde{x}_0$ and let $x_0(t)$ denote a solution of* (1.1). *Moreover, assume* (3.3) *and*

$$\begin{cases} \text{if } \tau \in D, \text{ then } \forall\, \xi \in \mathbb{R}^n \text{ and } \forall\, \varepsilon > 0\ \exists\, \delta > 0 \text{ such that} \\ \forall\, \delta' \in (0,\delta)\ \exists\, k_0 \in \mathbb{N} \text{ such that the relation} \\ |u_k(\tau) - u_k(\tau-\delta') - \Delta^- A(\tau)[\mathrm{I} - \Delta^- A(\tau)]^{-1}\xi \\ \qquad -[\mathrm{I} - \Delta^- A(\tau)]^{-1}\Delta^- f(\tau)| < \varepsilon \\ \text{is satisfied } \forall\, k \geq k_0 \text{ and } \forall\, u_k \text{ fulfilling (3.5)} \\ \text{and such that } |\xi - u_k(\tau-\delta')| \leq \delta. \end{cases} \tag{3.9}$$

Then for a.a. $k \in \mathbb{N}$ the solution x_k of (3.1) *exists on $[a,b]$ and* (3.7) *holds for any $t \in [a,b]$, where $x_0(t)$ is the solution of* (1.1). *Moreover,* (3.7) *holds locally uniformly on $[a,b] \setminus D$.*

Proof. Since we have $\Delta^+ A(t) = 0$ and $\Delta^+ A_k(t) = 0$ for all $k \in \mathbb{N}$ and $t \in [a,b]$, the condition (3.4) reduces to (3.9). $\square$

Consider initial value problems for ordinary differential equations

$$x' = B_k(t)\,x + g_k(t), \quad x(0) = \widetilde{x}_k, \quad k \in \mathbb{N}_\tau, \tag{3.10}$$

where $0 < \tau < 1$, $\mathbb{N}_\tau = \left\{ k \in \mathbb{N},\, k \geq \dfrac{1}{1-\tau} \right\}$,

$$\begin{cases} B_k(t) = P(t) + k\,\chi_{(\tau,\tau+\frac{1}{k})}(t)\ \mathrm{I},\ \ g_k(t) = g(t) + k\,\chi_{(\tau,\tau+\frac{1}{k})}(t)\, r \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } t \in [0,1], k \in \mathbb{N}_\tau, \\ P\colon [0,1] \to \mathbb{R}^{n\times n} \text{ and } g\colon [0,1] \to \mathbb{R}^n \text{ are Lebesgue integrable,} \\ P(t)P(s) = P(s)P(t) \qquad \text{for } t,s \in [0,1] \\ r \in \mathbb{R}^n,\ \widetilde{x}_k, \widetilde{x}_0 \in \mathbb{R}^n \quad \text{and} \quad \widetilde{x}_k \to \widetilde{x}_0. \end{cases} \tag{3.11}$$

3.4 Corollary. *Assume $0 < \tau < 1$ and* (3.11). *Then the solutions x_k of* (3.10) *are defined on $[0,1]$ for all $k \in \mathbb{N}_\tau$ and there exist functions $A \in \mathbf{BV}^{n\times n}[0,1]$ and $f \in \mathbf{BV}^n[0,1]$ such that* (3.7) *holds for any $t \in [0,1]$, where $x_0(t)$ is a solution of the generalized differential equation* (1.1). *Moreover,* (3.7) *holds locally uniformly on $[0,1] \setminus \{\tau\}$.*

Proof. For $t \in [0,1]$ and $k \in \mathbb{N}_\tau$ put

$$A_k(t) = \int_0^t B_k(s)\,\mathrm{d}s \quad \text{and} \quad f_k(t) = \int_0^t g_k(s)\,\mathrm{d}s\,,$$
$$A(t) = \int_0^t P(s)\,\mathrm{d}s + \chi_{(\tau,1]}(t)\,D^+ \quad \text{and} \quad f(t) = \int_0^t g(s)\,\mathrm{d}s + \chi_{(\tau,1]}(t)\,d^+\,,$$

where $D^+ \in \mathbb{R}^{n\times n}$ and $d^+ \in \mathbb{R}^n$ are to be determined later.

The problems (3.10) can be equivalently reformulated in the form

$$x(t) = \widetilde{x}_k + \int_0^t \mathrm{d}[A_k]\,x + f_k(t) - f_k(0)\,, \quad k \in \mathbb{N}_\tau\,,\ t \in [0,1]\,.$$

Clearly, for each $k \in \mathbb{N}_\tau$ the problem (3.10) has a unique solution x_k on $[0,1]$.

We have

$$\operatorname{var}_0^1 A_k \le \int_0^1 |P(s)|\,\mathrm{d}s + k\int_\tau^{\tau+\frac{1}{k}} \mathrm{d}s = \int_0^1 |P(s)|\,\mathrm{d}s + 1 < \infty \quad \text{for all } k \in \mathbb{N}_\tau\,. \tag{3.12}$$

Furthermore,

$$\begin{aligned} \lim_{k\to\infty} A_k(t) &= \int_0^t P(s)\,\mathrm{d}s + \chi_{(\tau,1]}(t)\ \mathrm{I}\,,\\ \lim_{k\to\infty} f_k(t) &= \int_0^t g(s)\,\mathrm{d}s + \chi_{(\tau,1]}(t)\,r \end{aligned} \tag{3.13}$$

holds for each $t \in [0,1]$. Moreover,

$$A_k^J \rightrightarrows A^J \quad \text{and} \quad f_k^J \rightrightarrows f^J \quad \text{holds for each compact} \ \ J \subset [0,1]\setminus\{\tau\}$$

and $A_k \rightrightarrows A$ and $f_k \rightrightarrows f$ on $[0,\tau]$. Therefore, due to Corollary 2.19, we have $x_k \rightrightarrows x_0$ on $[0,\tau]$, where x_0 is a solution of

$$x(t) = \widetilde{x}_0 + \int_0^t \mathrm{d}[A]\,x + f(t) - f(0), \quad t \in [0,1]\,. \tag{3.14}$$

Since $B_k(t)\,B_k(s) = B_k(s)\,B_k(t)$ holds for each $t, s \in [0,1]$ and each $k \in \mathbb{N}_\tau$, the fundamental matrices V_k for $v' = B_k(t)\,v$ fulfilling the condition $V_k(\tau) = \mathrm{I}$ are for $t \in [0,1]$ and $k \in \mathbb{N}_\tau$ given by

$$V_k(t) = \exp\left(\int_\tau^t B_k(s)\,\mathrm{d}s\right) = \Phi(t)\,\Phi^{-1}(\tau)\begin{cases} 1 & \text{if } \ t \le \tau,\\ \mathrm{e}^{k\,(t-\tau)} & \text{if } \ t \in (\tau, \tau+\frac{1}{k}),\\ \mathrm{e} & \text{if } \ t \ge \tau + \frac{1}{k}, \end{cases}$$

where $\Phi(t) = \exp\big(\int_0^t P(s)\mathrm{d}s\big)$, $t \in [0,1]$, is the fundamental matrix for the ordinary differential equation $x' = P(t)\,x$ fulfilling $\Phi(0) = \mathrm{I}$.

Now, let $k \in \mathbb{N}_\tau$, $\varepsilon > 0$, $\delta' > 0$, $\xi \in \mathbb{R}^n$, $\widetilde{v}_k \in \mathbb{R}^n$ and let v_k be a solution of

$$v' = B_k(t)\, v + g_k(t), \quad v(\tau) = \widetilde{v}_k \tag{3.15}$$

on $[\tau, 1]$. Then

$$v_k(\tau+\delta') = \Phi(\tau+\delta')\,\Phi^{-1}(\tau+\frac{1}{k})\, v_k(\tau+\frac{1}{k}) + \Phi(\tau+\delta') \int_{\tau+\frac{1}{k}}^{\tau+\delta'} \Phi^{-1}(s)\, g(s)\, \mathrm{d}s$$

and

$$v_k(\tau+\frac{1}{k}) = \mathrm{e}\,\Phi(\tau+\frac{1}{k})\,\Phi^{-1}(\tau)\,\widetilde{v}_k + \Phi(\tau+\frac{1}{k}) \int_{\tau}^{\tau+\frac{1}{k}} \mathrm{e}^{1-k\,(s-\tau)}\,\Phi^{-1}(s)\, g(s)\, \mathrm{d}s$$
$$+ k\,\Phi(\tau+\frac{1}{k}) \left(\int_{\tau}^{\tau+\frac{1}{k}} \mathrm{e}^{1-k\,(s-\tau)}\,\Phi^{-1}(s)\, \mathrm{d}s \right) r\,,$$

which yields

$$\left| v_k(\tau+\delta') - \widetilde{v}_k - \Delta^+ A(\tau)\,\xi - \Delta^+ f(\tau) \right| \le \left| \left[\mathrm{e}\,\Phi(\tau+\delta')\,\Phi^{-1}(\tau) - \mathrm{I} \right] \widetilde{v}_k - D^+\xi \right|$$
$$+ \left|\Phi(\tau+\delta')\right| \left| \int_{\tau}^{\tau+\frac{1}{k}} \mathrm{e}^{1-k\,(s-\tau)}\,\Phi^{-1}(s)\, g(s)\, \mathrm{d}s \right| + \left| \Phi(\tau+\delta') \int_{\tau+\frac{1}{k}}^{\tau+\delta'} \Phi^{-1}(s)\, g(s)\, \mathrm{d}s \right|$$
$$+ \left| k\,\Phi(\tau+\delta') \int_{\tau}^{\tau+\frac{1}{k}} \mathrm{e}^{1-k\,(s-\tau)}\,\Phi^{-1}(s)\, \mathrm{d}s - d^+\,\mathrm{I} \right| |r|\,.$$

It is easy to see that we can choose $\delta_1 > 0$ and $k_1 \in \mathbb{N}_\tau$ so that the second and third terms on the right hand side of the above inequality are less than $\frac{\varepsilon}{5}$ for all $\delta' \in (0, \delta_1)$ and $k \in \mathbb{N}_\tau \cap (k_0, \infty)$. Furthermore, since

$$\lim_{\delta' \to 0} \Phi(\tau+\delta')\,\Phi^{-1}(\tau) = \mathrm{I},$$

we can see that when choosing $D^+ = (\mathrm{e}-1)\,\mathrm{I}$, we can find $\delta_2 \in (0, \delta_1)$ such that the first term becomes smaller than $\frac{\varepsilon}{5}$ whenever $\delta' \in (0, \delta_2)$ and $|\widetilde{v}_k - \xi| < \delta_2$. Finally, observing that

$$\lim_{k\to\infty} k \int_{\tau}^{\tau+\frac{1}{k}} \mathrm{e}^{1-k\,(s-\tau)}\,\Phi^{-1}(s)\, \mathrm{d}s = \mathrm{e}\,\Phi^{-1}(\tau)\,,$$

we can conclude that, when setting $d^+ = \mathrm{e}$, we can choose $k_0 \in \mathbb{N}_\tau \cap (k_1, \infty)$ and $\delta \in (0, \delta_0)$ so that also the fourth term becomes smaller than $\frac{\varepsilon}{5}$ whenever $k \in \mathbb{N}_\tau \cap (k_0, \infty)$ and $\delta' \in (0, \delta)$. To summarize, the assumption of (3.8) is satisfied if we define

$$\begin{cases} A(t) = \displaystyle\int_0^t P(s)\, \mathrm{d}s + \chi_{(\tau,1]}(t)\,(\mathrm{e}-1)\,\mathrm{I}\,, \\ f(t) = \displaystyle\int_0^t g(s)\, \mathrm{d}s + \chi_{(\tau,1]}(t)\,\mathrm{e}\, r \end{cases} \quad \text{for } t \in [0,1]. \tag{3.16}$$

Furthermore, as

$$\det[\mathrm{I}+\Delta^+ A(t)] = \det\big((\mathrm{e}-1)\,\mathrm{I}\big) = \mathrm{e}-1 \neq 0,$$

by Corollary 3.2, the relation (3.7) holds for each $t \in [0,1]$, where x_0 is solution of (3.14) with A and f defined by (3.16). □

4 Inverse fundamental matrices

The equation (2.17), which is satisfied by the matrix function X^{-1}, is not a generalized linear differential equation of the type (2.10). This leads us to the consideration of *adjoint equations*, i.e. the equations of the form

$$y^T(t) = y^T(s) - y^T(t)\,A(t) + y^T(s)\,A(s) + \int_s^t \mathrm{d}[y^T]\,A\,. \tag{4.1}$$

4.1 Theorem. ([41, Theorem 2.7]) *Let $A \in \mathbf{BV}^{n\times n}[a,b]$ satisfy* (2.12) *and* (2.13). *Then the initial value problem* (4.1), *$y^T(a) = \tilde{y}^T$ has for every $\tilde{y} \in \mathbb{R}^n$ a unique solution y on $[a,b]$. This solution is of bounded variation on $[a,b]$ and is given on $[a,b]$ by*

$$y^T(t) = \tilde{y}^T X(a) X^{-1}(t)\,. \tag{4.2}$$

Moreover, every solution $y^T(t)$ of the equation (4.1) *on $[a,b]$ possesses the one-sided limits $y^T(t+)$, $y^T(t-)$ where the relations*

$$\begin{cases} y^T(t+) = y^T(t) - y^T(t+)\,\Delta^+ A(t) & \text{for all } t \in [a,b)\,, \\ y^T(t-) = y^T(t) + y^T(t-)\,\Delta^- A(t) & \text{for all } t \in (a,b] \end{cases} \tag{4.3}$$

hold.

Theorem 2.21 deals with a sequence of fundamental matrices. According to definition, each fundamental matrix corresponding to a given matrix function A fulfils for all $s,t \in [a,b]$ the equation $X(t) = X(s) + \int_s^t \mathrm{d}[A]\,X$. This fact is essentially used in the proof of Theorem 4.2. Furthermore, we take into account that the inverse of fundamental matrix $X^{-1}(t)$ satisfies on $[a,b]$ the relation

$$X^{-1}(t) = X^{-1}(a) - X^{-1}(t)\,A(t) + X^{-1}(a)\,A(a) + \int_a^t \mathrm{d}[X^{-1}]\,A\,, \tag{4.4}$$

which is adjoint to (2.16), cf. (2.17) and (4.1).

We want to prove assertion analogous to Theorem 2.21 for inverses of fundamental matrices. To this aim it is necessary to suppose also the regularity of $[\mathrm{I}+\Delta^+ A(t)]$ for each $t \in [a,b)$ and condition (4.5) which is a modification of (2.37). This is the main result of this section.

4.2 Theorem. *Let the assumptions of Theorem 2.21 be satisfied. Furthermore assume that (2.13) and the following conditions hold:*

$$\begin{cases} \text{if } \tau \in D, \text{ then } \forall\, \eta \in \mathbb{R}^n \text{ and } \forall\, \varepsilon > 0\ \exists\, \delta > 0 \text{ such that} \\ \forall\, \delta' \in (0,\delta)\ \exists\, k_0 \in \mathbb{N} \text{ such that the relations} \\ \qquad |w_k^T(\tau) - w_k^T(\tau-\delta') + \eta^T\,\Delta^- A(\tau)| < \varepsilon\,, \\ \qquad |z_k^T(\tau+\delta') - z_k^T(\tau) + \eta^T\,[\mathrm{I} + \Delta^+ A(\tau)]^{-1}\,\Delta^+ A(\tau)| < \varepsilon \\ \text{are satisfied } \forall\, k \geq k_0 \text{ and } \forall\, w_k,\ z_k \in \mathbb{R}^n \text{ fulfilling (4.6), (4.7)} \\ \text{and such that } |\eta^T - w_k^T(\tau-\delta')| \leq \delta,\ |\eta^T - z_k^T(\tau)| \leq \delta\,, \end{cases} \tag{4.5}$$

where

$$\begin{cases} w_k^T(t) = w_k^T(\tau-\delta') - w_k^T(t)\,A_k(t) + w_k^T(\tau-\delta')\,A_k(\tau-\delta') \\ \qquad\qquad + \displaystyle\int_{\tau-\delta'}^{t} \mathrm{d}[w_k^T]\,A_k \quad \text{on } [\tau-\delta',\tau]\,, \end{cases} \tag{4.6}$$

$$z_k^T(t) = z_k^T(\tau) - z_k^T(t)\,A_k(t) + z_k^T(\tau)\,A_k(\tau) + \int_{\tau}^{t} \mathrm{d}[z_k^T]\,A_k \quad \text{on } [\tau,\tau+\delta']\,. \tag{4.7}$$

Then for a.a. $k \in \mathbb{N}$ the fundamental matrices X_k corresponding to A_k and their inverses X_k^{-1} are defined on $[a,b]$,

$$\lim_{k\to\infty} X_k(t) = X_0(t) \quad \text{on } [a,b] \tag{4.8}$$

and

$$\lim_{k\to\infty} X_k^{-1}(t) = X_0^{-1}(t) \tag{4.9}$$

on $[a,b]$, where X_0 be the fundamental matrix corresponding to A. Moreover, (4.9) holds locally uniformly on $[a,b] \setminus D$.

Proof. First notice that Lemma 2.24 implies that (4.9) holds locally uniformly on $[a,b] \setminus D$ and (4.8) immediately follows from Theorem 2.21.

Assume that $D = \{\tau\}$, where $\tau \in (a,b)$; i.e. D consists of one point $\tau \in (a,b)$ only and $m = 1$.

Recall that the existence of the fundamental matrices X_k for a.a. $k \in \mathbb{N}$ and (4.8) immediately follows from Theorem 2.21. Since each fundamental matrix is regular, we get the existence of X_k^{-1} for a.a. $k \in \mathbb{N}$.

For $\tilde{y}_k \in \mathbb{R}^n$ and for a.a. $k \in \mathbb{N}$, denote by y_k the solution of the equation

$$y_k^T(t) = \tilde{y}_k^T - y_k^T(t)\,A_k(t) + \tilde{y}^T\,A_k(a) + \int_a^t \mathrm{d}[y_k^T]\,A_k \quad \text{on } [a,b]\,. \tag{4.10}$$

We will prove that

$$y_k \to y_0 \quad \text{on } [a,b] \tag{4.11}$$

provided $\tilde{y}_k \to \tilde{y}_0$. This will be done in three steps. First, we prove that (4.11) is true for $t \in [a,\tau)$, then for $t = \tau$ and finally for $t \in (\tau, b]$.

Step 1. Let $\alpha \in (a,\tau)$ be given. Then by Lemma 2.24 the relation (4.9) holds uniformly on $[a,\alpha]$. Therefore (4.9) is true for any $t \in [a,\tau)$ and by Theorem 4.1 we get that $y_k \to y_0$ on $[a,\tau)$.

Step 2. Now we will prove, that (4.11) is true also for $t = \tau$. For each $\delta' \in (0,\tau)$ and $k \in \mathbb{N}$ we get using (4.3) the estimate

$$\begin{aligned} |y_0^T(\tau) - y_k^T(\tau)| &\le |y_0^T(\tau) + y_0^T(\tau-)\,\Delta^- A(\tau) - y_0^T(\tau-\delta')| \\ &\quad + |y_0^T(\tau-\delta') - y_k^T(\tau-\delta')| + |y_k^T(\tau-\delta') - y_0^T(\tau-)\,\Delta^- A(\tau) - y_k^T(\tau)| \\ &= |y_0^T(\tau-) - y_0^T(\tau-\delta')| + |y_0^T(\tau-\delta') - y_k^T(\tau-\delta')| \\ &\quad + |y_k^T(\tau) - y_k^T(\tau-\delta') + y_0^T(\tau-)\,\Delta^- A(\tau)|\,. \end{aligned}$$

Let $\varepsilon > 0$ be given. According to (4.5) we can choose $\delta \in (0,\varepsilon)$ in such a way that for all $\delta' \in (0,\delta)$ there exists $k_1 \in \mathbb{N}$ such that

$$|w_k^T(\tau) - w_k^T(\tau-\delta') + y_0^T(\tau-)\,\Delta^- A(\tau)| < \varepsilon \tag{4.12}$$

holds for any $k \ge k_1$ and for each solution $w_k^T(t)$ of (4.6) fulfilling

$$|y_0^T(\tau-) - w_k^T(\tau-\delta')| \le \delta\,.$$

Set $w_k^T(t) = y_k^T(t)$ on $[\tau-\delta', \tau]$. Choose $\delta' \in (0,\delta)$ such that

$$|y_0^T(\tau-) - y_0^T(\tau-\delta')| < \frac{\delta}{2}.$$

Considering that $y_k^T(t) \to y_0^T(t)$ on $[a,\tau)$ as $k \to \infty$ we get the existence of a $k_0 \in \mathbb{N}$, $k_0 \ge k_1$ such that

$$|y_0^T(\tau-\delta') - y_k^T(\tau-\delta')| < \frac{\delta}{2} \qquad \text{for all } k \ge k_0.$$

Therefore the estimate

$$|y_0^T(\tau-) - y_k^T(\tau-\delta')| \le |y_0^T(\tau-) - y_0^T(\tau-\delta')| + |y_0^T(\tau-\delta') - y_k^T(\tau-\delta')| < \delta$$

is true for $k \ge k_0$. By (4.12) we have

$$|y_k^T(\tau) - y_k^T(\tau-\delta') + y_0^T(\tau-)\,\Delta^- A(\tau)| < \varepsilon\,.$$

To summarize,

$$|y_0^T(\tau) - y_k^T(\tau)| < \frac{\delta}{2} + \frac{\delta}{2} + \varepsilon < 2\,\varepsilon \quad \text{for all } k \geq k_0\,,$$

i.e. $y_k^T(\tau) \to y_0^T(\tau)$ for $k \to \infty$.

Step 3. Proof of the convergence on $(\tau, b]$ consists of two parts. First, we show that there is a $\delta > 0$ such that $y_k^T(t) \to y_0^T(t)$ on $(\tau, \tau+\delta)$ as $k \to \infty$. Then we extend this result to the whole interval $(\tau, b]$.

Let $\varepsilon > 0$ be given and let $\delta_0 \in (0, \varepsilon)$ be such that

$$|y_0^T(s) - y_0^T(\tau+)| < \varepsilon \quad \text{for all } s \in (\tau, \tau+\delta_0).$$

By the assumption (4.5), there exists $\delta \in (0, \delta_0)$ such that for all $\delta' \in (0, \delta)$ there exists $k_1 = k_1(\delta') \in \mathbb{N}$ and such that

$$|z_k^T(\tau+\delta') - z_k^T(\tau) + y_0^T(\tau)\,[\mathrm{I} + \Delta^+ A(\tau)]^{-1}\,\Delta^+ A(\tau)| < \varepsilon \tag{4.13}$$

is true for each solution $z_k^T(t)$ of (4.7) with the property $|y_0^T(\tau) - z_k^T(\tau)| \leq \delta$. Now the distance between $y_0^T(\tau+\delta')$ and $y_k^T(\tau+\delta')$ can be estimated. In view of (4.3) we get

$$\begin{aligned}
|y_0^T(\tau+\delta') - y_k^T(\tau+\delta')| &\leq |y_0^T(\tau+\delta') - y_0^T(\tau) + y_0^T(\tau+)\,\Delta^+ A(\tau)| \\
&\quad + |y_0^T(\tau) - y_k^T(\tau)| + |y_k^T(\tau) - y_0^T(\tau+)\,\Delta^+ A(\tau) - y_k^T(\tau+\delta')| \\
= |y_0^T(\tau+\delta') - y_0^T(\tau+)| &+ |y_0^T(\tau) - y_k^T(\tau)| + |y_k^T(\tau) - y_0^T(\tau+)\,\Delta^+ A(\tau) - y_k^T(\tau+\delta')|\,.
\end{aligned}$$

Considering that $y_k^T(\tau) \to y_0^T(\tau)$ for $k \to \infty$, we get the existence of $k_0 \in \mathbb{N}$, $k_0 \geq k_1$ such that $|y_0^T(\tau) - y_k^T(\tau)| < \delta$ for all $k \geq k_0$. Since $\tau + \delta' \in (\tau, \tau+\delta_0)$, we have $|y_0^T(\tau+\delta') - y_0^T(\tau+)| < \varepsilon$. Setting $z_k^T(t) = y_k^T(t)$ on $[\tau, \tau+\delta']$, we get by (4.13) the relation

$$|y_k^T(\tau) - y_0^T(\tau+)\,\Delta^+ A(\tau) - y_k^T(\tau+\delta')| < \varepsilon \quad \text{for all } k \geq k_0\,.$$

To summarize, for any $k \geq k_0$ the estimate

$$|y_0^T(\tau+\delta') - y_k^T(\tau+\delta')| \leq \varepsilon + \delta + \varepsilon < 3\,\varepsilon$$

is valid, as well. Therefore $y_k^T(t) \to y_0^T(t)$ on $(\tau, \tau+\delta)$ as $k \to \infty$.

Now, choose an arbitrary σ in $(\tau, \tau+\delta)$. Making use of Lemma 2.24 with σ in place of a the proof of this step can be completed.

Having in mind that $y_k^T(t) = \tilde{y}_k^T\, X_k(a)\, X_k^{-1}(t)$ on $[a, b]$ hold for all $k \in \mathbb{N}_0$, we can see that for each $i \in \{1, 2, \ldots, n\}$ it is always possible to choose $\tilde{y}_k^T$ in such a way that $y_k^T(t)$ is the i-th row of $X_k^{-1}(t)$ and $\tilde{y}_k^T \to \tilde{y}_0^T$. This consideration completes the proof of the validity of (4.9) for any $t \in [a, b]$.

The extension to the case $m > 1$ is obvious. □

5 Approximated solutions

5.1 Introduction

Let $A \in \mathbf{BV}^{n\times n}[a,b]$, $f \in \mathbf{G}^n[a,b]$ and $\widetilde{x}_0 \in \mathbb{R}^n$. Consider the equation

$$x(t) = \widetilde{x}_0 + \int_a^t \mathrm{d}[A]\, x + f(t) - f(a). \tag{5.1}$$

The aim of this section is to find conditions which enable to approximate the solutions of GLODE (5.1) by solutions of linear ordinary differential equations.

Let us recall some basic properties of matrix exponentials. Considering the norm $|\cdot|$ on $\mathbb{R}^{n\times n}$, it is well known that for each matrix $B \in \mathbb{R}^{n\times n}$ the series

$$\mathrm{I} + \sum_{i=1}^{\infty} \frac{B^i}{i!}$$

converges and the matrix exponential of B

$$\exp(B) = \mathrm{I} + \sum_{i=1}^{\infty} \frac{B^i}{i!}$$

is defined for each $B \in \mathbb{R}^{n\times n}$.

Obviously, $\exp(0) = \mathrm{I}$. Furthermore, for each $B \in \mathbb{R}^{n\times n}$

$$|\exp(B)| \le \exp(|B|).$$

If $A \in \mathbb{R}^{n\times n}$ is such that $AB = BA$, then

$$\exp(A)\ \exp(B) = \exp(B + A) = \exp(B)\ \exp(A).$$

This implies that for each $B \in \mathbb{R}^{n\times n}$ the matrix $\exp(B)$ has its inverse $[\exp(B)]^{-1}$ and

$$[\exp(B)]^{-1} = \exp(-B).$$

5.1 Lemma. *Let $C,\ D \in \mathbb{R}^{n\times n}$, then*

$$|\exp\big(C\big) - \exp\big(D\big)| \le |C - D|\ \exp\big(\max(|C|, |D|)\big).$$

Proof. Without loss of generality we can assume that $|C| > |D|$. Denote $\Delta = C - D$. According to definition of matrix exponential we have

$$\exp(C) = \mathrm{I} + \sum_{k=1}^{\infty} \frac{C^k}{k!}$$

and

$$\exp(D) = \exp(C - \Delta) = \mathrm{I} + \sum_{k=1}^{\infty} \frac{(C-\Delta)^k}{k!}\,.$$

Subtracting the previous formulas, we obtain

$$\exp(C) - \exp(D) = \Delta + \sum_{k=2}^{\infty} \frac{C^k - (C-\Delta)^k}{k!}\,.$$

Since

$$\left|C^k - (C-\Delta)^k\right| \le k\,|\Delta|\,|C|^{k-1} \quad \text{for } k = 2,3,\dots$$

we get

$$\begin{aligned}
|\exp(C) - \exp(D)| &\le |\Delta| + \sum_{k=2}^{\infty} \frac{k\,|\Delta|\,|C|^{k-1}}{k!} \\
&= |\Delta| + |\Delta| \sum_{k=1}^{\infty} \frac{|C|^k}{k!} = |\Delta| \exp(|C|)\,,
\end{aligned}$$

from which assertion of lemma follows. □

The next assertion which seem not to be available in the literature will be useful for our purposes.

5.2 Lemma. *Let* $A \in \mathbf{BV}^{n\times n}[a,b]$. *Then*

$$\left\{\begin{aligned}
&\lim_{s\to t-} \frac{1}{t-s}\left(\int_s^t \exp\left([A(t)-A(s)]\frac{t-r}{t-s}\right)\,\mathrm{d}r\right) \\
&\quad = \lim_{s\to t-} \frac{1}{t-s}\left(\int_s^t \exp\left(\Delta^- A(t)\frac{t-r}{t-s}\right)\,\mathrm{d}r\right) \quad \textit{if } \ t\in(a,b]
\end{aligned}\right. \tag{5.2}$$

and

$$\left\{\begin{aligned}
&\lim_{s\to t+} \frac{1}{s-t}\left(\int_t^s \exp\left([A(s)-A(t)]\frac{s-r}{s-t}\right)\,\mathrm{d}r\right) \\
&\quad = \lim_{s\to t+} \frac{1}{s-t}\left(\int_t^s \exp\left(\Delta^+ A(t)\frac{s-r}{s-t}\right)\,\mathrm{d}r\right) \quad \textit{if } \ t\in[a,b).
\end{aligned}\right. \tag{5.3}$$

Proof. (i) Let $t\in(a,b]$, $s\in[a,t)$ and let $\varepsilon>0$ be given. Then there is a $\delta>0$ such that

$$|A(t-) - A(s)| < \eta \quad \text{whenever} \quad t - s < \varepsilon.$$

Now, using Lemma 5.1 we get

$$\begin{aligned}
&\left|\frac{1}{t-s}\int_s^t \left[\exp\left([A(t)-A(s)]\frac{t-r}{t-s}\right) - \exp\left(\Delta^- A(t)\frac{t-r}{t-s}\right)\right]\,\mathrm{d}r\right| \\
&\quad \le \frac{1}{t-s}\,|A(t-) - A(s)| \int_s^t \exp\left(|\Delta^- A(t)|\right)\,\mathrm{d}r \\
&\quad = \varepsilon\,|A(t-) - A(s)|\,\exp\left(|\Delta^- A(t)|\right) \le \varepsilon\,\exp\left(|\Delta^- A(t)|\right)
\end{aligned}$$

for $t - s < \delta$, wherefrom the validity of (5.2) immediately follows.

(ii) Similarly we would justify the relation (5.3). $\square$

5.3 Notation. In what follows, for $k \in \mathbb{N}$ we denote by D_k a division of $[a,b]$ given by

$$\begin{cases} D_k = \{\alpha_0^k, \alpha_1^k, \dots, \alpha_{2^k}^k\}, & \text{where} \\ \alpha_i^k = a + \frac{i\,(b-a)}{2^k} & \text{for} \;\; i = 0, 1, \dots, 2^k. \end{cases} \tag{5.4}$$

For $A \in \mathbf{BV}^{n \times n}[a,b]$, $f \in \mathbf{G}^n[a,b]$ and $k \in \mathbb{N}$, we define

$$A_k(t) = \begin{cases} A(t) & \text{if} \;\; t \in D_k, \\ A(\alpha_{i-1}^k) + \dfrac{A(\alpha_i^k) - A(\alpha_{i-1}^k)}{\alpha_i^k - \alpha_{i-1}^k}\,(t - \alpha_{i-1}^k) & \text{if} \;\; t \in (\alpha_{i-1}^k, \alpha_i^k), \end{cases} \tag{5.5}$$

and

$$f_k(t) = \begin{cases} f(t) & \text{if} \;\; t \in D_k, \\ f(\alpha_{i-1}^k) + \dfrac{f(\alpha_i^k) - f(\alpha_{i-1}^k)}{\alpha_i^k - \alpha_{i-1}^k}\,(t - \alpha_{i-1}^k) & \text{if} \;\; t \in (\alpha_{i-1}^k, \alpha_i^k). \end{cases} \tag{5.6}$$

Then, obviously, $\{A_k\} \subset \mathbf{AC}^{n \times n}[a,b]$ and $\{f_k\} \subset \mathbf{AC}^n[a,b]$. Moreover, we have

5.4 Lemma. *Let sequences* $\{D_k\} \subset \mathcal{D}[a,b]$ *and* $\{A_k\} \subset \mathbf{AC}^{n \times n}[a,b]$ *be defined by* (5.4) *and* (5.5), *respectively. Then*

$$\operatorname{var}_a^b A_k \le \operatorname{var}_a^b A \quad \textit{for all} \;\; k \in \mathbb{N}.$$

Proof. Since

$$\operatorname{var}_{\alpha_{\ell-1}^k}^{\alpha_\ell^k} A_k = \big|A(\alpha_\ell^k) - A(\alpha_{\ell-1}^k)\big| \le \operatorname{var}_{\alpha_{\ell-1}^k}^{\alpha_\ell^k} A$$

for each $k \in \mathbb{N}$ and $\ell = 1, 2, \dots, 2^k$, we have

$$\operatorname{var}_a^b A_k = \sum_{\ell=1}^{2^k} \operatorname{var}_{\alpha_{\ell-1}^k}^{\alpha_\ell^k} A_k \le \sum_{\ell=1}^{2^k} \operatorname{var}_{\alpha_{\ell-1}^k}^{\alpha_\ell^k} A = \operatorname{var}_a^b A.$$

$\square$

Equations

$$x(t) = \widetilde{x}_k + \int_a^t \mathrm{d}[A_k]\,x + f_k(t) - f_k(a) \tag{5.7}$$

with A_k and f_k given by (5.5) and (5.6) are just initial value problems for linear ordinary differential systems

$$x' = A_k'(t)\,x + f_k'(t), \quad x(a) = \widetilde{x}_k. \tag{5.8}$$

In this view, the next Theorem 5.5 says that the solutions of (5.1) can be uniformly approximated by solutions of linear ordinary differential equations with piecewise constant coefficients provided the functions A and f are continuous.

5.5 Theorem. *Assume that* $A \in \mathbf{BV}^{n\times n}[a,b] \cap \mathbf{C}^{n\times n}[a,b]$ *and* $f \in \mathbf{C}^n[a,b]$*. Let* $\widetilde{x}_0$ *and* $\widetilde{x}_k \in \mathbb{R}^n$, $k \in \mathbb{N}$, *be such that* (2.30) *holds. Furthermore, let the sequence* $\{D_k\}$ *of divisions of the interval* $[a,b]$ *be given by* (5.4) *and let sequences* $\{A_k\} \subset \mathbf{AC}^{n\times n}[a,b]$, $\{f_k\} \subset \mathbf{AC}^n[a,b]$ *be defined by* (5.5) *and* (5.6), *respectively.*

Then equation (5.1) *has a unique solution* x *on* $[a,b]$*. Furthermore, for each* $k \in \mathbb{N}$, *equation* (5.7) *has a solution* x_k *on* $[a,b]$ *and* (2.31) *holds.*

Proof. STEP 1. Since A is uniformly continuous on $[a,b]$, we have:

$$\begin{cases} \text{for each } \ \varepsilon > 0 \text{ there is a } \delta > 0 \text{ such that} \\ |A(t) - A(s)| < \frac{\varepsilon}{2} \\ \text{holds for all } \ t, s \in [a,b] \ \text{ such that } \ |t-s| < \delta\,. \end{cases} \tag{5.9}$$

Let $\frac{1}{2^{k_0}} < \delta$ and let t be an arbitrary point of $[a,b]$. Furthermore, let

$$\alpha_{\ell-1}, \alpha_\ell \in \mathcal{P}_{k_0} = \{\alpha_0, \alpha_1, \ldots, \alpha_{p_{k_0}}\} \quad \text{and} \quad t \in [\alpha_{\ell-1}, \alpha_\ell].$$

Then

$$|\alpha_\ell - \alpha_{\ell-1}| = \frac{1}{2^{k_0}} < \delta$$

and, according to (5.4), (5.5) and (5.9), we get for $k \geq k_0$

$$\begin{aligned} |A_k(t) - A(t)| &= |A_k(t) - A_k(\alpha_{\ell-1}) + A(\alpha_{\ell-1}) - A(t)| \\ &\leq |A_k(t) - A_k(\alpha_{\ell-1})| + \frac{\varepsilon}{2} \\ &\leq \left| A(\alpha_{\ell-1}) + [A(\alpha_\ell) - A(\alpha_{\ell-1})] \left[\frac{t-\alpha_{\ell-1}}{\alpha_\ell - \alpha_{\ell-1}}\right] - A(\alpha_{\ell-1}) \right| + \frac{\varepsilon}{2} \\ &= |A(\alpha_\ell) - A(\alpha_{\ell-1})| \left[\frac{t-\alpha_{\ell-1}}{\alpha_\ell - \alpha_{\ell-1}}\right] + \frac{\varepsilon}{2} \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

As k_0 was chosen independently of t, we can conclude that (2.7) is true.

STEP 2. Analogously we can show that (2.29) holds for $\{f_k\}$ and f.

STEP 3. By Lemma 5.4, (2.6) holds. Moreover, as A and A_k, $k \in \mathbb{N}$, are continuous, by Theorem 2.10 the equations (5.1) and (5.7), $k \in \mathbb{N}$, have unique solutions and we can complete the proof using Theorem 2.18. □

5.2 Approximated solutions

In this section we will continue the consideration of the topics mentioned at the close of the previous section. Our aim is to disclose the relationship between solutions of generalized linear differential equation and limits of solutions of corresponding approximating sequences of linear ordinary differential equations.

We start by introducing the notions of *piecewise linear approximation* and *approximated solution* to the generalized linear differential equation (5.1). Recall that the divisions D_k occurring below have been introduced in Notation 5.3. Furthermore, in addition to (2.1) we will use the following notations.

5.6 Notation. For given $A \in \mathbf{BV}^{n\times n}[a,b]$, $f \in \mathbf{G}^n[a,b]$ and $k \in \mathbb{N}$, we denote

$$\begin{cases} \mathfrak{S}^+(f;[a,b]) & = \{t \in [a,b] \colon \Delta^+ f(t) \neq 0\}, \\ \mathfrak{S}^-(f;[a,b]) & = \{t \in [a,b] \colon \Delta^- f(t) \neq 0\}, \\ \mathfrak{S}(f;[a,b]) & = \mathfrak{S}^+(f;[a,b]) \cup \mathfrak{S}^-(f;[a,b]), \\ \mathfrak{S}(A,f;[a,b]) & = \mathfrak{S}(A;[a,b]) \cup \mathfrak{S}(f;[a,b]). \end{cases}$$

and

$$\begin{cases} \mathfrak{U}^+(A,k;[a,b]) & = \{t \in [a,b] \colon |\Delta^+ A(t)| \geq \frac{1}{k}\}, \\ \mathfrak{U}^-(A,k;[a,b]) & = \{t \in [a,b] \colon |\Delta^- A(t)| \geq \frac{1}{k}\}, \\ \mathfrak{U}(A,k;[a,b]) & = \mathfrak{U}^+(A,k;[a,b]) \cup \mathfrak{U}^-(A,k;[a,b]), \\ \mathfrak{U}^+(f,k;[a,b]) & = \{t \in [a,b] \colon |\Delta^+ A(t)| \geq \frac{1}{k}\}, \\ \mathfrak{U}^-(f,k;[a,b]) & = \{t \in [a,b] \colon |\Delta^- A(t)| \geq \frac{1}{k}\}, \\ \mathfrak{U}(f,k;[a,b]) & = \mathfrak{U}^+(f,k;[a,b]) \cup \mathfrak{U}^-(f,k;[a,b]), \\ \mathfrak{U}(A,f,k;[a,b]) & = \mathfrak{U}(A,k;[a,b]) \cup \mathfrak{U}(f,k;[a,b]). \end{cases}$$

5.7 Remark. In particular, we have

$$\mathfrak{S}(A;[a,b]) = \bigcup_{k=1}^{\infty} \mathfrak{U}(A,k;[a,b]).$$

5.8 Definition. Let $A \in \mathbf{BV}^{n\times n}[a,b]$, $f \in \mathbf{G}^n[a,b]$ and let $D_k \in \mathcal{D}[a,b]$ be given by (5.4). We say that the sequence $\{A_k,\ f_k\} \subset \mathbf{AC}^{n\times n}[a,b] \times \mathbf{AC}^n[a,b]$ is a *piecewise linear approximation* ($p\,\ell$-approximation) of $(A,\ f)$ if there exists a sequence $\{\mathcal{P}_k\}$ of divisions of the interval $[a,b]$ such that

$$\mathcal{P}_k \supset D_k \cup \mathfrak{U}(A,f,k;[a,b]) \quad \text{for } k \in \mathbb{N} \tag{5.10}$$

and $A_k,\ f_k$ are for $k \in \mathbb{N}$ defined by (5.5) and (5.6).

5.9 Remark. Let $\{A_k,\ f_k\}$ be a $p\,\ell$-approximation of $(A,\ f)$. Then (2.6) is true due to Lemma 5.4. Furthermore, as A_k are continuous, due to (2.22), we have $c_{A_k}=1$ for $k\in\mathbb{N}$. Hence, Corollary 2.15 yields

$$\operatorname{var}_a^b(x_k-f_k)\le\alpha^*\,(|\widetilde{x}_0|+2\|f_k\|)\quad\text{for all}\quad k\in\mathbb{N}$$

and, by Helly's Theorem, there is a subsequence $\{x_{k_m}-f_{k_m}\}$ of $\{x_k-f_k\}$ and $y\in\mathbf{G}^n[a,b]$ and such that

$$\lim_{m\to\infty}(x_{k_m}(t)-f_{k_m}(t))=y(t)+f(t)\quad\text{for each}\quad t\in[a,b].$$

In particular,

$$\lim_{m\to\infty}x_{k_m}(t)=w(t)+f(t)$$

for all $t\in[a,b]$ such that $\lim_{m\to\infty}f_{k_m}(t)=f(t)$.

Notice that if the set $\mathfrak{S}(f;[a,b])$ has at most a finite number of elements, then

$$\lim_{k\to\infty}f_k(t)=f(t)\quad\text{for all}\quad t\in[a,b].\tag{5.11}$$

5.10 Definition. Let $A\in\mathbf{BV}^{n\times n}[a,b]$, $f\in\mathbf{G}^n[a,b]$ and $\widetilde{x}_0\in\mathbb{R}^n$. We say that $y\colon\ [a,b]\to\mathbb{R}^n$ is an *approximated solution* to equation (5.1) on the interval $[a,b]$ if there is a $p\,\ell$-approximation $\{A_k,\ f_k\}\in\mathbf{AC}^{n\times n}[a,b]\times\mathbf{AC}^n[a,b]$ of $(A,\ f)$ such that

$$\lim_{k\to\infty}x_k(t)=y(t)\quad\text{on }[a,b]\tag{5.12}$$

holds for solutions x_k, $k\in\mathbb{N}$, of the corresponding approximating initial value problems (5.8).

5.11 Remark. Notice that, using the language of Definitions 5.8 and 5.10, we can translate Theorem 5.5 into the following form:

Assume that $A\in\mathbf{BV}^{n\times n}[a,b]\cap\mathbf{C}^{n\times n}[a,b]$ and $f\in\mathbf{C}^n[a,b]$. Then, the equation (5.1) *has a unique approximated solution y on $[a,b]$ and y coincides on $[a,b]$ with the solution of* (5.1).

In the rest of this section we consider the case when the set $\mathfrak{S}(A,f;[a,b])$ of discontinuities of the coefficients A, f is non empty. We will start with the simplest case $\mathfrak{S}(A,f;[a,b])=\{b\}$.

5.12 Lemma. *Let $A\in\mathbf{BV}^{n\times n}[a,b]$ and $f\in\mathbf{G}^n[a,b]$ be continuous on $[a,b)$ and such that*

$$|\Delta^-A(b)|\,|\Delta^-f(b)|=0\tag{5.13}$$

and let $\widetilde{x}_0\in\mathbb{R}^n$.

Then the equation (5.1) *has a unique approximated solution y on $[a,b]$. Furthermore, y is continuous on $[a,b)$*

$$y(b)=\exp\big(\Delta^-A(b)\big)\ y(b-)+\Delta^-f(b)\tag{5.14}$$

and y coincides with the solution of (5.1) *on $[a,b)$.*

Proof. STEP 1. Let $\{A_k,\, f_k\}$ be an arbitrary $p\,\ell$-approximation of $\{A,\, f\}$ and let $\{\mathcal{P}_k\}$ be the corresponding sequence of divisions of $[a,b]$ fulfilling (5.5) and (5.6). Notice that, under our assumptions, $\mathcal{P}_k = D_k$ for $k \in \mathbb{N}$. For $k \in \mathbb{N}$, put

$$\tau_k = \max\{t \in D_k\colon\, t < b\}.$$

By (5.4) we have $b - \frac{b-a}{2^k} < \tau_k < b$ for $k \in \mathbb{N}$, and hence

$$\lim_{k\to\infty} \tau_k = b. \tag{5.15}$$

For $k \in \mathbb{N}$ and $t \in [a,b]$, define

$$\widetilde{A}_k(t) = \begin{cases} A_k(t) & \text{if } \ t \in [a,\tau_k], \\[2ex] A(\tau_k) + \dfrac{A(b-) - A(\tau_k)}{b-\tau_k}\,(t-\tau_k) & \text{if } \ t \in (\tau_k, b], \end{cases}$$

$$\widetilde{f}_k(t) = \begin{cases} f_k(t) & \text{if } \ t \in [a,\tau_k], \\[2ex] f(\tau_k) + \dfrac{f(b-) - f(\tau_k)}{b-\tau_k}\,(t-\tau_k) & \text{if } \ t \in (\tau_k, b]. \end{cases}$$

Furthermore, let

$$\widetilde{A}(t) = \begin{cases} A(t) & \text{if } \ t \in [a,b), \\[1ex] A(b-) & \text{if } \ t = b, \end{cases} \qquad \widetilde{f}(t) = \begin{cases} f(t) & \text{if } \ t \in [a,b), \\[1ex] f(b-) & \text{if } \ t = b. \end{cases} \tag{5.16}$$

We have $\widetilde{A}_k \in \mathbf{AC}^{n\times n}[a,b]$, $\widetilde{f}_k \in \mathbf{AC}^{n}[a,b]$ for $k \in \mathbb{N}$, $\widetilde{A} \in \mathbf{BV}^{n\times n}[a,b] \cap \mathbf{C}^{n\times n}[a,b]$ and $\widetilde{f} \in \mathbf{C}^{n}[a,b]$.

Consider problems (5.1), (5.8) and

$$u_k' = \widetilde{A}_k'(t)\,u_k + \widetilde{f}_k'(t), \quad u_k(a) = \widetilde{x}_0, \quad k \in \mathbb{N}, \tag{5.17}$$

and

$$u(t) = \widetilde{x}_0 + \int_a^t \mathrm{d}[\widetilde{A}]\,u + \widetilde{f}(t) - \widetilde{f}(a). \tag{5.18}$$

Let $\{x_k\}$ and $\{u_k\}$ be the sequences of solutions on $[a,b]$ of problems (5.8) and (5.17), respectively. We can see that, for each $k \in \mathbb{N}$, u_k coincides with x_k on $[a,\tau_k]$. Furthermore, by Theorem 2.10, equation (5.18) possesses a unique solution u on $[a,b]$ a u is continuous on $[a,b]$. It's easy to see that the relations

$$\widetilde{A}_k \rightrightarrows \widetilde{A} \quad \text{and} \quad \widetilde{f}_k \rightrightarrows \widetilde{f} \quad \text{on } [a,b]$$

are true. Therefore, by Theorem 2.18, we get

$$u_k \rightrightarrows u \quad \text{on } [a,b]. \tag{5.19}$$

Since $x_k = u_k$ on $[a,\tau_k]$, and due to (5.15), we have

$$\lim_{k\to\infty} x_k(t) = u(t) \quad \text{for } t \in [a,b). \tag{5.20}$$

STEP 2. Next we will prove that

$$\lim_{k\to\infty} x_k(\tau_k) = u(b). \tag{5.21}$$

Indeed, let $\varepsilon > 0$ be given and let $\delta > 0$ be such that

$$|u(t) - u(b)| < \frac{\varepsilon}{2} \quad \text{for} \quad t \in [b-\delta, b]$$

Further, by (5.19), there is a $k_0 \in \mathbb{N}$ such that

$$\tau_k \in [b-\delta, b) \quad \text{and} \quad \|u_k - u\| < \frac{\varepsilon}{2} \quad \text{whenever} \quad k \geq k_0.$$

Consequently,

$$\begin{aligned} |x_k(\tau_k) - u(b-)| &\leq |x_k(\tau_k) - u(\tau_k)| + |u(\tau_k) - x(b-)| \\ &= |u_k(\tau_k) - u(\tau_k)| + |u(\tau_k) - x(b-)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

holds for $k \geq k_0$. This completes the proof of (5.21).

STEP 3. On the intervals $[\tau_k, b]$, the equations from (5.8) reduce to the equations with constant coefficients

$$x_k' = B_k\, x_k + e_k,$$

where

$$B_k = \frac{A_k(b) - A_k(\tau_k)}{b - \tau_k} \quad \text{and} \quad e_k = \frac{f_k(b) - f_k(\tau_k)}{b - \tau_k}.$$

Their solutions x_k are on $[\tau_k, b]$ given by

$$x_k(t) = \exp\left(B_k\,(t-\tau_k)\right) x_k(\tau_k) + \left(\int_{\tau_k}^t \exp\left(B_k\,(t-r)\right) \mathrm{d}r\right) e_k.$$

In particular,

$$\begin{aligned} x_k(b) &= \exp\left(A(b) - A(\tau_k)\right) x_k(\tau_k) \\ &\quad + \frac{1}{b-\tau_k}\left(\int_{\tau_k}^b \exp\left([A(b) - A(\tau_k)]\,\frac{b-r}{b-\tau_k}\right) \mathrm{d}r\right) [f_k(b) - f_k(\tau_k)]. \end{aligned}$$

By Lemma 5.2, we have

$$\begin{aligned} &\lim_{k\to\infty} \frac{1}{b-\tau_k}\left(\int_{\tau_k}^b \exp\left([A(b) - A(\tau_k)]\frac{b-r}{b-\tau_k}\right)\, \mathrm{d}r\right) [f_k(b) - f_k(\tau_k)] \\ &\quad = \lim_{k\to\infty} \frac{1}{b-\tau_k}\left(\int_{\tau_k}^b \exp\left(\Delta^- A(b)\frac{b-r}{b-\tau_k}\right)\, \mathrm{d}r\right) \Delta^- f(b). \end{aligned}$$

In particular, having in mind (5.21), we obtain

$$\lim_{k\to\infty} x_k(b) = \begin{cases} \exp\left(\Delta^- A(b)\right) u(b) & \text{if } \ \Delta^- f(b) = 0, \\ u(b) + \Delta^- f(b) & \text{if } \ \Delta^- A(b) = 0. \end{cases}$$

So, in view of the assumption (5.13), we can conclude that the relation

$$\lim_{k\to\infty} x_k(b) = \exp\big(\Delta^- A(b)\big)\, u(b) + \Delta^- f(b) \tag{5.22}$$

is true.

Step 4. Define

$$y(t) = \begin{cases} u(t) & \text{if } \ t \in [a,b), \\ \exp\left(\Delta^- A(b)\right) u(b) + \Delta^- f(b) & \text{if } \ t = b. \end{cases}$$

Then $y(b-) = u(b)$, $y(t) = \lim\limits_{k\to\infty} x_k(t)$ for $t \in [a,b)$ due to (5.21) and $y(b) = \lim\limits_{k\to\infty} x_k(b)$ due to (5.22). Therefore, y is a $p\,\ell$-approximated solution of (5.1). Since it does not depend upon the choice of the approximating sequence $\{A_k,\, f_k\}$, we can see that y is also the unique approximated solution of (5.1). This completes the proof. □

The following assertion can be related to Lemma 5.12 by introducing new independent variable s by a substitution $s = a + b - t$. Nevertheless, we prefer to give here its direct proof, though little bit more concise.

5.13 Lemma. *Let $A \in \mathbf{BV}^{n\times n}[a,b]$ and $f \in \mathbf{G}^n[a,b]$ be continuous on $(a,b]$ and such that*

$$|\Delta^+ A(a)|\, |\Delta^+ f(a)| = 0 \tag{5.23}$$

and let $\widetilde{x}_0 \in \mathbb{R}^n$.

Then the equation (5.1) has a unique approximated solution y on $[a,b]$. Furthermore, y is continuous on $(a,b]$,

$$y(a+) = \exp\big(\Delta^+ A(a)\big)\, \widetilde{x}_0 + \Delta^+ f(a)$$

and y coincides on $(a,b]$ with the solution of the equation

$$y(t) = \widetilde{y} + \int_a^t \mathrm{d}[\widetilde{A}]\, y + \widetilde{f}(t) - \widetilde{f}(a), \tag{5.24}$$

where

$$\widetilde{y} = \exp\big(\Delta^+ A(a)\big)\, \widetilde{x}_0 + \Delta^+ f(a),$$

and

$$\widetilde{A}(t) = \begin{cases} A(a+) & \text{if } \ t = a, \\ A(t) & \text{if } \ t \in (a,b] \end{cases} \quad \text{and} \quad \widetilde{f}(t) = \begin{cases} f(a+) & \text{if } \ t = a, \\ f(t) & \text{if } \ t \in (a,b]. \end{cases}$$

Proof. STEP 1. On the intervals $[a, \tau_k]$, the equations from (5.8) reduce to equations with constant coefficients

$$A_k'(t) = \frac{A_k(\tau_k) - A_k(a)}{\tau_k - a}, \quad f_k'(t) = \frac{f_k(\tau_k) - f_k(a)}{\tau_k - a}.$$

Their solutions x_k are on $[a, \tau_k]$ given by

$$\begin{aligned} x_k(t) &= \exp\left(\frac{A_k(\tau_k) - A_k(a)}{\tau_k - a}\,(t-a)\right)\widetilde{x}_0 \\ &\quad + \left(\int_a^t \exp\left(\frac{A_k(\tau_k) - A_k(a)}{\tau_k - a}\,(t-r)\right)\mathrm{d}r\right)\frac{f_k(\tau_k) - f_k(a)}{\tau_k - a}. \end{aligned}$$

In particular,

$$\begin{aligned} x_k(\tau_k) &= \exp\left(A(\tau_k) - A(a)\right) x_k(\tau_k) \\ &\quad + \frac{1}{\tau_k - a}\left(\int_a^{\tau_k} \exp\left([A(\tau_k) - A(a)]\,\frac{\tau_k - r}{\tau_k - a}\right)\mathrm{d}r\right)[f_k(\tau_k) - f_k(\tau_k)]. \end{aligned}$$

By Lemma 5.2, we have

$$\begin{aligned} &\lim_{k\to\infty} \frac{1}{\tau_k - a}\left(\int_a^{\tau_k} \exp\left([A(\tau_k) - A(a)]\,\frac{\tau_k - r}{\tau_k - a}\right)\mathrm{d}r\right)[f_k(\tau_k) - f_k(a)] \\ &\quad = \lim_{k\to\infty} \frac{1}{\tau_k - a}\left(\int_a^{\tau_k} \exp\left(\Delta^+ A(a)\frac{\tau_k - r}{\tau_k - a}\right)\,\mathrm{d}r\right)\Delta^+ f(b). \end{aligned}$$

Thus,

$$\lim_{k\to\infty} x_k(a) = \begin{cases} \exp\left(\Delta^+ A(a)\right)\widetilde{x}_0 & \text{if } \ \Delta^+ f(a) = 0, \\ \widetilde{x}_0 + \Delta^+ f(a) & \text{if } \ \Delta^+ A(a) = 0. \end{cases}$$

With respect to the assumption (5.13), we can conclude that the relation

$$\lim_{k\to\infty} x_k(\tau_k) = \exp\left(\Delta^+ A(a)\right)\widetilde{x}_0 + \Delta^+ f(a) = \widetilde{y} \tag{5.25}$$

is true.

STEP 2. Let $\{A_k,\, f_k\}$ be an arbitrary $p\,\ell$-approximation of $\{A,\, f\}$ and let $\{D_k\}$ be the corresponding sequence of divisions of $[a, b]$ fulfilling (5.5) and (5.6). Let $\{x_k\}$ be a sequence of solutions of the approximating initial value problems (5.8) on $[a, b]$. Consider equation (5.24). By Theorem 2.10, it has a unique solution u on $[a, b]$, u is continuous on $[a, b]$ and, by an argument analogous to that used in STEP 1 of the proof of Lemma L3.6, we can show that the relation

$$\lim_{k\to\infty} x_k(t) = u(t) \quad \text{for} \quad t \in (a, b] \tag{5.26}$$

is true. Finally, notice that, with respect to (5.25), we have also

$$\lim_{k\to\infty} x_k(\tau_k) = u(a).$$

STEP 3. Analogously to STEP 4 of the proof of Lemma 5.12, we can complete the proof by showing that the function

$$y(t) = \begin{cases} \widetilde{x}_0 & \text{if } t = a, \\ u(t) & \text{if } t \in (a,b], \end{cases}$$

is the unique approximated solution of (5.1). □

5.14 Remark. First, let us notice that if $a < c < b$ and the functions y_1 and y_2 are respectively $p\,\ell$-approximated solutions to

$$x(t) = \widetilde{x}_1 + \int_a^t \mathrm{d}[A]\,x + f(t) - f(a), \quad t \in [a,c]$$

and

$$x(t) = \widetilde{x}_2 + \int_c^t \mathrm{d}[A]\,x + f(t) - f(c), \quad t \in [c,b],$$

where $\widetilde{x}_2 = y_1(c)$, then the function

$$y(t) = \begin{cases} y_1(t) & \text{if } t \in [a,c], \\ y_2(t) & \text{if } t \in (c,b] \end{cases}$$

is a $p\,\ell$-approximated solution to (5.1).

The main result of this section is the following Theorem 5.15.

5.15 Theorem. *Assume that* $A \in \mathbf{BV}^{n\times n}[a,b]$, $f \in \mathbf{G}^n[a,b]$, $s_1, s_2, \dots, s_m \in (a,b)$, $\mathfrak{S}(A,f;[a,b]) = \{s_1, s_2, \dots, s_m\}$ *and*

$$\begin{cases} \mathfrak{S}^-(A;[a,b]) \cap \mathfrak{S}^-(f;[a,b]) = \emptyset, \\ \mathfrak{S}^+(A;[a,b]) \cap \mathfrak{S}^+(f;[a,b]) = \emptyset. \end{cases} \tag{5.27}$$

Then, for each $\widetilde{x}_0 \in \mathbb{R}^n$, *there is exactly one approximated solution* y *of equation* (5.1) *on* $[a,b]$. *Furthermore,* $\mathfrak{S}(y;[a,b]) = \{s_1, s_2, \dots, s_m\}$ *and*

$$y(t) = \exp\big(\Delta^- A(t)\big)\, y(t-) + \Delta^- f(t) \qquad \text{if } t \in \{s_1, s_2, \dots, s_m\},$$

$$y(t+) = \exp\big(\Delta^+ A(t)\big)\, y(t) + \Delta^+ f(t) \qquad \text{if } t \in \{s_1, s_2, \dots, s_m\},$$

$$y(t) = y(s_{i-1}+) + \int_{s_{i-1}}^t \mathrm{d}[\widetilde{A}^{[i]}]\, y + f(t) - f(s_{i-1}+) \text{ if } t \in (s_{i-1}, s_i),$$

hold for $t \in [a,b]$ *and* $i = 1, 2, \ldots, m+1$, *where* $s_0 = a$, $s_{m+1} = b$ *and*

$$\widetilde{A}^{[i]}(t) = \begin{cases} A(s_{i-1}+) & \text{if } t = s_{i-1}, \\ A(t) & \text{if } t \in (s_{i-1}, s_i), \\ A(s_i-) & \text{if } t = s_i\,. \end{cases}$$

Proof. Having in mind Remark 5.14, we deduce the assertion of Theorem 5.15 by a successive use of Lemmas 5.12 and 5.13. To this aim it is sufficient to choose a division $D = \{\alpha_0, \alpha_1, \ldots, \alpha_r\}$ of $[a,b]$ such that for each subinterval $[\alpha_{k-1}, \alpha_k]$, $k = 1, 2, \ldots, r$, either the assumptions of Lemma 5.12 or the assumptions of Lemma 5.13 are satisfied with with α_{k-1} in place of a and α_k in place of b. □

5.16 Remark. It is natural to expect that Theorem 5.15 can be also obtained as a Corollary of our Theorem 3.1. However, at this moment we are able to justify such a hypothesis only assuming that $f \in \mathbf{BV}^n[a,b]$.

Part II
Singular Periodic Impulse Problems

Part II is devoted to periodic impulse problems for nonlinear second order impulsive differential equations of the form

$$u'' = f(t, u, u'), \tag{6.1}$$

$$\begin{cases} u(t_i+) = u(t_i) + J_i(u, u'), \\ u'(t_i+) = u'(t_i) + M_i(u, u'), \quad i = 1, 2, \dots, m, \end{cases} \tag{6.2}$$

$$u(0) = u(T), \quad u'(0) = u'(T) \tag{6.3}$$

where the function $f\colon [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ fulfils the Carathéodory conditions,

$$0 < t_1 < t_2 < \ldots < t_m < T \quad \text{are fixed points of the interval } [0,T] \tag{6.4}$$

and the functionals $J_i,\ M_i\colon \mathbf{G}[0,T] \times \mathbf{G}[0,T] \to \mathbb{R}$, $i = 1, 2, \dots, m$, are continuous.

6 Preliminaries

In this part we establish an existence principle suitable for solving singular impulsive periodic problems.

6.1 Notation. Throughout this part we keep the following notation and conventions: for a real valued function u defined a.e. on $[0,T]$, we put

$$\|u\|_\infty = \sup \operatorname{ess}_{t\in[0,T]} |u(t)| \quad \text{and} \quad \|u\|_1 = \int_0^T |u(s)| \, \mathrm{d}s.$$

For a given interval $J \subset \mathbb{R}$, by $\mathbf{C}(J)$ we denote the set of real valued functions which are continuous on J. Furthermore, $\mathbf{C}^1(J)$ is the set of functions having continuous first derivatives on J and $L_1(J)$ is the set of functions which are Lebesgue integrable on J.

Any function $x\colon [0,T] \to \mathbb{R}$ which possesses finite limits

$$x(t+) = \lim_{\tau\to t+} x(\tau) \quad \text{and} \quad x(s-) = \lim_{\tau\to s-} x(\tau)$$

for all $t \in [0,T)$ and $s \in (0,T]$ is said to be regulated on $[0,T]$. The linear space of functions regulated on $[0,T]$ is denoted by $\mathbf{G}[0,T]$. It is well known that $\mathbf{G}[0,T]$ is a Banach space with respect to the norm $x \in \mathbf{G}[0,T] \to \|x\|_\infty$ (cf. [14, Theorem I.3.6]).

Let $m \in \mathbb{N}$ and let $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = T$ be a division of the interval $[0,T]$. We denote $D = \{t_1, t_2, \ldots, t_m\}$ and define $\mathbf{C}^1_D[0,T]$ as the set of functions $u\colon [0,T] \to \mathbb{R}$ such that

$$u(t) = \begin{cases} u_{[0]}(t) & \text{if } t \in [0,t_1], \\ u_{[1]}(t) & \text{if } t \in (t_1,t_2], \\ \vdots & \vdots \\ u_{[m]}(t) & \text{if } t \in (t_m,T], \end{cases}$$

where $u_{[i]} \in \mathbf{C}^1[t_i,t_{i+1}]$ for $i = 0,1,\ldots,m$. In particular, if $u \in \mathbf{C}^1_D[0,T]$, then u' possesses finite one-sided limits

$$u'(t-) := \lim_{\tau \to t-} u'(\tau) \quad \text{and} \quad u'(s+) := \lim_{\tau \to s+} u'(\tau)$$

for each $t \in (0,T]$ and $s \in [0,T)$. Moreover, $u'(t-) = u'(t)$ for all $t \in (0,T]$ and $u'(0+) = u'(0)$. For $u \in \mathbf{C}^1_D[0,T]$ we put

$$\|u\|_D = \|u\|_\infty + \|u'\|_\infty.$$

Then $\mathbf{C}^1_D[0,T]$ becomes a Banach space when endowed with the norm $\|.\|_D$. Furthermore, by $\mathbf{AC}^1_D[0,T]$ we denote the set of functions $u \in \mathbf{C}^1_D[0,T]$ having first derivatives absolutely continuous on each subinterval (t_i,t_{i+1}), $i = 1,2,\ldots,m+1$.

We say that $f : [0,T] \times \mathbb{R}^2 \mapsto \mathbb{R}$ satisfies on $[0,T] \times \mathbb{R}^2$ the *Carathéodory conditions* if

(i) for each $x \in \mathbb{R}$ and $y \in \mathbb{R}$ the function $f(.,x,y)$ is measurable on $[0,T]$;

(ii) for almost every $t \in [0,T]$ the function $f(t,.,.)$ is continuous on $\mathbb{R}^2$;

(iii) for each compact set $K \subset \mathbb{R}^2$ there is a function $m_K(t) \in L[0,T]$ such that $|f(t,x,y)| \le m_K(t)$ holds for a.e. $t \in [0,T]$ and all $(x,y) \in K$.

The set of functions satisfying the Carathéodory conditions on $[0,T] \times \mathbb{R}^2$ is denoted by $Car([0,T] \times \mathbb{R}^2)$.

Given a subset Ω of a Banach space X, its closure is denoted by $\overline{\Omega}$. As usual, the symbol I stands for the identity operator or the identity matrix. Finally, we will write $\bar{e}$ instead of $\dfrac{1}{T}\displaystyle\int_0^T e(s)\,\mathrm{d}s$ and $\Delta^+ u(t)$ instead of $u(t+) - u(t)$.

If $f \in Car([0,T]\times\mathbb{R}^2)$, problem (6.1)–(6.3) is said to be *regular* and a function $u \in \mathbf{AC}^1_D[0,T]$ is its solution if

$$u''(t) = f(t,u(t),u'(t)) \quad \text{holds for a.e.} \quad t \in [0,T]$$

and conditions (6.2) and (6.3) are satisfied. If $f \notin Car([0,T]\times\mathbb{R}^2)$, problem (6.1)–(6.3) is said to be *singular*.

In this part we will deal with rather simplified, however the most typical, case of the singular problem with

$$f(t,x,y) = c\,y + g(x) + e(t) \text{ for } x \in (0,\infty),\ y \in \mathbb{R} \text{ and a.e. } t \in [0,T],$$

where

$$c \in \mathbb{R}, \quad g \in \mathbf{C}(0,\infty), \quad e \in L_1[0,T]. \tag{6.5}$$

6.2 Definition. A function $u \in \mathbf{AC}^1_D[0,T]$ is called a solution of problem

$$u'' + c\,u' = g(u) + e(t), \quad (6.2),\ (6.3) \tag{6.6}$$

if $u > 0$ a.e. on $[0,T]$,

$$u''(t) + c\ u'(t) = g(u(t)) + e(t) \quad \text{for a.e.} \quad t \in [0,T],$$

and conditions (6.2) and (6.3) are satisfied.

7 Operator representation for impulsive problems

For our purposes an appropriate choice of the operator representation of (6.1)–(6.3) is important. To this aim, let us consider the following impulsive problem with nonlinear two-point boundary conditions

$$u'' + a_2(t)\,u' + a_1(t)\,u = f(t,u,u') \text{ a.e. on } [0,T], \tag{7.1}$$

$$\Delta^+ u(t_i) = J_i(u,u'), \quad \Delta^+ u'(t_i) = M_i(u,u'), \quad i = 1,2,\ldots,m, \tag{7.2}$$

$$P\begin{pmatrix} u(0) \\ u'(0) \end{pmatrix} + Q\begin{pmatrix} u(T) \\ u'(T) \end{pmatrix} = R(u,u'), \tag{7.3}$$

and its linearized version

$$u'' + a_2(t)\,u' + a_1(t)\,u = h(t) \text{ a.e. on } [0,T], \tag{7.4}$$

$$\Delta^+ u(t_i) = d_i, \quad \Delta^+ u'(t_i) = d'_i, \quad i = 1,2,\ldots,m, \tag{7.5}$$

$$P\begin{pmatrix} u(0) \\ u'(0) \end{pmatrix} + Q\begin{pmatrix} u(T) \\ u'(T) \end{pmatrix} = \delta, \tag{7.6}$$

where

$$\begin{cases} J_i \text{ and } M_i\colon \mathbf{G}[0,T]\times\mathbf{G}[0,T]\to\mathbb{R},\ i=1,2,\dots,m, \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{are continuous mappings,} \\ J_i(u,u') = M_i(u,u') = 0, \qquad i=1,2,\dots,m, \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } \ u(t)\equiv u(0) \quad \text{on } [0,T] \end{cases} \tag{7.7}$$

and

$$\begin{cases} a_1,\, h\in L[0,T],\ a_2\in \mathbf{C}[0,T],\ f\in Car([0,T]\times\mathbb{R}^2), \\ \delta\in\mathbb{R}^2,\ d_i,\ d'_i\in\mathbb{R},\ i=1,2,\dots,m, \\ P,\ Q \text{ are real } \ 2\times 2-\text{matrices}, \ \operatorname{rank}(P,Q)=2, \\ R\colon \mathbf{G}[0,T]\times\mathbf{G}[0,T]\to\mathbb{R}^2 \text{ is a continuous mapping.} \end{cases} \tag{7.8}$$

Solutions of problems (7.1)–(7.3) and (7.4)–(7.6) are defined in a natural way quite analogously to the above mentioned definition of regular periodic problems. Problem (7.4)–(7.6) is equivalent to the two-point problem for a special case of generalized linear differential systems of the form

$$x(t) - x(0) - \int_0^t A(s)\,x(s)\,\mathrm{d}s = b(t) - b(0) \quad \text{for } t\in[0,T], \tag{7.9}$$

$$P\,x(0) + Q\,x(T) = \delta, \tag{7.10}$$

where

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t)\end{pmatrix} = \begin{pmatrix} u(t) \\ u'(t)\end{pmatrix}, \quad A(t) = \begin{pmatrix} 0 & 1 \\ -a_1(s) & -a_2(s)\end{pmatrix}, \tag{7.11}$$

$$b(t) = \int_0^t \begin{pmatrix} 0 \\ h(s)\end{pmatrix}\,\mathrm{d}s + \sum_{i=1}^m \begin{pmatrix} d_i \\ d'_i\end{pmatrix}\,\chi_{(t_i,T]}(t), \quad t\in[0,T],$$

and $\chi_{(t_i,T]}(t) = 1$ if $t\in(t_i,T]$, $\chi_{(t_i,T]}(t) = 0$ otherwise. Solutions of (7.9), (7.10) are 2-vector functions of bounded variation on $[0,T]$ satisfying the two-point condition (7.10) and fulfilling the integral equation (7.9) for all $t\in[0,T]$, cf. e.g. [42]. Assume that the homogeneous problem

$$u'' + a_2(t)\,u' + a_1(t)\,u = 0, \quad P\begin{pmatrix} u(0) \\ u'(0)\end{pmatrix} + Q\begin{pmatrix} u(T) \\ u'(T)\end{pmatrix} = 0 \tag{7.12}$$

has only the trivial solution. Then, obviously, the homogeneous problem corresponding to (7.9), (7.10) has also only the trivial solution. In view of [51, Theorems 4.2 and 4.3] (see also [41, Theorem 4.1]), problem (7.9), (7.10) has a unique solution x and it is given by

$$x(t) = X(t)\,D^{-1}\,\delta + \int_0^T \Gamma(t,s)\,\mathrm{d}[b(s)], \quad t\in[0,T], \tag{7.13}$$

where X is the fundamental matrix solution of the homogeneous equation $x' - A(t)\,x = 0$ fulfilling the condition $X(0) = \mathrm{I}$, $D = P\,X(0) + Q\,X(T)$ and

$$\Gamma(t,s) = (\gamma_{i,j}(t,s))_{i,j=1,2}$$

is Green's matrix for the problem

$$x' - A(t)\,x = 0, \quad P\,x(0) + Q\,x(T) = 0.$$

Recall that, for each $s \in (0,T)$, the matrix function $t \to \Gamma(t,s)$ is absolutely continuous on $[0,T] \setminus \{s\}$ and

$$\begin{aligned}
&\frac{\partial}{\partial t}\Gamma(t,s) - A(t)\,\Gamma(t,s) = 0 \quad \text{for a.e.} \quad t \in [0,T],\\
&P\,\Gamma(0,s) + Q\,\Gamma(T,s) = 0,\\
&\Gamma(t+,t) - \Gamma(t-,t) = \mathrm{I} \quad \text{for} \quad t \in (0,T).
\end{aligned}$$

Moreover, the component $\gamma_{1,2}$ of Γ is absolutely continuous on $[0,T]$ for each $s \in (0,T)$ and

$$\frac{\partial}{\partial t}\gamma_{1,2}(t,s) = \gamma_{2,2}(t,s) \quad \text{for a.e.} \quad t \in [0,T].$$

Denote $G(t,s) = \gamma_{1,2}(t,s)$. Then $G(t,s)$ is Green's function of (7.12). Furthermore, we have

$$\frac{\partial}{\partial s}\Gamma(t,s) = -\Gamma(t,s)\,A(s) \quad \text{for all} \;\; t \in (0,T) \text{ and a.e. } s \in [0,T].$$

In particular,

$$\gamma_{1,1}(t,s) = -\frac{\partial}{\partial s}G(t,s) + a_1(s)\,G(t,s) \quad \text{for all} \;\; t \in [0,T] \text{ and a.e. } s \in [0,T].$$

Inserting (7.11) into (7.13) we get that, for each $h \in L[0,T]$, c, d_i, $d_i' \in \mathbb{R}$ and $i = 1,2,\dots,m$, the unique solution u of problem (7.4)–(7.6) is given by

$$\begin{cases} u(t) = U(t)\,\delta + \displaystyle\int_0^T G(t,s)\,h(s)\,\mathrm{d}s \\ \qquad + \displaystyle\sum_{i=1}^m \Big(-\frac{\partial}{\partial s}G(t,t_i) + a_1(t)\,G(t,t_i)\Big)\,d_i + \sum_{i=1}^m G(t,t_i)\,d_i' \\ \qquad \text{for} \;\; t \in [0,T], \end{cases} \tag{7.14}$$

where $U(t) = \big(u_{11}(t), u_{12}(t)\big)$ is the first row of the matrix $X(t)\,D^{-1}$. Now, choose an arbitrary $w \in \mathbf{C}^1_D[0,T]$ and put

$$\begin{cases} h(t) = f(t, w(t), w'(t)) \quad \text{for a.e.} \quad t \in [0,T], \\ d_i = J_i(w,w'), \;\; d_i' = M_i(w,w'),\; i = 1,2,\dots,m, \\ \delta = R(w,w'). \end{cases}$$

Then $h \in L[0,T]$, c, d_i, $d'_i \in \mathbb{R}$, $i = 1,2,\dots,m$, and there is a unique $u \in \mathbf{AC}^1_D[0,T]$ fulfilling (7.4)–(7.6) and it is given by (7.14). Therefore, we conclude that $u \in \mathbf{C}^1_D[0,T]$ is a solution to (7.1)–(7.3) if and only if

$$\begin{cases} u(t) = U(t)\,R(u,u') + \displaystyle\int_0^T G(t,s)\,f(s,u(s),u'(s))\,\mathrm{d}s \\ \qquad + \displaystyle\sum_{i=1}^m \Big(-\frac{\partial}{\partial s}\,G(t,t_i) + a_1(t)\,G(t,t_i)\Big)\,J_i(u,u') \\ \qquad + \displaystyle\sum_{i=1}^m G(t,t_i)\,M_i(u,u') \quad \text{for} \ \ t \in [0,T]. \end{cases} \tag{7.15}$$

Let us define operators F_1 and $F_2 \colon \mathbf{C}^1_D[0,T] \to \mathbf{C}^1_D[0,T]$ by

$$F_1(u)(t) = \int_0^T G(t,s)\,f(s,u(s),u'(s))\,\mathrm{d}s, \quad t \in [0,T]$$

and

$$F_2(u)(t) = U(t)\,R(u,u') + \sum_{i=1}^m \Big(-\frac{\partial}{\partial s}\,G(t,t_i) + a_1(t)\,G(t,t_i)\Big)\,J_i(u,u') + \sum_{i=1}^m G(t,t_i)\,M_i(u,u'), \quad t \in [0,T].$$

The former one, F_1, is a composition of the Green type operator

$$h \in L_1[0,T] \to \int_0^T G(t,s)\,h(s)\,\mathrm{d}s \in \mathbf{C}^1[0,T],$$

which is known to map equiintegrable subsets[2] of $L_1[0,T]$ onto relatively compact subsets of $\mathbf{C}^1[0,T] \subset \mathbf{C}^1_D[0,T]$, and of the superposition operator generated by $f \in Car([0,T] \times \mathbb{R}^2)$, which, similarly to the classical setting, maps bounded subsets of $\mathbf{C}^1_D[0,T]$ to equiintegrable subsets of $L_1[0,T]$. Therefore, it is easy to see that F_1 is completely continuous. Furthermore, since R, J_i, M_i, $i = 1,2,\dots,m$, are continuous mappings, the operator F_2 is continuous as well. Having in mind that F_2 maps bounded sets onto bounded sets and its values are contained in a $2(m{+}1)$-dimensional subspace[3] of $\mathbf{C}^1_D[0,T]$, we conclude that the operators F_2 and $F = F_1 + F_2$ are completely continuous as well.

So, we have the following assertion.

7.1 Theorem. *Assume* (6.4), (7.7) *and* (7.8). *Furthermore, let problem* (7.12) *have Green's function $G(t,s)$ and let $U \in \mathbf{AC}^1_D[0,T]$ have the same meaning as in*

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[2] i.e. sets of functions having a common integrable majorant

[3] i.e. spanned over the set

$$\{u_{11},\, u_{12},\, \delta,\, G(.,t_i),\, \big(-\tfrac{\partial}{\partial s}\,G(.,t_i) + a_1\,G(.,t_i)\big)\,;\ i = 1,2,\dots,m\}$$

(7.16). *Then* $u \in \mathbf{AC}^1_D$ *is a solution to* (7.1)–(7.3) *if and only if* $u = F(u)$, *where* $F\colon \mathbf{C}^1_D[0,T] \to \mathbf{C}^1_D[0,T]$ *is the completely continuous operator given by*

$$\begin{cases} F(u)(t) = U(t)\,R(u,u') \\ \quad + \displaystyle\int_0^T G(t,s)\,(f(t,u(s),u'(s)) - a_1(s)\,u(s) - a_2(s)\,u'(s))\,\mathrm{d}s \\ \quad + \displaystyle\sum_{i=1}^m \left(-\frac{\partial}{\partial s}\,G(t,t_i) + a_1(t)G(t,t_i)\right) J_i(u,u') \\ \quad + \displaystyle\sum_{i=1}^m G(t,t_i)\,M_i(u,u'),\ t \in [0,T]. \end{cases} \tag{7.16}$$

In particular, if $a_1(t) = a_2(t) = 0$ on $[0,T]$,

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

then problem (7.12) reduces to the simple Dirichlet problem

$$u'' = 0, \quad u(0) = u(T) = 0$$

and its Green's function is well-known:

$$G(t,s) = \begin{cases} \dfrac{s\,(t-T)}{T} & \text{if } 0 \le s < t \le T, \\ \dfrac{t\,(s-T)}{T} & \text{if } 0 \le t \le s \le T \end{cases} \tag{7.17}$$

and

$$\frac{\partial}{\partial s}\,G(t,s) = \begin{cases} \dfrac{T-t}{T} & \text{if } 0 \le s < t \le T, \\ -\dfrac{t}{T} & \text{if } 0 \le t \le s \le T. \end{cases}$$

Furthermore, it is easy to verify that

$$X(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \quad \text{for } t \in [0,T], \quad D^{-1} = \frac{1}{T}\begin{pmatrix} T & 0 \\ -1 & 1 \end{pmatrix}$$

and

$$U(t) = \frac{1}{T}\,(T-t,t) \quad \text{for } t \in [0,T].$$

Consequently,

$$U(t)\,\delta = d \quad \text{holds for each } d \in \mathbb{R} \text{ and } \delta = \begin{pmatrix} d \\ d \end{pmatrix}.$$

Now, notice that the periodic boundary conditions (6.3) can be reformulated as

$$u(0) = u(0) + u'(0) - u'(T), \quad u(T) = u(0) + u'(0) - u'(T),$$

i.e., in the form (7.3), where

$$R(u,v) = \begin{pmatrix} u(0) + v(0) - v(T) \\ u(0) + v(0) - v(T) \end{pmatrix} \quad \text{for } \ u, v \in \mathbf{G}[0,T].$$

In particular,

$$U(t)\, R(u,u') = u(0) + u'(0) - u'(T) \text{ for each } t \in [0,T] \text{ and each } u \in \mathbf{G}[0,T].$$

To summarize, the following assertion is a corollary of Theorem 7.1:

7.2 Proposition. *Assume* (6.4) *and* (7.7)*. Let* $f \in Car([0,T] \times \mathbb{R}^2)$ *and let the function* $G(t,s)$ *be given by* (7.17)*. Then* $u \in \mathbf{AC}^1_D$ *is a solution to* (6.1)–(6.3) *if and only if* $u = F(u)$*, where* $F\colon \mathbf{C}^1_D[0,T] \to \mathbf{C}^1_D[0,T]$ *is the completely continuous operator given by*

$$\begin{cases} F(u)(t) = u(0) + u'(0) - u'(T) + \displaystyle\int_0^T G(t,s)\, f(t,u(s),u'(s))\, \mathrm{d}s \\ \quad - \displaystyle\sum_{i=1}^m \frac{\partial}{\partial s} G(t,t_i)\, J_i(u,u') + \sum_{i=1}^m G(t,t_i)\, M_i(u,u'), \ t \in [0,T]. \end{cases} \tag{7.18}$$

7.3 Remark. Similarly, $u \in \mathbf{AC}^1_D$ is a solution to the impulsive Dirichlet problem (6.1), (6.2), $u(0) = u(T) = c$ if and only if $u = F_{dir}\, u$, where

$$\begin{cases} F_{dir}(u)(t) = c + \displaystyle\int_0^T G(t,s)\, f(t,u(s),u'(s))\, \mathrm{d}s \\ \quad - \displaystyle\sum_{i=1}^m \frac{\partial}{\partial s} G(t,t_i)\, J_i(u,u') + \sum_{i=1}^m G(t,t_i)\, M_i(u,u'), \ t \in [0,T]. \end{cases}$$

8 Existence principle

8.1 Theorem. *Let assumptions* (6.4), (6.5) *and* (7.7) *hold. Furthermore, assume that there exist* $r \in (0,\infty)$, $R \in (r,\infty)$ *and* $R' \in (0,\infty)$ *such that*

(i) $r < v < R$ *on* $[0,T]$ *and* $\|v'\|_\infty < R'$ *for each* $\lambda \in (0,1]$ *and for each positive solution* v *of the problem*

$$v''(t) = \lambda\,(-c\,v'(t) + g(v(t)) + e(t)) \quad \textit{for a.e. } t \in [0,T], \tag{8.1}$$

$$\Delta^+ v(t_i) = \lambda\, J_i(v,v'), \qquad i = 1,2,\ldots,m, \tag{8.2}$$

$$\Delta^+ v'(t_i) = \lambda\, M_i(v,v'), \quad i = 1,2,\ldots,m, \tag{8.3}$$

$$v(0) = v(T), \quad v'(0) = v'(T); \tag{8.4}$$

(ii) $(g(x)+\bar e=0) \quad \Longrightarrow \quad r<x<R$;

(iii) $(g(r)+\bar e)\,(g(R)+\bar e)<0$.

Then problem (6.6) *has a solution u such that*

$$r<u<R \ \text{ on } [0,T] \ \text{ and } \ \|u'\|_\infty<R'.$$

Proof. STEP 1. For $\lambda\in[0,1]$ and $v\in\mathbf{C}^1_D[0,T]$ denote

$$\begin{cases} \Xi_\lambda(v)=\displaystyle\int_0^T g(v(s))\,\mathrm{d}s+T\,\bar e \\ \qquad\qquad +\displaystyle\sum_{i=1}^m M_i(v,v')+\lambda\,c\sum_{i=1}^m J_i(v,v'). \end{cases} \tag{8.5}$$

Notice that

$$\Xi_\lambda(v)=0 \quad \text{holds for all solutions } \ v\in\mathbf{C}^1_D[0,T] \text{ of } (8.1)--(8.4). \tag{8.6}$$

Indeed, let $v\in\mathbf{C}^1_D[0,T]$ be a solution to (8.1)–(8.4). Then

$$\begin{aligned}\int_0^T v''(s)\,\mathrm{d}s&=\sum_{i=0}^m\int_{t_i}^{t_{i+1}} v''(s)\,\mathrm{d}s=\sum_{i=0}^m\big[v'(t_{i+1})-v'(t_i+)\big]\\ &=v'(T)-v'(0)-\sum_{i=1}^m\Delta^+v'(t_i)=-\lambda\sum_{i=1}^m M_i(v,v')\end{aligned}$$

and

$$\begin{aligned}\int_0^T c\,v'(s)\,\mathrm{d}s&=c\sum_{i=0}^m\int_{t_i}^{t_{i+1}} v'(s)\,\mathrm{d}s=c\sum_{i=0}^m\big[v(t_{i+1})-v(t_i+)\big]\\ &=c\Big[v(T)-v(0)-\sum_{i=1}^m\Delta^+v(t_i)\Big]=-\lambda\,c\sum_{i=1}^m J_i(v,v').\end{aligned}$$

Thus, integrating (8.1) over $[0,T]$ gives (8.6).

STEP 2. Consider system (8.7), (8.2), (8.4), where (8.7) is the functional-differential equation

$$v''=\lambda\,[-c\,v'+g(v)+e(t)]+(1-\lambda)\,\frac{1}{T}\,\Xi_\lambda(v). \tag{8.7}$$

Due to (8.6), we can see that for each $\lambda\in[0,1]$ the problems (8.1)–(8.4) and (8.7), (8.2)–(8.4) are equivalent. Moreover, for $\lambda=1$, problem (8.7), (8.2), (8.4) reduces to the given problem (6.6) (with u replaced by v).

Now, notice that in view of (7.17) we have

$$\int_0^T G(t,s)\,\mathrm{d}s = \frac{1}{2}\,t\,(t-T) \quad \text{for } t\in[0,T]$$

and define for $\lambda\in[0,1]$, $u\in\mathbf{C}^1_D[0,T]$, $u>0$ on $[0,T]$, and $t\in[0,T]$

$$\left\{\begin{aligned} &F_\lambda(u)(t) = u(0)+u'(0)-u'(T)\\ &\quad+\lambda\int_0^T G(t,s)\,\big[-c u'(s)+g(u(s))+e(s)\big]\,\mathrm{d}s\\ &\quad+(1-\lambda)\,\frac{t\,(t-T)}{2\,T}\,\Xi_\lambda(u)\\ &\quad-\lambda\sum_{i=1}^m \frac{\partial}{\partial s}G(t,t_i)\,J_i(u,u')+\lambda\sum_{i=1}^m G(t,t_i)\,M_i(u,u'). \end{aligned}\right. \tag{8.8}$$

In particular, if $\lambda=0$, then

$$F_0(u)(t) = u(0)+u'(0)-u'(T)+\frac{t\,(t-T)}{2\,T}\,\Xi_0(u) \quad \text{for} \ \ t\in[0,T].$$

Let us put

$$\Omega=\{u\in\mathbf{C}^1_D[0,T]:\ r<u<R \text{ on } \ [0,T] \text{ and } \ \|u'\|_\infty<R'\,\}.$$

Arguing similarly to the regular case (see Proposition 7.2), we can conclude that for each $\lambda\in[0,1]$ the operator $F_\lambda\colon\overline{\Omega}\subset\mathbf{C}^1_D[0,T]\to\mathbf{C}^1_D[0,T]$ is completely continuous and a function $v\in\overline{\Omega}$ is a solution of (8.7), (8.2)–(8.4) if and only if it is a fixed point of F_λ. In particular,

$$u\in\overline{\Omega} \text{ is a solution to (6.6) if and only if } \ F_1(u)=u. \tag{8.9}$$

Step 3. We will show that

$$F_\lambda(u)\neq u \quad \text{for all} \ \ u\in\partial\,\Omega \quad \text{and} \quad \lambda\in[0,1]. \tag{8.10}$$

Indeed, for $\lambda\in(0,1]$ relation (8.10) follows immediately from assumption (i), while for $\lambda=0$ it is a corollary of assumption (ii) and of the following claim.

Claim. *$u\in\overline{\Omega}$ is a fixed point of F_0 if and only if there is $x\in\mathbb{R}$ such that $u(t)\equiv x$ on $[0,T]$, $x\in(r,R)$ and*

$$g(x)+\bar{e}=0. \tag{8.11}$$

Proof of Claim. Let $u\in\overline{\Omega}$ be a fixed point of F_0, i.e.

$$u(t)=u(0)+u'(0)-u'(T)+\frac{t\,(t-T)}{2\,T}\,\Xi_0(u) \quad \text{for all} \ \ t\in[0,T]. \tag{8.12}$$

Inserting $t = 0$ into (8.12), we get $u(0) = u(0) + u'(0) - u'(T)$, which implies that $u'(0) = u'(T)$. Similarly, inserting $t = T$ we get $u(T) = u(0)$. Furthermore,

$$u'(t) = \frac{2\,t - T}{2T}\,\Xi_0(u) \quad \text{for} \quad t \in [0, T].$$

Since $u'(0) = u'(T)$, it follows that $\Xi_0(u) = 0$. This means that u is constant on $[0, T]$. Denote $x = u(0)$. Then $0 = \Xi_0(u) = T\,(g(x) + \bar{e})$, i.e., (8.11) is true. On the other hand, it is easy to see that if $x \in \mathbb{R}$ is such that (8.11) holds and $u(t) \equiv x$ on $[0, T]$, then $u \in \overline{\Omega}$ is a fixed point of F_0. This completes the proof of Claim.

Step 4. By Step 3 and by the invariance under homotopy property of the topological degree, we have

$$\deg(\mathrm{I} - F_1, \Omega) = \deg(\mathrm{I} - F_0, \Omega). \tag{8.13}$$

Step 5. Let us denote

$$\mathbb{X} = \{u \in \mathbf{C}^1_D[0, T] : \, u(t) \equiv u(0) \text{ on } [0, T]\} \quad \text{and} \quad \Omega_0 = \Omega \cap \mathbb{X}.$$

Notice that $\Omega_0 = \{u \in \mathbb{X} : \, r < u(0) < R\}$ and $\overline{\Omega}_0 = \{u \in \mathbb{X} : \, r \le u(0) \le R\}$. By Claim in Step 3, all fixed points of F_0 belong to Ω_0. Hence, by the excision property of the topological degree we have

$$\deg(\mathrm{I} - F_0, \Omega) = \deg(\mathrm{I} - F_0, \Omega_0). \tag{8.14}$$

Step 6. Define

$$\begin{cases} \widetilde{F}_\mu(u)(t) = u(0) + \Big[1 - \mu + \dfrac{\mu}{2}\,t\,(t - T)\Big]\,\big(g(u(0) + \bar{e}\big) \\ \qquad\qquad\qquad\qquad \text{for} \quad t \in [0, T],\ u \in \overline{\Omega}_0 \quad \text{and} \quad \mu \in [0, 1]. \end{cases} \tag{8.15}$$

We have

$$\widetilde{F}_0(u) = u(0) + g(u(0)) + \bar{e} \quad \text{and} \quad \widetilde{F}_1(u) = F_0(u) \quad \text{for each} \quad u \in \mathbb{X}.$$

Similarly to F_λ, the operators $\widetilde{F}_\mu$, $\mu \in [0, 1]$, are also completely continuous and, by Claim in Step 3, we have

$$\widetilde{F}_1(u) \ne u \quad \text{for all} \quad u \in \partial\,\Omega_0.$$

Let i and i_{-1} be respectively the natural isometrical isomorphism $\mathbb{R} \to \mathbb{X}$ and its inverse, i.e.

$$i(x)(t) \equiv u \ \text{ for } \ x \in \mathbb{R} \qquad \text{and} \qquad i_{-1}(u) = u(0) \ \text{ for } \ u \in \mathbb{X},$$

and assume that $\mu \in [0, 1)$, $x \in (0, \infty)$, $u = i(x)$ and $\widetilde{F}_\mu(u) = u$. Then

$$\Big[1 - \mu + \frac{\mu}{2}\,t\,(T - t)\Big]\,\big(g(x) + \bar{e}\big) = 0 \quad \text{for all} \quad t \in [0, T].$$

If $t = 0$, this relation reduces to $g(x) + \overline{e} = 0$, which is due to assumption (ii) possible only if $x \in (r, R)$. To summarize, we have

$$\widetilde{F}_\mu(u) \neq u \quad \text{for all} \quad u \in \partial\,\Omega_0 \quad \text{and all} \quad \mu \in [0,1].$$

Hence, using the invariance under homotopy property of the topological degree and taking into account that $\dim \mathbb{X} = 1$, we conclude that

$$\deg(\mathrm{I} - F_0, \Omega_0) = \deg(\mathrm{I} - \widetilde{F}_1, \Omega_0) = \mathrm{d}_B(\mathrm{I} - \widetilde{F}_0, \Omega_0), \tag{8.16}$$

where $\mathrm{d}_B(\mathrm{I} - \widetilde{F}_0, \Omega_0)$ stands for the Brouwer degree of $\mathrm{I} - \widetilde{F}_0$ with respect to the set Ω_0 (and the point 0).

Step 7. Define $\Phi\colon x \in (0,\infty) \to g(x) + \bar{e} \in \mathbb{R}$. Then

$$(\mathrm{I} - \widetilde{F}_0)(i(x)) = i(\Phi(x)) \quad \text{for each} \quad x \in (0,\infty).$$

In other words, $\Phi = i_{-1} \circ (\mathrm{I} - \widetilde{F}_0) \circ i$ on $(0,\infty)$. Consequently,

$$\mathrm{d}_B(\mathrm{I} - \widetilde{F}_0, \Omega_0) = \mathrm{d}_B(\Phi, (r,R)). \tag{8.17}$$

Now, put

$$\Psi(x) = \Phi(r)\,\frac{R-x}{R-r} + \Phi(R)\,\frac{x-r}{R-r}\,.$$

We can see that Ψ has a unique zero $x_0 \in (r,R)$ and

$$\Psi'(x_0) = \frac{\Phi(R) - \Phi(r)}{R-r}.$$

Hence, by the definition of the Brouwer degree in $\mathbb{R}$ we have

$$\mathrm{d}_B(\Psi, (r,R)) = \operatorname{sign} \Psi'(x_0) = \operatorname{sign}\left(\Phi(R) - \Phi(r)\right).$$

By the homotopy property and thanks to our assumption (iii), we conclude that

$$\mathrm{d}_B(\Phi, (r,R)) = \mathrm{d}_B(\Psi, (r,R)) = \operatorname{sign}\left(\Phi(R) - \Phi(r)\right) \neq 0. \tag{8.18}$$

Step 8. To summarize, by (8.13)–(8.18) we have

$$\deg(\mathrm{I} - F_1, \Omega) \neq 0,$$

which, in view of the existence property of the topological degree, shows that F_1 has a fixed point $u \in \Omega$. By Step 1 this means that problem (6.6) has a solution. $\square$

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