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Periodic Boundary Value Problems for Nonlinear Second Order Differential Equations with Impulses - Part II

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Summary. In this paper, using the lower/upper functions argument, we establish new existence results for the nonlinear impulsive periodic boundary value problem

(1.1) $$u'' = f(t, u, u'),$$

(1.2) $$u(t_i+) = \mathrm{J}_i(u(t_i)), \quad u'(t_i+) = \mathrm{M}_i(u'(t_i)), \quad i = 1, 2, \ldots, m,$$

(1.3) $$u(0) = u(T), \quad u'(0) = u'(T),$$

where $f \in \mathrm{Car}([0,T] \times \mathbb{R}^2)$ and J_i, $\mathrm{M}_i \in \mathbb{C}(\mathbb{R})$. The main goal of the paper is to obtain the results in the case that the lower/upper functions σ_1/σ_2 associated with the problem are not well-ordered, i.e. $\sigma_1 \not\leq \sigma_2$ on $[0,T]$.
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0. Introduction

The paper is a continuation of [12], where we have proved solvability of the problem (1.1) - (1.3) provided there exists a well-ordered pair $\sigma_1 \leq \sigma_2$ on $[0,T]$ of lower/upper

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functions associated with the problem. Moreover, in [12], the monotonicity of the impulse functions J_i, M_i required in [1], [3], [5] - [9] and [13] has been replaced by weaker conditions. Here we extend the results of [12] to the case of lower/upper functions which are not well-ordered; i.e.,

$$\sigma_1(\tau) > \sigma_2(\tau) \quad \text{for some} \quad \tau \in [0,T]. \tag{0.1}$$

As far as we know, there is no existence result for impulsive periodic problems having only lower/upper functions satisfying (0.1). The first step in this direction we did in [11] where we worked with strict lower/upper functions and with the case $m = 1$.

Throughout the paper we keep the following notation and conventions: For a real valued function u defined a.e. on $[0,T]$, we put

$$\|u\|_\infty = \sup_{t\in[0,T]} |u(t)| \quad \text{and} \quad \|u\|_1 = \int_0^T |u(s)|\,\mathrm{d}s.$$

For a given interval $J \subset \mathbb{R}$, by $\mathbb{C}(J)$ we denote the set of real valued functions which are continuous on J. Furthermore, $\mathbb{C}^1(J)$ is the set of functions having continuous first derivatives on J and $\mathbb{L}(J)$ is the set of functions which are Lebesgue integrable on J.

Let $m \in \mathbb{N}$ and let $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = T$ be a division of the interval $[0,T]$. We denote $\mathrm{D} = \{t_1, t_2, \dots, t_m\}$ and define $\mathbb{C}^1_{\mathrm{D}}[0,T]$ as the set of functions $u : [0,T] \mapsto \mathbb{R}$,

$$u(t) = \begin{cases} u_{[0]}(t) & \text{if } t \in [0,t_1], \\ u_{[1]}(t) & \text{if } t \in (t_1,t_2], \\ \dots & \dots \\ u_{[m]}(t) & \text{if } t \in (t_m,T], \end{cases}$$

where $u_{[i]} \in \mathbb{C}^1[t_i,t_{i+1}]$ for $i = 0,1,\dots,m$. Moreover, $\mathbb{AC}^1_{\mathrm{D}}[0,T]$ stands for the set of functions $u \in \mathbb{C}^1_{\mathrm{D}}[0,T]$ having first derivatives absolutely continuous on each subinterval (t_i,t_{i+1}), $i = 0,1,\dots,m$. For $u \in \mathbb{C}^1_{\mathrm{D}}[0,T]$ and $i = 1,2,\dots,m+1$ we write

$$u'(t_i) = u'(t_i-) = \lim_{t\to t_i-} u'(t), \quad u'(0) = u'(0+) = \lim_{t\to 0+} u'(t) \tag{0.2}$$

and $\|u\|_{\mathrm{D}} = \|u\|_\infty + \|u'\|_\infty$. Note that the set $\mathbb{C}^1_{\mathrm{D}}[0,T]$ becomes a Banach space when equipped with the norm $\|.\|_{\mathrm{D}}$ and with the usual algebraic operations.

We say that $f : [0,T] \times \mathbb{R}^2 \mapsto \mathbb{R}$ satisfies the *Carathéodory conditions* on $[0,T] \times \mathbb{R}^2$ if (i) for each $x \in \mathbb{R}$ and $y \in \mathbb{R}$ the function $f(.,x,y)$ is measurable on $[0,T]$; (ii) for almost every $t \in [0,T]$ the function $f(t,.,.)$ is continuous on $\mathbb{R}^2$; (iii) for each compact set $K \subset \mathbb{R}^2$ there is a function $m_K(t) \in \mathbb{L}[0,T]$ such that $|f(t,x,y)| \le$

$m_K(t)$ holds for a.e. $t \in [0,T]$ and all $(x,y) \in K$. The set of functions satisfying the Carathéodory conditions on $[0,T] \times \mathbb{R}^2$ will be denoted by $\mathrm{Car}([0,T] \times \mathbb{R}^2)$.

Given a Banach space $\mathbb{X}$ and its subset M, let $\mathrm{cl}(M)$ and ∂M denote the closure and the boundary of M, respectively.

Let Ω be an open bounded subset of $\mathbb{X}$. Assume that the operator $\mathrm{F} : \mathrm{cl}(\Omega) \mapsto \mathbb{X}$ is completely continuous and $\mathrm{F}\, u \neq u$ for all $u \in \partial\,\Omega$. Then $\deg(\mathrm{I} - \mathrm{F}, \Omega)$ denotes the *Leray-Schauder topological degree* of $\mathrm{I} - \mathrm{F}$ with respect to Ω, where I is the identity operator on $\mathbb{X}$. For the definition and properties of the degree see e.g. [4] or [10].

1. Formulation of the problem and main assumptions

Here we study the existence of solutions to the problem

$$u'' = f(t,u,u'), \tag{1.1}$$

$$u(t_i+) = \mathrm{J}_i(u(t_i)), \quad u'(t_i+) = \mathrm{M}_i(u'(t_i)), \quad i = 1,2,\dots,m, \tag{1.2}$$

$$u(0) = u(T), \quad u'(0) = u'(T), \tag{1.3}$$

where $u'(t_i)$ are understood in the sense of (0.2), $f \in \mathrm{Car}([0,T] \times \mathbb{R}^2)$, $\mathrm{J}_i \in \mathbb{C}(\mathbb{R})$ and $\mathrm{M}_i \in \mathbb{C}(\mathbb{R})$.

1.1. Definition. A *solution of the problem* (1.1) - (1.3) is a function $u \in \mathbb{AC}^1_{\mathrm{D}}[0,T]$ which satisfies the impulsive conditions (1.2), the periodic conditions (1.3) and for a.e. $t \in [0,T]$ fulfils the equation (1.1).

1.2. Definition. A function $\sigma_1 \in \mathbb{AC}^1_{\mathrm{D}}[0,T]$ is called a *lower function of the problem* (1.1) - (1.3) if

$$\sigma_1''(t) \geq f(t,\sigma_1(t),\sigma_1'(t)) \quad \text{for a.e. } t \in [0,T], \tag{1.4}$$

$$\sigma_1(t_i+) = \mathrm{J}_i(\sigma_1(t_i)), \quad \sigma_1'(t_i+) \geq \mathrm{M}_i(\sigma_1'(t_i)), \quad i = 1,2,\dots,m, \tag{1.5}$$

$$\sigma_1(0) = \sigma_1(T), \quad \sigma_1'(0) \geq \sigma_1'(T). \tag{1.6}$$

Similarly, a function $\sigma_2 \in \mathbb{AC}^1_{\mathrm{D}}[0,T]$ is an *upper function of the problem* (1.1) - (1.3) if

$$\sigma_2''(t) \leq f(t,\sigma_2(t),\sigma_2'(t)) \quad \text{for a.e. } t \in [0,T], \tag{1.7}$$

$$\sigma_2(t_i+) = \mathrm{J}_i(\sigma_2(t_i)), \quad \sigma_2'(t_i+) \leq \mathrm{M}_i(\sigma_2'(t_i)), \quad i = 1,2,\dots,m, \tag{1.8}$$

$$\sigma_2(0) = \sigma_2(T), \quad \sigma_2'(0) \leq \sigma_2'(T). \tag{1.9}$$

A straightforward illustration of Definition 1.2 is the following proposition providing a simplest example of conditions ensuring the existence of lower and upper functions for (1.1) - (1.3).

1.3. Proposition. *Let* $\alpha_0 \in R$. *For* $i = 1, 2, \ldots, m$ *assume that* $\mathrm{M}_i(0) = 0$, $\alpha_i = \mathrm{J}_i(\alpha_{i-1})$ *where* $\alpha_m = \alpha_0$, $f(t, \alpha_i, 0) \le 0$ *for a.e.* $t \in (t_i, t_{i+1})$, *and put* $\sigma_1(t) = \alpha_i$ *on* $(t_i, t_{i+1}]$, $\sigma_1(t) = \alpha_0$ *on* $[0, t_1]$. *Then* σ_1 *is a lower function of* (1.1)-(1.3).

Let $\beta_0 \in R$. *For* $i = 1, 2, \ldots, m$ *assume that* $\mathrm{M}_i(0) = 0$, $\beta_i = \mathrm{J}_i(\beta_{i-1})$ *where* $\beta_m = \beta_0$, $f(t, \beta_i, 0) \ge 0$ *for a.e.* $t \in (t_i, t_{i+1})$, *and put* $\sigma_2(t) = \beta_i$ *on* $(t_i, t_{i+1}]$, $\sigma_2(t) = \beta_0$ *on* $[0, t_1]$. *Then* σ_2 *is an upper function of* (1.1)-(1.3).

1.4. Remark. In particular, if $\mathrm{M}_i(0) = 0$, $\mathrm{J}_i(\alpha_0) = \alpha_0$, $\mathrm{J}_i(\beta_0) = \beta_0$ for $i = 1, 2, \ldots, m$ and $f(t, \alpha_0, 0) \le 0$, $f(t, \beta_0, 0) \ge 0$ for a.e. $t \in [0, T]$, then $\sigma_1(t) = \alpha_0$ and $\sigma_2(t) = \beta_0$, $t \in [0, T]$, are respectively lower and upper functions of (1.1) - (1.3).

1.5. Assumptions. In the paper we work with the following assumptions:

$$\text{(1.10)} \quad \left\{ \begin{array}{l} 0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T < \infty, \ \mathrm{D} = \{t_1, t_2, \ldots, t_m\}, \\ f \in \mathrm{Car}([0, T] \times \mathbb{R}^2), \ \mathrm{J}_i \in \mathbb{C}(\mathbb{R}), \ \mathrm{M}_i \in \mathbb{C}(\mathbb{R}), \ i = 1, 2, \ldots, m; \end{array} \right.$$

(1.11) σ_1 and σ_2 are respectively lower and upper functions of (1.1) - (1.3);

$$\text{(1.12)} \quad \left\{ \begin{array}{l} x > \sigma_1(t_i) \implies \mathrm{J}_i(x) > \mathrm{J}_i(\sigma_1(t_i)), \\ x < \sigma_2(t_i) \implies \mathrm{J}_i(x) < \mathrm{J}_i(\sigma_2(t_i)), \quad i = 1, 2, \ldots, m; \end{array} \right.$$

$$\text{(1.13)} \quad \left\{ \begin{array}{l} y \le \sigma_1'(t_i) \implies \mathrm{M}_i(y) \le \mathrm{M}_i(\sigma_1'(t_i)), \\ y \ge \sigma_2'(t_i) \implies \mathrm{M}_i(y) \ge \mathrm{M}_i(\sigma_2'(t_i)), \quad i = 1, 2, \ldots, m. \end{array} \right.$$

1.6. Operator reformulation of (1.1)-(1.3). Put

$$G(t, s) = \left\{ \begin{array}{ll} \dfrac{t\,(s - T)}{T} & \text{if } 0 \le t \le s \le T, \\ \dfrac{s\,(t - T)}{T} & \text{if } 0 \le s < t \le T, \end{array} \right.$$

and define an operator $\mathrm{F} : \mathbb{C}^1_{\mathrm{D}}[0, T] \mapsto \mathbb{C}^1_{\mathrm{D}}[0, T]$ by

$$\text{(1.14)} \quad (\mathrm{F}\,u)(t) = u(0) + u'(0) - u'(T) + \int_0^T G(t, s)\, f(s, u(s), u'(s))\,\mathrm{d}s - \sum_{i=1}^m \frac{\partial G}{\partial s}(t, t_i)\,(\mathrm{J}_i(u(t_i)) - u(t_i)) + \sum_{i=1}^m G(t, t_i)\,(\mathrm{M}_i(u'(t_i)) - u'(t_i)).$$

Then, as in [11, Lemma 3.1], we get that F is completely continuous and u is a solution of (1.1) - (1.3) if and only if $\mathrm{F}\,u = u$.

In the proof of our main result we will need the next proposition which concerns the case of well-ordered lower/upper functions and which follows from [12, Corollary 3.5].

1.7. Proposition. *Assume that* (1.10) *holds and let* α *and* β *be respectively lower and upper functions of* (1.1) - (1.3) *such that*

$$\alpha(t) < \beta(t) \;\; for\ t \in [0,T] \quad and \quad \alpha(\tau+) < \beta(\tau+) \;\; for\ \tau \in \mathrm{D}, \tag{1.15}$$

$$\alpha(t_i) < x < \beta(t_i) \implies \mathrm{J}_i(\alpha(t_i)) < \mathrm{J}_i(x) < \mathrm{J}_i(\beta(t_i)), \quad i = 1, 2, \dots, m \tag{1.16}$$

and

$$\begin{cases} y \le \alpha'(t_i) \implies \mathrm{M}_i(y) \le \mathrm{M}_i(\alpha'(t_i)), \\ y \ge \beta'(t_i) \implies \mathrm{M}_i(y) \ge \mathrm{M}_i(\beta'(t_i)), \quad i = 1, 2, \dots, m. \end{cases} \tag{1.17}$$

Further, let $h \in \mathbb{L}[0,T]$ *be such that*

$$|f(t,x,y)| \le h(t) \qquad for\ a.e.\ t \in [0,T] \;\; and\ all \;\; (x,y) \in [\alpha(t), \beta(t)] \times \mathbb{R} \tag{1.18}$$

and let the operator F *be defined by* (1.14). *Finally, for* $r \in (0,\infty)$ *denote*

$$\begin{aligned} \Omega(\alpha,\beta,r) = \{u \in \mathbb{C}^1_{\mathrm{D}}[0,T] :\ & \alpha(t) < u(t) < \beta(t) \;\; for\ t \in [0,T], \\ & \alpha(\tau+) < u(\tau+) < \beta(\tau+) \;\; for\ \tau \in \mathrm{D},\ \|u'\|_\infty < r\}. \end{aligned} \tag{1.19}$$

Then $\deg(\mathrm{I} - \mathrm{F}, \Omega(\alpha,\beta,r)) = 1$ *whenever* $\mathrm{F}\,u \ne u$ *on* $\partial\Omega(\alpha,\beta,r)$ *and*

$$r > \|h\|_1 + \frac{\|\alpha\|_\infty + \|\beta\|_\infty}{\Delta}, \quad where \;\; \Delta = \min_{i=1,2,\dots,m+1} (t_i - t_{i-1}). \tag{1.20}$$

Proof. Using the Mean Value Theorem, we can show that

$$\|u'\|_\infty \le \|h\|_1 + \frac{\|\alpha\|_\infty + \|\beta\|_\infty}{\Delta} \tag{1.21}$$

holds for each $u \in \mathbb{C}^1_{\mathrm{D}}[0,T]$ fulfilling $\alpha(t) < u(t) < \beta(t)$ for $t \in [0,T]$ and $\alpha(\tau+) < u(\tau+) < \beta(\tau+)$ for $\tau \in \mathrm{D}$. Thus, if we denote by c the right-hand side of (1.21), we can follow the proof of [12, Corollary 3.5]. □

2. A priori estimates

The proof of our main existence result (Theorem 3.1) is based on the evaluation of the topological degree of a proper auxiliary operator by means of Proposition 1.7. To this aim we need a priori estimates for certain sets of functions which are provided in this section.

2.1. Lemma. *Let* $\rho_1 \in (0,\infty)$, $\widetilde{h} \in \mathbb{L}[0,T]$, $\mathrm{M}_i \in \mathbb{C}(\mathbb{R})$, $i = 1,2,\dots,m$. *Then there exists* $d \in (\rho_1,\infty)$ *such that the estimate*

$$\|u'\|_\infty < d \tag{2.1}$$

is valid for each function $u \in \mathbb{AC}^1_{\mathrm{D}}[0,T]$ *satisfying* (1.3),

$$|u'(\xi_u)| < \rho_1 \quad \textit{for some} \;\; \xi_u \in [0,T], \tag{2.2}$$

$$u'(t_i+) = \mathrm{M}_i(u'(t_i)), \quad i = 1,2,\dots,m, \tag{2.3}$$

and

$$|u''(t)| < \widetilde{h}(t) \;\; \textit{for a.e.} \;\; t \in [0,T]. \tag{2.4}$$

Proof. Suppose that u satisfies (1.3) and (2.2) - (2.4). Since $\mathrm{M}_i \in \mathbb{C}(\mathbb{R})$ for $i = 1,2,\dots,m$, we have

$$b_i(a) := \sup_{|y|<a} |\,\mathrm{M}_i(y)| < \infty \quad \text{for} \;\; a \in (0,\infty), \;\; i = 1,2,\dots,m. \tag{2.5}$$

Furthermore, due to (1.3), we can assume that $\xi_u \in (0,T]$, i.e. there is $j \in \{1,2,\dots,m+1\}$ such that $\xi_u \in (t_{j-1},t_j]$. We will distinguish 3 cases: either $j = 1$ or $j = m+1$ or $1 < j < m+1$.

Let $j = 1$. Then, using (2.2) and (2.4), we obtain

$$|u'(t)| < a_1 \;\; \text{on} \;\; [0,t_1], \tag{2.6}$$

where $a_1 = \rho_1 + \|\widetilde{h}\|_1$. Hence, in view of (2.5), we have $|u'(t_1+)| < b_1(a_1)$, wherefrom, using (2.4), we deduce that $|u'(t)| < b_1(a_1) + \|\widetilde{h}\|_1$ for $t \in (t_1,t_2]$. Continuing by induction, we get $|u'(t)| < a_{i+1} = b_i(a_i) + \|\widetilde{h}\|_1$ on $(t_i,t_{i+1}]$ for $i = 2,\dots,m$, i.e.

$$\|u'\|_\infty < d := \max\{a_i : \; i = 1,2,\dots,m+1\}. \tag{2.7}$$

Assume that $j = m+1$. Then, using (2.2) and (2.4), we obtain

$$|u'(t)| < a_{m+1} \;\; \text{on} \;\; (t_m,T], \tag{2.8}$$

where $a_{m+1} = \rho_1 + \|\widetilde{h}\|_1$. Furthermore, due to (1.3), we have $|u'(0)| < a_{m+1}$ which together with (2.4) yields that (2.6) is true with $a_1 = a_{m+1} + \|\widetilde{h}\|_1$. Now, proceeding as in the case $j = 1$, we show that (2.7) is true also for $j = m+1$.

Assume that $1 < j < m+1$. Then (2.2) and (2.4) yield $|u'(t)| < a_{j+1} = \rho_1 + \|\widetilde{h}\|_1$ on $(t_j,t_{j+1}]$. If $j < m$, then $|u'(t)| < a_{j+2} = b_{j+1}(a_{j+1}) + \|\widetilde{h}\|_1$ on $(t_{j+1},t_{j+2}]$. Proceeding by induction we get (2.8) with $a_{m+1} = b_m(a_m) + \|\widetilde{h}\|_1$, wherefrom (2.7) again follows as in the previous case. □

2.2. Lemma. *Let $\rho_0, d \in (0,\infty)$ and $\mathrm{J}_i \in \mathbb{C}(\mathbb{R})$, $i = 1,2,\dots,m$. Then there exists $c \in (\rho_0,\infty)$ such that the estimate*

$$\|u\|_\infty < c \tag{2.9}$$

is valid for each function $u \in \mathbb{C}^1_{\mathrm{D}}[0,T]$ satisfying (1.3), (2.1),

$$u(t_i+) = \widetilde{\mathrm{J}}_i(u(t_i)), \quad i = 1,2,\dots,m, \tag{2.10}$$

and

$$|u(\tau_u)| < \rho_0 \quad \textit{for some} \quad \tau_u \in [0,T] \tag{2.11}$$

and for each $\widetilde{\mathrm{J}}_i \in \mathbb{C}(\mathbb{R})$, $i = 1,2,\dots,m$, such that

$$\begin{gathered}\mathrm{J}_i(-a,a) \subset (-b,b) \Longrightarrow \widetilde{\mathrm{J}}_i(-a,a) \subset (-b,b) \\ \textit{for} \quad i = 1,2,\dots,m, \ a \in (0,\infty), \ b \in (a,\infty).\end{gathered} \tag{2.12}$$

Proof. We will argue similarly as in the proof of Lemma 2.1. Suppose that u satisfies (1.3), (2.1), (2.10), (2.11) and that $\widetilde{\mathrm{J}}_i$, $i = 1,2,\dots,m$, satisfy (2.12). Due to (1.3) we can assume that $\tau_u \in (0,T]$, i.e. there is $j \in \{1,2,\dots,m+1\}$ such that $\tau_u \in (t_{j-1},t_j]$. We will consider three cases: $j = 1$, $j = m+1$, $1 < j < m+1$. If $j = 1$, then (2.1) and (2.11) yield $|u(t)| < a_1 = \rho_0 + dT$ on $[0,t_1]$. In particular, $|u(t_1)| < a_1$. Since $\mathrm{J}_1 \in \mathbb{C}(\mathbb{R})$, we can find $b_1(a_1) \in (0,\infty)$ such that $|\,\mathrm{J}_1(x)| < b_1(a_1)$ for all $x \in (-a_1,a_1)$ and consequently, by (2.12), also $|\widetilde{\mathrm{J}}_1(x)| < b_1(a_1)$ for all $x \in (-a_1,a_1)$. Therefore, by (2.1), $|u(t)| < |u(t_1+)| + dT = |\widetilde{\mathrm{J}}_1(u(t_1))| + dT < a_2 = b_1(a_1) + dT$ on $(t_1,t_2]$. Proceeding by induction we get $|u(t)| < a_{i+1} = b_i(a_i) + dT$ for $t \in (t_i,t_{i+1}]$ and $i = 2,\dots,m$. As a result, (2.9) is true with $c = \max\{a_i : i = 1,2,\dots,m+1\}$. Analogously we would proceed in the remaining cases $j = m+1$ or $1 < j < m+1$. ☐

Finally, we will need two estimates for functions u satisfying one of the following conditions:

$$u(s_u) < \sigma_1(s_u) \quad \text{and} \quad u(t_u) > \sigma_2(t_u) \quad \text{for some} \quad s_u, t_u \in [0,T], \tag{2.13}$$

$$u \ge \sigma_1 \quad \text{on} \quad [0,T] \quad \text{and} \quad \inf_{t\in[0,T]} |u(t) - \sigma_1(t)| = 0, \tag{2.14}$$

$$u \le \sigma_2 \quad \text{on} \quad [0,T] \quad \text{and} \quad \inf_{t\in[0,T]} |u(t) - \sigma_2(t)| = 0. \tag{2.15}$$

Let us denote

$$\begin{aligned}B = \{u \in \mathbb{C}^1_{\mathrm{D}}[0,T] : u \text{ satisfies } &(1.3), (2.10), (2.3) \text{ and one} \\ &\text{of the conditions } (2.13), (2.14), (2.15)\}.\end{aligned} \tag{2.16}$$

2.3. Lemma. *Assume that* $\sigma_1, \sigma_2 \in \mathbb{AC}^1_{\mathrm{D}}[0,T]$, J_i, M_i, $\widetilde{\mathrm{J}}_i \in \mathbb{C}(\mathbb{R})$, $i = 1,2,\ldots,m$, *satisfy* (1.12), (1.13) *and*

$$\left\{ \begin{array}{l} x > \sigma_1(t_i) \implies \widetilde{\mathrm{J}}_i(x) > \mathrm{J}_i(\sigma_1(t_i)), \\ x < \sigma_2(t_i) \implies \widetilde{\mathrm{J}}_i(x) < \mathrm{J}_i(\sigma_2(t_i)), \quad i = 1,2,\ldots,m. \end{array} \right. \tag{2.17}$$

Let the set B *be defined by* (2.16). *Then each function* $u \in B$ *satisfies*

$$\left\{ \begin{array}{l} |u'(\xi_u)| < \rho_1 \text{ for some } \xi_u \in [0,T], \quad \text{where} \\ \rho_1 = \dfrac{2}{t_1}\,(\|\sigma_1\|_\infty + \|\sigma_2\|_\infty) + \|\sigma_1'\|_\infty + \|\sigma_2'\|_\infty + 1. \end{array} \right. \tag{2.18}$$

Proof. • PART 1. Assume that $u \in B$ satisfies (2.13). There are 3 cases to consider:

CASE A. If $\min\{\sigma_1(t), \sigma_2(t)\} \le u(t) \le \max\{\sigma_1(t), \sigma_2(t)\}$ for $t \in [0,T]$, then, by the Mean Value Theorem, there is $\xi_u \in (0, t_1)$ such that

$$|u'(\xi_u)| \le \frac{2}{t_1}\,(\|\sigma_1\|_\infty + \|\sigma_2\|_\infty). \tag{2.19}$$

CASE B. Assume that $u(s) > \sigma_1(s)$ for some $s \in [0,T]$ and denote $v = u - \sigma_1$. Due to (2.13) we have

$$v_* = \inf_{t\in[0,T]} v(t) < 0 \quad \text{and} \quad v^* = \sup_{t\in[0,T]} v(t) > 0. \tag{2.20}$$

We are going to prove that

$$v'(\alpha) = 0 \text{ for some } \alpha \in [0,T] \text{ or } v'(t_j+) = 0 \text{ for some } t_j \in \mathrm{D}. \tag{2.21}$$

Suppose, on the contrary, that (2.21) does not hold.

Let $v'(0) > 0$. Then, according to (1.3) and (1.6), $v'(T) > 0$, as well. Due to the assumption that (2.21) does not hold, this together with (1.5) yields that

$$0 < v'(t_m+) = u'(t_m+) - \sigma_1'(t_m+) \le \mathrm{M}_m(u'(t_m)) - \mathrm{M}_m(\sigma_1'(t_m)),$$

which is by (1.13) possible only if $u'(t_m) > \sigma_1'(t_m)$, i.e. $v'(t_m) > 0$. Continuing in this way on each $(t_i, t_{i+1}]$, $i = 0,1,\ldots,m-1$, we get

$$v'(t) > 0 \text{ for } t \in [0,T] \quad \text{and} \quad v'(\tau+) > 0 \text{ for } \tau \in \mathrm{D}. \tag{2.22}$$

If $v(0) \ge 0$, then $v(t) > 0$ on $(0, t_1]$ due to (2.22). Further, it follows by (1.5), (2.10) and (2.17) that $u(t_1+) > \sigma_1(t_1+)$, i.e. $v(t_1+) > 0$. Continuing by induction we deduce that $v \ge 0$ on $[0,T]$, contrary to (2.20).

If $v(0) < 0$, then by (1.3) and (1.6) we have $v(T) < 0$. Further, by virtue of (2.22) we obtain $v < 0$ on $(t_m, T]$ and, in particular, $v(t_m+) < 0$. So, $\widetilde{\mathrm{J}}_m(u(t_m)) < \mathrm{J}_m(\sigma_1(t_m))$ wherefrom $u(t_m) \le \sigma_1(t_m)$ follows, due to (2.17). Thus, we have $v < 0$ on (t_{m-1}, t_m). Continuing by induction we get $v \le 0$ on $[0, T]$, contrary to (2.20).

Now, assume that $v'(0) < 0$. Then $v'(t_1) < 0$, i.e. $u'(t_1) < \sigma_1'(t_1)$ wherefrom, by (1.5), (1.13) and the assumption that (2.21) does not hold, the inequality $v'(t_1+) = u'(t_1+) - \sigma_1'(t_1+) < 0$ follows. Similarly as in the proof of (2.22) we show that

$$v'(t) < 0 \quad \text{for } t \in [0, T] \quad \text{and} \quad v'(\tau+) < 0 \quad \text{for } \tau \in \mathrm{D}\,. \tag{2.23}$$

Now, having (2.23), we consider as above two cases: $v(0) \ge 0$ and $v(0) < 0$, and construct a contradiction by means of analogous arguments.

So we have proved that (2.21) is true, which yields the existence of $\xi_u \in [0, T]$ having the property

$$|u'(\xi_u)| < \|\sigma_1'\|_\infty + 1. \tag{2.24}$$

Case C. If $u(s) < \sigma_2(s)$ for some $s \in [0, T]$, we put $v = u - \sigma_2$ and, using the properties of σ_2 instead of σ_1, we can argue as in Case B and show that there exists $\xi_u \in [0, T]$ such that

$$|u'(\xi_u)| < \|\sigma_2'\|_\infty + 1. \tag{2.25}$$

Taking into account (2.19), (2.24) and (2.25) we conclude that (2.18) is valid for any $u \in B$ fulfilling (2.13).

• Part 2. Let $u \in B$ satisfy (2.14). Then $u \ge \sigma_1$ on $[0, T]$ and either there is $\alpha_u \in [0, T]$ such that $u(\alpha_u) = \sigma_1(\alpha_u)$ or there is $t_j \in \mathrm{D}$ such that $u(t_j+) = \sigma_1(t_j+)$.

Case A. Let the first possibility occur. If $\alpha_u \in (0, T) \setminus \mathrm{D}$, then necessarily $u'(\alpha_u) = \sigma_1'(\alpha_u)$. Consequently, the estimate (2.24) is valid. If $\alpha_u = 0$, then $\inf\{u(t) - \sigma_1(t) : t \in [0, T]\} = u(0) - \sigma_1(0) = u(T) - \sigma_1(T) = 0$, which, by virtue of (1.3) and (1.6), implies $0 \le u'(0) - \sigma_1'(0) \le u'(T) - \sigma_1'(T) \le 0$, i.e. $u'(0) = \sigma_1'(0)$ and the estimate (2.24) is valid with $\xi_u = 0$. If $\alpha_u = t_j$ for some $t_j \in \mathrm{D}$, then $0 = u(t_j) - \sigma_1(t_j) = u(t_j+) - \sigma_1(t_j+)$. Having in mind that $u \ge \sigma_1$ on $[0, T]$, we get $u'(t_j+) \ge \sigma_1'(t_j+)$ and $u'(t_j) \le \sigma_1'(t_j)$. On the other hand, with respect to (1.13), the last inequality gives also $\mathrm{M}_j(u'(t_j)) \le \mathrm{M}_j(\sigma_1'(t_j))$, which leads to $\sigma_1'(t_j+) = u'(t_j+)$. Thus, (2.24) is fulfilled for some $\xi_u \in (t_j, t_{j+1})$ which is sufficiently close to t_j.

Case B. Let the second possibility occur, i.e. $u(t_j+) = \sigma_1(t_j+)$ for some $t_j \in \mathrm{D}$. According to (1.5) and (2.10), we have $\widetilde{\mathrm{J}}_j(u(t_j)) = \mathrm{J}_j(\sigma_1(t_j))$. Taking into account (2.17), we see that this can occur only if $u(t_j) \le \sigma_1(t_j)$. On the other hand, by the assumption (2.14) we have $u \ge \sigma_1$ on $[0, T]$. Hence we conclude that $u(t_j) = \sigma_1(t_j)$ and so, arguing as before, we get (2.24) again.

To summarize: (2.18) holds for any $u \in B$ fulfilling (2.14).

• PART 3. Let $u \in B$ satisfy (2.15). Then using the properties of σ_2 instead of σ_1, we argue analogously to PART 2 and prove that (2.25) is valid for each $u \in B$ which satisfies (2.15). In particular, (2.18) holds for any $u \in B$ fulfilling (2.15). □

2.4. Lemma. *Each $u \in B$ satisfies the condition*

$$\min\{\sigma_1(\tau_u+),\sigma_2(\tau_u+)\} \le u(\tau_u+) \le \max\{\sigma_1(\tau_u+),\sigma_2(\tau_u+)\} \quad \text{for some } \tau_u \in [0,T). \tag{2.26}$$

Proof. Assume, on the contrary, that there is $u \in B$ for which (2.26) does not hold. If $u(0) < \min\{\sigma_1(0),\sigma_2(0)\}$ then, taking into account the continuity of the functions u, σ_1 and σ_2 on $[0,t_1]$, we deduce that $u(t) < \min\{\sigma_1(t),\sigma_2(t)\}$ is true for each $t \in [0,t_1]$. Consequently, due to (1.12), we have $u(t_1+) < \min\{\sigma_1(t_1+),\sigma_2(t_1+)\}$. It is easy to see that proceeding by induction we get

$$u(t) < \min\{\sigma_1(t),\sigma_2(t)\} \quad \text{and} \quad u(\tau+) < \min\{\sigma_1(\tau+),\sigma_2(\tau+)\}$$

for each $t \in [0,T) \setminus \mathrm{D}$ and $\tau \in \mathrm{D}$, a contradiction to (2.13). Similarly, we can see that $u(0) > \max\{\sigma_1(0),\sigma_2(0)\}$ implies that

$$u(t) > \max\{\sigma_1(t),\sigma_2(t)\} \quad \text{and} \quad u(\tau+) > \max\{\sigma_1(\tau+),\sigma_2(\tau+)\}$$

hold for each $t \in [0,T) \setminus \mathrm{D}$ and $\tau \in \mathrm{D}$, again a contradiction to (2.13). The proof will be completed by an obvious observation that u can satisfy neither (2.14) nor (2.15) whenever it does not satisfy (2.26). □

3. Main results

Our main result is the following theorem which is the first known existence result for impulsive periodic problems with nonordered lower and upper functions.

3.1. Theorem. *Assume that* (1.10) - (1.13) *and* (0.1) *hold and let $h \in \mathbb{L}[0,T]$ be such that*

$$|f(t,x,y)| \le h(t) \quad \text{for a.e. } t \in [0,T] \text{ and all } (x,y) \in \mathbb{R}^2. \tag{3.1}$$

Further, let

$$y\,\mathrm{M}_i(y) \ge 0 \quad \text{for } y \in \mathbb{R} \quad \text{and} \quad i = 1,2,\dots,m. \tag{3.2}$$

Then the problem (1.1) - (1.3) *has a solution u satisfying* (2.26).

Proof. • STEP 1. *We construct a proper auxiliary problem.*

Let σ_1 and σ_2 be respectively lower and upper functions of (1.1)-(1.3) and let ρ_1 be associated with them as in (2.18). Put

$$\widetilde{h}(t) = 2\,h(t) + 1 \quad \text{for a.e. } t \in [0, T]$$

and, by Lemma 2.1, find $d \in (\rho_1, \infty)$ satisfying (2.1). Furthermore, put

$$\rho_0 = \|\sigma_1\|_\infty + \|\sigma_2\|_\infty + 1$$

and, by Lemma 2.2, find $c \in (\rho_0, \infty)$ fulfilling (2.9). In particular, we have

$$c > \|\sigma_1\|_\infty + \|\sigma_2\|_\infty + 1. \tag{3.3}$$

Finally, for a.e. $t \in [0, T]$ and all $x, y \in \mathbb{R}$ define functions

$$\widetilde{f}(t,x,y) = \begin{cases} f(t,x,y) - h(t) - 1 & \text{if } x \le -c-1, \\ f(t,x,y) + (x+c)\,(h(t)+1) & \text{if } -c-1 < x < -c, \\ f(t,x,y) & \text{if } -c \le x \le c, \\ f(t,x,y) + (x-c)\,(h(t)+1) & \text{if } c < x < c+1, \\ f(t,x,y) + h(t) + 1 & \text{if } x \ge c+1, \end{cases} \tag{3.4}$$

$$\widetilde{\mathrm{J}}_i(x) = \begin{cases} x & \text{if } x \le -c-1, \\ \mathrm{J}_i(-c)\,(c+1+x) - x\,(x+c) & \text{if } -c-1 < x < -c, \\ \mathrm{J}_i(x) & \text{if } -c \le x \le c, \\ \mathrm{J}_i(c)\,(c+1-x) + x\,(x-c) & \text{if } c < x < c+1, \\ x & \text{if } x \ge c+1, \quad i = 1, 2, \ldots, m, \end{cases} \tag{3.5}$$

and consider an auxiliary problem

$$u'' = \widetilde{f}(t, u, u'), \quad (2.10), \quad (2.3), \quad (1.3). \tag{3.6}$$

Due to (1.10), $\widetilde{f} \in \mathrm{Car}([0,T] \times \mathbb{R})$ and $\widetilde{\mathrm{J}}_i \in \mathbb{C}(\mathbb{R})$ for $i = 1, 2, \ldots, m$. Since $c > \rho_0$, according to (3.3) - (3.5) the functions σ_1 and σ_2 are respectively lower and upper functions of (3.6). By (3.1) we have

$$|\widetilde{f}(t,x,y)| \le \widetilde{h}(t) \quad \text{for a.e. } t \in [0,T] \text{ and all } (x,y) \in \mathbb{R}^2 \tag{3.7}$$

and

$$\begin{cases} \widetilde{f}(t,x,y) < 0 & \text{for a.e. } t \in [0,T] \text{ and all } (x,y) \in (-\infty, -c-1] \times \mathbb{R}, \\ \widetilde{f}(t,x,y) > 0 & \text{for a.e. } t \in [0,T] \text{ and all } (x,y) \in [c+1, \infty) \times \mathbb{R}. \end{cases} \tag{3.8}$$

Furthermore, in view of (3.5), it is easy to check that the condition (2.12) is satisfied. Moreover, due to (1.12), we see that (2.17) holds if $|x| \le c$. We are going to show that (2.17) is valid also for $|x| > c$. First, assume that $x > c$. In this case it suffices to verify the first condition in (2.17). Let $i \in \{1, 2, \dots, m\}$ be given. Notice that, due to (3.3) and (1.12), we have

$$c > \max\{\sigma_1(t_i), \sigma_1(t_i+)\} \ge \mathrm{J}_i(\sigma_1(t_i)) \quad \text{and} \quad \mathrm{J}_i(c) > \mathrm{J}_i(\sigma_1(t_i)). \tag{3.9}$$

In view of (1.5), (3.3) and (3.5), this yields that

$$\widetilde{\mathrm{J}}_i(x) = x > \sigma_1(t_i+) = \mathrm{J}_i(\sigma_1(t_i))$$

holds for $x > c+1$, i.e. the first condition in (2.17) is satisfied also for $x > c+1$. If $x \in (c, c+1]$, then the values $\widetilde{\mathrm{J}}_i(x)$ are convex combinations of the values $\mathrm{J}_i(c)$ and x, which both are according to (3.9) greater than $\mathrm{J}_i(\sigma_1(t_i))$, and so we can conclude that the first condition in (2.17) is satisfied for all $x \in (c, \infty)$. Similarly, we can prove that the second condition in (2.17) is satisfied for $x \in (-\infty, -c)$.

Now, put

$$A^* = 1 + \sum_{i=1}^{m} \max_{|x| \le c+1} |\widetilde{\mathrm{J}}_i(x)| \quad \text{and} \quad \sigma_3(t) \equiv -A^*, \ \sigma_4(t) \equiv A^* \ \text{ on } [0, T]. \tag{3.10}$$

By (3.5) and (3.10) we have $A^* \ge c + 2$ and

$$\widetilde{\mathrm{J}}_i(x) = A^* \quad \text{if and only if} \quad x = A^* \tag{3.11}$$

is true for any $i = 1, 2, \dots, m$. According to Remark 1.4, (3.2) and (3.8), the functions σ_3 and σ_4 are respectively lower and upper functions of (3.6) which are well-ordered, i.e.

$$\sigma_3(t) < \sigma_4(t) \ \text{ for } t \in [0, T] \quad \text{and} \quad \sigma_3(\tau+) < \sigma_4(\tau+) \ \text{ for } \tau \in \mathrm{D}\,.$$

Similarly, since $A^* \ge c + 2 > \rho_0$, we get by (3.3) the relations

$$\sigma_3(t) < \sigma_2(t) \ \text{ for } t \in [0, T], \quad \sigma_3(\tau+) < \sigma_2(\tau+) \ \text{ for } \tau \in \mathrm{D}$$

and

$$\sigma_1(t) < \sigma_4(t) \ \text{ for } t \in [0, T],, \quad \sigma_1(\tau+) < \sigma_4(\tau+) \ \text{ for } \tau \in \mathrm{D}\,.$$

To summarize, we have three pairs $\{\sigma_3, \sigma_4\}$, $\{\sigma_3, \sigma_2\}$ and $\{\sigma_1, \sigma_4\}$ of well-ordered lower and upper functions of the problem (3.6).

Having G from (1.14), define an operator $\widetilde{\mathrm{F}} : \mathbb{C}^1_{\mathrm{D}}[0, T] \mapsto \mathbb{C}^1_{\mathrm{D}}[0, T]$ by

$$\begin{aligned}(\widetilde{\mathrm{F}}u)(t) = u(0) + u'(0) - u'(T) + \int_0^T G(t, s)\, \widetilde{f}(s, u(s), u'(s))\, \mathrm{d}s \\ - \sum_{i=1}^{m} \frac{\partial G}{\partial s}(t, t_i)\, (\widetilde{\mathrm{J}}_i(u(t_i)) - u(t_i)) + \sum_{i=1}^{m} G(t, t_i)\, (\mathrm{M}_i(u'(t_i)) - u'(t_i)).\end{aligned} \tag{3.12}$$

Then, by [11, Lemma 3.1], $\widetilde{\mathrm{F}}$ is completely continuous and u is a solution of (3.6) if and only if it is a fixed point of $\widetilde{\mathrm{F}}$.

• STEP 2. *We prove the first a priori estimate for solutions of* (3.6).

Denote

$$\Omega_0 = \{u \in \mathbb{C}^1_{\mathrm{D}}[0,T] : \|u\|_\infty < A^*, \|u'\|_\infty < C^*\}, \tag{3.13}$$

$$\text{where } C^* = \frac{2A^*}{\Delta} + \|\widetilde{h}\|_1 + 1 \quad \text{and} \quad \Delta \text{ is defined in } (1.20).$$

By virtue of (1.19) and (3.10), we have $\Omega_0 = \Omega(\sigma_3, \sigma_4, C^*)$. We are going to prove that for each solution u of (3.6), the estimate

$$u \in \mathrm{cl}(\Omega_0) \implies u \in \Omega_0 \tag{3.14}$$

is true. To this aim, suppose that u is a solution of (3.6) and $u \in \mathrm{cl}(\Omega_0)$, i.e. $\|u\|_\infty \le A^*$ and $\|u'\|_\infty \le C^*$. By the Mean Value Theorem, there are $\xi_i \in (t_i, t_{i+1})$, $i = 1, 2, \ldots, m$, such that $|u'(\xi_i)| \le 2\,A^*/\Delta$. Hence, by (3.7), we get

$$\|u'\|_\infty < C^*, \tag{3.15}$$

where C^* is defined in (3.13). It remains to show that $\|u\|_\infty < A^*$. Assuming the contrary there are two cases to distinguish:

CASE A. Let

$$\sup\{u(t) :\ t \in [0,T]\} = A^*. \tag{3.16}$$

Then, due to (3.11), there is $\tau \in [0,T)$ such that

$$u(\tau) = u(\tau+) = A^*. \tag{3.17}$$

Recall that $A^* \ge c + 2$. Consequently, (3.17) implies that

$$u(t) > c + 1 \quad \text{for} \quad t \in [\tau, \tau + \delta] \tag{3.18}$$

is true for some $\delta > 0$. Furthermore, we have

$$u'(\tau+) = 0. \tag{3.19}$$

Indeed, if $\tau = 0$, then (1.3) and (3.16) give $u(0) = u(T) = A^*$ and $0 \ge u'(\tau+) = u'(0) = u'(T) \ge 0$. If $\tau \in \mathrm{D}$, then (3.16) and (3.17) yield $u'(\tau+) \le 0$ and $u'(\tau) \ge 0$, wherefrom, in view of (3.2), $u'(\tau+) \ge 0$. So, (3.19) holds. Finally, if $\tau \in (0,T) \setminus \mathrm{D}$, then the validity of (3.19) is evident.

Now, by (3.8) and (3.18), we obtain that $u''(t) > 0$ holds a.e. on $[\tau, \tau + \delta]$. Consequently, in view of (3.19), we have $u'(t) > u'(\tau+) = 0$ on $(\tau, \tau + \delta)$, a contradiction to (3.16) and (3.17).

Case B. If $\inf\{u(t):\, t\in[0,T]\} = -A^*$, we construct a contradiction similarly as in Case A.

Therefore, $\|u\|_\infty < A^*$ holds for each solution u of (3.6). This together with (3.15) shows that the estimate (3.14) is valid for each solution u of (3.6).

• Step 3. *We prove the second a priori estimate for solutions of the problem* (3.6). Define sets

$$\Omega_1 = \{u\in\Omega_0:\, u(t) > \sigma_1(t) \text{ for } t\in[0,T],\, u(\tau+) > \sigma_1(\tau+) \text{ for } \tau\in\mathrm{D}\},$$

$$\Omega_2 = \{u\in\Omega_0:\, u(t) < \sigma_2(t) \text{ for } t\in[0,T],\, u(\tau+) < \sigma_2(\tau+) \text{ for } \tau\in\mathrm{D}\}$$

and $\widetilde{\Omega} = \Omega_0 \setminus \mathrm{cl}(\Omega_1\cup\Omega_2)$. Then $\Omega_1\cap\Omega_2=\emptyset$ and

$$\widetilde{\Omega} = \{u\in\Omega_0:\, u \text{ satisfies } (2.13)\}. \tag{3.20}$$

Furthermore, with respect to (1.19) and (3.10) we have $\Omega_1 = \Omega(\sigma_1,\sigma_4,C^*)$ and $\Omega_2 = \Omega(\sigma_3,\sigma_2,C^*)$.

We are going to prove that the estimate

$$u\in\mathrm{cl}(\widetilde{\Omega}) \implies \|u\|_\infty < c \tag{3.21}$$

is valid for each solution u of (3.6). So, assume that u is a solution of (3.6) and $u\in\mathrm{cl}(\widetilde{\Omega})$. Then, due to (3.14), u fulfils one of the conditions (2.13), (2.14), (2.15) and so, by (2.16), $u\in B$. Since we have already proved that (2.17) holds, we can use Lemma 2.3 and get $\xi_u\in[0,T]$ such that (2.18) is true. Further, due to (1.3), (2.3) and (3.7), we can apply Lemma 2.1 to show that u satisfies the estimate (2.1). Finally, by Lemma 2.4 and (3.3), u satisfies also (2.11). Moreover, let us recall that $\widetilde{\mathrm{J}}_i$, $i=1,2,\dots,m$, verify the condition (2.12). Hence, by Lemma 2.2, we have (2.9), i.e. each solution u of (3.6) satisfies (3.21).

• Step 4. *We prove the existence of a solution to the problem* (1.1) – (1.3).

Consider the operator $\widetilde{\mathrm{F}}$ defined by (3.12). We distinguish two cases: either $\widetilde{\mathrm{F}}$ has a fixed point in $\partial\widetilde{\Omega}$ or it has no fixed point in $\partial\widetilde{\Omega}$.

Assume that $\widetilde{\mathrm{F}}u = u$ for some $u\in\partial\widetilde{\Omega}$. Then u is a solution of (3.6) and, with respect to (3.21), we have $\|u\|_\infty < c$, which by (3.4) and (3.5) means that u is a solution of (1.1) - (1.3). Furthermore, due to (3.14), u satisfies (2.14) or (2.15), which directly implies that it satisfies (2.26) (cf. also Lemma 2.4).

Now, assume that $\widetilde{\mathrm{F}}u \neq u$ for all $u\in\partial\widetilde{\Omega}$. Then $\widetilde{\mathrm{F}}u\neq u$ for all $u\in\partial\Omega_0\cup\partial\Omega_1\cup\partial\Omega_2$. If we replace f, h, J_i, $i=1,2,\dots,m$, α, β and r respectively by $\widetilde{f}$, $\widetilde{h}$, $\widetilde{\mathrm{J}}_i$, $i=1,2,\dots,m$, σ_3, σ_4 and C^* in Proposition 1.7, we see that the assumptions (1.15)-(1.18) and (1.20) are satisfied. Thus, by Proposition 1.7, we obtain that

$$\deg(\mathrm{I}-\widetilde{\mathrm{F}},\Omega(\sigma_3,\sigma_4,C^*)) = \deg(\mathrm{I}-\widetilde{\mathrm{F}},\Omega_0) = 1. \tag{3.22}$$

Similarly, we can apply Proposition 1.7 to show that

$$\deg(\mathrm{I} - \widetilde{\mathrm{F}}, \Omega(\sigma_1, \sigma_4, C^*)) = \deg(\mathrm{I} - \widetilde{\mathrm{F}}, \Omega_1) = 1 \tag{3.23}$$

and

$$\deg(\mathrm{I} - \widetilde{\mathrm{F}}, \Omega(\sigma_3, \sigma_2, C^*)) = \deg(\mathrm{I} - \widetilde{\mathrm{F}}, \Omega_2) = 1. \tag{3.24}$$

Using the additivity property of the Leray - Schauder topological degree we derive from (3.22) - (3.24) that

$$\deg(\mathrm{I} - \widetilde{\mathrm{F}}, \widetilde{\Omega}) = -1.$$

Therefore, $\widetilde{\mathrm{F}}$ has a fixed point $u \in \widetilde{\Omega}$. By (3.20), (3.21) and Lemma 2.4 we have (2.26) and $\|u\|_\infty < c$. This together with (3.4) and (3.5) yields that u is a solution to (1.1) - (1.3) fulfilling (2.26). □

We close this paper by two simple examples of effective existence criteria which are straightforward corollaries of Theorem 3.1 and Proposition 1.3.

3.2. Corollary. *Let* (1.10), (3.1) *and* (3.2) *hold and let* α_i, $\beta_i \in \mathbb{R}$, $i = 0, 1, \ldots, m$, *fulfil the assumptions of Proposition* 1.3. *Furthermore, assume that the implications*

$$x > \alpha_{i-1} \implies \mathrm{J}_i(x) > \mathrm{J}_i(\alpha_{i-1}) \quad \textit{and} \quad x < \beta_{i-1} \implies \mathrm{J}_i(x) < \mathrm{J}_i(\beta_{i-1})$$

are true for $i = 1, 2, \ldots, m$. *Then the problem* (1.1) - (1.3) *has a solution.*

Proof. Let the functions σ_1 and σ_2 be defined as in Proposition 1.3. By this proposition they are respectively lower and upper functions of (1.1) - (1.3). If $\alpha_i \le \beta_i$ for all $i = 0, 1, \ldots, m$, then $\sigma_1 \le \sigma_2$ on $[0, T]$ and, by [11, Proposition 3.2], the problem (1.1) - (1.3) has a solution u such that $\sigma_1 \le u \le \sigma_2$ on $[0, T]$. If $\alpha_j > \beta_j$ for some $j \in \{0, 1, \ldots, m\}$, then the existence of a solution u to (1.1) - (1.3) follows by means of Theorem 3.1. □

3.3. Remark. Notice that in the case $\sigma_2 \le \sigma_1$ on $[0, T]$, the property (2.26) reduces to

$$\sigma_2(\tau_u+) \le u(\tau_u+) \le \sigma_1(\tau_u+) \quad \text{for some} \ \ \tau_u \in [0, T).$$

The next assertion follows from Theorem 3.1 if we take into account Remarks 1.4 and 3.3.

3.4. Corollary. *Let* (1.10), (3.1) *and* (3.2) *hold. Assume that there are* r_1, $r_2 \in \mathbb{R}$ *such that* $f(t, r_1, 0) \le 0 \le f(t, r_2, 0)$ *for a.e.* $t \in [0, T]$. *Further, let the relations* $\mathrm{J}_i(r_1) = r_1 > r_2 = \mathrm{J}_i(r_2)$,

$$x > r_1 \implies \mathrm{J}_i(x) > \mathrm{J}_i(r_1) \qquad \textit{and} \qquad x < r_2 \implies \mathrm{J}_i(x) < \mathrm{J}_i(r_2)$$

be true for $i = 1, 2, \ldots, m$. *Then the problem* (1.1)-(1.3) *has a solution* u *such that* $r_2 \le u(t_u+) \le r_1$ *for some* $t_u \in [0, T)$. □

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