ACADEMY OF SCIENCES
OF THE CZECH REPUBLIC

MATHEMATICAL INSTITUTE

CONSTRUCTION OF A LYAPUNOV FUNCTIONAL FOR 1D-VISCOUS COMPRESSIBLE BAROTROPIC FLUID EQUATIONS ADMITTING VACUA

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(preprint)

164

2006

CONSTRUCTION OF A LYAPUNOV FUNCTIONAL FOR 1D-VISCOUS COMPRESSIBLE BAROTROPIC FLUID EQUATIONS ADMITTING VACUA

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Abstract

The Navier-Stokes equations for a compressible barotropic fluid in 1D with zero velocity boundary conditions are considered. We study the case of large initial data in H^1 as well as the mass force such that the stationary density is uniquely determined but admits vacua. Missing uniform lower bound for the density is compensated by a careful modification of the construction procedure for a Lyapunov functional known for the case of solutions which are globally away from zero [9]. An immediate consequence of this construction is a decay rate estimate for this highly singular problem. The results are proved in the Eulerian coordinates for a large class of increasing state functions including $p(\rho) = a\rho^\gamma$ with any $\gamma > 0$ ($a > 0$ a constant).

Mathematics Subject Classification (2000): 35Q30, 35B40, 76N15

Keywords: Compressible fluid, Navier-Stokes equations, asymptotic behavior

0 Introduction

The purpose of this study is *construction of a Lyapunov functional* for $1D$ Navier-Stokes equations of a viscous compressible barotropic fluid under the influence of a large mass force in the case when the *stationary density admits vacua.* We assume standard initial-boundary value problem with zero velocity boundary conditions as in (1.1) – (1.3) below. An immediate product of our construction is a result on a decay rate of evolutionary solution to the stationary one as time tends to infinity (see Theorem 1.1).

There are many results about the global behavior of solutions to equations (1.1), (1.2) below under different boundary conditions and other data and we refer e.g. to [3], [6], [7], [9], [10] and [11] and the references therein, see also results and comments in a recent monograph [5], Chapter 8.

In this work we continue the research results of which are summarized in [9], where a Lyapunov functional has been constructed for the case of *positive stationary density* given by equations (1.16), (1.17) below. Note that the explicit necessary and sufficient conditions for such a positivity are known (see Proposition 1.3). Since for large class of external forces the stationary densities can contain vacua zones while being uniquely determined, we believe that Lyapunov analysis is important also for this case. To our knowledge, the only result in this direction and generality is in [12], where an analogous problem with a free boundary has been tackled.

The free boundary condition allows us to derive a *global lower bound for the density* in terms of the stationary density which we are *not* able to find for the Dirichlet boundary condition and thus have to find an alternative argument. This argument is given by a careful use of a *comparison quasistationary density* approximating the original one. Two crucial apriori estimates play decisive role in the construction. An appropriate form of the energy equality and an estimate utilizing the monotonicity of the state function and the analysis of approximative relation between the quasistationary density and the original density ρ.

Despite of the singularity of the problem, a large class of mass forces and state functions is admitted. First, we give a survey of already known results which play an important role in the following arguments. Then we present the construction of a special differential equality including the velocity, the density, stationary density and quasistationary density. The terms including quasistationary density are carefully analysed with the aim to exclude it from the differential equality and modify it to a differential inequality including a suitable Lyapunov functional. Resolving the Lyapunov differential inequality we obtain a decay rate for the convergence of the evolutionary solution to the stationary one.

1 Basic known facts and the main result

We consider the following system of equations describing 1D-flow of a viscous compressible barotropic fluid

$$\rho_t + (\rho u)_x = 0, \tag{1.1}$$

$$(\rho u)_t + (\rho u^2)_x - (\mu u_x - p(\rho))_x = \rho f \tag{1.2}$$

in the domain $Q = (0, \ell) \times (0, \infty)$ with the boundary and initial conditions

$$u|_{x=0,\ell} = 0;\ \rho|_{t=0} = \rho^0(x),\ u|_{t=0} = u^0(x) \text{ in } (0, \ell). \tag{1.3}$$

Suppose that

$$f(x, t) = f_\infty(x) + g(x, t) \text{ with } f_\infty \in W^{1,\infty}(0, \ell) \text{ and } g \in L^{\infty,2}(Q_\infty). \tag{1.4}$$

Here $Q_T = (0, \ell) \times (0, T)$. Throughout the paper we use the anisotropic Lebesgue space $L^{q,s}(Q)$ equipped with the norm $\|w\|_{L^{q,s}(Q)} := \|\|w\|_{L^q(0,\ell)}\|_{L^s(0,\infty)}$, $W^{k,p}$ means the usual Sobolev space. Let the initial functions satisfy

$$\rho^0, u^0 \in H^1(0, \ell), \quad 0 < \underline{\rho}^0 \le \rho^0, \quad u^0\Big|_{x=0,\ell} = 0. \tag{1.5}$$

Our main requirements on the state function p are as follows.

$$p \text{ is } \textit{continuous, increasing} \text{ function on } [0, \infty),\ p(0) = 0,\ p(\infty) = \infty; \tag{1.6}$$

$$p' \in L^\infty_{\text{loc}}(0, \infty), \quad p'(r) > 0,\ r > 0; \tag{1.7}$$

$$p(r) \sim r^\gamma \text{ as } r \to 0^+ \text{ with a } \gamma > 0; \tag{1.8}$$

$$rp'(r) \le \text{const} \quad \text{as } r \to 0^+. \tag{1.9}$$

We shall study the asymptotic behavior of *the strong generalized solution* to problem (1.1) – (1.3) having the following properties: $\rho \in C\left(\overline{Q}_T\right)$, $\rho_x, \rho_t \in L^{2,\infty}(Q_T)$, $\rho > 0$ and $u \in H^1(Q_T) \cap L^2(0, T; \overset{\circ}{H}{}^1(0, \ell))$, $u_{xx} \in L^2(Q_T)$ for any $T > 0$.

Define also

$$P(r) := r \int_1^r \frac{p(s) - p(1)}{s^2}\, ds, \tag{1.10}$$

$$\Pi(r,s) = \int_s^r \frac{p(\sigma) - p(s)}{\sigma^2} d\sigma \quad r, s \geq 0$$

and $F := If_\infty$. We use the notation $Ih := \int_0^x h(y)\, dy$ for any function $h \in L^1(0,\ell)$.

First of all, we remind the mass and energy conservation laws:

$$\int_0^\ell \rho(x,t)\, dx = \int_0^\ell \rho^0(x)\, dx =: m, \tag{1.11}$$

$$\frac{d}{dt} \int_0^\ell \left(\frac{1}{2}\rho u^2 + P(\rho) - \rho F \right) dx + \mu \int_0^\ell (u_x)^2\, dx = \int_0^\ell \rho g u\, dx \tag{1.12}$$

or (1.13)

$$\frac{d}{dt} \int_0^\ell \frac{1}{2}\rho u^2\, dx + \mu \int_0^\ell (u_x)^2\, dx = \int_0^\ell (\rho f_\infty u + \rho g u - p(\rho)_x u)\, dx. \tag{1.14}$$

Denote the initial total energy by

$$E_0 := \int_0^\ell \Big(\frac{\rho_0 u_0^2}{2} + P(\rho_0) - \rho_0 F \Big)\, dx. \tag{1.15}$$

In the whole paper we will assume that the stationary problem which is given by

$$p(\rho_\infty)_x = \rho_\infty f_\infty \quad \text{on } (0,\ell), \tag{1.16}$$

$$\int_0^\ell \rho_\infty(x)\, dx = m, \quad \rho_\infty \geq 0 \tag{1.17}$$

has a unique solution $\rho_\infty \in L^\infty(0,\ell)$.

Our main result is contained in the following theorem.

Theorem 1.1 (Main result) *Let conditions (1.4)-(1.8) be satisfied and the stationary problem (1.16), (1.17)have a unique solution* $\rho_\infty \in L^\infty(0,\ell)$. *Then for any* $t_0 \geq 0$ *there are positive constants* $K := K(t_0, \ell, m, \mu, E_0, \|f_\infty\|_{W^{1,\infty}(0,\ell)})$ *and* $\alpha := \alpha(t_0, \ell, m, \mu, E_0, \|f_\infty\|_{W^{1,\infty}(0,\ell)})$ *such that*

$$\int_0^\ell \Big(\rho u^2 + \rho \Pi(\rho, \rho_\infty) + |\rho - \rho_\infty|^\beta + (p(\rho) - p(\overline{\rho}))^2 \Big)(x,t)\, dx \tag{1.18}$$

$$\leq K \Big\{ e^{-\alpha(t-t_0)} \Big[1 + \int_{t_0}^t e^{\alpha s} \|g(s)\|_2^2\, ds \Big] + \int_t^\infty \|g(s)\|_2^2\, ds \Big\}, \quad t \geq t_0,$$

where $\overline{\rho}$ *is given by (1.27) below and* $\beta \geq 2$ *if* $\gamma < 2$ *or* $\beta \geq \gamma$ *if* $\gamma \geq 2$ *is arbitrary but fixed.*

Theorem 1.1 will be proved in Section 2 after the following preliminaries.

First, a well-known consequence of energy equation (1.12) is

Proposition 1.2 *([9]) Suppose in addition to (1.6), (1.7) that the conditions*

$$0 < \rho^0 \leq N, \ \|u^0\|_{L^2(0,\ell)} \leq N, \ \|f_\infty\|_{L^\infty(0,\ell)} \leq N, \tag{1.19}$$

$$\|g\|_{L^{\infty,2}(Q)} \leq N \tag{1.20}$$

and $\|P(\rho^0)\|_{L^1(0,\ell)} \leq N$ *are satisfied. Then we have*

(i)

$$\|\sqrt{\rho}u\|_{L^{2,\infty}(Q)} + \|P(\rho)\|_{L^{1,\infty}(Q)} + \|u_x\|_{L^2(Q)} \le K(N); \tag{1.21}$$

(ii)

$$\rho(x,t) \le \tilde{\rho} =: K(N) \tag{1.22}$$

holds, and

(iii)

$$\frac{1}{2}\int_0^\ell (\rho u^2)(x,t)\,dx \to 0 \quad \text{as } t \to \infty. \tag{1.23}$$

It was already mentioned that there is a necessary and sufficient condition for the solution $\rho_\infty \in C([0,\ell])$ of (1.16), (1.17) such that $p(\rho_\infty)_x \in L^\infty(0,\ell)$ to be *positive* (i.e., $\rho_\infty > 0$). Denoting

$$F_{\min} := \min_{[0,\ell]} F(x),\ \ F_{\max} := \max_{[0,\ell]} F(x),\ \ C_p := \int_0^1 \frac{p(r)}{r^2}\,dr \le \infty,$$

this condition reads:

Proposition 1.3 *([9]) Let (1.6) be satisfied and $f_\infty \in L^\infty(0,\ell)$. Then the positive solution ρ_∞ to the problem (1.16), (1.17) exists if and only if*

$$C_p = \infty \quad \text{or} \quad \left.\begin{array}{r} C_p < \infty \\ F_{\max} - F_{\min} < \Psi(\infty) \\ \frac{1}{m}\int_0^\ell \Psi^{-1}(F(x) - F_{\min})\,dx < 1 \end{array}\right\}, \tag{1.24}$$

where $\Psi(r) := \frac{p(r)}{r} + \int_0^r \frac{p(s)}{s^2}\,ds$ for $r > 0$ and $\Psi(0) = 0$, with Ψ^{-1} being the inverse of Ψ. Moreover, for $C_p < \infty$, the function Ψ is continuous and increasing on $[0,\infty)$.

In addition, the positive solution is unique.

Proposition 1.4 *([9]) Let conditions (1.4) – (1.7) be satisfied and $p(\cdot)$, $f_\infty \in BV([0,\ell])$ and $m > 0$ be such that there is a unique solution of (1.16), (1.17). Then*

$$\|p(\rho(t)) - p(\overline{\rho}(t))\|_{L^q(0,\ell)} + \|\rho(t) - \rho_\infty\|_{L^q(0,\ell)} \to 0 \quad \text{as} \quad t \to \infty,\ \forall\, q \in [1,\infty) \tag{1.25}$$

and

$$\|p(\overline{\rho}(t)) - p(\rho_\infty)\|_{C([0,\ell])} \to 0 \quad \text{as } t \to \infty, \tag{1.26}$$

where $\overline{\rho} = \overline{\rho}(x,t)$ is such that

$$p(\overline{\rho}(x,t)) = \frac{1}{\ell}\int_0^\ell p(\rho(\xi,t))\,d\xi + \frac{1}{\ell}\int_0^\ell \int_\xi^\ell \rho(\eta,t)f_\infty(\eta)d\eta\,d\xi - \int_x^\ell \rho(\xi,t)f_\infty(\xi)\,d\xi. \tag{1.27}$$

Notice, that $\overline{\rho}$ satisfies

$$p(\overline{\rho})_x = \rho f_\infty, \quad x \in (0,\ell),\ t > 0, \qquad \int_0^\ell p(\overline{\rho})\,dx = \int_0^\ell p(\rho)\,dx,\ t > 0. \tag{1.28}$$

Let us note that the idea with "quasistationary density" $\overline{\rho}$ was for the first time used for stabilization in [4], where the case of 2 and 3 space variables has been treated.

Next Proposition shows that there are fairly general explicit conditions for uniqueness of the solution to equations (1.16), (1.17). We refer in this respect to ([1]) and the references therein.

Proposition 1.5 *([2],[1]) Let in addition to (1.7) we have* $p \in C([0,\infty)) \cap C^1(0,\infty)$ *and* $F = If$ *be locally Lipschitz continuous on* $(0,\ell)$.

If $\int_0^1 \frac{dp(s)}{s} < \infty$, *assume in addition, that the upper level sets* $\{x \in (0,\ell); f(x) > k\}$ *are connected in* $(0,\ell)$ *for any constant* $k \in R$.

Then, given $m > 0$, *there is at most one function* $\rho_\infty \in L^\infty_{\text{loc}}(0,\ell)$ *satisfying (1.16), (1.17) in the sense of distributions.*

Moreover, if such a function exists, it is given by the formula

$$\rho_\infty = \Psi^{-1}([f(x) - k_\ell]^+)$$

for a certain constant k_ℓ. *(Here* $[z]^+ := \max\{z, 0\}$.)

We will also need the following elementary lemma.

Lemma 1.6 *Let* $r_0 > 0$ *and* $s_0 > 0$ *be arbitrary fixed numbers and assume* $p(r) \sim r^\gamma$ *as* $r \to 0^+$ *with a constant* $\gamma > 0$. *Let* $\beta \geq 2$ *if* $\gamma < 2$ *and* $\beta \geq \gamma$ *if* $\gamma \geq 2$. *Then there is a constant* $k = k(\beta)$ *such that*

$$k(\beta)|r - s|^\beta \leq r\Pi(r,s)) \quad \text{for all } r \in (0, r_0],\ s \in [0, s_0]. \tag{1.29}$$

Proof. First, let $s > 0$ be fixed. Then by the l'Hospital rule

$$\lim_{r \to s} \frac{|r - s|^\beta}{(r\Pi(r,s))} \tag{1.30}$$

$$= \frac{\beta s}{p'(s)} \lim_{r \to s} |s - r|^{\beta - 2} = \begin{cases} \infty \text{ if } \beta < 2 \\ \frac{\beta s}{p'(s)} \text{ if } \beta = 2 \\ 0 \text{ if } \beta > 2. \end{cases}$$

Let now $s = 0$ and use the assumption $p(r) \sim r^\gamma$ near zero. Then

$$\lim_{r \to 0^+} \frac{r^\beta}{r\Pi(r,0))} = \lim_{r \to 0^+} (\beta - 1)\frac{r^\beta}{p(r)} = \begin{cases} \infty \text{ if } \beta < \gamma \\ \beta - 1 \text{ if } \beta = \gamma \\ 0 \text{ if } \beta > \gamma. \end{cases} \tag{1.31}$$

The result immediately follows. □

2 Construction of a Lyapunov functional

Let us subtract the differential equation in (1.28) from equation (1.2). We obtain the relation

$$(\rho u)_t + (\rho u^2)_x - \mu u_{xx} + p(\rho)_x - p(\overline{\rho})_x = \rho g. \tag{2.1}$$

Multiply (2.1) by $-\varepsilon I(p(\rho)-p(\overline{\rho}))$ and integrate over $(0,\ell)$:

$$
\begin{aligned}
&-\varepsilon\frac{d}{dt}\int_0^\ell \rho u I(p(\rho)-p(\overline{\rho}))\,dx+\varepsilon\int_0^\ell \rho u I(p(\rho)_t-p(\overline{\rho})_t)\,dx\\
&+\varepsilon\int_0^\ell(\rho u^2-\mu u_x)(p(\rho)-p(\overline{\rho}))\,dx+\varepsilon\int_0^\ell(p(\rho)-p(\overline{\rho}))^2\,dx \qquad (2.2)\\
&=\varepsilon\int_0^\ell \rho g I(p(\overline{\rho})-p(\rho))\,dx.
\end{aligned}
$$

Adding (1.14) multiplied by a positive parameter $\eta>0$ and (2.2) we find

$$
\begin{aligned}
&\frac{d}{dt}\int_0^\ell(\frac{\eta\rho u^2}{2}-\varepsilon\rho u I(p(\rho)-p(\overline{\rho})))\,dx+\eta\int_0^\ell(p(\overline{\rho})-p(\rho))u_x\,dx+\varepsilon\int_0^\ell \rho u I(p(\rho)_t-p(\overline{\rho})_t)\,dx\\
&+\varepsilon\int_0^\ell(\rho u^2-\eta\mu u_x)(p(\rho)-p(\overline{\rho}))\,dx+\varepsilon\int_0^\ell(p(\rho)-p(\overline{\rho}))^2\,dx+\eta\mu\int_0^\ell u_x^2\,dx \qquad (2.3)\\
&=\eta\int_0^\ell \rho u g\,dx+\varepsilon\int_0^\ell \rho g I(p(\overline{\rho})-p(\rho))\,dx.
\end{aligned}
$$

Next, we also have

$$
\frac{1}{2}\frac{d}{dt}\int_0^\ell(p(\rho)-p(\overline{\rho}))^2=\int_0^\ell(p(\rho)_t-p(\overline{\rho})_t)(p(\rho)-p(\overline{\rho}))\,dx, \qquad (2.4)
$$

where (by the equation of continuity and (1.27))

$$
\begin{aligned}
&p(\rho)_t-p(\overline{\rho})_t=-(p(\rho)u)_x+(p(\rho)-\rho p'(\rho))u_x \qquad (2.5)\\
&+\frac{1}{\ell}\int_0^\ell(\rho p'(\rho)-p(\rho))u_x\,dx+\int_0^\ell\frac{1}{\ell}I^*((\rho u)_x f_\infty)\,dx-I^*((\rho u)_x f_\infty)
\end{aligned}
$$

Further we have (notice that $\int_0^\ell(\int_0^\ell h d\xi)(p(\rho)-p(\overline{\rho}))dx=0$ since $\int_0^\ell p(\rho)dx=\int_0^\ell p(\overline{\rho})dx$)

$$
\begin{aligned}
&-\int_0^\ell(p(\rho)u)_x(p(\rho)-p(\overline{\rho}))\,dx=-\frac{1}{2}\int_0^\ell p(\rho)^2u_x\,dx-\int_0^\ell p(\rho)up(\overline{\rho})_x\,dx \qquad (2.6)\\
&=\frac{1}{2}\int_0^\ell(p(\overline{\rho})^2-p(\rho)^2)u_x\,dx-\frac{1}{2}\int_0^\ell p(\overline{\rho})^2u_x\,dx+\int_0^\ell(p(\overline{\rho})-p(\rho))up(\overline{\rho})_x\,dx\\
&-\int_0^\ell p(\overline{\rho})p(\overline{\rho})_x u\,dx=\int_0^\ell(p(\overline{\rho})-p(\rho))u\rho f_\infty\,dx+\frac{1}{2}\int_0^\ell(p(\overline{\rho})^2-p(\rho)^2)u_x\,dx,
\end{aligned}
$$

and

$$
I^*((\rho u)_x f_\infty)=-\rho u f_\infty-\int_x^\ell \rho u f'_\infty\,dx. \qquad (2.7)
$$

Summarizing (2.4)–(2.7) we get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^\ell(p(\rho)-p(\overline{\rho}))^2\,dx=\int_0^\ell(p(\overline{\rho})-p(\rho))u\rho f_\infty\,dx+\frac{1}{2}\int_0^\ell(p(\overline{\rho})^2-p(\rho)^2)u_x\,dx\\
&+\int_0^\ell(p(\rho)-p(\overline{\rho}))(p(\rho)-\rho p'(\rho))u_x\,dx+\int_0^\ell(p(\rho)-p(\overline{\rho}))(\rho u f_\infty+I^*(\rho u f'_\infty))\,dx\\
&=\frac{1}{2}\int_0^\ell(p(\overline{\rho})^2-p(\rho)^2)u_x\,dx+\int_0^\ell(p(\rho)-p(\overline{\rho}))(p(\rho)-\rho p'(\rho))u_x\,dx \qquad (2.8)\\
&+\int_0^\ell(p(\rho)-p(\overline{\rho}))I^*(\rho u f'_\infty)\,dx.
\end{aligned}
$$

Multiply equality (2.8) by a parameter $\delta > 0$ and add to (2.3):

$$\begin{aligned}
&\frac{d}{dt}\int_0^\ell \big(\frac{\eta\rho u^2}{2} + \frac{\delta}{2}(p(\overline{\rho}) - p(\rho))^2 + \varepsilon\rho u I(p(\overline{\rho}) - p(\rho))\big)\,dx + \eta\int_0^\ell (p(\overline{\rho}) - p(\rho))u_x\,dx \\
&+\varepsilon\int_0^\ell \rho u I(p(\rho)_t - p(\overline{\rho})_t)\,dx + \varepsilon\int_0^\ell (\rho u^2 - \eta\mu u_x)(p(\rho) - p(\overline{\rho}))\,dx \\
&+\varepsilon\int_0^\ell (p(\rho) - p(\overline{\rho}))^2\,dx + \eta\mu\int_0^\ell u_x^2\,dx \qquad (2.9)\\
&+\delta\Big[\frac{1}{2}\int_0^\ell (p(\overline{\rho})^2 - p(\rho)^2)u_x\,dx + \int_0^\ell (p(\rho) - p(\overline{\rho}))(p(\rho) - \rho p'(\rho))u_x\,dx \\
&+\int_0^\ell (p(\rho) - p(\overline{\rho}))I^*(\rho u f'_\infty)\,dx\Big] \\
&= \eta\int_0^\ell \rho u g\,dx + \varepsilon\int_0^\ell \rho g I(p(\overline{\rho}) - p(\rho))\,dx.
\end{aligned}$$

Our intention now is to compare the integral under $\frac{d}{dt}$ with the remaining terms in equality (2.3).

Lemma 2.1 *The following inequality holds true:*

$$\begin{aligned}
V_{\varepsilon,\delta}(t) &:= \int_0^\ell \Big(\frac{\eta\rho u^2}{2} + \frac{\delta}{2}(p(\rho) - p(\overline{\rho}))^2 + \varepsilon\rho u I(p(\rho) - p(\overline{\rho}))\Big)\,dx \qquad (2.10)\\
&\geq (\frac{\eta}{2} - \varepsilon m\beta)\int_0^\ell \rho u^2\,dx + (\frac{\delta}{2} - \varepsilon\ell m\beta^{-1})\int_0^\ell (p(\rho) - p(\overline{\rho}))^2\,dx.
\end{aligned}$$

Proof Indeed, we have

$$\begin{aligned}
&\int_0^\ell \rho u I(p(\overline{\rho}) - p(\rho))\,dx \leq \|\rho\|_1\Big(\int_0^\ell \rho u^2\,dx\Big)^{1/2}\|p(\overline{\rho}) - p(\rho)\|_1 \qquad (2.11)\\
&\leq m\Big(\beta\int_0^\ell \rho u^2\,dx + \frac{\ell}{\beta}\|p(\overline{\rho}) - p(\rho)\|_2^2\Big)
\end{aligned}$$

with any positive constant β. Now estimate (2.11) immediately follows. □

Lemma 2.2 *The following inequality holds true:*

$$\varepsilon\Big|\int_0^\ell \rho u I(p(\rho)_t - p(\overline{\rho})_t)\,dx\Big| \leq \varepsilon\, c(\ell, m, \mu, E_0, \|f_\infty\|_{W^1_\infty(0,\ell)})\,\|u_x\|_2^2, \qquad (2.12)$$

Proof First, by the renormalized equation of continuity,

$$\int_0^x p(\rho)_t\,d\xi = -p(\rho)u + \int_0^x (p(\rho) - \rho p'(\rho))u_x\,d\xi. \qquad (2.13)$$

Secondly, by (1.27) we have

$$Ip(\overline{\rho})_t = \frac{x}{\ell}\int_0^\ell p(\rho)_t\,d\xi + \frac{x}{\ell}\int_0^\ell\int_\xi^\ell \rho_t f_\infty\,d\eta\,d\xi - \int_0^x\int_\xi^\ell \rho_t f_\infty\,d\eta\,d\xi. \qquad (2.14)$$

Then, again by the equation of continuity and by the help of several integrations by parts we finally obtain

$$\begin{aligned} Ip(\overline{\rho})_t = \frac{x}{\ell}\int_0^\ell (p(\rho) - \rho p'(\rho))u_x \, d\xi + \frac{x}{\ell}\int_0^\ell \rho u(\xi f_\infty)_\xi \, d\xi \\ -x\rho u f_\infty + \int_0^x (\rho u)(\xi f_\infty)_\xi \, d\xi. \end{aligned} \tag{2.15}$$

Thus

$$\begin{aligned} &\int_0^\ell \rho u I(p(\rho)_t - p(\overline{\rho})_t)\, dx = -\int_0^\ell \rho u^2 p(\rho)\, dx + \int_0^\ell \rho u \int_0^x (p(\rho) - \rho p'(\rho))u_x \, d\xi dx \\ &- \int_0^\ell \frac{x}{\ell}\rho u \int_0^\ell (p(\rho) - \rho p'(\rho))u_x \, d\xi dx - \int_0^\ell \frac{x}{\ell}\rho u \int_0^\ell \rho u(\xi f_\infty)_\xi \, d\xi dx \\ &- \int_0^\ell x\rho u \int_x^\ell \rho u f_\infty \, d\xi dx + \int_0^\ell \rho u \int_0^\xi \rho u(\xi f_\infty)_\xi \, d\xi dx. \end{aligned} \tag{2.16}$$

Further we have

$$\Big|\int_0^\ell \rho p(\rho) u^2 \, dx\Big| \le \sup_{x,t} \rho p(\rho)\, \|u\|_2^2 \le c\|u_x\|_2^2,$$

since by Proposition 1.2, ρ is globally bounded. Notice that $p(\overline{\rho})$ is globally bounded:

$$|p(\overline{\rho})(x,t)| \le \frac{1}{\ell}\int_0^\ell p(\rho)\, dx + 2m\|f_\infty\|_\infty \le c(\ell, m, E_0, \|f_\infty\|_{W^1_\infty(0,\ell)}) \quad \forall x,t. \tag{2.17}$$

Similarly we have

$$\Big|\int_0^\ell \rho u \int_0^x (p(\rho) - \rho p'(\rho))u_x \, d\xi dx\Big| \le m\sqrt{\ell}\|u_x\|_2 \Big|\int_0^\ell (p(\rho) - \rho p'(\rho))u_x \, dx\Big| \le c\|u_x\|_2^2, \tag{2.18}$$

where, for brevity, we do not mark the arguments of $c(\cdot)$. Similarly we have

$$\begin{aligned} &\Big|\int_0^\ell \frac{x}{\ell}\rho u \int_0^\ell (p(\rho) - \rho p'(\rho))u_x \, d\xi dx\Big| \le c\|u_x\|_2^2, \\ &\Big|\int_0^\ell \frac{x}{\ell}\rho u \int_0^\ell \rho u(\xi f_\infty)_\xi \, d\xi dx \le \sqrt{\ell} m^2 \|u_x\|^2 \|(\xi f_\infty)_\xi\|_\infty \le c\|u_x\|_2^2, \\ &\Big|\int_0^\ell x\rho u \int_x^\ell \rho u f_\infty \, d\xi dx\Big| \le \ell^2 m^2 \|f_\infty\|_\infty \|u_x\|_2^2 \le c\|u_x\|_2^2, \\ &\Big|\int_0^\ell \rho u \int_0^x \rho u(\xi f_\infty)_\xi \, d\xi dx\Big| \le m^2 \ell \|(\xi f_\infty)_\xi\|_\infty \|u_x\|_2^2 \le c\|u_x\|_2^2. \end{aligned} \tag{2.19}$$

The inequality (2.12) immediately follows. □

According to Lemma 2.2 the term on the right-hand side of (2.12) can be made subordinate to the term $\mu \int_0^\ell u_x^2 \, dx$ when taking ε small enough, in particular if

$$\varepsilon c(\ell, m, \mu, E_0, \|f_\infty\|_{W^1_\infty(0,\ell)}) < \mu. \tag{2.20}$$

Proceeding in (2.9) to the estimate of the term $\int_0^\ell (\rho u^2 - \mu u_x)(p(\rho) - p(\overline{\rho}))\,dx$ we first observe that

$$\int_0^\ell \rho u^2 (p(\rho) - p(\overline{\rho}))\,dx \le \Big(\ell \int_0^\ell (p(\rho) - p(\overline{\rho}))(\rho - \overline{\rho})\,dx + \sup_{x,t} \overline{\rho}\ \|p(\rho) - p(\overline{\rho})\|_1\Big)\|u_x\|_2^2$$
$$\le \eta(t)\|u_x\|_2^2, \tag{2.21}$$

where $\eta(t) \to 0$ as $t \to \infty$ by Proposition 1.4. Next we observe in (2.9) that the term

$$\eta \int_0^\ell (p(\overline{\rho}) - p(\rho))u_x\,dx$$

can be compounded with the term $-\varepsilon\eta\mu \int_0^\ell (p(\rho) - p(\overline{\rho}))u_x\,dx$ to obtain $(\eta + \varepsilon\eta\mu)\int_0^\ell (p(\overline{\rho}) - p(\rho))u_x\,dx$ which we estimate as

$$\Big|(\eta + \varepsilon\eta\mu)\int_0^\ell (p(\overline{\rho}) - p(\rho))u_x\,dx\Big| \le (\eta + \varepsilon\eta\mu)(\lambda_1\|u_x\|_2^2 + \lambda_1^{-1}\|p(\overline{\rho}) - p(\rho)\|_2^2). \tag{2.22}$$

Quite analogously is estimated the last inconvenient term on the left-hand side of (2.9):

$$\begin{aligned}&\delta\Big|\frac{1}{2}\int_0^\ell (p(\overline{\rho})^2 - p(\rho)^2)u_x\,dx + \int_0^\ell (p(\rho) - p(\overline{\rho}))(p(\rho) - \rho p'(\rho))u_x\,dx \\ &\quad + \int_0^\ell (p(\rho) - p(\overline{\rho}))I^*(\rho u f'_\infty)\,dx\Big| \\ &\quad \le \delta c(\ell, m, \mu, E_0, \|f_\infty\|_{W^{1,\infty}(0,\ell)})(\lambda_2\|u_x\|_2^2 + \lambda_2^{-1}\|p(\rho) - p(\overline{\rho})\|_2^2).\end{aligned} \tag{2.23}$$

Finally,

$$\begin{aligned}\Big|\int_0^\ell \rho u g\,dx\Big| &\le \sqrt{\ell}(\sup_{x,t\ge t_0} \rho)\|u_x\|_2\|g\|_2 \\ &\le \lambda_3\|u_x\|_2^2 + c(\ell, m, \mu, E_0, \|f_\infty\|_{W^{1,\infty}(0,\ell)}))\lambda_3^{-1}\|g\|_2^2,\end{aligned} \tag{2.24}$$

and

$$\begin{aligned}\varepsilon\Big|\int_0^\ell \rho g I(p(\rho) - p(\overline{\rho}))\,dx\Big| &\le \varepsilon\sqrt{\ell}(\sup_{x,t} \rho)\|g\|_2\|p(\rho) - p(\overline{\rho})\|_2 \\ &\le \varepsilon(\lambda_4\|p(\rho) - p(\overline{\rho})\|_2^2 + c(\ell, m, \mu, E_0, \|f_\infty\|_{W^{1,\infty}(0,\ell)})\lambda_4^{-1}\|g\|_2^2).\end{aligned} \tag{2.25}$$

Using estimates (2.12), (2.21), (2.22), (2.23), (2.24) and (2.25) in (2.9) we obtain

$$\begin{aligned}&\frac{d}{dt}\int_0^\ell \Big(\frac{\eta\rho u^2}{2} + \frac{\delta}{2}(p(\rho) - p(\overline{\rho}))^2 + \varepsilon\rho u I(p(\rho) - p(\overline{\rho}))\Big)\,dx \\ &\quad + (\eta\mu - \varepsilon c - \eta(t) - (\eta + \varepsilon\eta\mu)\lambda_1 - c\delta\lambda_2 - \eta\lambda_3)\|u_x\|_2^2 \\ &\quad + (\varepsilon - (\eta + \varepsilon\eta\mu)\lambda_1^{-1} - c\delta\lambda_2^{-1} - \varepsilon\lambda_4)\|p(\rho) - p(\overline{\rho})\|_2^2 \\ &\quad \le c(\lambda_3^{-1}\eta + \varepsilon\lambda_4^{-1})\|g\|_2^2.\end{aligned} \tag{2.26}$$

To get a decay of the functional $V_{\varepsilon,\delta}(t)$ defined by (2.10) we need (observe that $\eta(t) \to 0$ as $t \to \infty$)

$$\begin{aligned}&\eta\mu > c\varepsilon + \lambda_1(\eta + \varepsilon\eta\mu) + c\delta\lambda_2 + \eta\lambda_3, \\ &\varepsilon > \lambda_1^{-1}(\eta + \varepsilon\eta\mu) + c\lambda_2^{-1}\delta + \varepsilon\lambda_4.\end{aligned} \tag{2.27}$$

Since the parameters λ_3, λ_4 can be chosen independently, so that, for example, sufficiently small, it suffices, instead of (2.27), to consider conditions

$$\begin{aligned} &\eta\mu > c\varepsilon + \lambda_1(\eta + \varepsilon\eta\mu) + c\delta\lambda_2 \\ &\varepsilon > \lambda_1^{-1}(\eta + \varepsilon\eta\mu) + c\lambda_2^{-1}\delta. \end{aligned} \tag{2.28}$$

From (2.10) we get additional conditions for positivity of $V_{\varepsilon,\delta}(t)$, namely

$$\varepsilon\beta\sqrt{\ell m} < \frac{\eta}{2}, \quad \varepsilon\beta^{-1}\sqrt{\ell m} < \frac{\delta}{2}. \tag{2.29}$$

The choice of β which obeys (2.29) is possible if and only if

$$4\ell m\varepsilon^2 < \eta\delta. \tag{2.30}$$

Next, the choice of λ_1 satisfying (2.28) is possible if

$$\varepsilon\lambda_2 > c\delta \quad \text{and} \quad \frac{(\eta + \varepsilon\eta\mu)^2}{\varepsilon - c\delta\lambda_2^{-1}} < \eta\mu - c\varepsilon - c\delta\lambda_2.$$

Now choose

$$\lambda_2 = 2c\delta\varepsilon^{-1}. \tag{2.31}$$

Then we have to require

$$2(\eta + \varepsilon\eta\mu)^2 < \varepsilon(\eta\mu - c\varepsilon - 2c^2\delta^2\varepsilon^{-1}). \tag{2.32}$$

Choose also

$$\delta = \varepsilon^{3/4}. \tag{2.33}$$

By (2.30) we have the constraint $4\ell m\varepsilon^{5/4} < \eta$.

Then we solve

$$2(\eta + \varepsilon\eta\mu)^2 < \varepsilon(\eta\mu - c\varepsilon - 2c^2\sqrt{\varepsilon}).$$

Choose ε so small that $\eta\mu - c\varepsilon - 2c^2\sqrt{\varepsilon} > \frac{\eta\mu}{2}$. Then it suffices to require

$$4\eta(1 + \varepsilon\mu)^2 < \varepsilon\mu. \tag{2.34}$$

Since ε may be chosen of order $\eta^{4/5}$, for sufficiently small η the last inequality can be satisfied. Then, choosing ε so small that (2.30) and (2.34) hold, and other parameters as above, we can achieve that in (2.26) the coefficients at $\|u_x\|_2^2$ and $\|p(\rho) - p(\overline{\rho})\|_2^2$ are positive. Then (2.26) implies

$$\begin{aligned} &\frac{d}{dt}\int_0^\ell \Big(\frac{\eta\rho u^2}{2} + \frac{\delta}{2}(p(\rho) - p(\overline{\rho}))^2 + \varepsilon\rho u I(p(\rho) - p(\overline{\rho}))\Big)(x,t)\,dx \\ &+a(\|u_x\|_2^2 + \|p(\rho) - p(\overline{\rho})\|_2^2) \le k\|g\|_2^2, \quad t \ge t_0 \end{aligned} \tag{2.35}$$

with some positive constants a, k and t_0.

Further, we have

$$V_{\varepsilon,\delta}(t) \equiv \int_0^\ell \Big(\frac{\eta\rho u^2}{2} + \frac{\delta}{2}(p(\rho)-p(\overline{\rho}))^2 + \varepsilon\rho u I(p(\rho)-p(\overline{\rho}))\Big)\,dx \tag{2.36}$$
$$\leq \frac{\eta m\ell}{2}\|u_x\|_2^2 + \frac{\delta}{2}\|p(\rho)-p(\overline{\rho})\|_2^2 + \varepsilon m\ell\|u_x\|_2\|p(\rho)-p(\overline{\rho})\|_2$$
$$\leq \frac{1}{2}(\eta m\ell + \delta + \varepsilon m\ell)(\|u_x\|_2^2 + \|p(\rho)-p(\overline{\rho})\|_2^2).$$

Putting

$$\alpha := \frac{2a}{\delta + m\ell(\eta+\varepsilon)} \tag{2.37}$$

we get from (2.35)

$$\frac{dV_{\varepsilon,\delta}}{dt}(t) + \alpha V_{\varepsilon,\delta}(t) \leq k\|g(t)\|_2^2, \quad t \geq t_0. \tag{2.38}$$

By integration of (2.38) over the interval (t_0, t) we arive at the inequality

$$V_{\varepsilon,\delta}(t) \leq ke^{-\alpha(t-t_0)}\Big(V_{\varepsilon,\delta}(t_0) + \int_{t_0}^t e^{\alpha s}\|g(s)\|_2^2\,ds\Big), \quad t \geq t_0 \tag{2.39}$$

with some constant $k \geq 1$. Note, that α, k, ε and δ are locally bounded functions of ℓ, m, μ, E_0 and $\|f_\infty\|_{W^1_\infty(0,\ell)}$ and $t_0 \geq 0$, previously sufficiently large, can be chosen arbitrary, since, due to the regularity of the solution, (2.39) holds on any finite interval $[0, T_0]$ (the constant k may eventually change). Now we need the following technical lemma.

Lemma 2.3 *Let the set* $\{x \in (0,\ell); \rho_\infty(x) = 0\}$ *be of measure zero and*

$$\limsup_{r\to 0^+} \int_0^r \frac{dp(s)}{s}\,ds < \infty.$$

Then

$$\frac{d}{dt}\int_0^\ell \rho\Pi(\rho,\rho_\infty)\,dx = \int_0^\ell (p(\overline{\rho}) - p(\rho))u_x\,dx. \tag{2.40}$$

Proof Let $\rho_n = \rho_\infty + \frac{1}{n}$. Then by (1.10) we have

$$\frac{d}{dt}\int_0^\ell \rho\Pi(\rho,\rho_n)\,dx = \int_0^\ell \Big(\Pi(\rho,\rho_n) + \rho\frac{p(\rho)-p(\rho_n)}{\rho^2}\Big)\rho_t\,dx$$
$$= -\int_0^\ell \Big(\Pi(\rho,\rho_n) + \frac{p(\rho)-p(\rho_n)}{\rho}\Big)(\rho u)_x\,dx \tag{2.41}$$
$$= \int_0^\ell \rho\cdot\Big(\Pi(\rho,\rho_n) + \frac{p(\rho)-p(\rho_n)}{\rho}\Big)_x\cdot u\,dx.$$

Further,

$$\rho\Big(\Pi(\rho,\rho_n) + \frac{p(\rho)-p(\rho_n)}{\rho}\Big)_x \tag{2.42}$$

$$= \rho\Big(\frac{p(\rho)-p(\rho_n)}{\rho^2}\rho_x - \int_{\rho_n}^{\rho}\frac{p(\rho_n)_x}{\sigma^2}\,d\sigma + \frac{p(\rho)_x - p(\rho_n)_x}{\rho} - \frac{p(\rho)-p(\rho_n)}{\rho^2}\rho_x\Big)$$
$$= \rho p(\rho_n)_x\Big(\frac{1}{\rho}-\frac{1}{\rho_n}\Big) + p(\rho)_x - p(\rho_n)_x = p(\rho)_x - \frac{\rho}{\rho_n}p(\rho_n)_x = p(\rho)_x - \rho\pi(\rho_n)_x,$$

where $\pi(r) = \int_0^r \frac{p'(s)}{s}\,ds$. Since $\{\rho_\infty > 0\}$ is an open set, we can write it in the form $\cup_{j\in S}(a_j, b_j)$, where $S \subset N$ is countable. Notice that $\rho_\infty(a_j), \rho_\infty(b_k) = 0$ as soon as $a_j, b_k \in (0,\ell)$. Let $\varphi \in C^\infty(0,\ell), \varphi(0) = \varphi(\ell) = 0$. Then

$$\int_0^\ell \pi(\rho_n)_x\rho\varphi\,dx = -\int_0^\ell \pi(\rho_n)(\rho\varphi)_x\,dx \to -\int_0^\ell \pi(\rho_\infty)(\rho\varphi)_x\,dx$$
$$= -\sum_{j\in S}\int_{a_j}^{b_j}\pi(\rho_\infty)(\rho\varphi)_x\,dx = \sum_{j\in S}\int_{a_j}^{b_j}\pi(\rho_\infty)_x\rho\varphi\,dx = \sum_{j\in S}\int_{a_j}^{b_j}\frac{p(\rho_\infty)_x}{\rho_\infty}\rho\varphi\,dx$$
$$= \int_{\rho_\infty>0}\rho f\varphi\,dx = \int_0^\ell \rho f\varphi\,dx.$$

The result immediately follows. □

By (2.40) and (2.39) we have

$$\Big|\frac{d}{dt}\int_0^\ell \rho\Pi(\rho,\rho_\infty)\,dx\Big| \le \|u_x\|_2\|p(\overline{\rho}) - p(\rho)\|_2 \tag{2.43}$$
$$\le \frac{2}{\delta}\|u_x\|_2 e^{\frac{-\alpha}{2}(t-t_0)}\Big(V_{\varepsilon,\delta}(t_0) + k\int_{t_0}^t e^{\alpha s}\|g(s)\|_2^2\,ds\Big)^{1/2}, \quad t \ge t_0.$$

Since by (1.25),

$$\lim_{t\to\infty}\int_0^\ell \rho(x,t)\Pi(\rho(x,t),\rho_\infty(x))\,dx = \int_0^\ell \rho_\infty(x)\Pi(\rho_\infty(x),\rho_\infty(x))\,dx = 0, \tag{2.44}$$

we find

$$\int_0^\ell \rho(t)\Pi(\rho(t),\rho_\infty)\,dx - \int_0^\ell \rho(s)\Pi(\rho(s),\rho_\infty)\,dx \tag{2.45}$$
$$= -\int_t^s \frac{d}{d\tau}\int_0^\ell \rho(\tau)\Pi(\rho(\tau),\rho_\infty)\,dx d\tau \le \int_t^s\Big|\frac{d}{d\tau}\int_0^\ell \rho(\tau)\Pi(\rho(\tau),\rho_\infty)\,dx\Big|d\tau$$
$$\le \frac{2\sqrt{k}}{\delta}\int_t^s \|u_x(\tau)\|_2 e^{\frac{-\alpha}{2}(\tau-t_0)}\Big(V_{\varepsilon,\delta}(t_0) + \int_{t_0}^\tau e^{\alpha\sigma}\|g(\sigma)\|_2^2\,d\sigma\Big)^{1/2}\,d\tau$$
$$\le \frac{2\sqrt{k}}{\delta}\Big(\int_t^s \|u_x(\tau)\|_2^2\,d\tau\Big)^{1/2}\Big[\int_t^s e^{-\alpha(\tau-t_0)}\Big(V_{\varepsilon,\delta}(t_0) + \int_{t_0}^\tau e^{\alpha\sigma}\|g(\sigma)\|_2^2\,d\sigma\Big)\,d\tau\Big]^{1/2}$$
$$= \frac{2\sqrt{k}}{\delta}\Big(\int_t^s \|u_x(\tau)\|_2^2\,d\tau\Big)^{1/2}\Big\{-\frac{1}{\alpha}\Big[\varepsilon^{-\alpha(\tau-t_0)}\int_{t_0}^\tau e^{\alpha\sigma}\|g(\sigma)\|_2^2\,d\sigma\Big]_{\tau=t}^s$$
$$+\frac{V(t_0)}{\alpha}\Big(e^{-\alpha(t-t_0)} - e^{-\alpha(s-t_0)}\Big) + \frac{1}{\alpha}\int_t^s e^{-\alpha(\tau-t_0)}e^{\alpha\tau}\|g(\tau)\|_2^2\,ds\Big\}$$
$$\le \frac{2\sqrt{k}}{\delta}\Big(\int_t^\infty \|u_x(\tau)\|_2^2\,d\tau\Big)^{1/2}\Big\{\frac{1}{\alpha}e^{-\alpha(t-t_0)}\int_{t_0}^t e^{\alpha\sigma}\|g(\sigma)\|_2^2\,d\sigma$$
$$+\frac{V(t_0)}{\alpha}e^{-\alpha(t-t_0)} + \frac{1}{\alpha}\int_t^\infty e^{\alpha t_0}\|g(\tau)\|_2^2\,d\tau\Big\}.$$

Sending $s \to \infty$ and using (2.44) we obtain

$$\int_0^\ell \rho(t)\Pi(\rho(t),\rho_\infty)\,dx \tag{2.46}$$
$$\le \kappa(t_0,\ell,m,\mu,E_0,\|f_\infty\|_{W^{1,\infty}(0,\ell)})\Big[e^{-\alpha(t-t_0)}\Big(1+\int_{t_0}^t e^{\alpha\sigma}\|g(\sigma)\|_2^2\,d\sigma\Big)$$
$$+\int_t^\infty \|g(\sigma)\|_2^2\,d\tau\Big].$$

By (2.10) and (2.39) we also have

$$\int_0^\ell \Big(\rho u^2 + (p(\rho)-p(\overline{\rho}))^2\Big)(x,t)\,dx \le a_0 V_{\varepsilon,\delta}(t)$$
$$\le a_0 e^{-\alpha(t-t_0)}\Big(V_{\varepsilon,\delta}(t_0) + k\int_{t_0}^t e^{\alpha s}\|g(s)\|_2^2\,ds\Big) \tag{2.47}$$
$$\le a_1 e^{-\alpha(t-t_0)}\Big[\int_0^\ell \Big(\rho u^2 + (p(\rho)-p(\overline{\rho}))^2\Big)(x,t_0)\,dx + \int_{t_0}^t e^{\alpha s}\|g(s\|_2^2\,ds\Big]$$

with constants $a_j = a_j(\ell,m,\mu,E_0,\|f_\infty\|_{W^{1,\infty}(0,\ell)})$, $j=0,1$. This together with (2.46) yields

$$\int_0^\ell \Big(\rho u^2 + \rho\Pi(\rho,\rho_\infty) + (p(\rho)-p(\overline{\rho}))^2\Big)(x,t)\,dx \tag{2.48}$$
$$\le K(t_0,\ell,m,\mu,E_0,\|f_\infty\|_{W^{1,\infty}(0,\ell)})\Big\{e^{-\alpha(t-t_0)}\Big[1+\int_{t_0}^t e^{\alpha s}\|g(s)\|_2^2\,ds\Big]$$
$$+\int_t^\infty \|g(\sigma)\|_2^2\,d\tau\Big\} \tag{2.49}$$

The estimate (2.48) in combination with Lemma 1.6 yields the desired estimate (1.18). □

Remark 2.4 *Let us note that since* $\lim_{t\to\infty}\int_0^t e^{-\alpha(t-s)}G(s)\,ds = 0$ *for all* $G \in L^q(R^+)$ *with* $\alpha > 0$ *and* $1 \le q < \infty$, *the right-hand side of (2.48) tends to zero as* $t \to \infty$. *If, moreover,* $\|e^{bt}g(x,t)\|_{L^2(Q)} \le N$ *with some* $b \in (0,\alpha]$ *(for example, if* $g \equiv 0$*), then the decay rate is exponential, i.e.,*

$$\int_0^\ell \Big(\rho u^2 + \rho\Pi(\rho,\rho_\infty) + |\rho-\rho_\infty|^\beta + \|p(\rho)-p(\overline{\rho})\|_2^2\Big)\,dx \le k(N)e^{-bt}, \quad t \ge 0. \quad \square$$

Acknowledgements

The research was supported by the University of Sud, Toulon Var, by the Grant Agency of the Czech Republic (grant No. 201/05/0005) and by the Academy of Sciences of the Czech Republic, Institutional Research Plan No. AVOZ1090503.

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