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On Some Aspects of the hp-FEM for Time-Harmonic Maxwell's Equations

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Midwest Numerical Analysis Conference,
May 20–22, 2005, University of Iowa

Time harmonic Maxwell's equations

$$\mathbf{curl}\left(\mu_{\mathrm{r}}^{-1}\,\mathrm{curl}\,\mathbf{E}\right) - \kappa^2\epsilon_{\mathrm{r}}\mathbf{E} = \mathbf{F} \quad \text{in } \Omega,$$

where

- $\mathbf{curl} = (\partial/\partial x_2, -\partial/\partial x_1)^\top$
- $\mathrm{curl}\,\mathbf{E} = \partial E_2/\partial x_1 - \partial E_1/\partial x_2$
- $\Omega \subset \mathbb{R}^2$
- $\mu_{\mathrm{r}} = \mu_{\mathrm{r}}(x) \in \mathbb{R}$ relative permeability
- $\epsilon_{\mathrm{r}} = \epsilon_{\mathrm{r}}(x) \in \mathbb{C}^{2\times 2}$ relative permittivity
- $\mathbf{E} = \mathbf{E}(x) \in \mathbb{C}^2$ phaser of the electric field intensity
- $\mathbf{F} = \mathbf{F}(x) \in \mathbb{C}^2$
- $\kappa \in \mathbb{R}$ the wave number

Time harmonic Maxwell's equations + boundary conditions

$$\mathbf{curl}\left(\mu_{\mathrm{r}}^{-1}\,\mathrm{curl}\,\mathbf{E}\right) - \kappa^2\epsilon_{\mathrm{r}}\mathbf{E} = \mathbf{F} \quad \text{in } \Omega$$

Perfect conducting boundary conditions:

$$\mathbf{E}\cdot\tau = 0, \quad \text{on } \Gamma_P.$$

Impedance boundary conditions:

$$\mu_{\mathrm{r}}^{-1}\,\mathrm{curl}\,\mathbf{E} - \mathrm{i}\kappa\lambda\mathbf{E}\cdot\tau = \mathbf{g}\cdot\tau \quad \text{on } \Gamma_I.$$

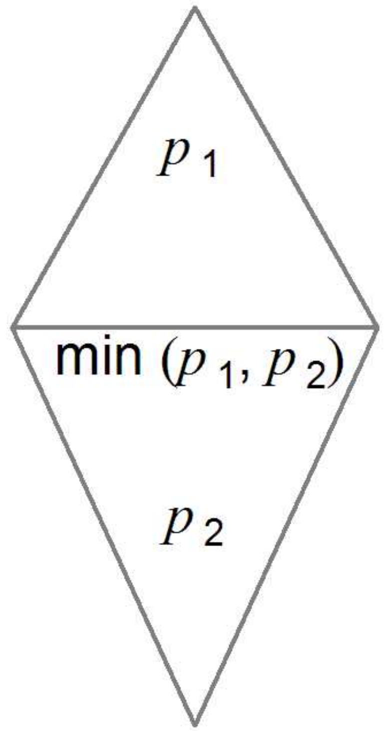
Here,

- $\tau = (-\nu_2, \nu_1)^\top$ positively oriented unit tangent vector
- $\lambda = \lambda(x) > 0$ impedance
- $\mathbf{g} = \mathbf{g}(x) \in \mathbb{C}^2$

Weak and FEM formulations

$$V = \{\mathbf{E} \in \mathbf{H}(\mathsf{curl}, \Omega) : \nu \times \mathbf{E} = 0 \text{ on } \Gamma_P\}$$

$$\mathbf{E} \in V : \quad \boxed{a(\mathbf{E}, \mathbf{\Phi}) = \mathcal{F}(\mathbf{\Phi})} \quad \forall \mathbf{\Phi} \in V$$



$$V_h = \Big\{\mathbf{E}_h \in V : \mathbf{E}_h|_{K_j} \in P^{p_j}(K_j) \text{ and } \mathbf{E}_h \cdot \tau_k \text{ is continuous on each edge } e_k\}$$

$$\mathbf{E}_h \in V_h : \quad \boxed{a(\mathbf{E}_h, \mathbf{\Phi}_h) = \mathcal{F}(\mathbf{\Phi}_h)} \quad \forall \mathbf{\Phi}_h \in V_h$$

$$a(\mathbf{E}, \mathbf{\Phi}) = \left(\mu_{\mathsf{r}}^{-1} \mathsf{curl}\, \mathbf{E}, \mathsf{curl}\, \mathbf{\Phi}\right) - \kappa^2 (\epsilon_{\mathsf{r}} \mathbf{E}, \mathbf{\Phi}) - \mathsf{i}\kappa \langle \lambda \mathbf{E} \cdot \tau, \mathbf{\Phi} \cdot \tau \rangle$$
$$\mathcal{F}(\mathbf{\Phi}) = (F, \mathbf{\Phi}) + \langle \mathrm{g}, \mathbf{\Phi} \cdot \tau \rangle$$

$$\boxed{\mathbf{E}_h = \sum_j^N \underbrace{c_j}_{\in \mathbb{C}} \mathbf{\Psi}_j} \qquad \mathbf{\Psi}_j \; \ldots \; \text{hierarchic basis}$$

Shape functions

Whitney functions:

$$\psi_0^{e_1} = \frac{\lambda_3 \mathbf{n}_2}{\mathbf{n}_2 \cdot \mathbf{t}_1} + \frac{\lambda_2 \mathbf{n}_3}{\mathbf{n}_3 \cdot \mathbf{t}_1}$$

$$\psi_0^{e_2} = \frac{\lambda_1 \mathbf{n}_3}{\mathbf{n}_3 \cdot \mathbf{t}_2} + \frac{\lambda_3 \mathbf{n}_1}{\mathbf{n}_1 \cdot \mathbf{t}_2}$$

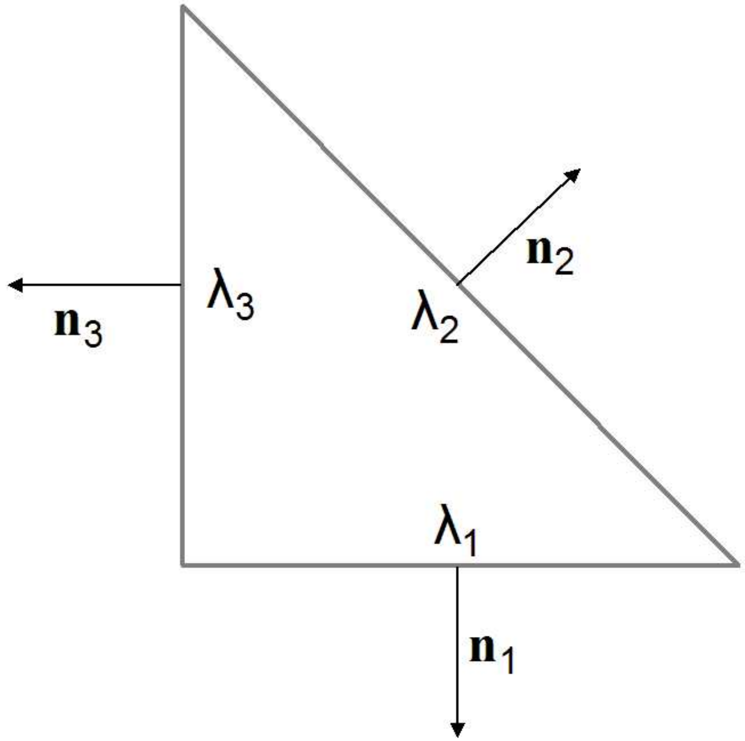
$$\psi_0^{e_3} = \frac{\lambda_2 \mathbf{n}_1}{\mathbf{n}_1 \cdot \mathbf{t}_3} + \frac{\lambda_1 \mathbf{n}_2}{\mathbf{n}_2 \cdot \mathbf{t}_3}$$

First order functions:

$$\psi_1^{e_1} = \frac{\lambda_3 \mathbf{n}_2}{\mathbf{n}_2 \cdot \mathbf{t}_1} - \frac{\lambda_2 \mathbf{n}_3}{\mathbf{n}_3 \cdot \mathbf{t}_1}$$

$$\psi_1^{e_2} = \frac{\lambda_1 \mathbf{n}_3}{\mathbf{n}_3 \cdot \mathbf{t}_2} - \frac{\lambda_3 \mathbf{n}_1}{\mathbf{n}_1 \cdot \mathbf{t}_2}$$

$$\psi_1^{e_3} = \frac{\lambda_2 \mathbf{n}_1}{\mathbf{n}_1 \cdot \mathbf{t}_3} - \frac{\lambda_1 \mathbf{n}_2}{\mathbf{n}_2 \cdot \mathbf{t}_3}$$



$$\mathbf{t}_i = \begin{bmatrix} -\mathbf{n}_{i,2} \\ \mathbf{n}_{i,1} \end{bmatrix}$$

Edge functions:

$$\psi_k^{e_1} = \frac{2k-1}{k} L_{k-1}(\lambda_3 - \lambda_2)\psi_1^{e_1} - \frac{k-1}{k} L_{k-2}(\lambda_3 - \lambda_2)\psi_0^{e_1},$$
$$\psi_k^{e_2} = \frac{2k-1}{k} L_{k-1}(\lambda_1 - \lambda_3)\psi_1^{e_1} - \frac{k-1}{k} L_{k-2}(\lambda_1 - \lambda_3)\psi_0^{e_1},$$
$$\psi_k^{e_2} = \frac{2k-1}{k} L_{k-1}(\lambda_2 - \lambda_1)\psi_1^{e_1} - \frac{k-1}{k} L_{k-2}(\lambda_2 - \lambda_1)\psi_0^{e_1}, \quad k = 2, 3, \ldots$$

Edge based bubble functions:

$$\psi_k^{b,e_1} = \lambda_3\lambda_2 L_{k-2}(\lambda_3 - \lambda_2)\mathbf{n}_1,$$
$$\psi_k^{b,e_2} = \lambda_1\lambda_3 L_{k-2}(\lambda_1 - \lambda_3)\mathbf{n}_2,$$
$$\psi_k^{b,e_3} = \lambda_2\lambda_1 L_{k-2}(\lambda_2 - \lambda_1)\mathbf{n}_3, \quad k = 2, 3, \ldots$$

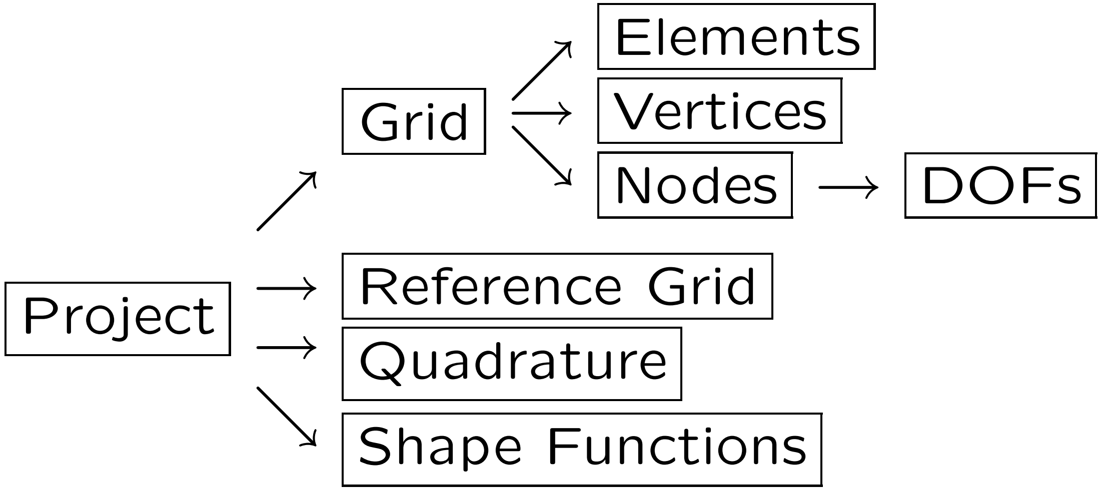
Genuine bubble functions:

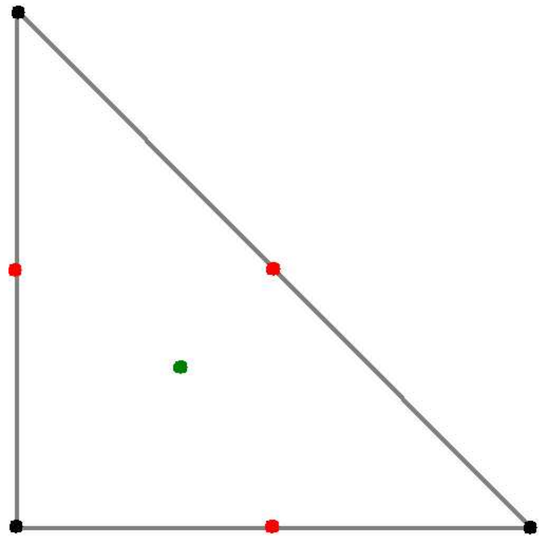
$$\psi_{n_1,n_2}^{b,1} = \lambda_1\lambda_2\lambda_3 L_{n_1-1}(\lambda_3 - \lambda_2) L_{n_2-1}(\lambda_2 - \lambda_1) \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$
$$\psi_{n_1,n_2}^{b,2} = \lambda_1\lambda_2\lambda_3 L_{n_1-1}(\lambda_3 - \lambda_2) L_{n_2-1}(\lambda_2 - \lambda_1) \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad 1 \le n_1, n_2$$

ELSYS_2D – hp-FEM solver

H^1	$\mathbf{H}(\text{curl})$	$\mathbf{H}(\text{div})$
• H^1 conforming elements • elliptic problems • linear – nonlinear • systems	• $\mathbf{H}(\text{curl})$ conforming elements • time harmonic Maxwell's equations	. . .

Structure of the C++ object oriented code





Modularity of the code

common core × equation dependent modules

`CProject` → `H1Project`, `HcurlProject`

`CGrid` → `H1Grid`, `HcurlGrid`

CProject (common core)	H1Project / HcurlProject	CGrid (common core)	H1Grid / HcurlGrid
File names	Assembling	Elements	Boundary cond.
CPU time	Solving	Vertices	DOFs Allocation
Input	Output	Nodes	
Output	Error computation	Read Grid file	
Quadrature		Preprocessing	
		Refinement	

Modularity of the code

`CElement` → `H1Element`, `HcurlElement`

`sMatrix`

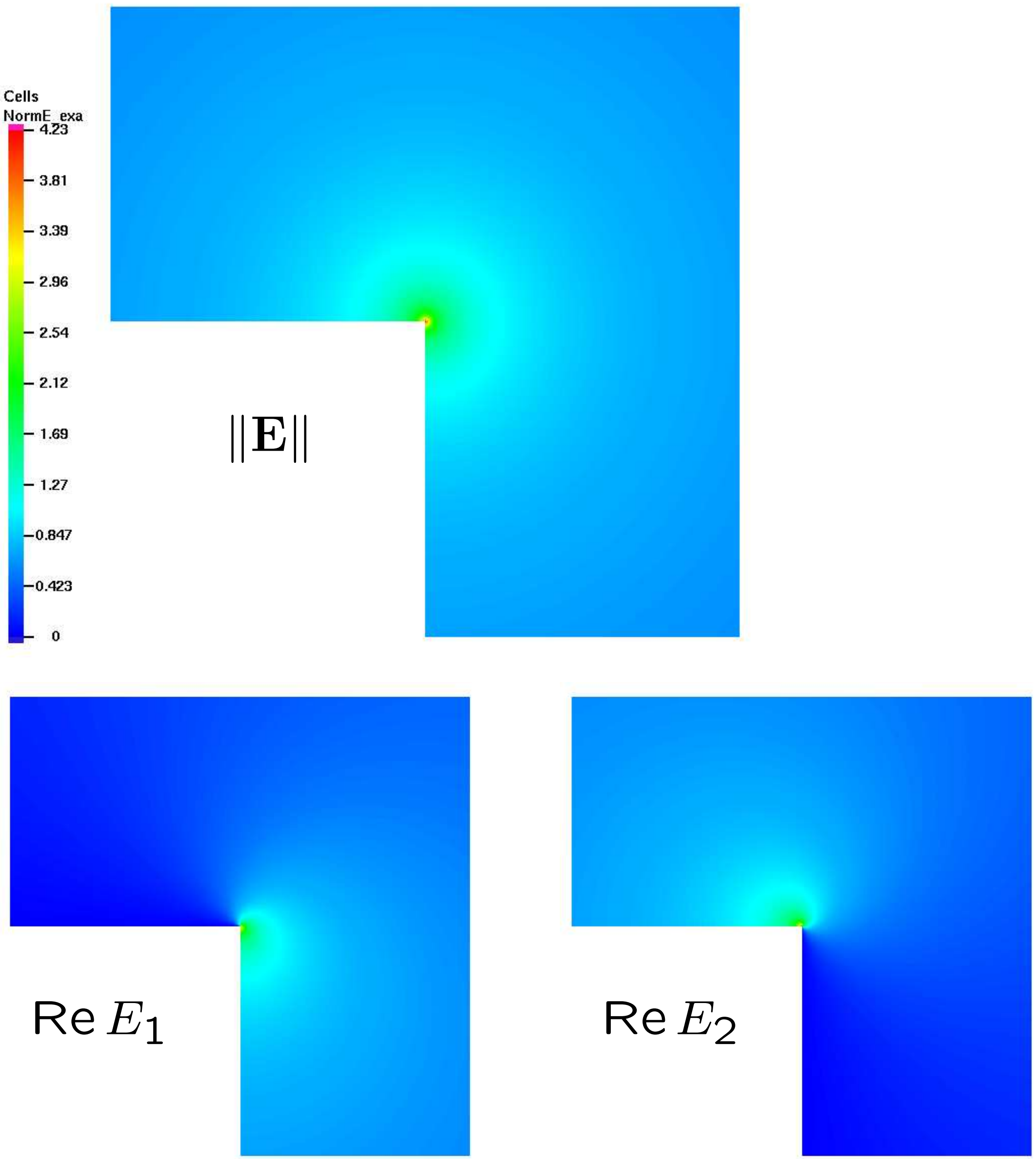
Vertices
Neighours
Transformation
Edge orientations

Nodes – DOFs
Solution

Sparse matrices
Iterative solvers
Interface for:

- `Trilinos`
- `PETSc`
- `UMFPACK`

Example 1



$$u = r^{\frac{2}{3}} \sin\left(\frac{2}{3}\theta + \frac{\pi}{3}\right)$$

$$\mathbf{E} = \nabla u$$

$$\mathbf{E} = \frac{2}{3} r^{-\frac{1}{3}} \begin{bmatrix} \cos\left(\frac{\pi}{6} + \frac{\theta}{3}\right) \\ \sin\left(\frac{\pi}{6} + \frac{\theta}{3}\right) \end{bmatrix}$$

$$\mathbf{F} = -\mathbf{E}$$

$$\mu_{\mathrm{r}} = 1$$

$$\epsilon_{\mathrm{r}} = I$$

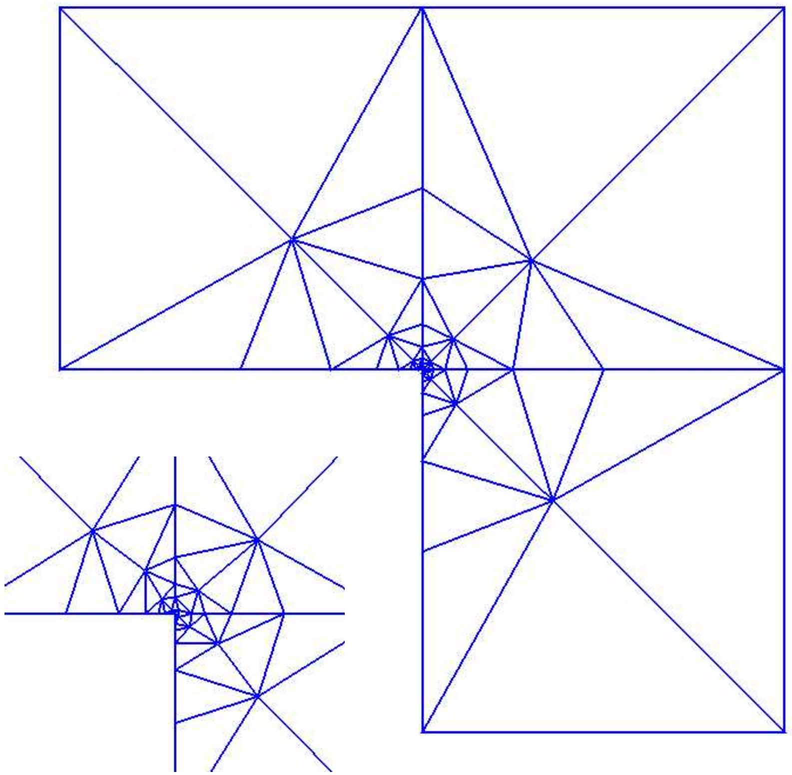
$$\kappa = 1$$

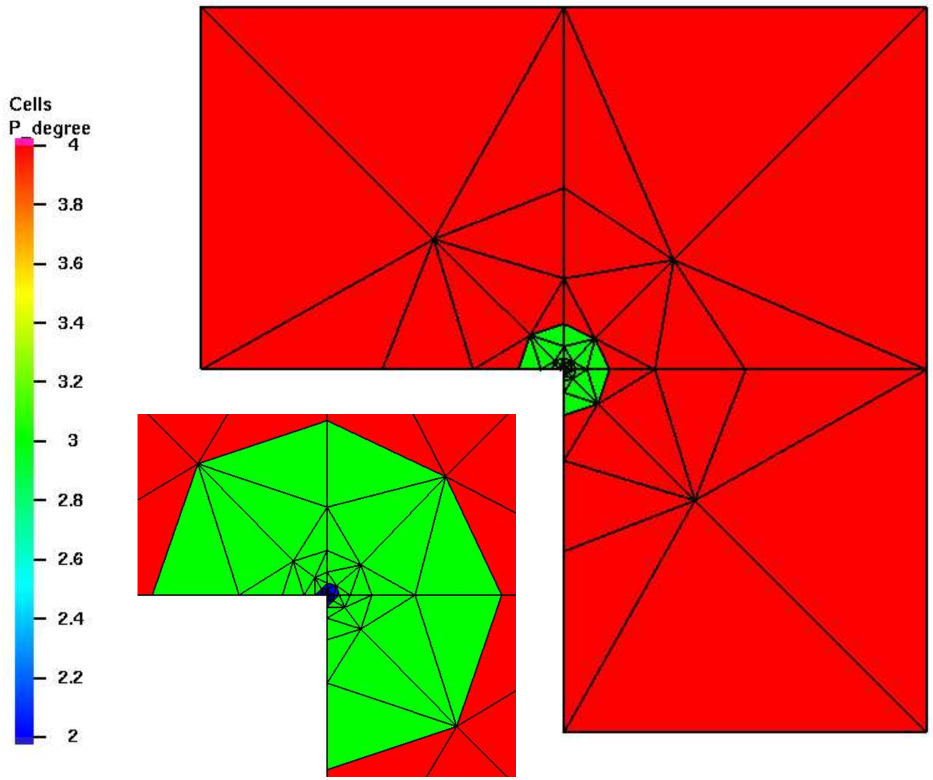
$$\lambda = 1$$

$$\mathbf{g} = \ldots$$

Example 1

	DOFs	CPU time	$\|\mathrm{Err}\|_{\mathbf{H}(\mathrm{curl})}/\|\mathbf{E}\|_{\mathbf{H}(\mathrm{curl})}$
$p = 0$	2 758 400	11 min 26 s	0.156 %
hp	2 732	0.55 s	0.138 %
Improvement	1 010×	1 247×	

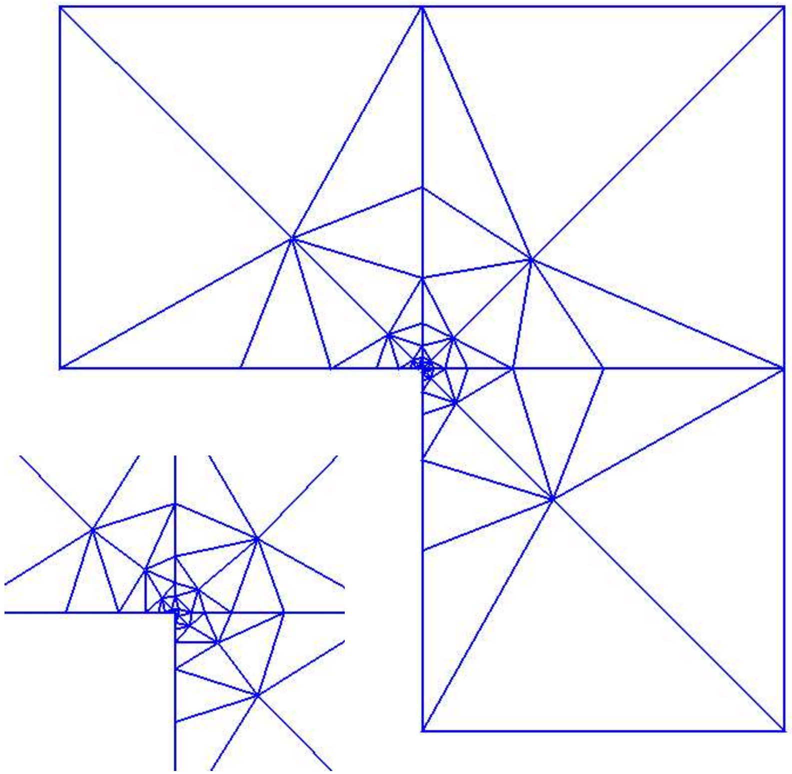


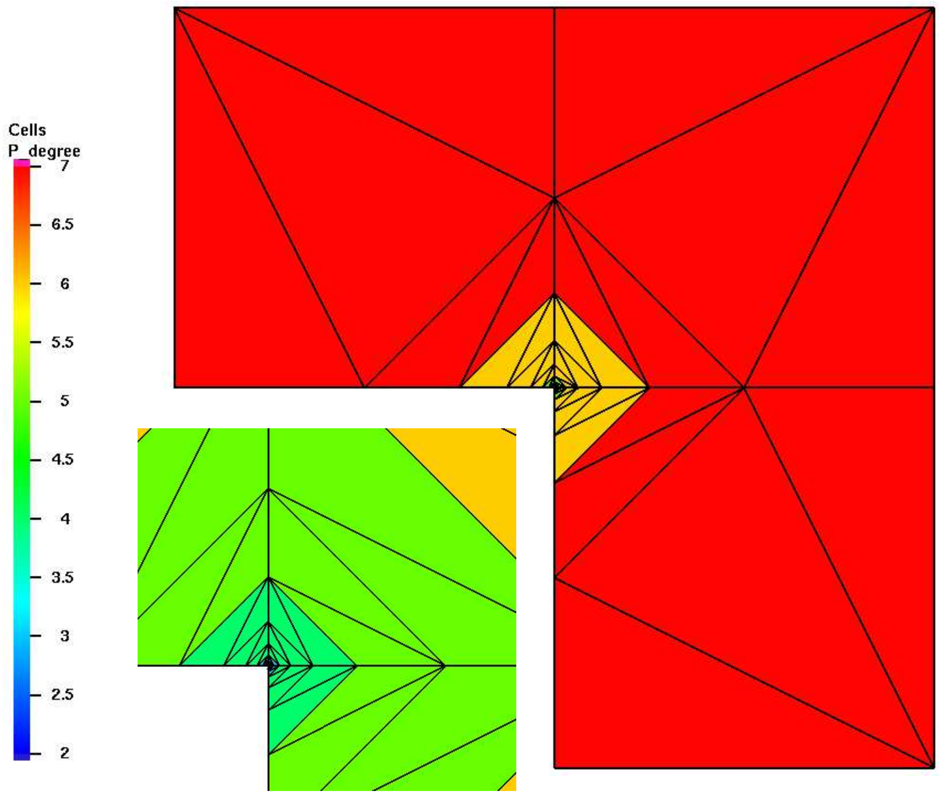


refinement 100

Example 1

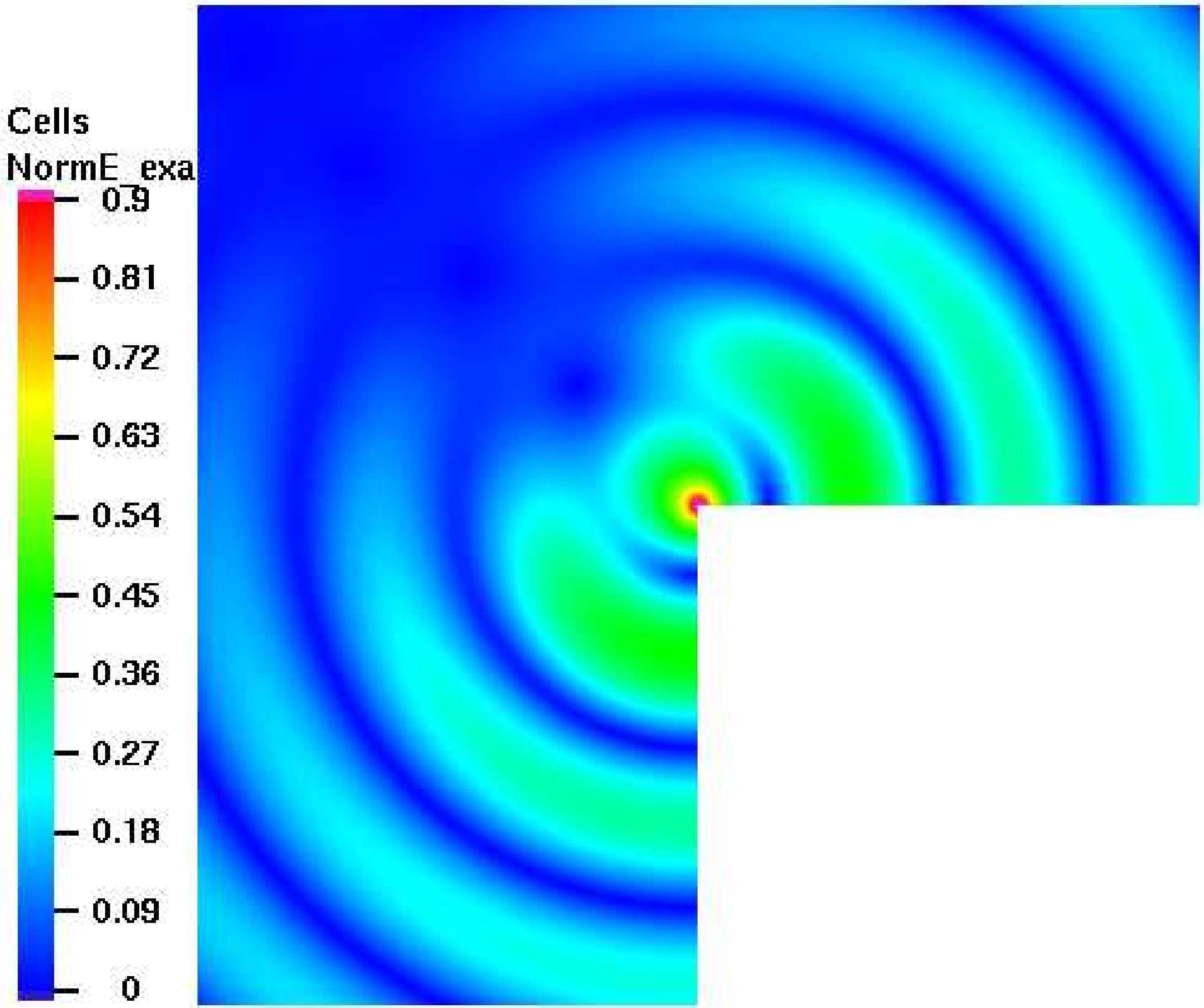
	DOFs	CPU time	$\|\text{Err}\|_{\mathbf{H}(\text{curl})}/\|\mathbf{E}\|_{\mathbf{H}(\text{curl})}$
$p = 1$	266 464	4 min 18 s	0.02612 %
hp	5 534	2.67 s	0.02608 %
Improvement	48×	97×	





refinement 22

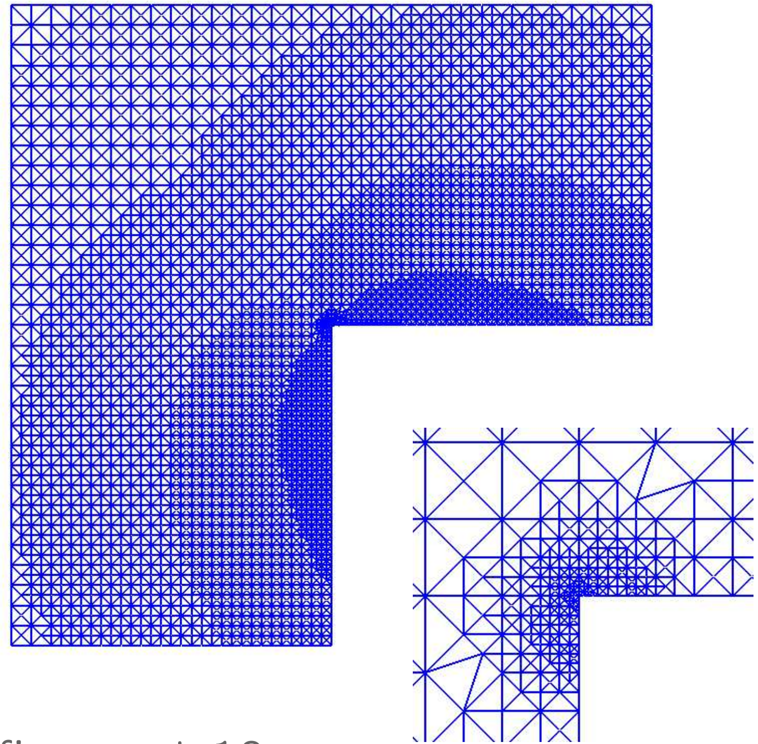
Example 2 (P. Monk, 2003)



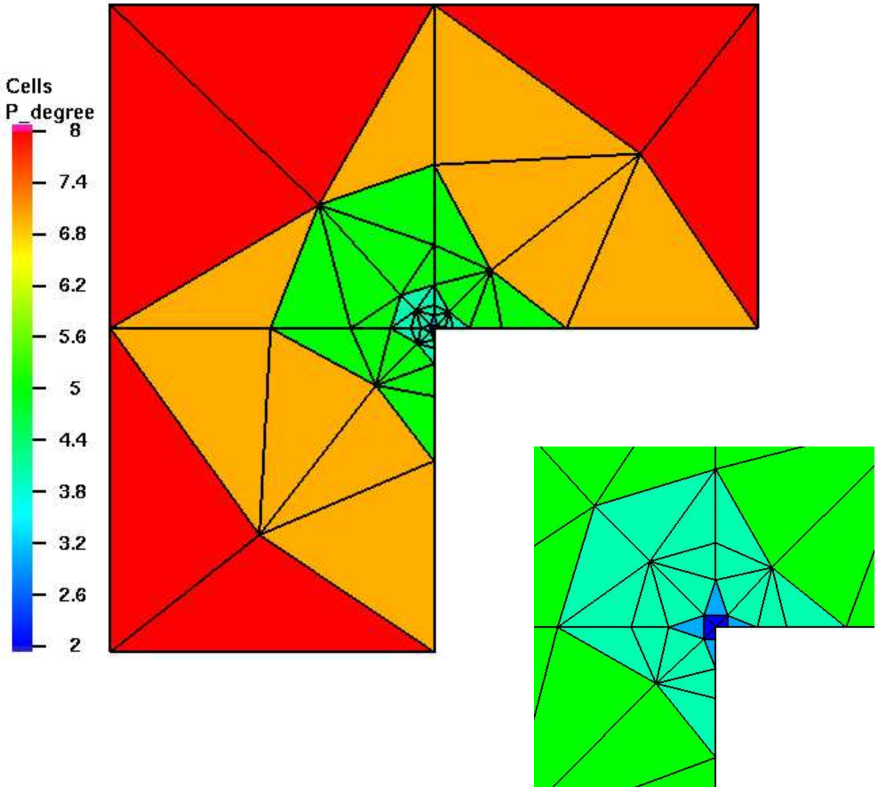
$$u = J_{\frac{2}{3}}(r)\cos\left(\tfrac{2}{3}\theta\right)$$
$$\mathbf{E} = \mathbf{curl}\, u$$
$$\mathbf{F} = 0$$
$$\mu_r = 1$$
$$\epsilon_r = I$$
$$\kappa = 1$$
$$\lambda = 1$$
$$\mathbf{g} = \dots$$

Example 2

	DOFs	CPU time	$\|Err\|_{\mathbf{H}(\text{curl})}/\|\mathbf{E}\|_{\mathbf{H}(\text{curl})}$
$p = 0$	2 586 540	21 min 12 s	0.645 %
hp	4 324	2.49 s	0.621 %
Improvement	598×	511×	

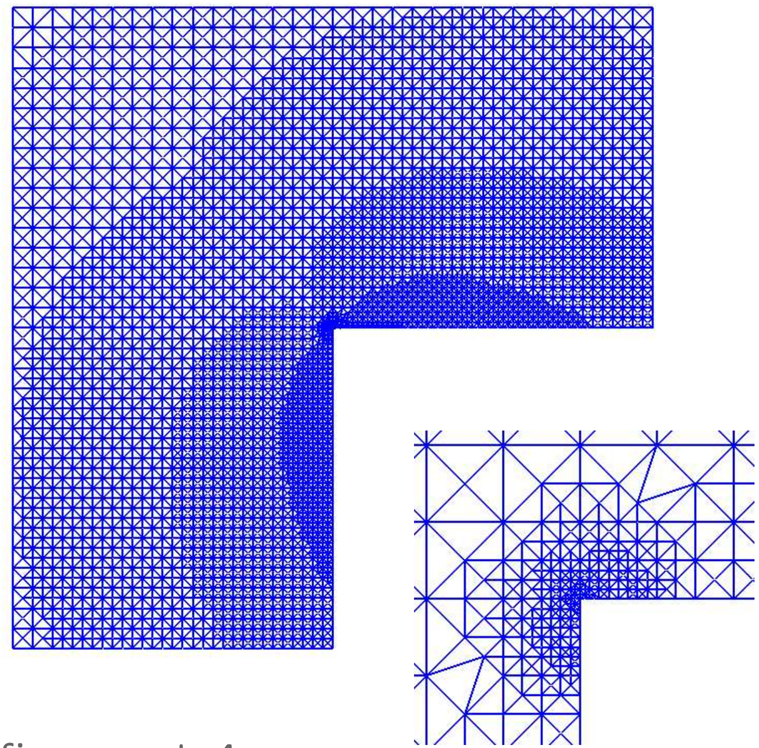


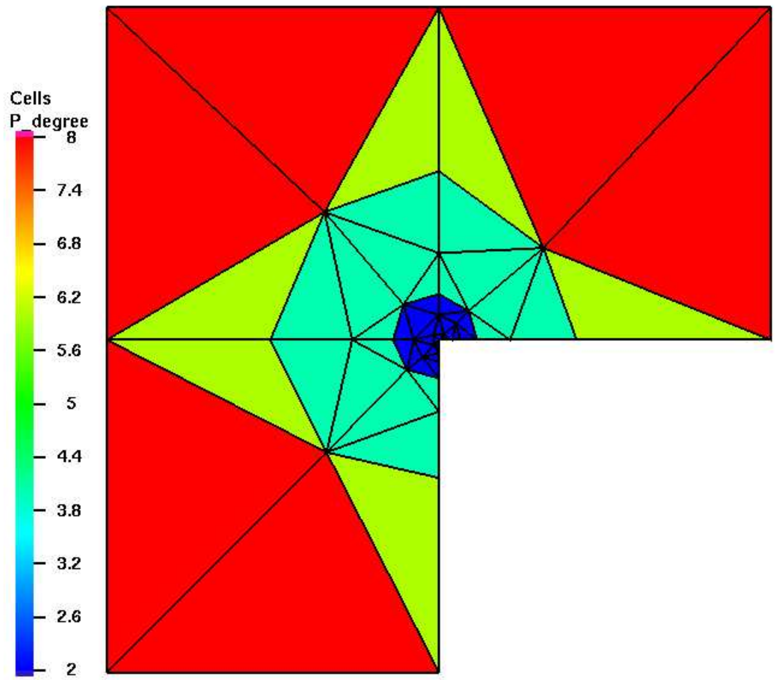
refinement 10



Example 2

	DOFs	CPU time	$\|\text{Err}\|_{\mathbf{H}(\text{curl})}/\|\mathbf{E}\|_{\mathbf{H}(\text{curl})}$
$p = 1$	827 664	7 min 3 s	1.068 %
hp	2 624	1.51 s	0.966 %
Improvement	315×	280×	





refinement 4

Outlook

- H^1 and $\mathbf{H}$(curl) conforming elements in 3D
- parallelization
- a posteriori error estimates
- automatic hp-adaptivity
- orthonormalization of the bubble functions
  (investigation of the non-affine hierarchic elements)
- . . . .

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Thank you for your attention.

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