Dimension-free imbeddings of Sobolev spaces

Miroslav Krbec* and Hans-Jürgen Schmeisser†

Abstract

We prove dimension-free imbedding theorems for Sobolev spaces using extrapolation means and the Gross logarithmic inequality.

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1 Introduction, notations

Our concern in this paper lies with the weighted inequalities

$$\left(\int_{\Omega} |f(x)|^p V(x)\, dx\right)^{1/p} \le c\left(\int_{\Omega} |\nabla f(x)|^p\, dx\right)^{1/p}, \qquad f \in W_0^{1,p}(\Omega), \quad (1.1)$$

and

$$\|f|L^p[\log(1+L)]^\alpha\| \le c\|\nabla f|L^p\|, \qquad f \in W_0^{1,p}(\Omega), \qquad (1.2)$$

*Institute of Mathematics, Academy of Sciences of the Czech Republic, Žitná 25, 115 67 Praha 1, Czech Republic (krbecm@math.cas.cz). The research of this author was supported by the Academy of Sciences of the Czech Republic, Institutional Research Plan no. AV0Z10190503, and by the Grant Agency of the Czech Republic under No. 201/06/0400. Also a support of the Necas Center for Mathematical Modeling is appreciated.

†Mathematisches Institut, Fakultät für Mathematik und Informatik, Friedrich-Schiller-Universität, Ernst-Abbe-Platz 1-2, 07743 Jena, Germany (mhj@minet.uni-jena.de). Both authors appreciate support of the PPP project D 15 - CZ 2/07-08.

where either Ω is a bounded domain in $\mathbb{R}^N$ or $\Omega = \mathbb{R}^N$ and V is a weight on Ω, that is, a.e. non-negative and locally integrable function on Ω, and the constant c on the right hand sides is independent of f and N. Both inequalities can naturally be interpreted as imbedding theorems independent of the dimension.

Variants and generalizations of the above inequalities in $\mathbb{R}^N$ or on domains in $\mathbb{R}^N$ have been intensively studied during last decades. They appear under various names as the *trace inequality* or the *uncertainty principle* and they have many relevant applications in analysis. It would be a difficult task to collect even the most important references and we shall make no attempt to do that. We shall just recall several basic facts and explain our motivation. Necessary and sufficient conditions for the imbedding of $W^{1,p}$ into $L^q(V)$ depending on the dimension have been studied e.g. in [1] (Adams' inequality), let us recall Maz'ya's works using capacities, [19], [20]. For $p = q = 2$ and $N \geq 3$, a necessary and sufficient condition is due to Kerman and Sawyer [13]—this is connected with Sawyer's necessary and sufficient conditions for validity of two weight inequalities for the Riesz potentials, see [22]. Observe that due to the nature of these two-weight conditions (which require an information on the acting of Riesz potentials on weights in question) and of capacities, of importance are sufficient conditions (close to necessary ones as much as possible of course) in amenable terms of various classes and/or spaces of functions. Fefferman in [6] gave the following sufficient condition: Let us recall that the *Fefferman-Phong class* F_p, $1 \leq p \leq N/2$, consists of functions V such that

$$\|V\|_{F_p} = \sup_{\substack{x \in \mathbb{R}^N \\ r > 0}} r^2 \left(\frac{1}{|B(x,r)|} \int_{B(x,r)} |V(y)|^p \, dy \right)^{1/p} < \infty.$$

Then (see [6]) for $N \geq 3$, $1 < p \leq N/2$, and $V \in F_p$ inequality (1.1) holds with $\mathbb{R}^N$ in place of Ω. Note that Chiarenza and Frasca [4] gave a very fine alternative proof making use of of the maximal operator.

For $N = 2$ and functions in $W_0^{1,2}(\Omega)$ (Ω a bounded smooth domain) there is the sufficient condition $V \in L \log L(\Omega)$ for (1.1) due to Gossez and Loulit in [8] and a more general condition in terms of Lorentz-Zygmund spaces based on a fine critical imbedding theorem due to Brezis and Wainger [3], see Krbec and Schott [16]; this is, however, strictly limited to planar domains.

Dimension free estimates answer the natural question about existence of some residual improvement of the integrability properties independent of the dimension. They are also linked with other interesting concepts concerning the Sobolev spaces, for instance, properties of contraction semigroups and find applications even in quantum physics (see e.g. [18] for some of the references). One of the major triggering moments was the celebrated Gross logarithmic inequality [9], generalized later in various directions by several authors, see, e.g. [11], [10]. Recall that the Gross logarithmic inequality (see e.g. [18] for a detailed discussion),

$$\int_{\mathbb{R}^N} |f(x)|^2 \log\left(\frac{|f(x)|^2}{\|f\|_2^2}\right) dx + N\|f\|_2^2 \le \frac{1}{\pi}\int_{\mathbb{R}^N} |\nabla f(x)|^2\, dx, \tag{1.3}$$

gives, for a function f living, say, in a bounded domain $\Omega \subset \mathbb{R}^N$ and with $\|f|W_0^{1,2}(\Omega)\| \le 1$,

$$\int_\Omega |f(x)|^2 \log|f(x)|\, dx \le \frac{1}{2\pi}\int_\Omega |\nabla f(x)|^2\, dx \tag{1.4}$$

(since under our assumptions $\log\|f\|_2 \le 0$). Note that one can formally put 0 in the integral on the left-hand side of (1.3) and (1.4) if $|f(x)| = 0$ (which corresponds well to elementary limit $\lim_{t\to 0_+} t^\delta \log t = 0$ for any $\delta > 0$). The left hand side of (1.3) contains generally both positive and negative values and the estimate says that the final balance of that, containing a logarithmic residuum integrability improvement is estimated by a multiple of the L^2-norm of the gradient.

Note also that in [2] Adams considered more general and dimension dependent inequalities (with norms taken with respect to the Gaussian measure $\exp(-|x|^2)\, dx$).

In [15] we have employed the Gross theorem to show that

$$\int_B |f(x)|^2 \log(1 + |f(x)|)\, dx \le c\|f|W_0^{1,2}(B)\|^2 \tag{1.5}$$

($W_0^{1,2}(B) = \overline{C_0^\infty(B)}^{W^{1,2}(B)}$, B being the unit ball in $\mathbb{R}^N$) with a constant c independent of f and N.

In this paper we will study the general form of (1.5), namely,

$$\int_\Omega |f(x)|^p [\log(1 + |f(x)|)]^\alpha\, dx \le c\|\nabla f|L^p(\Omega)\|^p \tag{1.6}$$

for $1 < p < \infty$, $\alpha > 0$, and $f \in W_0^{1,p}(\Omega)$, and also we will establish the weighted dimension-free imbedding of the form

$$\int_\Omega |f(x)|^p V(x)\,dx \le c\|\nabla f|L^p(\Omega)\|^p,$$

for $f \in W_0^{1,p}(\Omega)$, Ω being a bounded domain and/or $\mathbb{R}^N$.

We shall tacitly assume that all functions here are real-valued (complex-valued functions can be considered, too). Various constants independent of f will be denoted by the same generic symbol c, C etc. if no misunderstanding can arise.

We shall use the standard notation $\|\,.\,\|_{k,p}$ for the norm in $W^{k,p}$; if $k = 0$, then $W^{k,p} = L^p$ with the norm denoted by $\|\,.\,\|_p$; sometimes we shall use symbols like $\|f|L^2\|$ etc. for the sake of better legibility. If V is a weight in a domain $\Omega \subset \mathbb{R}^N$ then the *weighted Lebesgue space* $L^p(V) = L^p(\Omega, V)$ is defined as the space of all measurable f on Ω with the finite norm $\|f|L^p(V)\| = \left(\int_\Omega |f(x)|^p V(x)\,dx\right)^{1/p}$. If f is a measurable function in $\mathbb{R}^N$, then f^* will denote its *non-increasing rearrangement.* The symbol $L^{p,q}$ will stand for the usual *Lorentz space* ($1 \le p, q < \infty$, or $1 \le p < \infty$ and $q = \infty$).

If Φ is a Young function, that is, Φ is even, convex, $\Phi(0) = 0$, $\lim_{t\to\infty} \Phi(t)/t) = \infty$, and $\Omega \subset \mathbb{R}^N$ is measurable, then $m(\Phi, f) = \int_\Omega \Phi(f(x))\,dx$ is the *modular* and the (quasi)norm in the corresponding Orlicz space $L_\Phi = L_\Phi(\Omega)$ is the Minkowski functional of the modular unit ball, namely, $\|f|L_\Phi\| = \inf\{\lambda > 0 : m(\Phi, f/\lambda) \le 1\}$ (the Luxemburg norm). We refer to [14] and [21] for the theory of classical Orlicz spaces and of modular spaces, resp. We shall restrict ourselves to a characterization of weighted Orlicz spaces $L_\Phi(V) = L_\Phi(\Omega, V)$, generated by the modular $\int_\Omega \Phi(f(x))V(x)\,dx$ as special Musielak-Orlicz spaces. Let us recall the latter concept in a form adapted to our needs (see [21] for the general case). Let us assume that $\Phi = \Phi(x,t) : \Omega \times \mathbb{R} \to [0,\infty)$ is a Young function of the variable t for each fixed $x \in \Omega$ and a measurable function of the variable x for each fixed $t \in \mathbb{R}$. The function Φ with these properties is called the *generalized Young function* or the *Musielak-Orlicz function.* Then

$$\varrho(f) = \int_\Omega \Phi(x, f(x))\,dx$$

is a *modular* on the set of all measurable functions on Ω so that we can consider the corresponding Orlicz space.

The weighted Orlicz spaces can be described in this language. Let V be a weight on Ω and let Φ be a Young function. Define

$$\Phi_1(x,t) = \Phi(t)V(x), \qquad x \in \Omega,\ t \in \mathbb{R}.$$

Then Φ_1 is a generalized Young function and the resulting Musielak-Orlicz space $L_{\Phi_1}(\Omega)$ is nothing but the *weighted Orlicz space* $L_\Phi(\Omega, V)$ with the modular

$$\varrho(f,w) = \int_\Omega \Phi(f(x))V(x)\,dx,$$

with the corresponding Luxemburg norm, and usually denoted by $L_\Phi(V)$ in the following.

The symbol $L^p[\log(1+L)]^\alpha$ ($\alpha > 0$) will denote the Orlicz space with the generating Young function $t \mapsto |t|^p[\log(1+|t|)]^\alpha$, $t \in \mathbb{R}$, and $L_{\exp t^\alpha}$ for $\alpha > 0$ will stand for the space with the Young function $t \mapsto \exp(|t|^\alpha) - 1$, $t \in \mathbb{R}$. For $\alpha = 1$ we shall simply write $L^p \log(1+L)$ and $L_{\exp}$.

A very suitable tool in the following will be the general imbedding theorem due to Ishii (see [12] and [21]). We state it in a slightly modified form, suitable for our purposes. Note that the norm of the imbedding in the theorem is independent of the dimension since it is a reformulation of an abstract theorem, which holds true in general Musielak-Orlicz spaces.

Proposition 1.1 (Ishii)**.** *Let U and V be weights in a measurable set $G \subset \mathbb{R}^N$, and let Φ and Ψ be Young functions. Then $L_\Phi(G,U) \hookrightarrow L_\Psi(G,V)$ if and only if there exists $K > 1$ such that the function*

$$x \mapsto \sup_{t>0} \left[\Psi(t)V(x) - \Phi(Kt)U(x)\right], \qquad x \in G,$$

is integrable over G.

2 Imbeddings on bounded domains based on the Gross inequality

We shall first discuss weighted consequences of the Sobolev imbedding theorem and of the general Gross logarithmic inequality.

Since we are interested in large N's we shall tacitly assume that $N \geq 3$ in the following to avoid unnecessary technicalities.

First of all let us briefly discuss a straightforward approach based on Sobolev imbeddings. It is not difficult to see that $V \in L^{N/p}$ is a sufficient condition for (1.1). One can do a little bit better: Since $W^{1,p}(\Omega)$ is imbedded into the Lorentz space $L^{Np/(N-p),p}$ we have

$$\begin{aligned}\int_\Omega |f(x)|^p V(x)\,dx &\le \int_0^{|\Omega|} (f^*(t))^p V^*(t)\,dt\\ &\le \int_0^{|\Omega|} t^{(N-p)/N} f^*(t)^p t^{p/N} V^*(t)\,\frac{dt}{t}\\ &\le \sup_{0<s<\infty} s^{p/N} V^*(s) \int_0^{|\Omega|} \left(t^{(N-p)/Np} f^*(t)\right)^p \frac{dt}{t},\end{aligned}$$

where we have used the Hardy-Littlewood rearrangement inequality. Hence (1.1) holds if $V \in L^{N/p,\infty}$. In particular, $V \in L_{\exp}$ (or $L_{\exp t^\beta}$ with any $\beta \ge 1$) is sufficient for (1.1) in any $\mathbb{R}^N$. Nevertheless, a dimension-free imbedding would require a detailed inspection of the behaviour of the imbedding constants and also of the equivalence of the exponential norm of V with the asymptotic estimates for the $L^{N/p,\infty}$ norms in dependence on N. We shall not pursue this line here.

Instead, we shall employ the dimension-free estimates for functions in $W^{1,p}(\mathbb{R}^N)$, generalizing the Gross inequality. Recall that the original Gross theorem (see (1.3)) states that

$$\int_{\mathbb{R}^N} |f(x)|^2 \log\left(\frac{|f(x)|^2}{\|f\|_2^2}\right) dx + N\|f\|_2^2 \le \frac{1}{\pi}\int_{\mathbb{R}^N} |\nabla f(x)|^2\,dx \tag{2.1}$$

for all $f \in W^{1,2}(\mathbb{R}^N)$ If, say, $\|f|W^{1,2}(\mathbb{R}^N)\| \le 1$, we obtain from (2.1)

$$\int_{\mathbb{R}^N} |f(x)|^2 \log|f(x)|\,dx \le \frac{1}{2\pi}\int_{\mathbb{R}^N} |\nabla f(x)|^2\,dx \tag{2.2}$$

(since under our assumption $\log\|f\|_2 \le 0$). Note in passing that due to the presence of the log function the Gross inequality expresses a fine balance for the small and large values of $|f|$.

We shall start with the general form of (2.1) for $1 < p < \infty$, see Gunson [10]: It holds

$$\int_{\mathbb{R}^N} |f(x)|^p \log(|f(x)|)\,dx + \gamma_{N,p} \le \int_{\mathbb{R}^N} |\nabla f(x)|^p\,dx, \tag{2.3}$$

for all $f \in W^{1,p}(\mathbb{R}^N)$, $\|f\|_p = 1$, with

$$\begin{aligned}\gamma_{N,p} &= \frac{N}{p} + \frac{N\log\pi}{2p} + \frac{N\log p}{p^2} - \frac{N(p-1)\log(p-1)}{p^2} \\ &\quad - \frac{1}{p}\log\left(\frac{\Gamma(1+N/2)}{\Gamma(1+N/p')}\right) \\ &= T_1 + T_2 + T_3 - T_4 - T_5,\end{aligned} \tag{2.4}$$

where Γ is the Euler Gamma function and $p' = p/(p-1)$.

Substituting $f(x)/\|f\|_p$ into (2.3) we get the more usual Lebesgue norm form of the above inequality, namely,

$$\int_{\mathbb{R}^N} |f(x)|^p \log\frac{|f(x)|}{\|f\|_p}\,dx + \gamma_{N,p}\,\|f\|_p^p \le \int_{\mathbb{R}^N} |\nabla f(x)|^p\,dx. \tag{2.5}$$

In the remainder of this section Ω will be for simplicity the unit ball B in $\mathbb{R}^N$ and we shall consider functions in $W_0^{1,p} = W_0^{1,p}(B)$.

We wish to have an inequality analogous to (2.2). First of all

$$T_1 + T_2 + T_3 - T_4 = \frac{N}{p}\left(1 + \frac{\log\pi}{2} + \frac{\log p}{p} - \frac{\log(p-1)}{p'}\right) \tag{2.6}$$

so that

$$|T_1 + T_2 + T_3 - T_4| \le c_1(p)N.$$

Further,

$$\begin{aligned}\frac{\Gamma(1+N/2)}{\Gamma(1+N/p')} &\sim \frac{e^{N/p'}}{e^{N/2}} \cdot \frac{(N/2)^{N/2-1/2}}{(N/p')^{N/p'-1/2}} \\ &\sim \frac{e^{N/p'}}{e^{N/2}} \cdot \frac{(p')^{N/p'-1/2}}{2^{N/2-1/2}}\,\frac{N^{N/2-1/2}}{N^{N/p'-1/2}} \\ &\sim \frac{e^{N/p'}}{e^{N/2}} \cdot \frac{(p')^{N/p'}}{2^{N/2}} \cdot \frac{N^{N/2}}{N^{N/p'}} \\ &\sim \left(\frac{ep'}{N}\right)^{N/p'}\left(\frac{N}{2e}\right)^{N/2},\end{aligned}$$

which gives

$$|T_5| \sim N\left|\frac{1}{2}\log\frac{N}{2e} - \frac{1}{p'}\log\frac{N}{ep'}\right|, \tag{2.7}$$

hence $|T_5| \le c_2(p)N\log N$, and we get

$$\int_{\mathbb{R}^N} |f(x)|^p \log\frac{|f(x)|}{\|f\|_p}\,dx \le c(p)N\log N\|f\|_p^p + \int_{\mathbb{R}^N} |\nabla f(x)|^p\,dx, \tag{2.8}$$

for all $f \in W^{1,p}(\mathbb{R}^N)$.

Now we shall additionally need an asymptotic estimate for the best constant in the Sobolev inequality.

The Sobolev imbedding theorem states in particular that $W_0^{1,p} = W_0^{1,p}(\Omega)$, $1 \le p < N$, $N \ge 3$, where Ω is a domain in $\mathbb{R}^N$, is imbedded into $L^{Np/(N-p)}$. Moreover (see, e.g. [23]), the best constant in the corresponding inequality for spaces on $\mathbb{R}^N$ is well known: If $p < N$, then

$$\left(\int_{\mathbb{R}^N} |f(x)|^{Np/(N-p)}\,dx\right)^{(N-p)/Np} \le C\|\nabla f|L^p\|, \qquad f \in W^{1,p}(\mathbb{R}^N), \tag{2.9}$$

where

$$C = \sqrt{1/\pi}\frac{1}{N^{1/p}}\left(\frac{p-1}{N-p}\right)^{1-1/p}\left(\frac{\Gamma(N)\Gamma(1+N/2)}{\Gamma(N/p)\Gamma(1+N/p')}\right)^{1/N}.$$

Let $p \in (1,\infty)$ be fixed and $N > p$. Invoking Stirling's formula for the Gamma function we have $(\Gamma(\xi))^{1/\xi} \sim \xi$ as $\xi \to \infty$, hence

$$\begin{aligned} C &\sim \frac{1}{N^{1/p}}\left(\frac{p-1}{N-p}\right)^{1-1/p}\left(\frac{\Gamma(N)\Gamma(1+N/2)}{\Gamma(N/p)\Gamma(1+N/p')}\right)^{1/N} \\ &\sim \frac{1}{N^{1/p}N^{1/p'}}\left(\frac{\Gamma(N)\Gamma(1+N/2)}{\Gamma(N/p)\Gamma(1+N/p')}\right)^{1/N} \\ &\sim \frac{1}{N}\frac{N\left[(N/2)\Gamma(N/2)\right]^{1/N}(p')^{1/N}}{(\Gamma(N/p))^{(p/N)(1/p)}N^{1/N}(\Gamma(N/p')^{1/N}} \\ &\sim \left((\Gamma(N/2))^{2/N}\right)^{1/2}\frac{1}{(N/p)^{1/p}\left((\gamma(N/p'))^{p'/N}\right)^{1/p'}} \\ &\sim \left(\frac{N}{2}\right)^{1/2}\frac{1}{N^{1/p}N^{1/p'}} \sim \frac{1}{N^{1/2}}. \end{aligned} \tag{2.10}$$

Let $f \in W^{1,p}(\mathbb{R}^N)$, $\operatorname{supp} f \subset \overline{B}$.

Using (2.9), we get

$$\begin{aligned}\|f|L^p(B)\|^p &\leq \|f|L^{Np/(N-p)}(B)\|^p\,|B|^{p/N}\\ &\leq \frac{c}{N^{p/2}}\left(\frac{\pi^{N/2}}{\Gamma(1+N/2)}\right)^{p/N}\|\nabla f|L^p(\mathbb{R}^N)\|^p\\ &\leq \frac{c}{N^{p/2}}\left(\frac{2}{N\Gamma(N/2)}\right)^{p/N}\|\nabla f|L^p(\mathbb{R}^N)\|^p\\ &\sim \frac{c}{N^{p/2}}\,2^{p/N}\frac{1}{(N^{1/N})^p}\left(\frac{1}{\Gamma(N/2)}\right)^{(2/N)(p/2)}\|\nabla f|L^p(\mathbb{R}^N)\|^p\\ &\sim \frac{c}{N^{p/2}}\left(\frac{2}{N}\right)^{p/2}\|\nabla f|L^p(\mathbb{R}^N)\|^p\\ &\sim \frac{1}{N^p}\|\nabla f|L^p(\mathbb{R}^N)\|^p.\end{aligned} \tag{2.11}$$

Altogether

$$\begin{aligned}c(p)N\log N\,\|f|L^p\|^p &\leq (c_1(p)N + c_2(p)N\log N)\,\|f|L^p\|^p\\ &\leq \frac{c(p)\log N}{N^{p-1}}\|\nabla f|L^p\|^p.\end{aligned}$$

Since $p > 1$ the constant on the right hand side tends even to 0 as $N \to \infty$.

Inserting this estimate into (2.8) we get

Lemma 2.1. *Let $1 < p < \infty$. Then*

$$\int_B |f(x)|^p \log\frac{|f(x)|}{\|f\|_p}\,dx \leq c\|\nabla f|L^p\|^p, \tag{2.12}$$

for all $f \in W_0^{1,p}(B)$, with a constant c independent of f and N. The same is true for any fixed ball in $\mathbb{R}^N$ with a possibly different constant c, depending on this ball and independent of the dimension.

Now we are in position to prove the following theorem.

Theorem 2.2. *Let $N \geq 3$ and $1 < p < \infty$. Then there exists c independent of N such that*

$$\int_B |f(x)|^p \log(1+|f(x)|)\,dx \leq c\|\nabla f|L^p\|^p \tag{2.13}$$

for all $f \in W_0^{1,p}(B)$.

Moreover, if $V \in L_{\exp t}(B)$, *then there exists* $c > 0$ *independent of* N *such that*

$$\int_B |f(x)|^p V(x)\,dx \le c\|\nabla f|L^p\|^p$$

for all $f \in W_0^{1,p}(B)$.

Proof. Let $f \in W_0^{1,p}(B)$, $\|f|W_0^{1,p}(B)\| = 1/2$. Denote the extension of f by zero to the whole of $\mathbb{R}^N$ by the same symbol. Consider function $f_\varepsilon(x) = |f(x)| + \varepsilon^2 s_\varepsilon(x)$, with $s_\varepsilon(x) = 1$ if $|x| \le 1$, $s_\varepsilon(x) = 1 - (|x|-1)/\varepsilon$ if $1 < |x| < 1+\varepsilon$, that is, s_ε is radially decreasing from the value 1 to 0 for $1 \le |x| \le 1+\varepsilon$, and $s_\varepsilon(x) = 0$ if $|x| > 1+\varepsilon$. Our first step will be to show that

$$\begin{aligned}\int_{\mathbb{R}^N} (|f(x)| + \varepsilon^2 s_\varepsilon(x))^p \log(1 + |f(x)| + \varepsilon^2 s_\varepsilon(x))\,dx& \\ \le c\|\nabla f|L^p\| + \varepsilon c(N) < \infty&\end{aligned} \tag{2.14}$$

with some constant c independent of the dimension and (small) ε. The final step will be then to derive the desired weighted inequality from (2.13).

Let us turn our attention to (2.12). We have

$$\begin{aligned}&\int_{\mathbb{R}^N} (|f(x)| + \varepsilon^2 s_\varepsilon(x))^p \log(1 + |f(x)| + \varepsilon^2 s_\varepsilon(x))\,dx \\ &\quad\le \int_{|f(x)|\ge 2} (|f(x)| + \varepsilon^2)^p \log(1 + |f(x)| + \varepsilon^2)\,dx \\ &\qquad+ \int_{0<|f(x)|<2} (|f(x)| + \varepsilon^2)^p \log(1 + |f(x)| + \varepsilon^2)\,dx \\ &\qquad+ \int_B \varepsilon^{2p} \log(1+\varepsilon^2)\,dx + \int_{(1+\varepsilon)B\setminus B} \varepsilon^p \log(1+\varepsilon)\,dx \\ &\quad= I_1(\varepsilon) + I_2(\varepsilon) + I_3(\varepsilon) + I_4(\varepsilon).\end{aligned}$$

By virtue of (2.12) with $2B$ instead of B and ε small, since $\log(1 + |f(x)| + \varepsilon^2) \le 2\log(|f(x)| + \varepsilon^2)$ if $|f(x)| \ge 2$,

$$\begin{aligned}I_1(\varepsilon) &\le \int_{|f(x)|\ge 2} (|f(x)| + \varepsilon^2)^p \log(|f(x)| + \varepsilon^2)\,dx \\ &\le c\int_{\mathbb{R}^N} |\nabla(|f(x)| + \varepsilon^2 s_\varepsilon(x))|^p\,dx.\end{aligned}$$

For the right hand side there is the elementary estimate

$$\int_{\mathbb{R}^N} |\nabla(|f(x)| + \varepsilon^2 s_\varepsilon(x))|^p \, dx \leq c\|\nabla f|L^p\|^p + c\varepsilon^{2p} \int_{(1+\varepsilon)B\setminus B} |\nabla s_\varepsilon(x)|^p \, dx$$

(with c depending on p only). It is easy to estimate the last integral; we get

$$I_1(\varepsilon) \leq c\|\nabla f|L^p\|^p + c\varepsilon^{2p} \ |2B| \leq c\|\nabla f|L^p\| + 2^N c\varepsilon^{2p} \ |B|$$

provided $\varepsilon \leq 1$.

We estimate the second integral. Invoking the asymptotic estimate in (2.10) we have, by Hölder's inequality,

$$\begin{aligned} I_2(\varepsilon) &= \int_{0<|f(x)|<2} (|f(x)| + \varepsilon^2)^p \log(1 + |f(x)| + \varepsilon^2) \, dx \\ &\leq c\Big(\int_{0<|f(x)|<2} (|f(x)| + \varepsilon^2)^{Np/(N-p)}\Big)^{(N-p)/N} |B|^{p/N} \\ &\leq \frac{c}{N^{1/2}} \|\nabla(|f(x)| + \varepsilon^2 s_\varepsilon(x))|L^p\|^p \\ &\leq \frac{c}{N^{p/2}} \|\nabla f|L^p\|^p + c(N)\varepsilon^p. \end{aligned}$$

Finally, the third and the fourth integrals can be treated easily to show that

$$I_2 + I_4 \leq c\varepsilon$$

with some constant c independent of f and N (and ε); we omit the details.

Hence the left hand side of (2.14) is finite. Fatou's lemma gives

$$\int_B |f(x)|^p \log(1 + |f(x)|) \, dx \leq c\|\nabla f|L^p(B)\|^p,$$

where c is independent of N and f.

Since the modular and the norm convergence in $L^p \log L(B)$ are equivalent we arrive at the imbedding $W_0^{1,p}(B) \hookrightarrow L^p \log L$ in any $\mathbb{R}^N$, $N \in \mathbb{N}$, with the norm independent of N.

Now our problem reduces to establishing a sufficient condition for the imbedding $L^p \log L(B) \hookrightarrow L^p(B, V)$, where $L^p \log L(B)$ is the Orlicz space generated by the Young function $t \mapsto t^p \log(1 + |t|)$. Ishii's theorem gives a necessary and sufficient condition for that, namely, integrability of the function

$$\sup_{t>0}[t^p V(x) - Kt^p \log(1+Kt)], \qquad x \in B, \tag{2.15}$$

over B, for some $K > 1$. Let us rewrite the function in (2.15) as

$$\sup_{t>0}[tV(x) - Kt\log(1+Kt^{1/p})].$$

By virtue of the Young inequality the condition is $V \in L_{\widetilde{\Psi}}(B)$, where $\widetilde{\Psi}$ is the complementary function to $\Psi(t) = |t|\log(1+t^{1/p})$. Note that $\Psi(t) \sim |t|\log(1+t)$ and it is well known that the complementary function is equivalent to $t \mapsto \exp|t| - 1$. □

Remark 2.3. It is only a formal change to consider any bounded domain instead of the unit ball in previous considerations.

As to an unbounded Ω a closer inspection of (2.4) shows that the term $\gamma_{N,p}$ is non-negative if $p \in [2, p^*]$, where p^* is the unique solution of the equation

$$e\pi p^{1/p} = 2(p-1)^{1/p'}. \tag{2.16}$$

Indeed, the term $-T_5$ in (2.4) is non-negative if $p \geq 2$. As to remaining terms let us look at (2.6). An elementary calculation shows that the right hand side of (2.6) is non-negative provided $p \in (1, p^*)$, where p^* is the unique solution of (2.16). Consequently $\gamma_{N,p} \geq 0$ if $p \in (2, p^*)$.

3 Extrapolation of Sobolev imbeddings

In this section we will use extrapolation of Sobolev imbeddings to get the residual dimension-free imbeddings for functions with no constraints on their support.

The symbol $W_0^{1,p}$ will denote either $W_0^{1,p}(\Omega)$ with some domain $\Omega \subset \mathbb{R}^N$ or the space $W^{1,p}(\mathbb{R}^N)$ (which coincides with $W_0^{1,p}(\mathbb{R}^N)$).

Theorem 3.1. *Let $1 < p < N/3$. There exists a constant c independent of N, α, and $f \in W_0^{1,p}$ such that*

$$\|f|L^p(\log(1+L))^\alpha\| \leq c\|\nabla f|L^p\| \tag{3.1}$$

for every $f \in W_0^{1,p}$ and $\alpha \in (p^2/(N-p), p/2]$.

Corollary 3.2. *Let V be a weight function on Ω, $1 < p < N/3$, and assume that $V\chi_{\{V\geq 1\}} \in L_{\exp t^{1/\alpha}}$ with α as in Theorem 3.1 and $V\chi_{\{0<V<1\}} \in L^1$. Then there exists a constant c independent of N such that*

$$\|f|L^p(V)\| \leq c\|\nabla f|L^p\|$$

for every $f \in W_0^{1,p}(\Omega)$. If $|\Omega| < \infty$, it is enough to assume that $V \in L_{\exp t^{1/\alpha}}$.

Proof of Theorem 3.1. Let $\alpha > 0$. Hölder's inequality combined with the Sobolev imbedding gives

$$\begin{aligned}&\left(\int_\Omega |f(x)|^p \left[\log(1+||f(x)||)\right]^\alpha \, dx\right)^{1/p} \\ &\leq \left(\int_\Omega |f(x)|^{Np/(N-p)} \, dx\right)^{(N-p)/Np} \left(\int_\Omega \left[\log(1+|f(x)|)\right]^{N\alpha/p} \, dx\right)^{1/N} \\ &\leq \frac{c}{N^{1/2}}\|\nabla f|L^p\|\left(\int_\Omega \left[\log(1+|f(x)|)\right]^{N\alpha/p} \, dx\right)^{1/N}.\end{aligned} \tag{3.2}$$

It is not difficult to see that for $\varepsilon \in (0,1)$,

$$\log(1+\xi) \leq \frac{1}{\varepsilon}\xi^\varepsilon, \qquad \xi > 0.$$

Indeed, consider $h(\xi) = \log(1+\xi) - c_\varepsilon \xi^\varepsilon$ with c_ε to be specified later. Then $h(0) = 0$ and

$$h'(\xi) = \frac{1}{1+\xi} - \varepsilon c_\varepsilon \xi^{\varepsilon-1}.$$

We wish to find c_ε such that $h(\xi) \leq 0$, i.e.,

$$\varepsilon c_\varepsilon \xi^{\varepsilon-1} + \varepsilon c_\varepsilon \xi^\varepsilon \geq 1.$$

Plainly it is sufficient that $c_\varepsilon = 1/\varepsilon$ so that

$$\left[\log(1+|f(x)|)\right]^{N\alpha/p} \leq \left(\frac{1}{\varepsilon}\right)^{N\alpha/p} |f(x)|^{N\alpha\varepsilon/p},$$

and

$$\begin{aligned}&\left(\int_\Omega \left[\log(1+ \, |f(x)|)\right]^{N\alpha/p} \, dx\right)^{1/N} \\ &\qquad \leq \left(\frac{1}{\varepsilon}\right)^{\alpha/p} \left(\int_\Omega |f(x)|^{N\alpha\varepsilon/p} \, dx\right)^{1/N}.\end{aligned} \tag{3.3}$$

This means that the appropriate choice is

$$\frac{N\alpha\varepsilon}{p} = \frac{Np}{N-p},$$

in another terms,

$$\varepsilon = \frac{p^2}{\alpha(N-p)}.$$

Inserting this into (3.3) and applying Sobolev's inequality again we get

$$\begin{aligned}
&\left(\int_\Omega [\,\log(1+|f(x)|)]^{N\alpha/p}\,dx\right)^{1/N} \\
&\quad\le \left(\frac{1}{\varepsilon}\right)^{\alpha/p}\left(\int_\Omega |f(x)|^{N\alpha\varepsilon/p}\,dx\right)^{1/N} \\
&\quad\le \left(\frac{\alpha(N-p)}{p^2}\right)^{\alpha/p}\left(\int_\Omega |f(x)|^{Np/(N-p)}\,dx\right)^{1/N} \\
&\quad\le \left(\frac{\alpha(N-p)}{p^2}\right)^{\alpha/p}\left(\int_\Omega |f(x)|^{Np/(N-p)}\,dx\right)^{((N-p)/Np)(p/(N-p))} \\
&\quad\le \left(\frac{\alpha(N-p)}{p^2}\right)^{\alpha/p}\left(\frac{c}{N^{1/2}}\right)^{p/(N-p)}\|\nabla f|L^p\|^{p/(N-p)}.
\end{aligned}$$

Together with (3.2) this yields

$$\begin{aligned}
&\left(\int_\Omega |f(x)|^p\,[\log(1+|f(x)|)]^\alpha\,dx\right)^{1/p} \\
&\qquad\le \frac{c}{N^{1/2}}\left(\frac{\alpha(N-p)}{p^2}\right)^{\alpha/p}\left(\frac{1}{N^{1/2}}\right)^{p/(N-p)}\|\nabla f|L^p\|^{1+p/(N-p)}.
\end{aligned}$$

Hence for N large we have

$$\left(\int_\Omega |f(x)|^p\,[\log(1+|f(x)|)]^\alpha\,dx\right)^{1/p} \le c\frac{N^{\alpha/p}}{N^{1/2}}\|\nabla f|L^p\|^{1+p/(N-p)}$$

with some c independent of f and N. To achieve independence of the right hand side of N we have to choose

$$\alpha \le \frac{p}{2}.$$

Note that the best choice for spaces on bounded domains is $\alpha = p/2$ and we have than

$$\left(\int_\Omega |f(x)|^p[\log(1+|f(x)|)]^\alpha\,dx\right)^{1/p} \leq c\|\nabla f|L^p\|^{1+p/(N-p)}. \tag{3.4}$$

The power of the norm of the gradient on the right hand side of (3.4) might look strange at a first look but one should realize that the expression on the left hand side is not a norm but merely a power of the modular in $L^p[\log(1+L)]^\alpha$. In the following we shall derive the inequality (3.1).

Let us repeat the above procedure once more, this time with $f(x)/\lambda$ instead of f ($\lambda > 0$ arbitrary). We get

$$\frac{\lambda^{1+p/(N-p)}}{c}\left(\int_\Omega \left(\frac{f(x)}{\lambda}\right)^p \left[\log\left(1+\frac{|f(x)|}{\lambda}\right)\right]^\alpha\,dx\right)^{1/p} \leq \|\nabla f|L^p\|^{1+p/(N-p)}.$$

Let

$$\|\nabla f|L^p\| = 1$$

and

$$\lambda^{1+p/(N-p)} \geq c \quad (\text{that is, } \lambda \geq c^{1-p/N}),$$

then

$$\int_\Omega \left(\frac{|f(x)|}{\lambda}\right)^p \left[\log\left(1+\frac{|f(x)|}{\lambda}\right)\right]^\alpha\,dx \leq 1.$$

The last estimate says that $\|f|L^p[\log(1+L)]^\alpha\| \leq c^{1-p/N}$. Since c was independent of N, plainly also $\|f|L^p[\log(1+L)]^\alpha\|$ can be estimated from above independently of N. □

Proof of Corollary 3.2. According to Proposition 1.1 (Ishii's theorem) we have $L^p[\log(1+L)]^\alpha \hookrightarrow L^p(V)$ if and only if the function

$$H(x) = \sup_{t>0}\,(t^pV(x) - K^pt^p[\log(1+Kt)]^\alpha) \tag{3.5}$$

is integrable over B. But (3.5) can be rewritten as

$$H(x) = \sup_{t>0}\left(tV(x) - K^pt[\log(1+Kt^{1/p})]^\alpha\right) \tag{3.6}$$

hence the necessary and sufficient condition for the imbedding is

$$\int_\Omega \Psi(V(x))\,dx < \infty,$$

where Ψ is a Young function complementary to $K^p t[\log(1+Kt^{1/p})]^\alpha$.

The function $t \mapsto K^p \left[\log(1+Kt^{1/p})\right]^\alpha$ is an inverse to the Δ_3-function

$$\widetilde{\Psi}(\xi) = \frac{1}{K^p}\left[\exp\left(\frac{\xi^{1/\alpha}}{K^2}\right) - 1\right]^p;$$

so that (see [14, I/§6]) we have $\Psi(\xi) \sim \exp\left(\xi^{1/\alpha}\right) - 1$ for ξ bounded away from zero, say for $\xi \geq 1$ (in the sense of the equivalence of Young functions). As to values of V belonging to $(0,1)$ we have to look directly at the integrability of the function in (3.6). Elementary calculations show that if $V(x) \leq 1$, then the expression on the right hand side of (3.6) is negative if $t > K^{-p}\left(\exp K^{-p/\alpha} - 1\right)$ and the sup becomes a (fixed) multiple of $V(x)$.

□

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