On non-effective weights in Orlicz spaces

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Abstract

Given a weight w in $\Omega \subset \mathbb{R}^N$, $|\Omega| < \infty$ and a Young function Φ, we consider the weighted modular $\int_\Omega \Phi(f(x))w(x)\,dx$ and the resulting weighted Orlicz space $L_\Phi(w)$. For a Young function $\Phi \notin \Delta_2(\infty)$ we present a necessary and sufficient conditions in order that $L_\Phi(w) = L_\Phi(\chi_\Omega)$ up to the equivalence of norms. We find a necessary and sufficient condition for Φ in order that there exists an unbounded weight w such that the above equality of spaces holds. By way of applications we simplify criteria from [CHKM] for continuity of the composition operator from L_Φ into itself when $\Phi \notin \Delta_2(\infty)$ and obtain necessary and sufficient condition in order that the composition operator maps L_Φ continuously onto L_Φ.

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1 Preliminaries

In the sequel we assume that Ω is a subset of $\mathbb{R}^N$ with a finite Lebesgue measure; for simplicity we shall assume that $|\Omega| = 1$. If not otherwise specified, all the norms and all the spaces are tacitly assumed to concern Lebesgue measurable functions defined a.e. in Ω. All positive constants whose exact value is not important for our purposes will be denoted by c, C, and the like, occasionally with additional subscripts within the same formula or the same proof.

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A Young function will be an even convex function Φ on $\mathbb{R}$ such that $\Phi(0) = 0$, $\Phi(\infty) = \infty$. An N-function will be a Young function with zero derivative at the origin and with infinite limit of the derivative at infinity. We shall assume that the reader is familiar with the classical definition of a modular, of the Orlicz spaces and other basic facts from the Orlicz spaces theory—we refer to [KR] and [Mu]. If Φ is a Young function, then we shall denote by $L_\Phi(\Omega)$ or simply by L_Φ the corresponding Orlicz space and by $\|f|L_\Phi(\Omega)\|$ or simply by $\|f|L_\Phi\|$ or by $\|f\|_\Phi$ its norm. For our purposes it will be of no importance which norm (the Luxemburg norm, the Orlicz dual norm, the Amemyia norm) we use. Let us note that according to [HM] the Luxemburg and Amemyia norm even coincide in general. Observe also that the space and its norm remain to be the same (up to an equivalence as far as the norm is concerned) if the generating Young function is replaced by a Young function, which is equivalent to the original function near infinity. Let us observe that the equivalence of Young functions is a concept different of the standard equivalence for functions: The Young functions Φ_1 and Φ_2 are equivalent near infinity if there are finite positive constants c_1, c_2 and $t_0 > 0$ such that $\Phi_1(c_1 t) \le \Phi_2(t)$ and $\Phi_2(c_2 t) \le \Phi_1(t)$ for all $t > t_0$. This is sometimes formulated in terms of equivalent major parts of Young functions.

If $\Phi(t) = \exp t^\beta - t - 1$ (or $\Phi(t) = \exp t^\beta - 1$), $t \ge 0$, $\beta > 0$, then often the notation $L_{\exp t^\beta}$ is used ($L_{\exp} = L_{\exp t}$). Of course, near infinity all three functions are equivalent to the major part of the same Young function (one can take the first function, for instance).

These "classical" Orlicz spaces are a special case of more general *Musielak-Orlicz spaces*, which we recall now—adapted for our setting (see [Mu] for the general case). Let us assume that $\Phi = \Phi(x,t) : \Omega \times [0,\infty) \to [0,\infty)$ is a Young function of the variable t for each fixed $x \in \Omega$ and a measurable function of the variable x for each fixed $t \in \mathbb{R}$. The function Φ with these properties is called the *generalized Young function* or the *Musielak-Orlicz function*. Then

$$\varrho(f) = \int_\Omega \Phi(x, f(x))\, dx$$

is a *modular* on the set of all measurable functions on Ω so that we can consider the appropriate Orlicz space.

Note that the weighted Orlicz spaces can be described in this language. Let w be a weight in Ω, that is, an a.e. positive and locally integrable real function defined in Ω. Let Φ be a Young function and define

$$\Phi_1(x,t) = \Phi(t)w(x), \qquad x \in \Omega,\ t \in [0,\infty).$$

Then, plainly, Φ is a generalized Young function and the resulting Musielak-Orlicz space is nothing but the *weighted Orlicz space* with the modular

$$\varrho(f,w) = \int_\Omega \Phi(f(x))w(x)\, dx$$

and the corresponding Luxemburg (Amemyia) norm, and denoted by $L_\Phi(w)$ in the following. The definition can be formally generalized by replacing the Lebesgue measure by a general σ-finite measure ν in Ω. Then we shall write $L_\Phi(d\nu)$ and/or $L_\Phi(w\ d\nu)$ for the analogs of the above spaces.

We shall call a weight w *non-effective with respect to the measure* ν if $L_\Phi(w\, d\nu) = L_\Phi(d\nu)$. If $d\nu = dx$ we shall simply talk about a *non-effective weight.*

If $\Phi(t) = |t|^p$, then one gets the weighted Lebesgue space $L_p(w) = L_p(\Omega, w)$. More generally in this framework, $W^{k,p}(w) = W^{k,p}(\Omega, w)$ $(1 < p < \infty,\ k \in \mathbb{R}^N)$ will denote the weighted Sobolev space, consisting of all $f \in L_p(w)$ with distributional derivatives $D^\alpha f \in L_p(w)$, $|\alpha| \le k$.

2 Motivation and several examples

Let for a moment $\Omega \subset \mathbb{R}^N$ be additionally a Lipschitz domain. Denote by $W^{1,p} = W^{1,p}(\Omega)$, $1 \le p < \infty$, the usual Sobolev space, i.e. the space of all functions $f \in L_p(\Omega)$, whose distributional derivatives $\partial f/\partial x_i$ belong to $L_p(\Omega)$, $i = 1, \ldots, N$. If $p < N$, then it is well known that $W^{1,p}$ is imbedded into L_{p^*}, where $p^* = Np/(N-p)$ is the Sobolev exponent. Hence the space $W^{1,N}$ is imbedded into any $L_q(\Omega)$, $1 \le q < \infty$. On the other hand there are unbounded functions in $W^{1,N}$, see e.g. [Ad], [KJF]. This remarkable phenomenon has been thoroughly studied since the 1960s with a considerable revival of interest in last ten years or so. A careful analysis shows that if $f \in W^{1,N}$, then

$$\|f|L_q\| \le c q^{1/N'} \|f|W^{1,N}\|, \qquad 1 < q < \infty, \tag{2.1}$$

where $N' = N/(N-1)$ is the adjoint exponent to N and $c > 0$ is a constant independent of f and N. This is the key to basic refinements of Sobolev imbeddings in this so called first critical case (see Peetre [P], Trudinger [T] as the basic references). As is well known (see e.g. again [P] or [T]), the estimate (2.1) is nothing but a characterization of the fact that f belongs to the exponential space $L_\Phi(\Omega)$, where Φ is a Young function with the major part equivalent to the function $t \mapsto \exp t^{N'}$ near infinity.

In several recent papers (see [KS], [BuS]) the authors considered among other Riesz potentials in weighted Sobolev spaces $W^{k,p}(\Omega, w)$ and the asymptotic behaviour of norms of imbeddings into appropriate weighted Lebesgue spaces with the integrability exponent q, as $q \to \infty$, yielding a critical imbedding into a weighted exponential Orlicz space. We make simple calculation, using just Hölder's inequality. Assume that $f \in W^{1,N}(\Omega)$ and that w is a weight function in Ω, $w \in L_{1+\varepsilon}$ with some $\varepsilon > 0$. Then, for any $q > 1$,

$$\begin{aligned}\left(\int_\Omega |f(x)|^q w(x)\,dx\right)^{1/q} &\le \left(\|f|L_{q(1+\varepsilon)'}\|\,\|w|L_{1+\varepsilon}\|\right)^{1/q} \\ &\le c\left(q(1+\varepsilon)'\right)^{1-\frac{1}{q(1+\varepsilon)'}}\ \max(1, \|w|L_{1+\varepsilon}\|),\end{aligned}$$

hence f belongs to the weighted exponential space generated by the modular

$$\int_\Omega \Phi(f(x))w(x)\,dx,$$

where Φ is any Young function equivalent to exp at infinity (the extrapolation characterization of the exponential space carries over easily to this weighted case). A natural question arises whether this is an improvement of the critical imbedding theorem. An answer can hardly be expected in terms of spaces invariant with respect to rearrangements—plainly a weighted Lebesgue space is generally *not* a r.i. space. Later we shall see that weights in $L_{1+\varepsilon}$ for some $\varepsilon > 0$ do not affect the exponential Orlicz space. The following example shows that such an integrability condition for the weight should be close to a necessary condition.

Example 2.1. We pass to weights in $L(\log L)^\beta = L(\log L)^\beta(\Omega)$. Recall that $w \in L(\log L)^\beta$, $\beta \geq 1$, if and only if

$$\int_0^1 w^*(t)\left(\log\frac{1}{t}\right)^\beta dt < \infty \tag{2.2}$$

(see [BS]).

We give an example of a function $w \in L\log L = L(\log L)^1$ such that $L_{\exp} \setminus L_{\exp}(w) \neq \emptyset$.

We shall work directly on $(0,1)$ and we shall take w non-increasing (at least in some neighbourhood of zero) so that w can be assumed to coincide with its non-decreasing rearrangement. Let $\varepsilon > 0$, $w \in L\log L$, and let f be a non-negative locally integrable function (which will be determined later). Then

$$\int_0^1 \exp(\varepsilon f(t)) w(t)\, dt = \int_0^1 \frac{\exp(\varepsilon f(t))}{\log\frac{1}{t}} w(t)\log\frac{1}{t}\, dt.$$

For $\gamma > 0$ (to be determined later) put

$$w(t) = \frac{1}{t\left(\log\frac{1}{t}\right)^{2+\gamma}}.$$

Then $w \in L\log L$ in view of (2.2).

We shall try to find f such that

$$\exp(\varepsilon f(t)) \geq \left(\log\frac{1}{t}\right)^{1+\beta}$$

near the origin, that is,

$$\varepsilon f(t) \geq \log\left[\log\frac{1}{t}\right]^{1+\beta}$$

near zero. To this end it suffices that

$$\lim_{t\to 0_+} \frac{f(t)}{\log\left[\left(\log\frac{1}{t}\right)\right]^{1+\beta}} = \infty. \tag{2.3}$$

Then

$$\frac{\exp(\varepsilon f(t))}{\log\frac{1}{t}} \geq \left(\log\frac{1}{t}\right)^\beta$$

near zero and we shall have, for a suitable number $b = b(\varepsilon) \in (0,1)$ (the corresponding neighbourhood might depend on ε),

$$\begin{aligned}\int_0^b \exp(\varepsilon f(t)) w(t)\, dt &\geq \int_0^b \left(\log\frac{1}{t}\right)^\beta w(t)\log\frac{1}{t}\, dt = \int_0^b w(t)\left(\log\frac{1}{t}\right)^{1+\beta} dt\\ &= \int_0^b \frac{\left(\log\frac{1}{t}\right)^\beta}{t\left(\log\frac{1}{t}\right)^{1+\gamma}}\, dt.\end{aligned}$$

If $\beta \geq \gamma$, then the last integral is infinite, hence $f \notin L_{\exp}(w)$.

It remains to show that it is still possible to have $f \in L_{\exp}$. We put

$$f(t) = \left[\log\left(\log\frac{1}{t}\right)^{1+\beta}\right]^\alpha,$$

in $(0,1/e)$ for some $\alpha>1$ and define f e.g. as a non-zero constant on the rest of $(0,1)$. Then (2.3) is true. Let $\lambda>0$; we shall show that

$$\int_0^1 \exp\left\{\lambda\left[\log\left(\log\frac{1}{t}\right)^{1+\beta}\right]^\alpha\right\} dt<\infty.$$

We have

$$\begin{aligned} I&=\int_0^{1/e} \exp\left\{\lambda\left[\log\left(\log\frac{1}{t}\right)^{1+\beta}\right]^\alpha\right\} dt\\ &=\int_0^{1/e} \exp\left\{\left[\log\left(\log\frac{1}{t}\right)^{(1+\beta)\lambda^{1/\alpha}}\right]^\alpha\right\} dt\\ &=\int_0^{1/e} \exp\left(\left[\log\left(\log\frac{1}{t}\right)^{\delta}\right]^\alpha\right) dt, \end{aligned}$$

where we put $\delta=(1+\beta)\lambda^{1/\alpha}$. After the change of variables $\xi=(\log(1/t))^\delta$, that is, $t=\exp\left(-\xi^{1/\delta}\right)$, $dt=-\delta^{-1}\xi^{-1+1/\delta}\exp\left(-\xi^{1/\delta}\right)$, this becomes

$$I=\int_1^\infty \frac{\xi^{(1-\delta)/\delta}\exp\left(-\xi^{1/\delta}\right)}{\delta}\exp\left[(\log\xi)^\alpha\right]\,d\xi$$

and after another change of variables $\log\xi=y$

$$\begin{aligned} &=\frac{1}{\delta}\int_0^\infty \exp\left(\frac{y(1-\delta)}{\delta}\right)\exp\left(-e^{y/\delta}\right)\exp\left(y^\alpha\right)\exp y\,dy\\ &=\frac{1}{\delta}\int_0^\infty \exp\left(y^\alpha+y+\frac{y(1-\delta)}{\delta}-e^{y/\delta}\right)\,dy. \end{aligned}$$

The last integral converges for any $\delta>0$.

The same considerations are true if we take $w\in L(\log L)^q$, for any $q\geq 1$. Then instead of the above w we take

$$w(t)=\frac{1}{t\left(\log\frac{1}{t}\right)^{q+1+\gamma}}$$

and repeating the above estimates we arrive at the same condition for β and γ.

This seems to be a supporting argument for the conjecture that $L_\Phi(w)=L_\Phi(\chi_\Omega)$ iff $w\in L_{1+\varepsilon}$ for some $\varepsilon>0$. As we shall see later this is indeed the case in exponential Orlicz spaces: The condition $w\in L_{1+\varepsilon}$ is necessary and sufficient in order that $L_{\exp}(w)=L_{\exp}(\chi_\Omega)$. Since this will follow from our general Theorem 3.4 we omit the proof in this special case.

Nevertheless, in general Orlicz spaces the situation is more delicate. It is natural to expect that a weight w e.g. unbounded from above change the Orlicz space if its Young function satisfies the $\Delta_2(\infty)$-condition (we prove this in the following for completeness). For such weights an immediate guess might be to draw a border line between Young functions satisfying and not satisfying the $\Delta_2(\infty)$-condition to characterize spaces, which can and need not be affected by such w, respectively. We tackle this problem in Section 3 and give a simple characterization. It is rather surprising that the right criterion is *not* the $\Delta_2(\infty)$-condition; one has to consider a somewhat smaller class (see Theorem 3.13).

Another direct motivation came from the problems concerning composition operators in Orlicz spaces—see Section 4 for an extension of the recent paper [CHKM].

3 General theorems

Ishii [I] established an abstract imbedding theorem for the Orlicz-Musielak spaces; see also [Mu, Theorem 8.5] for a more general result concerning the Orlicz-Musielak spaces generated by the Orlicz-Musielak functions $\Phi(x,t)$ that for a fixed parameter t are non-decreasing but not necessarily convex on $\mathbb{R}_+$. In our further considerations the following reformulation will play an essential role.

Theorem 3.1. *Let Φ be a Young function (of one variable) and let w be a weight in Ω. Then*

(1) $L_\Phi \hookrightarrow L_\Phi(w)$ *if and only if there exist a constant $K > 0$ such that the the function*

$$x \mapsto \sup_{u \geq 0} \left(w(x)\Phi(u) - \Phi(Ku) \right) \tag{3.1}$$

is integrable in Ω

(2) $L_\Phi(w) \hookrightarrow L_\Phi$ *if and only if there exist a constant $K > 0$ such that the the function*

$$x \mapsto \sup_{u \geq 0} \left(\Phi(u) - \Phi(Ku)w(x) \right) \tag{3.2}$$

is integrable in Ω.

Note that the functions defined in (3.1) and (3.2) are non-negative because $\Phi(0) = 0$.

If $\Phi \in \Delta_2(\infty)$, then the situation is easy. We present the following theorem for completeness.

Theorem 3.2. *Let Φ be a Young function satisfying the $\Delta_2(\infty)$-condition and let w be a weight in Ω. Then $L_\Phi(w) = L_\Phi(\chi_\Omega)$ if and only if there exist constants $0 < c_1 < c_2 < \infty$ such that $c_1 \leq w(x) \leq c_2$ a.e. in Ω.*

Proof. If w is equivalent to a constant, then plainly both spaces $L_\Phi(w)$ and $L_\Phi(\chi_\Omega)$ coincide up to an equivalence of their norms.

Assume that $L_\Phi(w) = L_\Phi$. We know then that

$$\sup_{t \in W_+} \left[w(x)\Phi(t) - \Phi(Kt) \right] \in L_1$$

for some $K > 1$, where

$$W_+ = \{t;\ w(x)\Phi(t) > \Phi(Kt)\}.$$

Since $\Phi \in \Delta_2(\infty)$ we have $\Phi(Kt) \leq c\max(K^p, K^q)\Phi(t)$ for some c independent of $K > 1$ and t and some $p, q > 0$ independent of t (see e.g. [GP]); note that Φ can be redefined near the origin (without change of the resulting Orlicz space) in such way that it is between two powers. Define

$$A_{K,x,+} = \{t;\ w(x)\Phi(t) > c\max(K^p, K^q)\Phi(t)\}.$$

Then

$$w(x)\Phi(t) - \Phi(Kt) \geq [w(x) - c\max(K^p, K^q)]\Phi(t),$$

hence $A_{K,x,+} \subset W_+$ and

$$\begin{aligned}\sup_{t\in W_+}[w(x)\Phi(t) - \Phi(Kt)] &\geq \sup_{t\in A_{K,x,+}}[w(x)\Phi(t) - \Phi(Kt)] \\ &\geq \sup_{t\in A_{K,x,+}}[w(x) - c\max(K^p, K^q)]\Phi(t).\end{aligned}$$

But $w(x) > c\max(K^p, K^q)$ on a set of positive measure and we conclude that the function

$$\sup_{t\in A_{K,x,+}}[w(x)\Phi(t) - \Phi(Kt)]$$

is not integrable over the set $\{x \in \Omega;\ w(x) - c\max(K^p, K^q)\}$. This is a contradiction. □

Hence in the following we shall tackle weights not satisfying the assumption of the preceding Theorem. Let us first consider the case $w(x) \geq 1$ a.e. in Ω. Then we have

Theorem 3.3. *Let Φ be a Young function and for $K > 0$ put $L_K(t) = \Phi(K\Phi^{-1}(t))$. Let S_K be the complementary function to L_K and assume that $w \notin L_\infty$ is a weight function, bounded away from zero. Then $L_\Phi(\chi_\Omega) = L_\Phi(w)$ if and only if*

$$\int_\Omega S_K(w(x))\,dx < \infty. \tag{3.3}$$

Proof. The claim follows from Theorem 3.1, observing that the values of the function in (3.1) can be written as

$$\sup_{t>0}[w(x)t - \Phi(K\Phi^{-1}(t))]$$

(because Φ is surjective), which is nothing but $S_K(w(x))$ and the claim follows. □

Note that we have not used any special properties of the Lebesgue measure in the above theorem. The general imbedding theorem for the Musielak-Orlicz spaces in [Mu] holds for σ-finite and non-atomic weights. Hence at no actual extra effort we can reformulate the above theorem as

Theorem 3.4. *Let ν be a non-atomic and σ-finite measure in Ω and w a locally integrable with respect to ν, $w \geq 1$ ν-a.e. in Ω. Then $L_\Phi(d\nu) \hookrightarrow L_\Phi(w\,d\nu)$ if and only if for some $K > 1$,*

$$\int_\Omega S_K(w(x))d\nu < \infty,$$

where

$$S_K(t) = \sup_{u>0}\{u|t| - \Phi(K\Phi^{-1}(u)\}. \tag{3.4}$$

Corollary 3.5. *Let $0 < w \le 1$ a.e. in Ω, w measurable. Then $L_\Phi(w) \hookrightarrow L_\Phi$ if and only if there exists $K > 1$ such that*

$$\int_\Omega S_K\left(\frac{1}{w(x)}\right) w(x)\,dx < \infty, \tag{3.5}$$

where S_K is given by (3.4).

Proof. Put $d\nu(x) = w(x)\,dx$. Then the imbedding $L_\Phi(w) \hookrightarrow L_\Phi$ is the same as

$$L_\Phi(w) = L_\Phi(d\nu) \hookrightarrow L_\Phi(dx) = L_\Phi\left(\frac{d\nu}{w}\right).$$

By virtue of Theorem 3.3, the necessary and sufficient condition for

$$L_\Phi(d\nu) \hookrightarrow L_\Phi\left(\frac{d\nu}{w}\right)$$

is

$$\int_\Omega S_K\left(\frac{1}{w(x)}\right) w(x)\,dx < \infty$$

and we are done. □

Similar simple argument also yields

Corollary 3.6. *Let w_j be weight functions in Ω, $w_j \ge 1$ a.e. in Ω and et S_K be the complementary function from Theorem 3.3 with some $K > 1$. Then*

$$L_\Phi(w_1) \hookrightarrow L_\Phi(w_2)$$

if and only if

$$\int_\Omega S_K\left(\frac{w_2(x)}{w_1(x)}\right) w_1(x)\,dx < \infty.$$

Remark 3.7. Plainly the last Corollary admits a reformulation with a non-atomic σ-finite measure ν in Ω. We omit this.

Remark 3.8. For $\Phi(t) = \exp t - 1$ we have $\Phi(K\Phi^{-1}(\xi)) \sim (1+\xi)^K \sim \xi^K$ for large ξ. Hence $S_K(\xi) \sim \xi^{K'}$ near ∞, where $K' = K/(K-1)$. Hence condition (3.3) becomes $w \in L_{1+K'}$ and condition (3.5) reads as $w^{-1} \in L_{K'-1}$.

Remark 3.9. Corollary 3.6 admits a more general formulation, giving necessary and sufficient condition for the imbedding

$$L_\Psi(w_1) \hookrightarrow L_\Phi(w_2),$$

which reads

$$\int_\Omega S_{K,\Phi,\Psi}\left(\frac{w_2(x)}{w_1(x)}\right) w_1(x)\,dx < \infty,$$

where Φ and Ψ are Young functions and $S_{K,\Phi,\Psi}$ is the complementary function to $t \mapsto \Phi(K\Psi^{-1}(t))$. One can also consider measures absolutely continuous with respect to some measure instead of weight functions w_1 and w_2. In this manner we get imbedding theorems generalizing [Av], [K], [KP] and the "Orlicz part" in [KL]. We shall omit details.

Remark 3.10. It is natural to ask whether analogous theorems hold, when $L_\Phi(\chi_\Omega)$ and $L_\Phi(w)$ are replaced by $E_\Phi(\chi_\Omega)$ and $E_\Phi(w)$, that is, by the closures of the subset of the bounded functions in $L_\Phi(\chi_\Omega)$ and $L_\Phi(w)$, respectively. Let us observe that according to [I] and [Mu, Theorem 8.5] conditions (3.1) and/or (3.2) imply imbedding statements for the E-spaces analogous to those from Theorem 3.1. We shall show that the converse is true, too. Let, for instance, suppose that $E_\Phi(\chi_\Omega) \hookrightarrow E_\Phi(w)$, $w \geq 1$ a.e. in Ω. We prove that (3.1) is true. To this end it suffices to verify that $L_\Phi(\chi_\Omega) \hookrightarrow L_\Phi(w)$. Let $u \in L_\Phi(\chi_\Omega)$ and assume, without loss of generality that $u \geq 0$ a.e. in Ω. Let us define $u_n(x) = \min(n, u(x))$, $x \in \Omega$, $n \in \mathbb{N}$. Then $u_n \in E_\Phi(\chi_\Omega)$ and $\delta u_n \to \delta u$ in $L_\Phi(\chi_\Omega)$ for a suitable $\delta > 0$ in the modular sense, i.e. $\int_\Omega \Phi(\delta(u(x) - u_n(x)))\,dx \to 0$. We claim that $u \in L_\Phi(w)$. We have

$$\int_\Omega \Phi\left(\frac{\delta(u_m(x) - u_n(x))}{2}\right) \leq \frac{1}{2}\int_\Omega \Phi(\delta(u(x) - u_m(x)))\,dx + \frac{1}{2}\int_\Omega \Phi(\delta(u(x) - u_n(x)))\,dx$$

provided $m, n \to \infty$ and, say, $m > n$. Hence $\|u_m - u_n\|_{E_\Phi(\chi_\Omega)} \leq 2\delta^{-1}$ and consequently also $\|u_m - u_n\|_{E_\Phi(w)} \leq 2c\delta^{-1}$, where c is the norm of the imbedding in (3.1) and m, n are large enough. The set $\{u_m - u_n\}$ is therefore bounded in $L_\Phi(w)$, in particular, there is $M > 0$ such that

$$\int_\Omega \Phi\left(\frac{u_m(x) - u_n(x)}{M}\right) w(x)\,dx \leq 1$$

for all $m, n \in \mathbb{N}$. By Fatou's lemma,

$$\int_\Omega \Phi\left(\frac{u(x) - u_n(x)}{M}\right) w(x)\,dx \leq 1.$$

Hence for any $n \in \mathbb{N}$,

$$\|u\|_{L_\Phi(w)} \leq \|u - u_n\|_{L_\Phi(w)} + \|u_n\|_{L_\Phi(w)} < \infty.$$

We have proved that $u \in L_\Phi(w)$. This means that $L_\Phi(\chi_\Omega)$ is a subset of $L_\Phi(w)$. Since both the spaces $L_\Phi(\chi_\Omega)$ and $L_\Phi(w)$ are Banach function spaces we arrive at $L_\Phi(\chi_\Omega) \hookrightarrow L_\Phi(w)$, which by Theorem 3.1 means that condition (3.1) is true. One can proceed similarly in the case of condition (3.2).

Note that all claims following Theorem 3.1 have been derived from this theorem and inspecting the proofs more closely one can see that no special properties of the E_Φ- spaces have been used. For completeness we shall formulate it separately.

Corollary 3.11. *Theorems 3.1-3.4 and Corollaries 3.5 and 3.6 are true if the L_Φ-spaces are everywhere replaced by the E_Φ-spaces, that is, by closures of the set of the bounded functions in the respective norms.*

Remark 3.12. Let H be a Young function and w a weight in Ω. Define the Musielak-Orlicz function $\Psi(x,t) = w(x)H(t/w(x))$, $t \in \mathbb{R}$, $x \in \Omega$. Then by Corollary 3.11, Theorems 3.1-3.4 and Corollaries 3.5 and 3.6 have their dual counterpart for the couple of spaces $L_H(\chi_\Omega)$ and $L_\Psi(\chi_\Omega)$. This is easy to see since $\Lambda : (x,t) \mapsto H(t)w(x)$ and Ψ are complementary Musielak-Orlicz functions and the space $L_\Psi(\chi_\Omega)$ is the Köthe dual to $E_\Lambda(\chi_\Omega)$ (see [Mu]).

Next we shall consider the problem of a characterization of those Young functions for which there exists an unbounded weight w such that $L_\Phi = L_\Phi(w)$ (an/or $E_\Phi = E_\Phi(w)$). It was easy to show that this is impossible for $\Phi \in \Delta_2(\infty)$ unless the values of w are between two positive constants a.e. in Ω (see Theorem 3.2). Therefore a straightforward conjecture would be that there is a non-effective unbounded weight if and only if $\Phi \notin \Delta_2(\infty)$. Rather surprisingly, this is not true. Such an equality of spaces can occur only for a subclass of non-$\Delta_2(\infty)$ Young functions. We present a necessary and sufficient condition in the next theorem and, additionally, we give an example of $\Phi \notin \Delta_2(\infty)$, for which no non-effective unbounded weight exists.

Theorem 3.13. *Let Φ be a Young function. Then there exists an unbounded weight w such that $L_\Phi(\chi_\Omega) = L_\Phi(w)$ if and only if*

$$\liminf_{t\to\infty} \frac{\Phi(Kt)}{\Phi(t)} = \infty$$

for some $K > 1$.

Proof. By Theorem 3.3 for the equality $L_\Phi(\chi_\Omega) = L_\Phi(w)$ it is necessary that $\int_\Omega S_K(w(x))\,dx < \infty$ for some $K > 1$, where

$$S_K(v) = \sup_{t>0}\{ut - \Phi(K\Phi^{-1}(t))\}.$$

Assume that $\liminf_{t\to\infty} \Phi(Kt)/\Phi(t) < \infty$ for all $K > 1$, equivalently, $\liminf_{t\to\infty} \Phi(K\Phi^{-1}(t))/t < \infty$. So there is $c > 0$ and a sequence $t_n \to \infty$ such that $t_n \to \infty$ and $\lim_{n\to\infty} \Phi(K\Phi^{-1}(t_n))/t_n = c$. Consequently there is $c' \in (0, \infty)$ such that

$$\Phi(K\Phi^{-1}(t_n)) \le c't_n, \qquad n \in \mathbb{N}.$$

Assume that $v > c'$. Then for all $n \in \mathbb{N}$,

$$t_n v - \Phi(K\Phi^{-1}(t_n)) \ge t_n v - c't_n = (v - c')t_n.$$

Therefore

$$\begin{aligned} S_K(v) &= \sup_{u>0}\{uv - \Phi(K\Phi^{-1}(u))\} \ge \sup_n\{t_n v - \Phi(K\Phi^{-1}(t_n))\} \\ &\ge (v - c')\sup_n t_n = \infty. \end{aligned}$$

Hence, necessarily, $w(x) \le c'$.

Conversely, let us assume that $\liminf_{t\to\infty} \Phi(Kt)/\Phi(t) = \infty$. Then we have $\lim_{t\to\infty} \Phi(Kt)/\Phi(t) = \infty$, which is equivalent to

$$\lim_{t\to\infty} \frac{\Phi(K\Phi^{-1}(t))}{t} = \infty. \tag{3.6}$$

Hence for any $u, v > 0$ we have

$$uv - \Phi(K\Phi^{-1}(u)) = u\left(v - \frac{\Phi(K\Phi^{-1}(u))}{u}\right).$$

By (3.6) there is $u_v > 0$ such that

$$\frac{\Phi(K\Phi^{-1}(u))}{u} \ge v, \qquad u \ge u_v.$$

Therefore

$$\sup_{u>0}\{uv - \Phi(K\Phi^{-1}(u))\} = \sup_{u\in[0,u_v]}\{uv - \Phi(K\Phi^{-1}(u))\} < \infty$$

because a continuous function on a compact set is bounded from above. In such a way we have proved that the condition $\liminf_{t\to\infty} \Phi(Kt)/\Phi(t) = \infty$ guarantees that the function S_K complementary to $\Phi(K\Phi^{-1}(u))$ has finite values. Consequently there is an unbounded function w in Ω such that $\int_\Omega S_K(w(x))\,dx < \infty$, which, by virtue of Theorem 3.1 and Corollary 3.11, gives $L_\Phi = L_\Phi(w)$. ☐

We finish this section with presenting an example of a Young function $\Phi \notin \Delta_2(\infty)$ such that $L_\Phi(w) \neq L_\Phi(\chi_\Omega)$ and $E_\Phi(w) \neq E_\Phi(\chi_\Omega)$ for any unbounded weight w.

Example 3.14. There exists a Young function Φ such that for every $K > 1$,

$$\liminf_{t\to\infty} \frac{\Phi(Kt)}{\Phi(t)} < \infty \quad \text{and} \quad \limsup_{t\to\infty} \frac{\Phi(Kt)}{\Phi(t)} = \infty. \tag{3.7}$$

Define $p(t) = \Phi'(t)$ as follows:

$$p(t) = \begin{cases} e^t & \text{on } [(2n)^{2n}, (2n+1)^{2n+1}) = I_{2n},\ n \in \mathbb{N}, \\ \dfrac{e^{(2n+1)^{2n+1}}t}{(2n+1)^{2n+1}} & \text{on } [(2n+1)^{2n+1}, (2n+2)^{2n+2}) = I_{2n+1},\ n \in \mathbb{N}, \end{cases}$$

and define p to be linear and continuous on $[0, 4]$ with $p(0) = 0$. If we show that p is non-decreasing and satisfies condition (3.7) (with p instead of Φ), then we are done. Indeed, (3.7) for p and for Φ are plainly equivalent as we show below.

Given $K > 1$ there is an infinite number of intervals I_{2n} containing points t and Kt such that $p(Kt)/p(t) = e^{(K-1)t}$, which tends to ∞ as $t \to \infty$, and at the same time there is an infinite number of intervals I_{2n+1} containing points t, Kt (it suffices to take t close enough to the left end points of I_{2n+1}); then $p(Kt)/p(t) = K$ at these points.

It remains to show that Φ satisfies (3.7) for some $K > 1$ if p satisfies (3.7), and that p is non-decreasing. But if

$$\liminf_{t\to\infty} \frac{p(Kt)}{p(t)} < \infty \text{ for all } K > 1,$$

then

$$\liminf_{t\to\infty} \frac{\Phi(Kt)}{\Phi(t)} < \infty \text{ for all } K > 1$$

and vice versa. To see that note first that

$$\Phi(Kt) = \int_0^{Kt} p(s)\,ds \geq \int_{\sqrt{K}t}^{Kt} p(s)\,ds \geq (K - \sqrt{K})tp(\sqrt{K}t). \tag{3.8}$$

Hence for any $t > 0$ and $K > 1$,

$$\frac{\Phi(Kt)}{\Phi(t)} \geq \frac{(K-\sqrt{K})tp(\sqrt{K}t)}{tp(t)} = \frac{(K-\sqrt{K})p(\sqrt{K}t)}{p(t)}.$$

Further, it follows from (3.8) that

$$\frac{tp(t)}{\sqrt{K}} \le \frac{\Phi(\sqrt{K}t)}{K-\sqrt{K}}.$$

This yields

$$\frac{p(Kt)}{p(t)} \ge \frac{\Phi(Kt)}{Kt} \cdot \frac{K-\sqrt{K}}{\sqrt{K}} \cdot \frac{t}{\Phi(\sqrt{K}t)} = \frac{(K-\sqrt{K})}{K\sqrt{K}} \cdot \frac{\Phi(Kt)}{\Phi(\sqrt{K}t)}.$$

It remains to prove that p is non-decreasing. The situation is clear at the points $(2n+1)^{2n+1}$. We have to show that

$$p_-((2n+2)^{2n+2}) \le p_+((2n+2)^{2n+2}).$$

Plainly

$$\begin{aligned} p_-((2n+2)^{2n+2}) &\le \frac{\exp[(2n+1)^{2n+1}]}{(2n+1)^{2n+1}}(2n+2)^{2n+2}, \\ p_+((2n+2)^{2n+2}) &= \exp[(2n+2)^{2n+2}]. \end{aligned}$$

Hence we would like to show that

$$\frac{\exp[(2n+1)^{2n+1}]}{(2n+1)^{2n+1}}(2n+2)^{2n+2} \le \exp[(2n+2)^{2n+2}]. \tag{3.9}$$

For this it is enough that

$$(2n+2)^{2n+2}\exp[(2n+1)^{2n+1}] \le (2n+1)^{2n+1}\exp(2n+2)\exp[(2n+2)^{2n+1}]. \tag{3.10}$$

Since $\exp[(2n+1)^{2n+1}] \le \exp[(2n+2)^{2n+1}]$ sufficient for the (3.10) is that

$$\left(\frac{2n+2}{2n+1}\right)^{2n+1}(2n+2) \le \exp(2n+2). \tag{3.11}$$

The first term on the left is majorized by e, and clearly,

$$2n+2 \le e^{-1}e^{2n+2}$$

(which follows from $1+t \le e^t$ for all $t>0$). Hence (3.11) holds and going back we arrive at (3.9).

4 Applications to composition operators

Let $\Omega = (\Omega, \Sigma, \mu)$ be a σ-finite complete and non-atomic measure space and $\tau : \Omega \to \Omega$ be a measurable transformation, that is, $\tau^{-1}(A) \in \Sigma$ for every $A \in \Sigma$. The transformation τ is said to be *non-singular* if $\mu(\tau^{-1}(A)) = 0$ whenever $\mu(A) = 0$. Therefore, the non-singularity of τ gives absolute continuity of $\mu \circ \tau^{-1}$ with respect to μ and in this case there exists a function $w : \Omega \to \mathbb{R}_+$ such that

$$\mu \circ \tau^{-1}(A) = \int_A w(x)\, d\mu(x)$$

for any $A \in \Sigma$, namely, w is the *Radon-Nikodym derivative* of the measure $\mu \circ \tau^{-1}$ with respect to μ, denoted by $d(\mu \circ \tau^{-1})/d\mu$. We shall assume in the following that τ is a non-singular transformation. Any measurable transformation $\tau : \Omega \to \Omega$ generates the *composition operator* c_τ, taking $L^0(\Omega, \Sigma)$ (the space of all μ-measurable functions on Ω) into itself, defined by

$$c_\tau f(x) = f(\tau(x)), \qquad x \in \Omega,\ f \in L^0(\Omega, \Sigma).$$

The problem of continuity of c_τ in Orlicz spaces has been considered in [CHKM]: If $\Phi \in \Delta_2(\mathbb{R}_+)$, then the operator c_τ acts continuously from $L_\Phi(\Omega)$ into itself if and only if

1. there is $K > 1$ such that $\mu(\tau^{-1}(A)) \leq K\mu(A)$ for any $A \in \Sigma$ with $\mu(A) < \infty$.

Moreover, if $\Phi \in \Delta_2(\infty)$ (note that one need not assume that $\mu(\Omega) < \infty$) and if μ is non-atomic, then c_τ takes $L_\Phi(\Omega)$ continuously into itself if and only if

2. there is $K > 1$ such that

$$\Phi^{-1}(1/\mu(A)) \leq K\Phi^{-1}(1/\mu(\tau^{-1}(A)))$$

for any $A \in \Sigma$ with $0 < \mu(a) < \infty$ (note that in the case $\Phi \in \Delta_2(\infty) \setminus \Delta_2(\mathbb{R}_+)$ the condition for continuity of c_τ involves the function Φ).

In the case when μ is non-atomic and $\Phi \in \Delta_2(\infty) \setminus \Delta_2(\mathbb{R}_+)$, then the necessary and sufficient condition for continuity of c_τ from L_Φ into itself has been expressed in terms of the Radon-Nikodym derivative $d(\mu \circ \tau^{-1})/d\mu = w$ (see [CHKM, Theorem 2.5]) and this condition is in fact equivalent to (continuous) imbedding of L_Φ into $L_\Phi(w)$. Invoking results from the previous Section we almost immediately get a more precise characterization than that in Theorem 2.5 from [CHKM]. In addition to that, using our results on weighted imbeddings we can also formulate a necessary and sufficient condition in order that c_τ maps L_Φ continuously onto L_Φ.

Theorem 4.1. *Let (Ω, Σ, μ) be a non-atomic, complete, and finite measure space and $\tau : \Omega \to \Omega$ be a measurable non-singular transformation. Then*

1. *c_τ acts continuously from $L_\Phi(\Omega)$ into itself if and only if there exists $K > 1$ such that $\int_\Omega S_K(w(x))\, d\mu(x) < \infty$, where $w = d(\mu \circ \tau^{-1})/d\mu$ and S_K is the complementary Young function to $\Phi \circ (K\Phi^{-1})$.*
2. *c_τ acts continuously from $L_\Phi(\Omega)$ onto itself if and only if*

$$\int_\Omega S_K(w(x))\, d\mu(x) < \infty \quad \text{and} \quad \int_\Omega S_K(1/w(x))w(x)\, d\mu(x) < \infty, \tag{4.1}$$

where w and S_K are the same as in 1.

Remark 4.2. In the case when $\Phi \in \Delta_2(\infty)$, the function $w = d(\mu \circ \tau^{-1})/d\mu$ is in $L_\infty(\Omega)$ so that the continuity of c_τ follows easily. If $\Phi \notin \Delta_2(\infty)$, we are not able to prove that $w \in L_\infty$. Nevertheless, if $\liminf_{t\to\infty} \Phi(Kt)/\Phi(t) = \infty$ for some $K > 1$, there are unbounded weights such that (4.1) holds. This means that the Radon-Nikodym derivative $w = d(\mu \circ \tau^{-1})/d\mu$ can be unbounded and still giving continuity of the corresponding composition operator acting from an Orlicz space onto itself.

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