Singular Periodic Impulse Problems

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Abstract. Existence principle for the impulsive periodic boundary value problem $u'' + c\,u' = g(x) + e(t)$, $u(t_i+) = u(t_i) + J_i(u,u')$, $u'(t_i+) = u'(t_i) + M_i(u,u')$, $i = 1,\dots,m$, $u(0) = u(T)$, $u'(0) = u'(T)$ is established, where $g \in C(0,\infty)$ can have a strong singularity at the origin. Furthermore, we assume that $0 < t_1 < \dots < t_m < T$, $e \in L_1[0,T]$, $c \in \mathbb{R}$ and J_i, M_i, $i = 1,2,\dots,m$, are continuous mappings of $G[0,T] \times G[0,T]$ into $\mathbb{R}$, where $G[0,T]$ denotes the space of functions regulated on $[0,T]$.

The principle is based on an averaging procedure similar to that introduced by Manásevich and Mawhin for singular periodic problems with p – Laplacian in [11].

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1 Preliminaries

Starting with Hu and Lakshmikantham [7], periodic boundary value problems for nonlinear second order impulsive differential equations of the form

$$\begin{aligned}
&u'' = f(t,u,u'),\\
&\begin{cases} u(t_i+) = u(t_i) + J_i(u(t_i)), \\ u'(t_i+) = u'(t_i) + M_i(u'(t_i)), \end{cases} \qquad i = 1,2,\dots,m,\\
&u(0) = u(T), \quad u'(0) = u'(T)
\end{aligned}$$

have been studied by many authors. Usually it is assumed that the function $f\colon [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ fulfils the Carathéodory conditions,

$$0 < t_1 < t_2 < \dots < t_m < T \ \text{ are fixed points of the interval } [0,T] \tag{1.1}$$

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and J_i, $M_i : \mathbb{R} \to \mathbb{R}$, $i = 1, 2, \ldots, m$, are continuous functions and fulfil some monotonicity type conditions. A rather representative (however not complete) list of related papers is given in references. In particular, in [2], [3], [5], [9], [10] existence results in terms of lower/upper functions obtained by the monotone iterative method can be found. All of these results impose monotonicity of the impulse functions and existence of an associated pair of well-ordered lower/upper functions. The papers [4] and [30] are based on the method of bound sets, however the effective criteria contained therein correspond to the situation when there is a well-ordered pair of constant lower and upper functions. Existence results which apply also to the case when a pair of lower and upper functions which need not be well-ordered is assumed were provided only by Rachůnková and Tvrdý, see [18], [20]–[22]. Analogous results for impulsive problems with quasilinear differential operator were delivered by Rachůnková and Tvrdý in [23]–[25]. When no impulses are acting, periodic problems with singularities have been treated by many authors. For rather representative overview and references, see e.g. [15] or [16]. To our knowledge, up to now singular periodic impulsive problems have not been treated. For singular Dirichlet impulsive problems we refer to the papers by Rachůnková [14], Rachůnková and Tomeček [17] and Lee and Liu [8].

In this paper we establish an existence principle suitable for finding positive solutions to impulsive periodic problems of a more general form

$$u'' = f(t, u, u'), \tag{1.2}$$

$$\begin{cases} u(t_i+) = u(t_i) + J_i(u, u'), \\ u'(t_i+) = u'(t_i) + M_i(u, u'), \quad i = 1, 2, \ldots, m, \end{cases} \tag{1.3}$$

$$u(0) = u(T), \quad u'(0) = u'(T), \tag{1.4}$$

where J_i, M_i are continuous functionals and f can have a singularity for $u = 0$.

1.1. Notation. Throughout the paper we keep the following notation and conventions: for a real valued function u defined a.e. on $[0, T]$, we put

$$\|u\|_\infty = \sup \operatorname{ess}_{t \in [0,T]} |u(t)| \quad \text{and} \quad \|u\|_1 = \int_0^T |u(s)| \, \mathrm{d}s.$$

For a given interval $J \subset \mathbb{R}$, by $C(J)$ we denote the set of real valued functions which are continuous on J. Furthermore, $C^1(J)$ is the set of functions having continuous first derivatives on J and $L_1(J)$ is the set of functions which are Lebesgue integrable on J.

Any function $x : [0,T] \to \mathbb{R}$ which possesses finite limits

$$x(t+) = \lim_{\tau \to t+} x(\tau) \quad \text{and} \quad x(s-) = \lim_{\tau \to s-} x(\tau)$$

for all $t \in [0,T)$ and $s \in (0,T]$ is said to be regulated on $[0,T]$. The linear space of functions regulated on $[0,T]$ is denoted by $G[0,T]$. It is well known that $G[0,T]$ is a Banach space with respect to the norm $x \in G[0,T] \to \|x\|_\infty$ (cf. [6, Theorem I.3.6]).

Let $m \in \mathbb{N}$ and let $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = T$ be a division of the interval $[0,T]$. We denote $D = \{t_1, t_2, \ldots, t_m\}$ and define $C^1_D[0,T]$ as the set of functions $u\colon [0,T] \to \mathbb{R}$ such that

$$u(t) = \begin{cases} u_{[0]}(t) & \text{if } t \in [0,t_1], \\ u_{[1]}(t) & \text{if } t \in (t_1,t_2], \\ \vdots & \vdots \\ u_{[m]}(t) & \text{if } t \in (t_m,T], \end{cases}$$

where $u_{[i]} \in C^1[t_i, t_{i+1}]$ for $i = 0,1,\ldots,m$. In particular, if $u \in C^1_D[0,T]$, then u' possesses finite one-sided limits

$$u'(t-) := \lim_{\tau \to t-} u'(\tau) \quad \text{and} \quad u'(s+) := \lim_{\tau \to s+} u'(\tau)$$

for each $t \in (0,T]$ and $s \in [0,T)$. Moreover, $u'(t-) = u'(t)$ for all $t \in (0,T]$ and $u'(0+) = u'(0)$. For $u \in C^1_D[0,T]$ we put

$$\|u\|_D = \|u\|_\infty + \|u'\|_\infty.$$

Then $C^1_D[0,T]$ becomes a Banach space when endowed with the norm $\|.\|_D$. Furthermore, by $AC^1_D[0,T]$ we denote the set of functions $u \in C^1_D[0,T]$ having first derivatives absolutely continuous on each subinterval (t_i, t_{i+1}), $i = 1,2,\ldots,m+1$.

We say that $f : [0,T] \times \mathbb{R}^2 \mapsto \mathbb{R}$ satisfies the *Carathéodory conditions* on $[0,T] \times \mathbb{R}^2$ if (i) for each $x \in \mathbb{R}$ and $y \in \mathbb{R}$ the function $f(.,x,y)$ is measurable on $[0,T]$; (ii) for almost every $t \in [0,T]$ the function $f(t,.,.)$ is continuous on $\mathbb{R}^2$; (iii) for each compact set $K \subset \mathbb{R}^2$ there is a function $m_K(t) \in L[0,T]$ such that $|f(t,x,y)| \le m_K(t)$ holds for a.e. $t \in [0,T]$ and all $(x,y) \in K$. The set of functions satisfying the Carathéodory conditions on $[0,T] \times \mathbb{R}^2$ is denoted by $Car([0,T] \times \mathbb{R}^2)$.

Given a subset Ω of a Banach space X, its closure is denoted by $\overline{\Omega}$. As usual, the symbol I stands for the identity operator or the identity matrix. Finally, we will write $\bar{e}$ instead of $\dfrac{1}{T}\displaystyle\int_0^T e(s)\,\mathrm{d}s$ and $\Delta^+ u(t)$ instead of $u(t+) - u(t)$.

If $f \in Car([0,T] \times \mathbb{R}^2)$, problem (1.2)–(1.4) is said to be *regular* and a function $u \in AC^1_D[0,T]$ is its solutions if

$$u''(t) = f(t,u(t),u'(t)) \quad \text{holds for a.e. } t \in [0,T]$$

and conditions (1.3) and (1.4) are satisfied. If $f \notin Car([0,T] \times \mathbb{R}^2)$, problem (1.2)–(1.4) is said to be *singular*.

In this paper we will deal with rather simplified, however the most typical, case of the singular problem with

$$f(t,x,y) = c\,y + g(x) + e(t) \text{ for } x \in (0,\infty),\ y \in \mathbb{R} \text{ and a.e. } t \in [0,T],$$

where

$$c \in \mathbb{R}, \quad g \in C(0,\infty), \quad e \in L_1[0,T]. \tag{1.5}$$

1.2. Definition. A function $u \in AC^1_D[0,T]$ is called a solution of problem

$$u'' + c\,u' = g(u) + e(t), \quad (1.3),\ (1.4) \tag{1.6}$$

if $u > 0$ a.e. on $[0,T]$,

$$u''(t) + c\,u'(t) = g(u(t)) + e(t) \quad \text{for a.e. } t \in [0,T],$$

and conditions (1.3) and (1.4) are satisfied.

2 Green's functions and operator representations for impulsive two-point boundary value problems

For our purposes an appropriate choice of the operator representation of (1.2)–(1.4) is important. To this aim, let us consider the following impulsive problem with nonlinear two-point boundary conditions

$$u'' + a_2(t)\,u' + a_1(t)\,u = f(t,u,u') \text{ a.e. on } [0,T], \tag{2.1}$$

$$\Delta^+ u(t_i) = J_i(u,u'), \quad \Delta^+ u'(t_i) = M_i(u,u'), \quad i = 1,2,\ldots,m, \tag{2.2}$$

$$P\begin{pmatrix} u(0) \\ u'(0) \end{pmatrix} + Q\begin{pmatrix} u(T) \\ u'(T) \end{pmatrix} = R(u,u'), \tag{2.3}$$

and its linearized version

$$u'' + a_2(t)\,u' + a_1(t)\,u = h(t) \text{ a.e. on } [0,T], \tag{2.4}$$

$$\Delta^+ u(t_i) = d_i, \quad \Delta^+ u'(t_i) = d_i', \quad i = 1,2,\ldots,m, \tag{2.5}$$

$$P\begin{pmatrix} u(0) \\ u'(0) \end{pmatrix} + Q\begin{pmatrix} u(T) \\ u'(T) \end{pmatrix} = \delta, \tag{2.6}$$

where

$$\begin{cases} J_i \text{ and } M_i\colon\ G[0,T]\times G[0,T] \to \mathbb{R},\ i=1,2,\ldots,m, \\ \qquad\qquad\qquad\qquad \text{are continuous mappings,} \\ J_i(u,0) = M_i(u,0) = 0 \quad \text{for } u \in G[0,T] \text{ and } i=1,2,\ldots,m \end{cases} \tag{2.7}$$

and

$$\begin{cases} a_1,\ h \in L[0,T],\ a_2 \in C[0,T],\ f \in Car([0,T]\times\mathbb{R}^2), \\ \delta \in \mathbb{R}^2,\ d_i,\ d_i' \in \mathbb{R},\ i = 1,2,\ldots,m, \\ P,\ Q \text{ are real } 2\times 2 - \text{matrices, } \operatorname{rank}(P,Q) = 2, \\ R\colon G[0,T]\times G[0,T] \to \mathbb{R}^2 \text{ is a continuous mapping.} \end{cases} \tag{2.8}$$

Solutions of problems (2.1)–(2.3) and (2.4)–(2.6) are defined in a natural way quite analogously to the above mentioned definition of regular periodic problems. Problem (2.4)–(2.6) is equivalent to the two-point problem for a special case of generalized linear differential systems of the form

$$x(t) - x(0) - \int_0^t A(s)\, x(s)\, \mathrm{d}s = b(t) - b(0) \quad \text{for } t \in [0,T], \tag{2.9}$$

$$P\, x(0) + Q\, x(T) = \delta, \tag{2.10}$$

where

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} u(t) \\ u'(t) \end{pmatrix}, \quad A(t) = \begin{pmatrix} 0 & 1 \\ -a_1(s) & -a_2(s) \end{pmatrix}, \tag{2.11}$$

$$b(t) = \int_0^t \begin{pmatrix} 0 \\ h(s) \end{pmatrix} \mathrm{d}s + \sum_{i=1}^m \begin{pmatrix} d_i \\ d_i' \end{pmatrix} \chi_{(t_i,T]}(t), \quad t \in [0,T],$$

and $\chi_{(t_i,T]}(t) = 1$ if $t \in (t_i,T]$, $\chi_{(t_i,T]}(t) = 0$ otherwise. Solutions of (2.9), (2.10) are 2 -vector functions of bounded variation on $[0,T]$ satisfying the two-point condition (2.10) and fulfilling the integral equation (2.9) for all $t \in [0,T]$, cf. e.g. [28]. Assume that the homogeneous problem

$$u'' + a_2(t)\, u' + a_1(t)\, u = 0, \quad P \begin{pmatrix} u(0) \\ u'(0) \end{pmatrix} + Q \begin{pmatrix} u(T) \\ u'(T) \end{pmatrix} = 0 \tag{2.12}$$

has only the trivial solution. Then, obviously, the homogeneous problem corresponding to (2.9), (2.10) has also only the trivial solution. In view of [29, Theorems 4.2 and 4.3] (see also [27, Theorem 4.1]), problem (2.9), (2.10) has a unique solution x and it is given by

$$x(t) = X(t)\, D^{-1}\, \delta + \int_0^T \Gamma(t,s)\, \mathrm{d}[b(s)], \quad t \in [0,T], \tag{2.13}$$

where X is the fundamental matrix solution of the homogeneous equation $x' - A(t)\, x = 0$ fulfilling the condition $X(0) = I$, $D = P\, X(0) + Q\, X(T)$ and

$$\Gamma(t,s) = (\gamma_{i,j}(t,s))_{i,j=1,2}$$

is Green's matrix for the problem

$$x' - A(t)\, x = 0, \quad P\, x(0) + Q\, x(T) = 0.$$

Recall that, for each $s \in (0,T)$, the matrix function $t \to \Gamma(t,s)$ is absolutely continuous on $[0,T] \setminus \{s\}$ and

$$\frac{\partial}{\partial t}\Gamma(t,s) - A(t)\,\Gamma(t,s) = 0 \quad \text{for a.e. } \ t \in [0,T],$$

$$P\,\Gamma(0,s) + Q\,\Gamma(T,s) = 0,$$

$$\Gamma(t+,t) - \Gamma(t-,t) = I \quad \text{for } \ t \in (0,T).$$

Moreover, the component $\gamma_{1,2}$ of Γ is absolutely continuous on $[0,T]$ for each $s \in (0,T)$ and

$$\frac{\partial}{\partial t}\gamma_{1,2}(t,s) = \gamma_{2,2}(t,s) \quad \text{for a.e. } \ t \in [0,T].$$

Denote $G(t,s) = \gamma_{1,2}(t,s)$. Then $G(t,s)$ is Green's function of (2.12). Furthermore, we have

$$\frac{\partial}{\partial s}\Gamma(t,s) = -\Gamma(t,s)\,A(s) \quad \text{for all } \ t \in (0,T) \text{ and a.e. } s \in [0,T].$$

In particular,

$$\gamma_{1,1}(t,s) = -\frac{\partial}{\partial s}G(t,s) + a_1(s)\,G(t,s) \quad \text{for all } \ t \in [0,T] \text{ and a.e. } s \in [0,T].$$

Inserting (2.11) into (2.13) we get that, for each $h \in L[0,T]$, $c, d_i, d'_i \in R$, $i = 1, 2, \ldots, m$, the unique solution u of problem (2.4)–(2.6) is given by

$$\begin{cases} u(t) = U(t)\,\delta + \displaystyle\int_0^T G(t,s)\,h(s)\,\mathrm{d}s \\ \qquad\quad + \displaystyle\sum_{i=1}^m \Big(-\frac{\partial}{\partial s}G(t,t_i) + a_1(t)\,G(t,t_i)\Big)\,d_i + \sum_{i=1}^m G(t,t_i)\,d'_i \\ \qquad\quad \text{for } \ t \in [0,T], \end{cases} \tag{2.14}$$

where $U(t) = (u_{11}(t), u_{12}(t))$ is the first row of the matrix $X(t)\,D^{-1}$. Now, choose an arbitrary $w \in C^1_D[0,T]$ and put

$$\begin{cases} h(t) = f(t, w(t), w'(t)) \quad \text{for a.e. } \ t \in [0,T], \\ d_i = J_i(w,w'), \ d'_i = M_i(w,w'), \ i = 1, 2, \ldots, m, \\ \delta = R(w,w'). \end{cases}$$

Then $h \in L[0,T]$, $c, d_i, d'_i \in \mathbb{R}, i = 1, 2, \dots, m,$ and there is a unique $u \in AC^1_D[0,T]$ fulfilling (2.4)–(2.6) and it is given by (2.14). Therefore, we conclude that $u \in C^1_D[0,T]$ is a solution to (2.1)–(2.3) if and only if

$$\begin{cases} u(t) = U(t)\, R(u,u') + \displaystyle\int_0^T G(t,s)\, f(s,u(s),u'(s))\, \mathrm{d}s \\ \qquad + \displaystyle\sum_{i=1}^m \left(-\frac{\partial}{\partial s} G(t,t_i) + a_1(t)\, G(t,t_i)\right) J_i(u,u') \\ \qquad + \displaystyle\sum_{i=1}^m G(t,t_i)\, M_i(u,u') \quad \text{for } \ t \in [0,T]. \end{cases} \tag{2.15}$$

Let us define operators F_1 and $F_2\colon C^1_D[0,T] \to C^1_D[0,T]$ by

$$F_1(u)(t) = \int_0^T G(t,s)\, f(s,u(s),u'(s))\, \mathrm{d}s, \quad t \in [0,T]$$

and

$$F_2(u)(t) = U(t)\, R(u,u') + \sum_{i=1}^m \left(-\frac{\partial}{\partial s} G(t,t_i) + a_1(t)\, G(t,t_i)\right) J_i(u,u') + \sum_{i=1}^m G(t,t_i)\, M_i(u,u'), \quad t \in [0,T].$$

The former one, F_1, is a composition of the Green type operator

$$h \in L_1[0,T] \to \int_0^T G(t,s)\, h(s)\, \mathrm{d}s \in C^1[0,T],$$

which is known to map equiintegrable subsets[1] of $L_1[0,T]$ onto relatively compact subsets of $C^1[0,T] \subset C^1_D[0,T]$, and of the superposition operator generated by $f \in Car([0,T] \times \mathbb{R}^2)$, which, similarly to the classical setting, maps bounded subsets of $C^1_D[0,T]$ to equiintegrable subsets of $L_1[0,T]$. Therefore, it is easy to see that F_1 is completely continuous. Furthermore, since $R, J_i, M_i, i = 1, 2, \dots, m,$ are continuous mappings, the operator F_2 is continuous as well. Having in mind that F_2 maps bounded sets onto bounded sets and its values are contained in a $2(m+1)$ -dimensional subspace[2] of $C^1_D[0,T]$, we conclude that the operators F_2 and $F = F_1 + F_2$ are completely continuous as well.

[1] i.e. sets of functions having a common integrable majorant

[2] i.e. spanned over the set $\{u_{11}, u_{12}, \delta, G(.,t_i), \left(-\frac{\partial}{\partial s} G(.,t_i) + a_1 G(.,t_i)\right), i = 1, \dots, m\}$

So, we have the following assertion.

2.1. Proposition. *Assume* (1.1), (2.7) *and* (2.8). *Furthermore, let problem* (2.12) *have Green's function* $G(t,s)$ *and let* $U \in AC^1_D[0,T]$ *have the same meaning as in* (2.16). *Then* $u \in AC^1_D$ *is a solution to* (2.1)–(2.3) *if and only if* $u = F(u)$, *where* $F : C^1_D[0,T] \to C^1_D[0,T]$ *is the completely continuous operator given by*

$$\begin{cases} F(u)(t) = U(t)\, R(u,u') \\ \quad + \displaystyle\int_0^T G(t,s)\,(f(t,u(s),u'(s)) - a_1(s)\,u(s) - a_2(s)\,u'(s))\,\mathrm{d}s \\ \quad + \displaystyle\sum_{i=1}^m \left(-\frac{\partial}{\partial s}\,G(t,t_i) + a_1(t) G(t,t_i)\right) J_i(u,u') \\ \quad + \displaystyle\sum_{i=1}^m G(t,t_i)\,M_i(u,u'),\ \ t \in [0,T]. \end{cases} \tag{2.16}$$

In particular, if $a_1(t) = a_2(t) = 0$ on $[0,T]$,

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

then problem (2.12) reduces to the simple Dirichlet problem

$$u'' = 0, \quad u(0) = u(T) = 0$$

and its Green's function is well-known:

$$G(t,s) = \begin{cases} \dfrac{s\,(t-T)}{T} & \text{if } 0 \le s < t \le T, \\ \dfrac{t\,(s-T)}{T} & \text{if } 0 \le t \le s \le T \end{cases} \tag{2.17}$$

and

$$\frac{\partial}{\partial s}\,G(t,s) = \begin{cases} \dfrac{T-t}{T} & \text{if } 0 \le s < t \le T, \\ -\dfrac{t}{T} & \text{if } 0 \le t \le s \le T. \end{cases}$$

Furthermore, it is easy to verify that

$$X(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \quad \text{for} \ \ t \in [0,T], \quad D^{-1} = \frac{1}{T} \begin{pmatrix} T & 0 \\ -1 & 1 \end{pmatrix}$$

and

$$U(t) = \frac{1}{T}\,(T-t,t)\,.$$

Consequently,

$$U(t)\,\delta = d \quad \text{holds for each} \ \ d \in \mathbb{R} \ \ \text{and} \ \ \delta = \begin{pmatrix} d \\ d \end{pmatrix}.$$

Now, notice that the periodic boundary conditions (1.4) can be reformulated as

$$u(0) = u(0) + u'(0) - u'(T), \quad u(T) = u(0) + u'(0) - u'(T),$$

i.e., in the form (2.3), where

$$R(u,v) = \begin{pmatrix} u(0) + v(0) - v(T) \\ u(0) + v(0) - v(T) \end{pmatrix} \quad \text{for} \ \ u, v \in G[0,T].$$

In particular,

$$U(t)\,R(u,u') = u(0)+u'(0)-u'(T) \quad \text{for each} \ \ t \in [0,T] \text{ and each} \ \ u \in G[0,T].$$

To summarize, the following assertion is a corollary of Proposition 2.1:

2.2. Corollary. *Assume* (1.1), (2.7) *and* (2.8) *and let the function* $G(t,s)$ *be given by* (2.17). *Then* $u \in AC^1_D$ *is a solution to* (1.2)–(1.4) *if and only if* $u = F(u)$, *where* $F : C^1_D[0,T] \to C^1_D[0,T]$ *is the completely continuous operator given by*

$$\left\{ \begin{aligned} &(Fu)(t) = u(0) + u'(0) - u'(T) + \int_0^T G(t,s)\,f(t,u(s),u'(s))\,\mathrm{d}s \\ &\quad - \sum_{i=1}^{m} \frac{\partial}{\partial s} G(t,t_i)\,J_i(u,u') \\ &\quad + \sum_{i=1}^{m} G(t,t_i)\,M_i(u,u'), \ \ t \in [0,T]. \end{aligned} \right. \tag{2.18}$$

2.3. Remark. Similarly, $u \in AC^1_D$ is a solution to the impulsive Dirichlet problem (1.2), (1.3), $u(0) = u(T) = c$ if and only if $u = F_{dir}\, u$, where

$$\begin{cases} (F_{dir}u)(t) = c + \displaystyle\int_0^T G(t,s)\, f(t,u(s),u'(s))\,\mathrm{d}s \\ \qquad - \displaystyle\sum_{i=1}^m \frac{\partial}{\partial s} G(t,t_i)\, J_i(u,u') + \sum_{i=1}^m G(t,t_i)\, M_i(u,u'), \ t \in [0,T]. \end{cases}$$

3 Existence principle

3.1. Theorem. *Let assumptions* (1.1) , (1.5) *and* (2.7) *hold. Furthermore, assume that there exist* $r \in (0,\infty)$, $R \in (r,\infty)$ *and* $R' \in (0,\infty)$ *such that*

(i) $r < v < R$ *on* $[0,T]$ *and* $||v'||_\infty < R'$ *for each* $\lambda \in (0,1]$ *and for each positive solution* v *of the problem*

$$v''(t) = \lambda\, (-c\,v'(t) + g(v(t)) + e(t)) \quad \textit{for a.e. } t \in [0,T], \tag{3.1}$$

$$\Delta^+ v(t_i) = \lambda\, J_i(v,v'), \qquad i = 1,2,\ldots,m, \tag{3.2}$$

$$\Delta^+ v'(t_i) = \lambda\, M_i(v,v'), \quad i = 1,2,\ldots,m, \tag{3.3}$$

$$v(0) = v(T), \quad v'(0) = v'(T); \tag{3.4}$$

(ii) $(g(x) + \bar{e} = 0) \implies r < x < R$;

(iii) $(g(r) + \bar{e})\ (g(R) + \bar{e}) < 0$.

Then problem (1.6) *has a solution* u *such that*

$$r < u < R \ \ \textit{on } [0,T] \ \ \textit{and} \ \ \|u'\|_\infty < R'.$$

Proof. Step 1. For $\lambda \in [0,1]$ and $v \in C^1_D[0,T]$ denote

$$\begin{cases} \Xi_\lambda(v) = \displaystyle\int_0^T g(v(s))\,\mathrm{d}s + T\,\bar{e} \\ \qquad + \displaystyle\sum_{i=1}^m M_i(v,v') + \lambda\, c \sum_{i=1}^m J_i(v,v'). \end{cases} \tag{3.5}$$

Notice that

$$\Xi_\lambda(v) = 0 \quad \text{holds for all solutions} \;\; v \in C^1_D[0,T] \;\; \text{of (3.1)–(3.4).} \tag{3.6}$$

Indeed, let $v \in C^1_D[0,T]$ be a solution to (3.1)–(3.4). Then

$$\begin{aligned}\int_0^T v''(s)\,\mathrm{d}s &= \sum_{i=0}^m \int_{t_i}^{t_{i+1}} v''(s)\,\mathrm{d}s = \sum_{i=0}^m \big[v'(t_{i+1}) - v'(t_i+)\big] \\ &= v'(T) - v'(0) - \sum_{i=1}^m \Delta^+ v'(t_i) = -\lambda \sum_{i=1}^m M_i(v,v')\end{aligned}$$

and

$$\begin{aligned}\int_0^T c\,v'(s)\,\mathrm{d}s &= c\sum_{i=0}^m \int_{t_i}^{t_{i+1}} v'(s)\,\mathrm{d}s = c\sum_{i=0}^m \big[v(t_{i+1}) - v(t_i+)\big] \\ &= c\Big[v(T) - v(0) - \sum_{i=1}^m \Delta^+ v(t_i)\Big] = -\lambda\, c \sum_{i=1}^m J_i(v,v').\end{aligned}$$

Thus, integrating (3.1) over $[0,T]$ gives (3.6).

Step 2. Consider system (3.7), (3.2), (3.4), where (3.7) is the functional-differential equation

$$v'' = \lambda\,[-c\,v' + g(v) + e(t)] + (1-\lambda)\,\frac{1}{T}\,\Xi_\lambda(v). \tag{3.7}$$

Due to (3.6), we can see that for each $\lambda \in [0,1]$ the problems (3.1)–(3.4) and (3.7), (3.2)–(3.4) are equivalent. Moreover, for $\lambda = 1$, problem (3.7), (3.2), (3.4) reduces to the given problem (1.6) (with u replaced by v).

Now, notice that in view of (2.17) we have

$$\int_0^T G(t,s)\,\mathrm{d}s = \frac{1}{2}\,t\,(t-T) \quad \text{for } t \in [0,T]$$

and define for $\lambda \in [0,1]$, $u \in C^1_D[0,T]$, $u > 0$ on $[0,T]$, and $t \in [0,T]$

$$\begin{cases} F_\lambda(u)(t) = u(0) + u'(0) - u'(T) \\ \quad +\lambda \displaystyle\int_0^T G(t,s)\, [-cu'(s) + g(u(s)) + e(s)]\, \mathrm{d}s \\ \quad +(1-\lambda)\, \dfrac{t\,(t-T)}{2T}\, \Xi_\lambda(u) \\ \quad -\lambda \displaystyle\sum_{i=1}^m \frac{\partial}{\partial s} G(t,t_i)\, J_i(u,u') + \lambda \sum_{i=1}^m G(t,t_i)\, M_i(u,u'). \end{cases} \tag{3.8}$$

In particular, if $\lambda = 0$, then

$$F_0(u)(t) = u(0) + u'(0) - u'(T) + \frac{t\,(t-T)}{2T}\, \Xi_0(u) \quad \text{for } t \in [0,T].$$

Let us put

$$\Omega = \{u \in C^1_D[0,T] \colon\ r < u < R \text{ on } [0,T] \text{ and } \|u'\|_\infty < R'\}.$$

Arguing similarly to the regular case (see Corollary 2.2), we can conclude that for each $\lambda \in [0,1]$ the operator $F_\lambda \colon \overline{\Omega} \subset C^1_D[0,T] \to C^1_D[0,T]$ is completely continuous and a function $v \in \overline{\Omega}$ is a solution of (3.7), (3.2)–(3.4) if and only if it is a fixed point of F_λ. In particular,

$$u \in \overline{\Omega} \text{ is a solution to (1.6) if and only if } F_1(u) = u. \tag{3.9}$$

Step 3. We will show that

$$F_\lambda(u) \neq u \quad \text{for all } u \in \partial\Omega \quad \text{and} \quad \lambda \in [0,1]. \tag{3.10}$$

Indeed, for $\lambda \in (0,1]$ relation (3.10) follows immediately from assumption (i), while for $\lambda = 0$ it is a corollary of assumption (ii) and of the following claim.

Claim. *$u \in \overline{\Omega}$ is a fixed point of F_0 if and only if there is $x \in \mathbb{R}$ such that $u(t) \equiv x$ on $[0,T]$, $x \in (r,R)$ and*

$$g(x) + \bar{e} = 0. \tag{3.11}$$

Proof of Claim. Let $u \in \overline{\Omega}$ be a fixed point of F_0, i.e.

$$u(t) = u(0) + u'(0) - u'(T) + \frac{t\,(t-T)}{2T}\, \Xi_0(u) \quad \text{for all } t \in [0,T]. \tag{3.12}$$

Inserting $t = 0$ into (3.12), we get $u(0) = u(0) + u'(0) - u'(T)$, which implies that $u'(0) = u'(T)$. Similarly, inserting $t = T$ we get $u(T) = u(0)$. Furthermore,

$$u'(t) = \frac{2\,t - T}{2T}\,\Xi_0(u) \quad \text{for } \ t \in [0,T].$$

Since $u'(0) = u'(T)$, it follows that $\Xi_0(u) = 0$. This means that u is constant on $[0,T]$. Denote $x = u(0)$. Then $0 = \Xi_0(u) = T\,(g(x) + \bar{e})$, i.e., (3.11) is true. On the other hand, it is easy to see that if $x \in \mathbb{R}$ is such that (3.11) holds and $u(t) \equiv x$ on $[0,T]$, then $u \in \overline{\Omega}$ is a fixed point of F_0. This completes the proof of Claim.

Step 4. By Step 3 and by the invariance under homotopy property of the topological degree, we have

$$\deg(I - F_1, \Omega) = \deg(I - F_0, \Omega). \tag{3.13}$$

Step 5. Let us denote

$$\mathbb{X} = \{u \in C^1_D[0,T] : \, u(t) \equiv u(0) \text{ on } \ [0,T]\} \quad \text{and} \quad \Omega_0 = \Omega \cap \mathbb{X}.$$

Notice that $\Omega_0 = \{u \in \mathbb{X} \colon \, r < u(0) < R\}$ and $\overline{\Omega}_0 = \{u \in \mathbb{X} \colon \, r \le u(0) \le R\}$. By Claim in Step 3, all fixed points of F_0 belong to Ω_0. Hence, by the excision property of the topological degree we have

$$\deg(I - F_0, \Omega) = \deg(I - F_0, \Omega_0). \tag{3.14}$$

Step 6. Define

$$\begin{cases} \widetilde{F}_\mu(u)(t) = u(0) + \Big[1 - \mu + \dfrac{\mu}{2}\,t\,(t - T)\Big]\,\big(g(u(0) + \bar{e}\big) \\ \qquad\qquad\qquad\qquad \text{for } \ t \in [0,T],\, u \in \overline{\Omega}_0 \ \text{ and } \ \mu \in [0,1]. \end{cases} \tag{3.15}$$

We have

$$\widetilde{F}_0(u) = u(0) + g(u(0)) + \bar{e} \quad \text{and} \quad \widetilde{F}_1(u) = F_0(u) \quad \text{for each } \ u \in \mathbb{X}.$$

Similarly to F_λ, the operators $\widetilde{F}_\mu$, $\mu \in [0,1]$, are also completely continuous and, by Claim in Step 3, we have

$$\widetilde{F}_1(u) \ne u \quad \text{for all} \quad u \in \partial\,\Omega_0.$$

Let i and i_{-1} be respectively the natural isometrical isomorphism $\mathbb{R} \to \mathbb{X}$ and its inverse, i.e.

$$i(x)(t) \equiv u \quad \text{for} \quad x \in \mathbb{R} \qquad \text{and} \qquad i_{-1}(u) = u(0) \quad \text{for} \quad u \in \mathbb{X},$$

and assume that $\mu \in [0,1)$, $x \in (0,\infty)$, $u = i(x)$ and $\widetilde{F}_\mu(u) = u$. Then

$$\Big[1 - \mu + \frac{\mu}{2}\, t\,(T-t)\Big]\,\big(g(x) + \bar{e}\big) = 0 \quad \text{for all} \quad t \in [0,T].$$

If $t = 0$, this relation reduces to $g(x) + \bar{e} = 0$, which is due to assumption (ii) possible only if $x \in (r,R)$. To summarize, we have

$$\widetilde{F}_\mu(u) \neq u \quad \text{for all} \quad u \in \partial\,\Omega_0 \quad \text{and all} \quad \mu \in [0,1].$$

Hence, using the invariance under homotopy property of the topological degree and taking into account that $\dim \mathbb{X} = 1$, we conclude that

$$\deg(I - F_0, \Omega_0) = \deg(I - \widetilde{F}_1, \Omega_0) = \mathrm{d}_B(I - \widetilde{F}_0, \Omega_0), \tag{3.16}$$

where $\mathrm{d}_B(I - \widetilde{F}_0, \Omega_0)$ stands for the Brouwer degree of $I - \widetilde{F}_0$ with respect to the set Ω_0 (and the point 0).

Step 7. Define $\Phi\colon x \in (0,\infty) \to g(x) + \bar{e} \in \mathbb{R}$. Then

$$(I - \widetilde{F}_0)(i(x)) = i(\Phi(x)) \quad \text{for each} \quad x \in (0,\infty).$$

In other words, $\Phi = i_{-1} \circ (I - \widetilde{F}_0) \circ i$ on $(0,\infty)$. Consequently,

$$\mathrm{d}_B(I - \widetilde{F}_0, \Omega_0) = \mathrm{d}_B(\Phi, (r,R)). \tag{3.17}$$

Now, put

$$\Psi(x) = \Phi(r)\,\frac{R-x}{R-r} + \Phi(R)\,\frac{x-r}{R-r}\,.$$

We can see that Ψ has a unique zero $x_0 \in (r,R)$ and

$$\Psi'(x_0) = \frac{\Phi(R) - \Phi(r)}{R-r}.$$

Hence, by the definition of the Brouwer degree in $\mathbb{R}$ we have

$$\mathrm{d}_B(\Psi, (r,R)) = \operatorname{sign} \Psi'(x_0) = \operatorname{sign}\left(\Phi(R) - \Phi(r)\right).$$

By the homotopy property and thanks to our assumption (iii), we conclude that

$$\mathrm{d}_B(\Phi,(r,R)) = \mathrm{d}_B(\Psi,(r,R)) = \mathrm{sign}\,(\Phi(R)-\Phi(r)) \neq 0. \tag{3.18}$$

STEP 8. To summarize, by (3.13)–(3.18) we have

$$\deg(I-F_1,\Omega)\neq 0,$$

which, in view of the existence property of the topological degree, shows that F_1 has a fixed point $u\in\Omega$. By STEP 1 this means that problem (1.6) has a solution. □

References

[1] BAI CHUANZHI AND YANG DANDAN. Existence of solutions for second order nonlinear impulsive differential equations with periodic boundary value conditions. *Boundary Value Problems*, to appear.

[2] D. BAINOV AND P. SIMEONOV. *Impulsive Differential Equations: Periodic Solutions and Applications.* Longman Sci. Tech., Harlow, 1993.

[3] A. CABADA, J. J. NIETO, D. FRANCO AND S. I. TROFIMCHUK. A generalization of the monotone method for second order periodic boundary value problem with impulses at fixed points. *Dynam. Contin. Discrete Impuls. Systems* **7** (2000), 145–158.

[4] DONG YUJUN Periodic solutions for second order impulsive differential systems. *Nonlinear Anal.* **27** (1996), 811–820.

[5] L. H. ERBE AND LIU XINZHI. Existence results for boundary value problems of second order impulsive differential equations. *J. Math. Anal. Appl.* **149** (1990), 56–69.

[6] CH.S. HÖNIG. *Volterra Stieltjes-Integral Equations.* North Holland and American Elsevier, Mathematics Studies 16, Amsterdam and New York, 1975.

[7] HU SHOUCHUAN AND V. LAKSMIKANTHAM. Periodic boundary value problems for second order impulsive differential systems. *Nonlinear Anal.* **13** (1989), 75–85.

[8] LEE YONG-HOON AND LIU XINZHI. Study of singular boundary value problem for second order impulsive differential equations. *J, Math. Anal. Appl.* **331** (2007), 159–176.

[9] E. LIZ AND J. J. NIETO. Periodic solutions of discontinuous impulsive differential systems. *J. Math. Anal. Appl.* **161** (1991), 388–394.

[10] E. LIZ AND J. J. NIETO. The monotone iterative technique for periodic boundary value problems of second order impulsive differential equations. *Comment. Math. Univ. Carolin.* **34** (1993), 405–411.

[11] R. MANÁSEVICH AND J. MAWHIN. Periodic solutions for nonlinear systems with p-Laplacian-like operators. *J. Differential Equations* **145** (1998), 367–393.

[12] J. Mawhin. Topological degree methods in nonlinear boundary value problems. In: *Regional Conference Series in Mathematics.* No.40. R.I.: The American Mathematical Society (AMS). 1979, 122 p.

[13] J. Mawhin. Topological Degree and Boundary Value Problems for Nonlinear Differential Equations. In: *Topological methods for ordinary differential equations.* (M. Furi and P. Zecca, eds.) Lect. Notes Math. 1537, Springer, Berlin, 1993, pp. 73–142.

[14] I. Rachůnková. Singular Dirichlet second order boundary value problems with impulses. *J. Differential Equations* **193** (2003), 435–459.

[15] I. Rachůnková, S. Staněk and M. Tvrdý. Singularities and Laplacians in Boundary Value Problems for Nonlinear Ordinary Differential Equations. In: *Handbook of Differential Equations. Ordinary Differential Equations, vol.***3**. (A. Cañada, P. Drábek, A. Fonda, eds.) Elsevier 2006, pp. 607–723.

[16] I. Rachůnková, S. Staněk and M. Tvrdý. *Solvability of Nonlinear Singular Problems for Ordinary Differential Equations.* Hindawi [Contemporary Mathematics and Its Applications, Vol.5], in print.

[17] I. Rachůnková and J. Tomeček. Singular Dirichlet problem for ordinary differential equations with impulses. *Nonlinear Anal., Theory Methods Appl.* **65** (2006), 210–229.

[18] I. Rachůnková and M. Tvrdý. Impulsive Periodic Boundary Value Problem and Topological Degree. *Funct. Differ. Equ.* **9**, no.3-4, 471–498.

[19] I. Rachůnková and M. Tvrdý. Nonmonotone impulse effects in second order periodic boundary value problems. *Abstr. Anal. Appl.* **2004**: 7, 577–590.

[20] I. Rachůnková and M. Tvrdý. Non-ordered lower and upper functions in second order impulsive periodic problems. *Dyn. Contin. Discrete Impuls. Syst., Ser. A, Math. Anal.* **12** (2005), 397–415.

[21] I. Rachůnková and M. Tvrdý. Existence results for impulsive second order periodic problems, *Nonlinear Anal., Theory Methods Appl.* **59** (2004) 133–146.

[22] I. Rachůnková and M. Tvrdý. Method of lower and upper functions in impulsive periodic boundary value problems. In: *EQUADIFF 2003.* Proceedings of the International Conference on Differential Equations, Hasselt, Belgium, July 22-26,2003, ed. by F. Dumortier, H. Broer, J. Mawhin, A. Vanderbauwhede, S. Verduyn, Hackensack, NJ, World Scientific (2005), pp. 252–257.

[23] I. Rachůnková and M. Tvrdý. Second Order Periodic Problem with ϕ -Laplacian and Impulses - Part I. *Mathematical Institute of the Academy of Sciences of the Czech Republic, Preprint* **155/2004** [available as \ `http://www.math.cas.cz/ ~ tvrdy/lapl1.pdf` or \ `http://www.math.cas.cz/ ~ tvrdy/lapl1.ps`].

[24] I. Rachůnková and M. Tvrdý. Second Order Periodic Problem with ϕ -Laplacian and Impulses - Part II. *Mathematical Institute of the Academy of Sciences of the Czech Republic, Preprint* **156/2004** [available as \ `http://www.math.cas.cz/ ~ tvrdy/lapl2.pdf` or \ `http://www.math.cas.cz/ ~ tvrdy/lapl2.ps`].

[25] I. Rachůnková and M. Tvrdý. Second order periodic problem with phi-Laplacian and impulses. *Nonlinear Analysis, T.M.A.* **63** (2005), e257-e266.

[26] I. Rachůnková and M. Tvrdý. Periodic singular problem with quasilinear differential operator. *Mathematica Bohemica* **131** (2006), 321–336.

[27] Š. Schwabik and M. Tvrdý. Boundary value problems for generalized linear differential equations, *Czechoslovak Math. J.* **29 (104)** (1979), 451-477.

[28] Š. Schwabik, M. Tvrdý and O. Vejvoda. *Differential and Integral Equations: Boundary Value Problems and Adjoint.* Academia and D. Reidel, Praha and Dordrecht, 1979.

[29] M. Tvrdý. Fredholm-Stieltjes integral equations with linear constraints: duality theory and Green's function. *Časopis pěst. mat.* **104** (1979), 357–369.

[30] Zhang Zhitao Existence of solutions for second order impulsive differential equations. *Appl. Math., Ser. B (Engl. Ed.)* **12** (1997), 307–320.