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# Impulsive periodic boundary value problem and topological degree

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**Summary.** The paper deals with the impulsive periodic boundary value problem

$$u'' = f(t,u,u'), \quad u(t_1+) = \mathrm{J}(u(t_1)), \quad u'(t_1+) = \mathrm{M}(u'(t_1-)),$$
$$u(0) = u(T), \quad u'(0) = u'(T).$$

The problem is reformulated as an operator equation $u - \mathrm{F}\,u = 0$. Our main results are contained in Theorems 3.2 and 4.3, where the Leray-Schauder degree of the operator $\mathrm{I} - \mathrm{F}$ is determined with respect to certain open sets $\Omega_1$ or $\Omega_2$ which are given in terms of a strict lower function $\sigma_1$ and a strict upper function $\sigma_2$. We do not restrict ourselves to the well ordered $\sigma_1 < \sigma_2$ on $[0,T]$ but we study the nonordered $\sigma_1$ and $\sigma_2$ as well as the reversely ordered $\sigma_2 < \sigma_1$ on $[0,T]$.

These results are substantially used in our next papers, where we get rid of the assumption on the strictness of lower and upper functions and obtain new existence criteria for the given problem.

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**Keywords.** Second order nonlinear ordinary differential equation with impulses, periodic solutions, lower and upper functions, strict lower and upper functions, Leray - Schauder topological degree.

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# 1 . Preliminaries

**1.1. Introduction.** In this paper we study the impulsive periodic boundary value problem

$$u'' = f(t, u, u'), \tag{1.1}$$

$$u(t_1+) = \mathrm{J}(u(t_1)), \quad u'(t_1+) = \mathrm{M}(u'(t_1-)), \tag{1.2}$$

$$u(0) = u(T), \quad u'(0) = u'(T). \tag{1.3}$$

Our basic assumption is the existence of strict lower and upper functions $\sigma_1$ and $\sigma_2$ of the problem (1.1)-(1.3) (see Definitions 1.5 and 1.6). Using these functions we construct open bounded sets $\Omega_1$ and $\Omega_2$ and evaluate the Leray-Schauder topological degree of the operator $\mathrm{I} - \mathrm{F}$ generated by the problem (1.1)-(1.3) (see Theorems 3.2 and 4.3). We prove that $\deg(\mathrm{I} - \mathrm{F}, \Omega_1) = 1$ and $\deg(\mathrm{I} - \mathrm{F}, \Omega_2) = -1$, which immediately guarantees the existence of solutions of (1.1)-(1.3) lying in these sets (see Corollaries 3.3, 3.4, 4.4 and 4.5). Finally, in Section 5 we indicate possible generalizations of the existence results presented in Corollaries 3.3, 3.4, 4.4, 4.5.

In our considerations we distinguish two cases:

(i) functions $\sigma_1$ and $\sigma_2$ are *well ordered* (Section 3), i.e. $\sigma_1(t) < \sigma_2(t)$ for $t \in [0, T]$,

(ii) functions $\sigma_1$ and $\sigma_2$ are not well ordered (Section 4), i.e. either $\sigma_1(t) > \sigma_2(t)$ for $t \in [0, T]$ or $\sigma_1$, $\sigma_2$ are not ordered at all.

As we can see in References, there are several papers dealing with existence results for similar problems (cf. [1], [2], [4], [6]-[8], [10] and [13]). Some of them ([1], [2], [6]-[8]) work with upper and lower functions. All these works consider the case (i) only. The papers [4] and [13] are also well related to our work because, similarly as we do in this paper, they work with the topological degree of operators generated by periodic impulsive boundary value problems.

In [4], the vector case of (1.1)-(1.3) with $f$ continuous on $[0, T] \times \mathbb{R}^2$ is considered and existence results based on the curvature bound sets are proved there. In the scalar case these results yield the existence of solutions to (1.1)-(1.3) provided the well ordered constant strict lower and upper functions of (1.1)-(1.3) are available. The same is true for [13], where the impulse conditions differ a little bit from (1.2). However, also the existence results in [13] cover only the well ordered case (i). The authors believe that till now, there has been no existence result for the second order periodic impulsive boundary value problems in the case of the existence of non well ordered lower and upper functions. In this paper we want to start serious considerations of this case. To this aim we modify the methods developed in our previous paper [11].

**1.2. Notation.** For a real valued function $x$ defined a.e. on $[0,T]$, we put

$$\|x\|_\infty = \sup_{t\in[0,T]} \operatorname{ess} |x(t)| \quad \text{and} \quad \|x\|_1 = \int_0^T |x(s)|\,\mathrm{d}s.$$

By $\mathbb{G}[0,T]$ we denote the set of functions regulated on $[0,T]$, i.e. the set of functions $x:\ [0,T] \mapsto \mathbb{R}$ for which the one-sided limits

$$x(t+) = \lim_{\tau\to t+} x(\tau) \quad \text{and} \quad x(s-) = \lim_{\tau\to s-} x(\tau) \tag{1.4}$$

exist and are finite for $t \in [a,b)$ and $s \in (a,b]$. Furthermore,

$$\mathbb{G}_{\mathrm{L}}[0,T] = \{x \in \mathbb{G}[0,T] :\ x(t-) = x(t) \ \text{ for } \ t \in (0,T] \quad \text{and} \quad x(0+) = x(0)\}.$$

Having $x \in \mathbb{G}[0,T]$ and $t \in (0,T)$, we denote $\Delta x(t) = x(t+) - x(t-)$. Notice that if $x \in \mathbb{G}[0,T]$, then we have

$$\|x\|_\infty = \sup_{t\in[0,T]} |x(t)|.$$

By $\widetilde{\mathbb{C}}[0,T]$ we denote the set of functions $x \in \mathbb{G}_{\mathrm{L}}[0,T]$ which are continuous at every point $t \in (0,T) \setminus \{t_1\}$. It is well-known (cf. e.g. [5]) that $\mathbb{G}[0,T]$ is a Banach space with respect to the norm $\|.\|_\infty$ and $\mathbb{G}_{\mathrm{L}}[0,T]$ and $\widetilde{\mathbb{C}}[0,T]$ are its closed subspaces.

Furthermore, $\widetilde{\mathbb{G}}[0,T]$ is the set of functions $x :\ [0,T] \setminus \{t_1\} \mapsto \mathbb{R}$ for which the limits (1.4) exist and are finite for $t \in [a,b)$ and $s \in (a,b]$. By $\widetilde{\mathbb{C}}^1[0,T]$ we denote the set of functions $x \in \widetilde{\mathbb{C}}[0,T]$ with $x' \in \widetilde{\mathbb{G}}[0,T]$ and such that $x'$ is continuous at every point $t \in [0,T] \setminus \{t_1\}$. (We put $x'(0) = x'(0+)$ and $x'(T) = x'(T-)$.) It is easy to see that when equipped with the norm

$$\|x\|_{\widetilde{\mathbb{C}}^1} = \|x\|_\infty + \|x'\|_\infty,$$

$\widetilde{\mathbb{C}}^1[0,T]$ becomes a Banach space. Notice that if $x \in \widetilde{\mathbb{C}}^1[0,T]$, then we have

$$\|x'\|_\infty = \sup_{t\in[0,t_1)\cup(t_1,T]} |x'(t)|.$$

As usual, by $\mathbb{AC}[0,T]$ and $\mathbb{L}[0,T]$ we denote the spaces of functions $x :\ [0,T] \mapsto \mathbb{R}$ which are respectively absolutely continuous and Lebesgue integrable on $[0,T]$. $\widetilde{\mathbb{AC}}^1[0,T]$ stands for the set of functions $x \in \widetilde{\mathbb{C}}^1[0,T]$ which are absolutely continuous on $[0,t_1) \cup (t_1,T]$.

$\mathrm{Car}([0,T] \times \mathbb{R}^2)$ is the set of functions $f\ :\ [0,T] \times \mathbb{R}^2 \mapsto \mathbb{R}$ which fulfil the *Carathéodory conditions* on $[0,T] \times \mathbb{R}^2$, i.e. (i) for each $x \in \mathbb{R}$ and $y \in \mathbb{R}$ the function $f(.,x,y)$ is measurable on $[0,T]$; (ii) for almost every $t \in [0,T]$ the function $f(t,.,.)$ is continuous on $\mathbb{R}^2$; (iii) for each compact set $K \subset \mathbb{R}^2$ the function $m_K(t) = \sup_{(x,y)\in K} |f(t,x,y)|$ is Lebesgue integrable on $[a,b]$.

Having a Banach space $\mathbb{X}$ and its subset $M$, then $\mathrm{cl}(M)$ and $\partial M$ stand for the closure and the boundary of $M$, respectively.

If $\Omega$ is an open bounded subset in $\mathbb{X}$, the operator $\mathrm{F} : \mathrm{cl}(\Omega) \mapsto \mathbb{X}$ is completely continuous and $\mathrm{F}\,x \neq x$ for all $x \in \partial\,\Omega$, then $\deg(\mathrm{I} - \mathrm{F}, \Omega)$ denotes the *Leray-Schauder topological degree* of $\mathrm{I}-\mathrm{F}$ with respect to $\Omega$, where I is the identity operator on $\mathbb{X}$. For a definition and properties of the degree see e.g. [3] or [9].

**1.3. Assumptions.** Throughout the paper we assume

$$0 < t_1 < T, \tag{1.5}$$

$$f \in \mathrm{Car}([0,T] \times \mathbb{R}^2), \tag{1.6}$$

$$\mathrm{J} \quad \text{and} \quad \mathrm{M} \quad \text{are continuous mappings of } \mathbb{R} \text{ into } \mathbb{R}, \tag{1.7}$$

$$\mathrm{J} \text{ is increasing on } \mathbb{R} \quad \text{and} \quad \mathrm{M} \text{ is nondecreasing on } \mathbb{R}. \tag{1.8}$$

**1.4. Definition.** By a *solution of the impulsive problem* (1.1), (1.2) we understand a function $u \in \widetilde{\mathbb{AC}}^1[0,T]$ fulfilling (1.2) and such that

$$u''(t) = f(t, u(t), u'(t)) \quad \text{for a.e. } t \in [0,T]. \tag{1.9}$$

A solution $u \in \widetilde{\mathbb{AC}}^1[0,T]$ of the impulsive problem (1.1), (1.2) which satisfies (1.3) is called a *solution to the boundary value problem* (1.1)-(1.3).

**1.5. Definition.** A function $\sigma_1 \in \widetilde{\mathbb{AC}}^1[0,T]$ is a *lower function* of (1.1)-(1.3) if

$$\sigma_1''(t) \geq f(t, \sigma_1(t), \sigma_1'(t)) \quad \text{for a.e. } t \in [0,T], \tag{1.10}$$

$$\sigma_1(t_1+) = \mathrm{J}(\sigma_1(t_1)) \quad \text{and} \quad \sigma_1'(t_1+) \geq \mathrm{M}(\sigma_1'(t_1-)), \tag{1.11}$$

$$\sigma_1(0) = \sigma_1(T), \qquad \sigma_1'(0) \geq \sigma_1'(T). \tag{1.12}$$

Similarly, a function $\sigma_2 \in \widetilde{\mathbb{AC}}^1[0,T]$ is an *upper function of* (1.1)-(1.3) if

$$\sigma_2''(t) \leq f(t, \sigma_2(t), \sigma_2'(t)) \quad \text{for a.e. } t \in [0,T], \tag{1.13}$$

$$\sigma_2(t_1+) = \mathrm{J}(\sigma_2(t_1)) \quad \text{and} \quad \sigma_2'(t_1+) \leq \mathrm{M}(\sigma_2'(t_1-)), \tag{1.14}$$

$$\sigma_2(0) = \sigma_2(T), \qquad \sigma_2'(0) \leq \sigma_2'(T). \tag{1.15}$$

**1.6. Definition.** A lower function $\sigma_1$ of (1.1)-(1.3) which is not a solution of (1.1)-(1.3) is called a *strict lower function* of (1.1)-(1.3) if there exists $\varepsilon > 0$ such that

$$\sigma_1''(t) \geq f(t,x,y) \quad \text{for a.e. } \ t \in [0,T] \\ \text{and all } \ (x,y) \in [\sigma_1(t), \sigma_1(t)+\varepsilon] \times [\sigma_1'(t)-\varepsilon, \sigma_1'(t)+\varepsilon]. \tag{1.16}$$

Similarly, an upper function $\sigma_2$ of (1.1)-(1.3) which is not a solution of (1.1)-(1.3) is called a *strict upper function* of (1.1)-(1.3) if there exists $\varepsilon > 0$ such that

$$\sigma_2''(t) \leq f(t,x,y) \quad \text{for a.e. } \ t \in [0,T] \\ \text{and all } \ (x,y) \in [\sigma_2(t)-\varepsilon, \sigma_2(t)] \times [\sigma_2'(t)-\varepsilon, \sigma_2'(t)+\varepsilon]. \tag{1.17}$$

**1.7. Remark.** Assume (1.5)-(1.7) and

$$\mathrm{M}(0) = 0. \tag{1.18}$$

Furthermore, let $r_1 \in \mathbb{R}$, $\mathrm{J}(r_1) = r_1$ and $f(t, r_1, 0) \leq 0$ for a.e. $t \in [0,T]$. Then $\sigma_1(t) \equiv r_1$ is a lower function of the problem (1.1)-(1.3). If, moreover, there is $\varepsilon > 0$ such that

$$f(t,x,y) \leq 0 \ \text{ for a.e. } \ t \in [0,T] \ \text{ and all } \ (x,y) \in [r_1, r_1+\varepsilon] \times [-\varepsilon, \varepsilon] \tag{1.19}$$

then $\sigma_1$ is a strict lower function of (1.1)-(1.3).

Similarly, if $r_2 \in \mathbb{R}$ and $\varepsilon > 0$ are such that (1.18), $\mathrm{J}(r_2) = r_2$ and $f(t, r_2, 0) \geq 0$ for a.e. $t \in [0,T]$ or

$$f(t,x,y) \geq 0 \ \text{ for a.e. } \ t \in [0,T] \ \text{ and all } \ (x,y) \in [r_2-\varepsilon, r_2] \times [-\varepsilon, \varepsilon] \tag{1.20}$$

hold, then $\sigma_2(t) \equiv r_2$ is an upper function or a strict upper function of (1.1)-(1.3), respectively.

## 2 . Auxiliary assertions

In Section 3 we construct an operator representation $(\mathrm{I} - \mathrm{F})x = 0$ of the problem (1.1)-(1.3). To this aim we need an explicit form of a solution of the related linear impulsive problem (2.1)-(2.3) (see Lemma 2.1) and a priori estimates of solutions of nonlinear impulsive problems (Lemmas 2.3, 2.5, 2.7 and 2.8).

**2.1. Lemma.** *Let $h \in \mathbb{L}[0,T]$ and $c,\ d,\ e \in \mathbb{R}$. Then there is a unique function $\xi \in \widetilde{\mathbb{AC}}^1[0,T]$ fulfilling*

$$\xi''(t) = h(t) \quad \text{for a.e. } t \in [0,T], \quad \Delta\xi(t_1) = d, \quad \Delta\xi'(t_1) = e, \tag{2.1}$$

$$\xi(0) = \xi(T) = c. \tag{2.2}$$

*This function is given by*

$$\xi(t) = c + \widetilde{g}(t,t_1)\, d + g(t,t_1)\, e + \int_0^T g(t,s)\, h(s)\, ds \quad \text{on } [0,T], \tag{2.3}$$

*where*

$$\widetilde{g}(t,s) = \begin{cases} \dfrac{T-t}{T} & \text{if } 0 \le s < t \le T, \\ -\dfrac{t}{T} & \text{if } 0 \le t \le s \le T, \end{cases} \tag{2.4}$$

*and*

$$g(t,s) = \begin{cases} \dfrac{s\,(t-T)}{T} & \text{if } 0 \le s < t \le T, \\ \dfrac{t\,(s-T)}{T} & \text{if } 0 \le t \le s \le T. \end{cases} \tag{2.5}$$

*Proof.* Let us choose $\widetilde{x} \in \mathbb{R}$ and define

$$\xi(t) = \begin{cases} c + t\,\widetilde{x} + \displaystyle\int_0^t (t-s)\, h(s)\, \mathrm{d}s & \text{if } t \in [0,t_1], \\ c + t\,\widetilde{x} + d + e\,(t-t_1) + \displaystyle\int_0^t (t-s)\, h(s)\, \mathrm{d}s & \text{if } t \in (t_1,T]. \end{cases} \tag{2.6}$$

We have

$$\xi'(t) = \begin{cases} \widetilde{x} + \displaystyle\int_0^t h(s)\, \mathrm{d}s & \text{if } t \in [0,t_1), \\ \widetilde{x} + \displaystyle\int_0^t h(s)\, \mathrm{d}s + e & \text{if } t \in (t_1,T] \end{cases}$$

and it is easy to see that $\xi \in \widetilde{\mathbb{AC}}^1[0,T]$ fulfils (2.1) and $\xi(0) = c$. Furthermore, we will have $\xi(T) = c$ whenever

$$T\,\widetilde{x} = -d - e\,(T-t_1) - \int_0^T (T-s)\, h(s)\, \mathrm{d}s \tag{2.7}$$

is true. Inserting (2.7) into (2.6), we get

$$\xi(t) = c - \frac{t}{T}\,d + \frac{t\,(t_1 - T)}{T}\,e + \int_0^t \frac{(t-T)\,s}{T}\,h(s)\,\mathrm{d}s + \int_t^T \frac{t\,(s-T)}{T}\,h(s)\,\mathrm{d}s \qquad \text{for } t \in [0, t_1]$$

and

$$\xi(t) = c + \frac{T-t}{T}\,d + \frac{t_1\,(t-T)}{T}\,e + \int_0^t \frac{(t-T)\,s}{T}\,h(s)\,\mathrm{d}s + \int_t^T \frac{t\,(s-T)}{T}\,h(s)\,\mathrm{d}s \qquad \text{for } t \in (t_1, T],$$

wherefrom, taking into account (2.4)-(2.6), we obtain the representation (2.3).

On the other hand, if we had two functions $\xi_1$ and $\xi_2 \in \widetilde{\mathbb{AC}}^1[0,T]$ fulfilling (2.1)-(2.2), then for $\eta = \xi_1 - \xi_2$ we would have

$$\eta''(t) = 0 \quad \text{a.e. on} \quad [0,T], \quad \eta(0) = \eta(T) = 0,$$

i.e. $\eta(t) \equiv 0$ on $[0,T]$. □

After inserting $c = d = e = 0$, Lemma 2.1 reduces to

**2.2. Corollary.** *For every $h \in \mathbb{L}[0,T]$ the function*

$$z(t) = \int_0^T g(t,s)\,h(s)\,ds, \quad t \in [0,T]$$

*with $g(t,s)$ given by* (2.5) *is a unique solution of the Dirichlet boundary value problem*

$$z''(t) = h(t) \ \text{ for a.e. } t \in [0,T], \quad z(0) = z(T) = 0. \tag{2.8}$$

Using the properties of strict upper functions we prove inequalities (2.10) and (2.22) which enable us to estimate solutions of nonlinear impulsive problems from above.

**2.3. Lemma.** *Assume that* (1.5) − (1.8) *hold and let $\sigma_2$ be a strict upper function of* (1.1) − (1.3). *Furthermore, let $\widetilde{f} \in \mathrm{Car}([0,T] \times \mathbb{R}^2)$ and*

$$\widetilde{f}(t,x,y) > f(t,\sigma_2(t),y) \quad \text{for a.e. } t \in [0,T] \text{ and all } (x,y) \in (\sigma_2(t),\infty) \times \mathbb{R}. \tag{2.9}$$

*Then*

$$(2.10)\qquad u(t) \le \sigma_2(t) \quad \text{on} \ [0,T]$$

*holds for any solution* $u$ *of the impulsive problem* (1.2),

$$(2.11)\qquad u'' = \widetilde{f}(t,u,u')$$

*having the property* $u(0) = u(T) \le \sigma_2(0)$.

*Proof.* Let $u$ be a solution of (2.11), (1.2) and let $u(0) = u(T) \le \sigma_2(0)$. Denote $v(t) = u(t) - \sigma_2(t)$ for $t \in [0,T]$.

(i) First, we shall show that $v$ does not have any point of a positive local maximum in $[0,t_1) \cup (t_1,T]$.

Indeed, let $\varepsilon > 0$ be such that (1.17) is true and let $\alpha \in [0,t_1) \cup (t_1,T]$ be a point of a positive local maximum of $v$. We have

$$(2.12)\qquad v(0) = v(T) = u(0) - \sigma_2(0) \le 0.$$

Hence $\alpha \in (0,t_1) \cup (t_1,T)$. Moreover, $v(\alpha) > 0$ and $v'(\alpha) = 0$, which guarantees the existence of $\beta \in (\alpha,T]$ such that $[\alpha,\beta] \subset (0,t_1) \cup (t_1,T]$,

$$(2.13)\qquad |v'(t)| \le \varepsilon \quad \text{and} \quad v(t) > 0$$

on $[\alpha,\beta]$. Using (2.9) and (1.17), we get

$$v''(t) = u''(t) - \sigma_2''(t) = \widetilde{f}(t,u(t),u'(t)) - \sigma_2''(t) > f(t,\sigma_2(t),u'(t)) - \sigma_2''(t) \ge 0$$

for a.e. $t \in [\alpha,\beta]$. Hence,

$$0 < \int_\alpha^t v''(s)\,\mathrm{d}s = v'(t)$$

for all $t \in (\alpha,\beta]$, which contradicts the fact that $v$ has a local maximum at $\alpha$. This completes the proof of our claim.

(ii) Let

$$(2.14)\qquad v(t) \le v(t_1) \ \text{ for all } \ t \in [0,t_1) \quad \text{and} \quad v(t_1) > 0.$$

Then necessarily $v'(t_1-) \ge 0$. Furthermore, taking into account also (1.8), we have

$$(2.15)\qquad u(t_1) > \sigma_2(t_1) \implies \mathrm{J}(u(t_1)) > \mathrm{J}(\sigma_2(t_1)),$$

and

$$(2.16)\qquad u'(t_1-) \ge \sigma_2'(t_1-) \implies \mathrm{M}(u'(t_1-)) \ge \mathrm{M}(\sigma_2'(t_1-)),$$

which means that $v(t_1+) = \mathrm{J}(u(t_1)) - \mathrm{J}(\sigma_2(t_1)) > 0$ and $v'(t_1+) \geq \mathrm{M}(u'(t_1-)) - \mathrm{M}(\sigma_2'(t_1-)) \geq 0$. If $v'(t_1+) > 0$ were valid, then, in view of (2.12) and (2.14), $v$ would have a point of a positive local maximum in $(t_1, T]$, contrary to (i). Provided that $v'(t_1+) = 0$, we can find $\beta \in (t_1, T]$ such that (2.13) is satisfied for $t \in (t_1, \beta]$. Using (2.9) and (1.17) as in part (i), we get $v''(t) > 0$ for a.e. $t \in (t_1, \beta]$ and therefore $v'(t) > 0$ for all $t \in (t_1, \beta]$. Thus, by virtue of the fact that $v(T) \leq 0$, $v$ has a point of a positive local maximum in $(t_1, T]$, which is impossible due to (i). Therefore (2.14) cannot occur.

(iii) It remains to show that neither

$$v(t_1+) \geq v(t) \ \text{ for all } \ t \in (t_1, T] \quad \text{and} \quad v(t_1+) > 0 \tag{2.17}$$

can occur. Indeed, if (2.17) holds, then $v'(t_1+) \leq 0$. Moreover, by (1.8) we have

$$\mathrm{J}(u(t_1)) > \mathrm{J}(\sigma_2(t_1)) \implies u(t_1) > \sigma_2(t_1) \tag{2.18}$$

and thus $v(t_1-) > 0$. Assume that $v'(t_1+) = 0$. Then there is $\beta \in (t_1, T]$ such that (2.13) holds on $(t_1, \beta]$. As in part (i), this implies that $v'(t) > 0$ for all $t \in (t_1, \beta]$, contrary to (2.17). Hence, $v'(t_1+) < 0$. In view of (1.8) we have

$$\mathrm{M}(u'(t_1-)) < \mathrm{M}(\sigma_2'(t_1-)) \implies u'(t_1-) < \sigma_2'(t_1-), \tag{2.19}$$

i.e. $v'(t_1-) < 0$, which, in view of (2.12) and of the fact that $v(t_1-) > 0$, means that $v$ has a point of a positive local maximum in $[0, t_1)$, contrary to (i). □

**2.4. Remark.** In Lemma 2.3 we can weaken the assumption (1.8) concerning monotonicity of J and M on $\mathbb{R}$. In particular, the assertion of Lemma 2.3 remains true if we suppose instead of (1.8) that for each $x, y \in \mathbb{R}$ the conditions

$$\mathrm{J}(x) > \mathrm{J}(\sigma_2(t_1)) \iff x > \sigma_2(t_1), \tag{2.20}$$

$$y \geq \sigma_2'(t_1-) \implies \mathrm{M}(y) \geq \mathrm{M}(\sigma_2'(t_1-)) \tag{2.21}$$

are fulfilled. Indeed, if we consider the proof of Lemma 2.3, we see that we make use of (1.8) just to get (2.15), (2.16), (2.18) and (2.19). But all these conditions follow from (2.20) and (2.21), as well.

**2.5. Lemma.** *Assume that* (1.5) – (1.8) *hold and let* $\sigma_2$ *be a strict upper function of the problem* (1.1) – (1.3). *Then for any solution* $u$ *of* (1.1) – (1.3) *fulfilling* (2.10) *we have*

$$u(t) < \sigma_2(t) \ \text{ for } \ t \in [0, T] \quad \text{and} \quad u(t_1+) < \sigma_2(t_1+). \tag{2.22}$$

*Proof.* Denote $v = u - \sigma_2$. We have

$$v(t) \le 0 \quad \text{for} \quad t \in [0,T]. \tag{2.23}$$

First, let us notice that the following assertion is true:
*Let $\alpha \in [0,T)$ and*

$$v(\alpha+) = v'(\alpha+) = 0. \tag{2.24}$$

*Then there exists $\beta \in (\alpha,T]$ such that the identities*

$$v(t) = 0 \quad \text{and} \quad v'(t) = 0 \tag{2.25}$$

*are true for $t \in (\alpha,\beta]$.*

Indeed, let $\varepsilon > 0$ be such that (1.17) is true. According to (2.23) and (2.24), we can choose $\beta \in (\alpha,T]$ in such a way that

$$-\varepsilon < v(t) \le 0 \quad \text{and} \quad |v'(t)| \le \varepsilon \quad \text{for} \quad t \in (\alpha,\beta]. \tag{2.26}$$

By (1.17) and (2.26), we get

$$v''(t) = f(t,u(t),u'(t)) - \sigma_2''(t) \ge 0 \quad \text{a.e. on} \quad (\alpha,\beta].$$

If $v'(\tau) < 0$ held for some $\tau \in (\alpha,\beta]$, we would have

$$0 \le \int_\alpha^\tau v''(t)\,\mathrm{d}t = v'(\tau) < 0,$$

a contradiction. Thus, $0 \le v'(t) \le \varepsilon$ for $t \in (\alpha,\beta]$. On the other hand, if $v'(\tau) > 0$ held for some $\tau \in (\alpha,\beta]$, then we would have

$$v(\tau) = \int_\alpha^\tau v'(t)\,\mathrm{d}t > 0,$$

a contradiction with (2.23). Therefore $v'(t) = 0$ on $(\alpha,\beta]$ and, consequently, $v(t) = 0$ on $(\alpha,\beta]$. This completes the proof of our claim.

(i) Now, assume that $v(T) = 0$. Then, owing to the periodic conditions, we have $v(0) = 0$ and, in view of (2.23), also

$$0 \ge v'(0) = u'(0) - \sigma_2'(0) \ge u'(T) - \sigma_2'(T) = v'(T) \ge 0,$$

i.e. $v'(0) = v'(T) = 0$. Hence (2.24) is true with $\alpha = 0$ and we can choose $\beta \in (0,t_1)$ so that the relations (2.25) are satisfied on $[0,\beta]$. Let us put

$$t^* = \sup\Big\{s \in (0,t_1] : v(t) = 0 \quad \text{on} \quad [0,s]\Big\}. \tag{2.27}$$

Then $t^* \geq \beta$, $v(t^*) = 0$ and $v'(t^*-) = 0$. Let us assume that $t^* < t_1$. Then we have (2.24) with $\alpha = t^*$. Hence we can find $\beta^* \in (t^*, t_1]$ such that (2.25) is true for all $t \in [0, \beta^*]$, which contradicts (2.27). Therefore $t^* = t_1$ and we get

$$v(t) = 0 \quad \text{for} \quad t \in [0, t_1] \tag{2.28}$$

and $v'(t_1-) = 0$. Consequently,

$$v(t_1+) = \mathrm{J}(u(t_1)) - \mathrm{J}(\sigma_2(t_1)) = 0$$

and, by (1.14),

$$v'(t_1+) \geq \mathrm{M}(u'(t_1-) - \mathrm{M}(\sigma_2'(t_1-)) = 0.$$

Since $v'(t_1+) > 0$ implies $v(t) > 0$ for some $t \in (t_1, T]$, contrary to (2.23), we necessarily have $v'(t_1+) = 0$. To summarize, we have

$$v(t_1+) = v'(t_1+) = 0, \tag{2.29}$$

i.e. (2.24) with $\alpha = t_1$ is true. Hence there is $\beta \in (t_1, T]$ such that (2.25) holds for all $t \in (t_1, \beta]$. Put

$$r^* = \sup \Big\{ s \in (t_1, T] : \, v(t) = 0 \quad \text{on} \quad (t_1, s] \Big\}. \tag{2.30}$$

Then $r^* \geq \beta$ and $v(r^*) = v'(r^*-) = 0$. Thus, if $r^* < T$ were valid, we would have (2.24) with $\alpha = r^*$ and (2.25) on $(t_1, \beta]$ with $\beta \in (r^*, T]$, contrary to (2.30). Therefore, $r^* = T$ and, by virtue of (2.28), we get

$$u(t) = \sigma_2(t) \quad \text{for all} \quad t \in [0, T], \tag{2.31}$$

which contradicts Definition 1.6.

(ii) Let $v(\alpha) = 0$ for some $\alpha \in (t_1, T)$. Then $v'(\alpha) = 0$ and we can find $\beta \in (\alpha, T]$ such that (2.25) is valid for $t \in [\alpha, \beta]$. Define

$$r^* = \sup \Big\{ s \in (\alpha, T] : \, v(t) = 0 \quad \text{on} \quad [\alpha, s] \Big\}.$$

As in part (i) of this proof we get $r^* = T$. Therefore $v(T) = 0$ and we arrive at (2.31) as in part (i).

(iii) Let $v(t_1+) = 0$. Then, as (1.8) implies

$$\mathrm{J}(u(t_1)) = \mathrm{J}(\sigma_2(t_1)) \implies u(t_1) = \sigma_2(t_1), \tag{2.32}$$

we have $v(t_1) = 0$. Furthermore, (2.23) implies that both $v'(t_1-) \geq 0$ and $v'(t_1+) \leq 0$ are true. On the other hand, in view of (1.8) we have (2.16). Thus, using (1.14) and the fact that $v'(t_1-) \ge 0$ we get $v'(t_1+) \ge 0$, as well. Therefore (2.29) is true and we can follow part (i) to get (2.31).

(iv) If $v(t_1) = 0$, then also $v(t_1+) = 0$ and we can follow part (iii).

(v) If $v(\alpha) = 0$ for some $\alpha \in (0, t_1)$, then necessarily $v'(\alpha) = 0$ and we can find $\beta \in (\alpha, t_1]$ such that (2.25) is valid for all $t \in [\alpha, \beta]$. Analogously as before, put

$$t^* = \sup \Big\{ s \in (\alpha, t_1] : v(t) = 0 \ \text{ on } \ [\alpha, s] \Big\}.$$

As in part (i) we get $t^* = t_1$. This means that $v(t_1) = 0$ and hence, by part (iv), we can again conclude that (2.31) is true. □

**2.6. Remark.** The assertion of Lemma 2.5 remains valid if we replace the condition (1.8) with (2.21) and moreover suppose that

$$\mathrm{J}(x) = \mathrm{J}(\sigma_2(t_1)) \iff x = \sigma_2(t_1). \tag{2.33}$$

Going through the proof of Lemma 2.5 we observe that (1.8) is used for the validity of (2.16) and (2.32). But both of them are included in (2.21) and (2.33).

The proofs of the lower estimates dual to (2.10) and (2.22) are based on the properties of strict lower functions and they are omitted due to their similarity to the proofs of Lemmas 2.3 and 2.5.

**2.7. Lemma.** *Assume that* (1.5) – (1.8) *hold and let* $\sigma_1$ *be a strict lower function of* (1.1) – (1.3). *Furthermore, let* $\widetilde{f} \in \mathrm{Car}([0,T] \times \mathbb{R}^2)$ *and*

$$\widetilde{f}(t,x,y) < f(t,\sigma_1(t),y) \quad \textit{for a.e.} \quad t \in [0,T] \ \textit{and all} \ \ (x,y) \in (-\infty, \sigma_1(t)) \times \mathbb{R}.$$

*Then* $u(t) \ge \sigma_1(t)$ *on* $[0,T]$ *holds for any solution* $u$ *of the impulsive problem* (2.11), (1.2) *satisfying* $u(0) = u(T) \ge \sigma_1(0)$. □

**2.8. Lemma.** *Assume that* (1.5) – (1.8) *hold and let* $\sigma_1$ *be a strict lower function of the problem* (1.1) – (1.3). *Then for any solution* $u$ *of* (1.1) – (1.3) *fulfilling* $u(t) \ge \sigma_1(t)$ *for* $t \in [0,T]$ *we have* $u(t) > \sigma_1(t)$ *for* $t \in [0,T]$ *and* $u(t_1+) > \sigma_1(t_1+)$. □

**2.9. Remark.** Lemma 2.7 remains valid if instead of (1.8) we suppose that for each $x, y \in \mathbb{R}$ the conditions

$$\mathrm{J}(x) < \mathrm{J}(\sigma_1(t_1)) \iff x < \sigma_1(t_1),$$

$$y \le \sigma_1'(t_1-) \implies \mathrm{M}(y) \le \mathrm{M}(\sigma_1'(t_1-)) \tag{2.34}$$

are fulfilled. Similarly, in Lemma 2.8 we can replace (1.8) with the conditions (2.34) and

$$\mathrm{J}(x) = \mathrm{J}(\sigma_1(t_1)) \iff x = \sigma_1(t_1).$$

# 3 . Well ordered strict lower and upper functions and topological degree

This section is devoted to the evaluation of the Leray - Schauder topological degree of the operator $\mathrm{I} - \mathrm{F}$ with respect to an open bounded set $\Omega_1$ (see (3.1) and (3.8)). Here we consider the case that strict lower and upper functions for (1.1)-(1.3) exist and are well ordered, i.e. the condition (3.5) is satisfied. First, notice that the periodic conditions (1.3) can be written in the equivalent form

$$u(0) = u(0) + u'(0) - u'(T), \quad u(T) = u(0) + u'(0) - u'(T).$$

**3.1. Lemma.** *Assume that* (1.5) – (1.8) *hold. Let* $\mathrm{F} : \widetilde{\mathbb{C}}^1[0,T] \mapsto \widetilde{\mathbb{C}}^1[0,T]$ *be given by*

$$\begin{aligned}(\mathrm{F}\,x)(t) = x(0) + x'(0) - x'(T) + \int_0^T g(t,s)\, f(s, x(s), x'(s))\, ds \\ + \widetilde{g}(t,t_1)\,(\mathrm{J}(x(t_1)) - x(t_1)) \\ + g(t,t_1)\,(\mathrm{M}(x'(t_1-)) - x'(t_1-)), \quad t \in [0,T],\end{aligned} \tag{3.1}$$

*where* $g(t,s)$ *and* $\widetilde{g}(t,s)$ *are defined by* (2.5) *and* (2.4), *respectively.*

*Then* F *is completely continuous and a function* $u \in \widetilde{\mathbb{C}}^1[0,T]$ *is a solution of* (1.1) – (1.3) *if and only if* $u$ *is a fixed point of the operator* F .

*Proof.* Choose an arbitrary $y \in \widetilde{\mathbb{C}}^1[0,T]$ and consider an auxiliary linear problem

$$x''(t) = f(t, y(t), y'(t)) \ \text{ for a.e. } t \in [0,T], \tag{3.2}$$

$$\Delta x(t_1) = \mathrm{J}(y(t_1)) - y(t_1), \quad \Delta x'(t_1) = \mathrm{M}(y'(t_1-)) - y'(t_1-), \tag{3.3}$$

$$x(0) = y(0) + y'(0) - y'(T), \quad x(T) = y(0) + y'(0) - y'(T). \tag{3.4}$$

Clearly, $f(t, y(t), y'(t)) \in \mathbb{L}[0,T]$ and hence, by Lemma 2.1, the problem (3.2)-(3.4) has a unique solution $x \in \widetilde{\mathbb{AC}}^1[0,T]$. Furthermore, by virtue of (2.3) and (3.1), this solution is of the form

$$x(t) = (\mathrm{F}\,y)(t) \ \text{ for } t \in [0,T].$$

In particular, $u \in \widetilde{\mathbb{C}}^1[0,T]$ is a solution to (1.1)-(1.3) if and only if it is a fixed point of the operator F, i.e. if and only if it is a solution to the operator equation

$$u - \mathrm{F}\, u = 0.$$

Let an operator $\mathrm{F}_1 : \widetilde{\mathbb{C}}^1[0,T] \mapsto \widetilde{\mathbb{C}}^1[0,T]$ be defined by

$$(\mathrm{F}_1 y)(t) = \int_0^T g(t,s)\, f(s, y(s), y'(s))\, \mathrm{d}s, \quad t \in [0,T].$$

By Corollary 2.2 we have $\mathrm{F}_1\, y \in \mathbb{C}^1[0,T]$ for every $y \in \widetilde{\mathbb{C}}^1[0,T]$ and $\mathrm{F}_1$ is a composition of the Green type operator for the Dirichlet problem (2.8) and of the superposition operator generated by $f \in \mathrm{Car}([0,T] \times \mathbb{R}^2)$. Making use of the Lebesgue Dominated Convergence Theorem and the Arzelà-Ascoli Theorem, we get in a standard way that $\mathrm{F}_1$ is completely continuous. Furthermore, since J and M are continuous, the operator $\mathrm{F}_2 = \mathrm{F} - \mathrm{F}_1$ is continuous, as well. Finally, having in mind that $\mathrm{F}_2$ maps bounded sets onto bounded sets and its values are contained in a three-dimensional subspace of $\widetilde{\mathbb{C}}^1[0,T]$, we conclude that the operators $\mathrm{F}_2$ and $\mathrm{F} = \mathrm{F}_1 + \mathrm{F}_2$ are completely continuous. □

**3.2. Theorem.** *Assume that* (1.5) – (1.8) *hold. Let* $\sigma_1$ *and* $\sigma_2$ *be respectively strict lower and strict upper functions of* (1.1) – (1.3) *satisfying*

$$\sigma_1(t) < \sigma_2(t) \quad \textit{on} \ [0,T] \tag{3.5}$$

*and let* $m \in \mathbb{L}[0,T]$ *be such that*

$$|f(t,x,y)| \le m(t) \quad \textit{for a.e.} \ t \in [0,T] \quad \textit{and all} \ (x,y) \in [\sigma_1(t), \sigma_2(t)] \times \mathbb{R}. \tag{3.6}$$

*Then the operator* F *given by* (3.1) *satisfies*

$$\deg(\mathrm{I} - \mathrm{F}, \Omega_1) = 1, \tag{3.7}$$

*where*

$$\begin{aligned} \Omega_1 = \{x \in \widetilde{\mathbb{C}}^1[0,T] :\ & \sigma_1(t) < x(t) < \sigma_2(t) \quad \textit{on} \ [0,T], \\ & \sigma_1(t_1+) < x(t_1+) < \sigma_2(t_1+),\ \|x'\|_\infty < C\} \end{aligned} \tag{3.8}$$

*and*

$$C \ge C(m, \sigma_1, \sigma_2) := \|m\|_1 + \Big(\max\{\frac{1}{t_1}, \frac{1}{T - t_1}\} + 1\Big)\, (\|\sigma_1\|_{\widetilde{\mathbb{C}}^1} + \|\sigma_2\|_{\widetilde{\mathbb{C}}^1}). \tag{3.9}$$

*Proof.* Choose $C \in (0,\infty)$ fulfilling (3.9) and define an auxiliary operator $\widetilde{\mathrm{H}} : \widetilde{\mathbb{C}}^1[0,T] \mapsto \widetilde{\mathbb{C}}^1[0,T]$ by

$$(\widetilde{\mathrm{H}}x)(t) = \alpha_0(x(0)+x'(0)-x'(T)) + \int_0^T g(t,s)\,\widetilde{h}(s,x(s),x'(s))\,\mathrm{d}s \\ + \widetilde{g}(t,t_1)\,(\mathrm{J}(\alpha_1(x(t_1))) - \alpha_1(x(t_1))) \\ + g(t,t_1)\,(\mathrm{M}(\beta(x'(t_1-))) - \beta(x'(t_1-))) \quad \text{for } t\in[0,T], \tag{3.10}$$

where $\alpha_0(x)$, $\alpha_1(x)$, $\beta(y)$ and $\widetilde{h}(t,x,y)$ are given for a.e. $t \in [0,T]$ and all $(x,y) \in \mathbb{R}^2$ by

$$\alpha_0(x) = \begin{cases} \sigma_1(0) & \text{if } x < \sigma_1(0), \\ x & \text{if } \sigma_1(0) \le x \le \sigma_2(0), \\ \sigma_2(0) & \text{if } x > \sigma_2(0), \end{cases} \tag{3.11}$$

$$\alpha_1(x) = \begin{cases} \sigma_1(t_1) & \text{if } x < \sigma_1(t_1), \\ x & \text{if } \sigma_1(t_1) \le x \le \sigma_2(t_1), \\ \sigma_2(t_1) & \text{if } x > \sigma_2(t_1), \end{cases} \tag{3.12}$$

$$\beta(y) = \begin{cases} -C & \text{if } y < -C, \\ y & \text{if } |y| \le C, \\ C & \text{if } y > \ C, \end{cases} \tag{3.13}$$

$$h(t,x,y) = f(t,x,\beta(y)) \tag{3.14}$$

and

$$\widetilde{h}(t,x,y) = \begin{cases} h(t,\sigma_1(t),y) - \dfrac{\sigma_1(t)-x}{1+\sigma_1(t)-x} & \text{if } x < \sigma_1(t), \\ h(t,x,y) & \text{if } \sigma_1(t) \le x \le \sigma_2(t), \\ h(t,\sigma_2(t),y) + \dfrac{x-\sigma_2(t)}{1+x-\sigma_2(t)} & \text{if } x > \sigma_2(t). \end{cases} \tag{3.15}$$

We have $\widetilde{h} \in \mathrm{Car}([0,T]\times\mathbb{R}^2)$, $\alpha_0$, $\alpha_1$ and $\beta$ are continuous mappings of $\mathbb{R}$ into itself. Consequently, we argue as in the proof of Lemma 3.1 concluding that $\widetilde{\mathrm{H}}$ is completely continuous.

Now, consider the parameter system of operator equations

$$u - \lambda\,\widetilde{\mathrm{H}}u = 0, \quad \lambda \in [0,1]. \tag{3.16}$$

Taking into account (2.4), (2.5), (3.6), (3.10)-(3.15), we find $r \in (0,\infty)$ such that

$$\|\widetilde{\mathrm{H}}x\|_{\widetilde{\mathbb{C}}^1} \le r \quad \text{for} \quad x \in \widetilde{\mathbb{C}}^1[0,T].$$

Therefore, we can choose $R > 0$ in such a way that

$$\Omega_1 \subset \mathscr{K}(R) := \{x \in \widetilde{\mathbb{C}}^1[0,T] : \|x\|_{\widetilde{\mathbb{C}}^1} < R\}$$

and for any $\lambda \in [0,1]$ each solution $u$ of (3.16) belongs to $\mathscr{K}(R)$. This means that the operator $\mathrm{I} - \lambda\,\widetilde{\mathrm{H}}$ is a homotopy on $\mathscr{K}(R) \times [0,1]$ and

$$\deg(\mathrm{I} - \widetilde{\mathrm{H}},\, \mathscr{K}(R)) = \deg(\mathrm{I}, \mathscr{K}(R)\,) = 1. \tag{3.17}$$

Now, let $\lambda = 1$ and let $u$ be an arbitrary solution of the corresponding problem (3.16). According to Lemma 3.1, this is possible if and only if

$$u''(t) = \widetilde{h}(t, u(t), u'(t)) \quad \text{for a.e. } t \in [0,T], \tag{3.18}$$

$$u(t_1+) = \widetilde{\mathrm{J}}(u(t_1)), \quad u'(t_1+) = \widetilde{\mathrm{M}}(u'(t_1-)), \tag{3.19}$$

$$u(0) = u(T) = \alpha_0(u(0) + u'(0) - u'(T)), \tag{3.20}$$

where

$$\widetilde{\mathrm{J}}(x) = x + \mathrm{J}(\alpha_1(x)) - \alpha_1(x) \quad \text{for} \quad x \in \mathbb{R}$$

and

$$\widetilde{\mathrm{M}}(y) = y + \mathrm{M}(\beta(y)) - \beta(y) \quad \text{for} \quad y \in \mathbb{R}.$$

In view of (3.11) and (3.15), we have $\sigma_1(0) \le \alpha_0(u(0) + u'(0) - u'(T)) \le \sigma_2(0)$,

$$\widetilde{h}(t,x,y) > h(t,\sigma_2(t),y) \quad \text{for a.e.} \quad t \in [0,T]$$
$$\text{and all} \quad (x,y) \in (\sigma_2(t),\infty) \times \mathbb{R}$$

and

$$\widetilde{h}(t,x,y) < h(t,\sigma_1(t),y) \quad \text{for a.e.} \quad t \in [0,T]$$
$$\text{and all} \quad (x,y) \in (-\infty,\sigma_1(t)) \times \mathbb{R}.$$

Furthermore, both $\widetilde{\mathrm{J}}$ and $\widetilde{\mathrm{M}}$ are continuous on $\mathbb{R}$, $\widetilde{\mathrm{J}}$ is increasing on $\mathbb{R}$ and $\widetilde{\mathrm{M}}$ is nondecreasing on $\mathbb{R}$. Consider $\varepsilon > 0$ from Definition 1.6. Since $\|\sigma_1\|_{\widetilde{\mathbb{C}}^1} + \|\sigma_2\|_{\widetilde{\mathbb{C}}^1} < C$, we can find $\varepsilon_1 \in (0,\varepsilon)$ such that $\|\sigma_1'\|_\infty + \varepsilon_1 < C$, $\|\sigma_2'\|_\infty + \varepsilon_1 < C$ and so we have $h(t,x,y) = f(t,x,y)$ for a.e. $t \in [0,T]$ and for all $(x,y) \in M_1(t) \cup M_2(t)$, where $M_1(t) = [\sigma_1(t), \sigma_1(t) + \varepsilon_1] \times [\sigma_1'(t) - \varepsilon_1, \sigma_1'(t) + \varepsilon_1]$ and $M_2(t) = [\sigma_2(t) - \varepsilon_1, \sigma_2(t)] \times [\sigma_2'(t) - \varepsilon_1, \sigma_2'(t) + \varepsilon_1]$. Therefore, $\sigma_1$ and $\sigma_2$ are respectively strict lower and strict upper functions for $u'' = h(t, u, u')$, (1.2), (1.3). In view of (3.11) and (3.20) we have $\sigma_1(0) \le u(0) = u(T) \le \sigma_2(0)$. Thus, applying Lemmas 2.3 and 2.7 (with $h$ and $\widetilde{h}$ in the place of $f$ and $\widetilde{f}$ and with $\widetilde{\mathrm{J}}$ and $\widetilde{\mathrm{M}}$ in the place of J and M, respectively) we get

$$\sigma_1(t) \le u(t) \le \sigma_2(t) \quad \text{on} \;\; [0, T]. \tag{3.21}$$

Now, by the Mean Value Theorem there is $\xi_1 \in (0, t_1)$ such that

$$|u'(\xi_1)|\, t_1 = |u(t_1) - u(0)| \le (\|\sigma_1\|_\infty + \|\sigma_2\|_\infty)$$

and so (3.6), (3.15), (3.18) and (3.19) lead to

$$\begin{aligned} |u'(t)| &\le |u'(\xi_1)| + \Big| \int_{\xi_1}^t m(s)\, \mathrm{d}s \Big| \\ &\le \|m\|_1 + \frac{1}{t_1} (\|\sigma_1\|_\infty + \|\sigma_2\|_\infty) \quad \text{for all} \;\; t \in [0, t_1). \end{aligned}$$

Similarly,

$$|u'(t)| \le \|m\|_1 + \frac{1}{T - t_1} (\|\sigma_1\|_\infty + \|\sigma_1\|_\infty) \quad \text{for all} \;\; t \in (t_1, T]$$

and therefore, in view of (3.9),

$$\|u'\|_\infty < C. \tag{3.22}$$

By (3.12), (3.13), (3.21) and (3.22), we have

$$\alpha_1(u(t_1)) = u(t_1), \quad \beta(u'(t_1-)) = u'(t_1-)$$

and, according to (3.15),

$$\widetilde{h}(t, u(t), u'(t)) = f(t, u(t), u'(t)) \quad \text{for a.e.} \;\; t \in [0, T].$$

Hence, (3.18)-(3.20) imply that $u$ satisfies (1.9), (1.2) and $u(0) = u(T)$. To show that $u$ is a solution of (1.1)-(1.3) it remains to prove that $u'(0) = u'(T)$. In particular, we will prove that

$$\sigma_1(0) \le u(0) + u'(0) - u'(T) \le \sigma_2(0) \tag{3.23}$$

is valid. Suppose on the contrary

$$u(0) + u'(0) - u'(T) > \sigma_2(0). \tag{3.24}$$

Then by (3.11) we have $\alpha_0(u(0) + u'(0) - u'(T)) = \sigma_2(0)$. This together with (1.15) and (3.20) yields

$$u(0) = u(T) = \sigma_2(0) = \sigma_2(T). \tag{3.25}$$

Inserting (3.25) into (3.24) we get

$$u'(0) > u'(T). \tag{3.26}$$

On the other hand, (3.25) together with (3.21) and (3.26) implies that

$$\sigma_2'(0) \ge u'(0) > u'(T) \ge \sigma_2'(T),$$

which is a contradiction with (1.15). Now, assume

$$u(0) + u'(0) - u'(T) < \sigma_1(0). \tag{3.27}$$

Then $\alpha_0(u(0) + u'(0) - u'(T)) = \sigma_1(0)$ and (1.12) together with (3.20) gives

$$u(0) = u(T) = \sigma_1(0) = \sigma_1(T),$$

wherefrom, in view of (3.27), the inequality $u'(0) < u'(T)$ follows. On the other hand, taking into account (3.21), we can see that

$$\sigma_1'(0) \le u'(0) < u'(T) \le \sigma_1'(T)$$

must hold, which is a contradiction with (1.12). Thus, (3.23) is true. Therefore, by (3.11) and (3.20) we have $u'(0) = u'(T)$.

To summarize: we have shown that every solution $u$ of (3.16) with $\lambda = 1$ solves the problem (1.1)-(1.3) and satisfies (3.21). Consequently, we can make use of Lemmas 2.5 and 2.8 to show that

$$\sigma_1(t) < u(t) < \sigma_2(t) \ \text{ on } \ [0,T] \quad \text{and} \quad \sigma_1(t_1+) < u(t_1+) < \sigma_2(t_1+).$$

Hence, taking into account (3.22), we conclude that $u \in \Omega_1$. Now, since $\mathrm{F} = \widetilde{\mathrm{H}}$ on $\mathrm{cl}(\Omega_1)$ and $x \neq \mathrm{F}\, x$ for all $x \in \partial\Omega_1$, by (3.17) and by the excision property of the Leray-Schauder degree we conclude

$$\deg(\mathrm{I} - \mathrm{F}, \Omega_1) = \deg(\mathrm{I} - \widetilde{\mathrm{H}}, \Omega_1) = \deg(\mathrm{I} - \widetilde{\mathrm{H}}, \mathscr{K}(R)) = 1,$$

i.e. (3.7) is valid. □

Using Theorem 3.2 and the existence property of the Leray - Schauder degree we immediately get the following existence result.

**3.3. Corollary.** *Let all assumptions of Theorem* 3.2 *be satisfied. Then the problem* (1.1) – (1.3) *has a solution* $u$ *satisfying*

$$\sigma_1(t) < u(t) < \sigma_2(t) \ \text{ on } \ [0,T] \quad \text{and} \quad \sigma_1(t_1+) < u(t_1+) < \sigma_2(t_1+).$$

□

In particular, in view of Remark 1.7, we have the following simple existence criterion.

**3.4. Corollary.** *Assume that* (1.5) – (1.8), (1.18) *and* (3.6) *are satisfied. Furthermore, let there exist* $r_1$, $r_2 \in \mathbb{R}$ *and* $\varepsilon > 0$ *such that* $r_1 < r_2$, $\mathrm{J}(r_1) = r_1$, $\mathrm{J}(r_2) = r_2$, (1.19) *and* (1.20) *hold. Then the problem* (1.1) – (1.3) *has a solution* $u$ *satisfying*

$$r_1 < u(t) < r_2 \quad on \;\; [0,T] \quad and \quad r_1 < u(t_1+) < r_2. \qquad \square$$

## 4 . Topological degree and strict lower and upper functions which are not well ordered

This section deals with the Leray - Schauder degree of the operator $\mathrm{I} - \mathrm{F}$, but here, in contrast to the previous section, we consider the case that the relation (3.5) is not satisfied.

The main result is contained in Theorem 4.3, the proof of which is based on Theorem 3.2 and on the inequalities (4.1) and (4.3).

**4.1. Lemma.** *Let* (1.5) – (1.8) *be fulfilled and let* $\sigma_1$ *and* $\sigma_2$ *be respectively strict lower and strict upper functions of* (1.1) – (1.3) *such that* (3.5) *is not valid. Then the inequality*

$$\sigma_1(\tau) > \sigma_2(\tau) \quad \textit{for some} \;\; \tau \in [0,T] \tag{4.1}$$

*holds.*

*Proof.* We have to show that the case

$$\sigma_1(t) \le \sigma_2(t) \;\; \text{on} \;\; [0,T] \quad \text{and} \quad \sigma_1(s) = \sigma_2(s) \quad \text{for some} \;\; s \in [0,T]$$

cannot occur. To this aim, we use the arguments from the proof of Lemma 2.5, where we put $v = \sigma_1 - \sigma_2$ and work with $\sigma_1$ instead of $u$. $\square$

**4.2. Lemma.** *Let* (1.5) – (1.8) *be fulfilled and let* $\sigma_1$ *and* $\sigma_2$ *be respectively lower and upper functions of* (1.1) – (1.3). *Denote by* $\widetilde{\Omega}$ *the set of functions* $x \in \widetilde{\mathbb{C}}^1[0,T]$ *fulfilling* (1.2), (1.3) *and*

$$\sigma_2(t_x) < x(t_x) \quad and \quad x(s_x) < \sigma_1(t_x) \quad \textit{for some} \;\; t_x, s_x \in [0,T]. \tag{4.2}$$

*Then there is* $\rho \in (0,\infty)$ *such that the inequality*

$$|x'(\xi_x)| < \rho \quad \textit{for some} \;\; \xi_x \in [0,T] \tag{4.3}$$

*is valid for each* $x \in \widetilde{\Omega}$.

*Proof.* Let $x \in \widetilde{\Omega}$.
(i) First, suppose that

$$\min\{\sigma_1(t), \sigma_2(t)\} \le x(t) \le \max\{\sigma_1(t), \sigma_2(t)\} \quad \text{for} \;\; t \in [0,T].$$

Then there is $\xi \in (0, t_1)$ such that

$$|x'(\xi)| = \Big|\frac{x(t_1) - x(0)}{t_1}\Big| \le \frac{2\,\max\{\|\sigma_1\|_\infty, \|\sigma_2\|_\infty\}}{t_1}. \tag{4.4}$$

(ii) Assume that $x(s) > \sigma_1(s)$ for some $s \in [0,T]$. Denote $v = x - \sigma_1$. Since $x$ satisfies (4.2), the relations

$$v_* = \inf_{t\in[0,T]} v(t) < 0 \quad \text{and} \quad v^* = \sup_{t\in[0,T]} v(t) > 0$$

are true. Provided

$$v'(\xi) = 0 \quad \text{for some} \;\; \xi \in [0, t_1) \cup (t_1, T], \tag{4.5}$$

we have

$$|x'(\xi)| = |\sigma_1'(\xi)| \le \|\sigma_1'\|_\infty. \tag{4.6}$$

Now, suppose that (4.5) is not true. In view of (1.3) and (1.12), we have $v(0) = v(T)$ and $v'(T) \ge v'(0)$. This is possible only if $v'(t) < 0$ for $t \in [0, t_1)$ and $v'(t) > 0$ for $t \in (t_1, T]$. In particular, we have

$$v'(t_1-) \le 0 \tag{4.7}$$

and

$$v'(t_1+) \ge 0. \tag{4.8}$$

On the other hand, by (1.8), (1.11) and (4.7) we obtain

$$v'(t_1+) \le M(u'(t_1-)) - M(\sigma_1'(t_1-) \le 0,$$

which together with (4.8) yields $v'(t_1+) = 0$. Thus, $|x'(t_1+)| = |\sigma_1'(t_1+)|$ and, consequently,

$$|x'(\xi)| < \|\sigma_1'\|_\infty + 1 \quad \text{for some} \;\; \xi \in (t_1, T). \tag{4.9}$$

(iii) If $x(s) < \sigma_2(s)$ for some $s \in [0,T]$, then we put $v = x - \sigma_2$ and similarly to part (ii) we show that

$$|x'(\xi)| < \|\sigma_2'\|_\infty + 1 \quad \text{for some} \;\; \xi \in (0, t_1) \cup (t_1, T). \tag{4.10}$$

Taking into account (4.4) and (4.6)-(4.10), we conclude that (4.3) is valid for

$$\rho = \frac{2\,\max\{\|\sigma_1\|_\infty, \|\sigma_2\|_\infty\}}{t_1} + \|\sigma_1'\|_\infty + \|\sigma_2'\|_\infty + 1.$$

□

**4.3. Theorem.** *Assume that* (1.5) – (1.8) *hold,* $\mathrm{M}(0) = 0$ *and there is* $m \in \mathbb{L}[0,T]$ *such that*

$$|f(t,x,y)| \le m(t) \quad \text{for a.e. } t \in [0,T] \quad \text{and all } (x,y) \in \mathbb{R}^2. \tag{4.11}$$

*Let* $\sigma_1$ *and* $\sigma_2$ *be respectively strict lower and strict upper functions of* (1.1) – (1.3) *not satisfying* (3.5).

*Then there are* $A, B \in (0,\infty)$ *such that*

$$\deg(\mathrm{I} - \mathrm{F}, \Omega_2) = -1 \tag{4.12}$$

*holds for the operator* F *and the set* $\Omega_2$ *given respectively by* (3.1) *and*

$$\Omega_2 = \{x \in \widetilde{\mathbb{C}}^1[0,T] : \|x\|_\infty < A,\ \|x'\|_\infty < B \quad \text{and} \quad x \text{ satisfies } (4.2)\}.$$

*Proof.* Let $\rho \in (0,\infty)$ be associated with $\sigma_1$ and $\sigma_2$ by Lemma 4.2 and let us choose $A$, $B \in (0,\infty)$ such that

$$B > \rho + 2\,\|m\|_1 + 1 + \|\sigma_1'\|_\infty + \|\sigma_2'\|_\infty \tag{4.13}$$

and

$$A > \|\sigma_1\|_\infty + \|\sigma_2\|_\infty + B\,T. \tag{4.14}$$

Let us consider an auxiliary problem (1.3),

$$u'' = \widetilde{f}(t,u,u'), \tag{4.15}$$

$$u(t_1+) = \widetilde{\mathrm{J}}(u(t_1)), \quad u'(t_1+) = \mathrm{M}(u'(t_1-)), \tag{4.16}$$

where, for a.e. $t \in [0,T]$ and all $(x,y) \in \mathbb{R}^2$,

$$\widetilde{f}(t,x,y) = \begin{cases} f(t,x,y) - m(t) - 1 & \text{if } x \le -A-1, \\ f(t,x,y) + (A+x)\,(m(t)+1) & \text{if } -A-1 < x < -A, \\ f(t,x,y) & \text{if } -A \le x \le A, \\ f(t,x,y) + (x-A)\,(m(t)+1) & \text{if } A < x < A+1, \\ f(t,x,y) + m(t) + 1 & \text{if } x \ge A+1, \end{cases} \tag{4.17}$$

$$\widetilde{A} = \max\{A, \mathrm{J}(A), |\,\mathrm{J}(-A)|\} + 1, \tag{4.18}$$

$$\widetilde{\mathrm{J}}(x) = \begin{cases} x & \text{if } \ x \le -\widetilde{A}, \\[2mm] \dfrac{-\widetilde{A} - \mathrm{J}(-A)}{A - \widetilde{A}}\,(x + A) + \mathrm{J}(-A) & \text{if } \ -\widetilde{A} < x < -A, \\ \mathrm{J}(x) & \text{if } \ -A \le x \le A, \\[2mm] \dfrac{\widetilde{A} - \mathrm{J}(A)}{\widetilde{A} - A}\,(x - A) + \mathrm{J}(A) & \text{if } \ A < x < \widetilde{A}, \\ x & \text{if } \ x \ge \widetilde{A}. \end{cases} \tag{4.19}$$

We have $\widetilde{f} \in \mathrm{Car}([0,T] \times \mathbb{R}^2)$,

$$|\widetilde{f}(t,x,y)| \le \widetilde{m}(t) := 2\,m(t) + 1 \ \text{ for a.e. } \ t \in [0,T] \ \text{ and all } \ (x,y) \in \mathbb{R}^2, \tag{4.20}$$

$\widetilde{\mathrm{J}}$ being increasing on $\mathbb{R}$. Furthermore, $\sigma_1$ and $\sigma_2$ are respectively strict lower and strict upper functions of (1.3), (4.15), (4.16). According to (4.11), for a.e. $t \in [0,T]$ and all $(x,y) \in \mathbb{R}^2$ we have

$$\begin{aligned} \widetilde{f}(t,x,y) &\ge \ \ 1 \qquad \text{if } \ x \ge \ \ A + 1, \\ \widetilde{f}(t,x,y) &\le -1 \qquad \text{if } \ x \le -A - 1. \end{aligned}$$

Since we assume $\mathrm{M}(0) = 0$, it follows that $\sigma_3(t) \equiv -\widetilde{A} - 1$ and $\sigma_4(t) \equiv \widetilde{A} + 1$ are respectively strict lower and strict upper functions of (1.3), (4.15), (4.16) which are well ordered, i.e.

$$\sigma_3(t) < \sigma_4(t) \ \text{ on } \ [0,T]. \tag{4.21}$$

Moreover,

$$\sigma_3(t) < \sigma_2(t) \quad \text{and} \quad \sigma_1(t) < \sigma_4(t) \ \text{ on } \ [0,T]. \tag{4.22}$$

Let us define sets

$$\begin{aligned} \Omega &= \{x \in \widetilde{\mathbb{C}}^1[0,T] : \, \|x\|_\infty < \widetilde{A} + 1, \, \|x'\|_\infty < \widetilde{C}\}, \\ \Delta_1 &= \{x \in \Omega : \, \sigma_1(t) < x(t) \ \text{ for } \ t \in [0,T], \, \sigma_1(t_1+) < x(t_1+)\}, \\ \Delta_2 &= \{x \in \Omega : \, x(t) < \sigma_2(t) \ \text{ for } \ t \in [0,T], \, x(t_1+) < \sigma_2(t_1+)\}, \end{aligned}$$

where

$$\widetilde{C} = (2\,(\widetilde{A} + 1) + \|\sigma_1'\|_\infty + \|\sigma_2'\|_\infty)\,\big(\max\{\frac{1}{t_1}, \frac{1}{T - t_1}\} + 1\big) + \|\widetilde{m}\|_1$$

with $\widetilde{m}$ defined in (4.20). By virtue of (3.9) we have

$$\widetilde{C} \ge \max\{C(\widetilde{m},\sigma_3,\sigma_4), C(\widetilde{m},\sigma_3,\sigma_2), C(\widetilde{m},\sigma_1,\sigma_4)\}. \tag{4.23}$$

Now, consider an operator $\widetilde{\mathrm{F}}: \widetilde{\mathbb{C}}^1[0,T] \mapsto \widetilde{\mathbb{C}}^1[0,T]$ defined by

$$\begin{aligned}(\widetilde{\mathrm{F}}x)(t) = x(0) + x'(0) - x'(T) + \int_0^T g(t,s)\,\widetilde{f}(s,x(s),x'(s))\,\mathrm{d}s \\ + \widetilde{g}(t,t_1)\,(\widetilde{\mathrm{J}}(x(t_1)) - x(t_1)) \\ + g(t,t_1)\,(\mathrm{M}(x'(t_1-)) - x'(t_1-)) \quad \text{for } t \in [0,T],\end{aligned}$$

where $g$ and $\widetilde{g}$ are given by (2.5) and (2.4), respectively. By Lemma 3.1, the operator $\widetilde{\mathrm{F}}$ is completely continuous and the problem (1.3), (4.15), (4.16) is equivalent to the equation

$$u - \widetilde{\mathrm{F}}u = 0.$$

According to (4.21) and (4.22) we have 3 pairs $\{\sigma_3,\sigma_4\}$, $\{\sigma_3,\sigma_2\}$, $\{\sigma_1,\sigma_4\}$ of well ordered strict lower and strict upper functions of the problem (1.3), (4.15), (4.16). Thus, by virtue of (4.20) and (4.23), we can successively apply Theorem 3.2 to the operator $\widetilde{\mathrm{F}}$ and to the sets $\Omega$, $\Delta_1$, $\Delta_2$. In this way we get

$$\deg(\mathrm{I} - \widetilde{\mathrm{F}}, \Omega) = \deg(\mathrm{I} - \widetilde{\mathrm{F}}, \Delta_1) = \deg(\mathrm{I} - \widetilde{\mathrm{F}}, \Delta_2) = 1.$$

Let us denote $\Delta = \Omega \setminus \mathrm{cl}(\Delta_1 \cup \Delta_2)$. Then

$$\Delta = \{x \in \Omega : \sigma_2(t_x) < x(t_x),\ x(s_x) < \sigma_1(s_x) \quad \text{for some } \ t_x, s_x \in [0,T]\}.$$

Lemma 4.1 and the assumption that $\sigma_1$ and $\sigma_2$ do not satisfy (3.5) imply that the sets $\Delta_1$ and $\Delta_2$ are disjoint and so, by the additivity property of the degree, we have

$$\deg(\mathrm{I} - \widetilde{\mathrm{F}}, \Delta) = \deg(\mathrm{I} - \widetilde{\mathrm{F}}, \Omega) - \deg(\mathrm{I} - \widetilde{\mathrm{F}}, \Delta_1) - \deg(\mathrm{I} - \widetilde{\mathrm{F}}, \Delta_2) = -1. \tag{4.24}$$

Now, let $u$ be a solution of (1.3), (4.15), (4.16) and let $u \in \Delta$. By Lemma 4.2 we have $|u'(\xi)| < \rho$ for some $\xi \in [0,T]$. If $\xi \in [0,t_1)$, then, using (4.20), we obtain

$$|u'(t)| = \Big|u'(\xi) + \int_\xi^t u''(s)\,\mathrm{d}s\Big| < \rho + \int_0^{t_1} \widetilde{m}(s)\,\mathrm{d}s \quad \text{for } \ t \in [0,t_1).$$

Furthermore, in view of the periodic conditions (1.3), we have

$$|u'(T)| = |u'(0)| < \rho + \int_0^{t_1} \widetilde{m}(s)\,\mathrm{d}s$$

and

$$|u'(t)| = \left|u'(T) - \int_t^T u''(s)\,\mathrm{d}s\right| < \rho + \|\widetilde{m}\|_1 \quad \text{for } t \in (t_1, T].$$

Therefore

$$|u'(t)| < \rho + \|\widetilde{m}\|_1 \quad \text{for } t \in [0, T]. \tag{4.25}$$

Similarly we prove the inequality (4.25) provided $\xi \in (t_1, T]$. Hence, taking into account (4.13), we have proved that

$$\|u'\|_\infty < B \tag{4.26}$$

holds.

Since $u \in \Delta$, there are $t_u$, $s_u \in [0, T]$ such that

$$u(t_u) > \sigma_2(t_u) \quad \text{and} \quad u(s_u) < \sigma_1(s_u). \tag{4.27}$$

Let $t_u \in [0, t_1]$. Using (4.26), (4.27) and (1.3), we get

$$u(t) = u(t_u) + \int_{t_u}^t u'(s)\,\mathrm{d}s > \sigma_2(t_u) - B\,t_1 \quad \text{for } t \in [0, t_1]$$

and

$$u(T) = u(0) > \sigma_2(t_u) - B\,t_1.$$

Consequently,

$$u(t) = u(T) - \int_t^T u'(s)\,\mathrm{d}s > \sigma_2(t_u) - B\,(T - t_1) \quad \text{for } t \in (t_1, T],$$

i.e.

$$u(t) > -\|\sigma_2\|_\infty - B\,T \quad \text{for } t \in [0, T]. \tag{4.28}$$

Similarly, we can show that (4.28) holds in the case that $t_u \in (t_1, T]$.

If $s_u \in [0, t_1]$, then using (4.26), (4.27) and (1.3) again, we get

$$u(t) = u(s_u) + \int_{s_u}^t u'(s)\,\mathrm{d}s < \sigma_1(s_u) + B\,t_1 \quad \text{for } t \in [0, t_1],$$

$$u(T) = u(0) < \sigma_1(s_u) + B\,t_1$$

and

$$u(t) = u(T) - \int_t^T u'(s)\,\mathrm{d}s < \sigma_1(s_u) + B\,(T - t_1) \quad \text{for } t \in (t_1, T],$$

i.e.

$$u(t) < \|\sigma_1\|_\infty + BT \quad \text{for} \quad t \in [0,T]. \tag{4.29}$$

Assuming that $s_u \in (t_1, T]$ we argue similarly and get (4.29), as well. Thus, by virtue of (4.14), (4.28) and (4.29), we have

$$\|u\|_\infty < A. \tag{4.30}$$

To summarize: if $u \in \Delta$ solves (1.3), (4.15), (4.16), then it satisfies the inequalities (4.26), (4.27) and (4.30) and, in particular, $u \in \Omega_2$. Consequently, by (4.24) and by the excision property of the Leray-Schauder degree we have

$$\deg(\mathrm{I} - \widetilde{\mathrm{F}}, \Omega_2) = \deg(\mathrm{I} - \widetilde{\mathrm{F}}, \Delta) = -1.$$

Finally, (4.17)-(4.19) imply that $\mathrm{F} = \widetilde{\mathrm{F}}$ on $\mathrm{cl}(\Omega_2)$, which means that (4.12) is true. □

Analogously to the case of Corollary 3.3, the existence property of the Leray - Schauder degree and Theorem 4.3 yield the following existence result.

**4.4. Corollary.** *Let all assumptions of Theorem* 4.3 *be satisfied. Then the problem* (1.1) – (1.3) *has a solution* $u$ *satisfying* (4.27). □

Now, taking into account Remark 1.7, we immediately obtain the following new simple existence criterion.

**4.5. Corollary.** *Assume that* (1.5) – (1.8), (1.18) *and* (4.11) *hold. Furthermore, let there be* $r_1,\ r_2 \in \mathbb{R}$ *and* $\varepsilon > 0$ *such that* $r_1 > r_2$, $\mathrm{J}(r_1) = r_1$, $\mathrm{J}(r_2) = r_2$ *and the conditions* (1.19) *and* (1.20) *are satisfied. Then the problem* (1.1) – (1.3) *has a solution* $u$ *satisfying*

$$r_2 < u(t_u) < r_1 \quad \textit{for some} \quad t_u \in [0,T].$$

□

# 5 . Concluding remarks

The existence results presented in Corollaries 3.3 and 4.4 are valid under quite strong assumptions that $\sigma_1$ and $\sigma_2$ are strict lower and upper functions and that the inequalities (3.5) and (4.1) are strict, as well. But, having such corollaries and using proper limiting processes we can omit all these strictnesses and obtain better existence results which are formulated below as Theorems 5.1 and 5.2 and which will be proved in detail in our next paper [12]. Moreover, we can consider the impulsive problem with a finite number of impulses

$$u(t_i+) = \mathrm{J}_i(u(t_i)), \quad u'(t_i+) = \mathrm{M}_i(u'(t_i-)), \quad i = 1, 2, \dots, p, \tag{5.1}$$

where $p \in \mathbb{N}$ and $0 < t_1 < t_2 < \dots < t_p < T$.

**5.1. Theorem.** *Let us suppose that* $f \in \mathrm{Car}([0,T] \times \mathbb{R}^2)$, *that the operators* $\mathrm{J}_i$ $i = 1, 2, \dots, p$, *are increasing and continuous mappings of* $\mathbb{R}$ *into* $\mathbb{R}$ *and that the operators* $\mathrm{M}_i$, $i = 1, 2, \dots, p$, *are nondecreasing and continuous mappings of* $\mathbb{R}$ *into* $\mathbb{R}$. *Let* $\sigma_1$ *and* $\sigma_2$ *be respectively lower and upper functions of* (1.1), (5.1), (1.3) *satisfying*

$$\sigma_1(t) \le \sigma_2(t) \quad \textit{for } t \in [0,T] \tag{5.2}$$

*and let* $m \in \mathbb{L}[0,T]$ *be such that* (3.6) *is valid. Then the problem* (1.1), (5.1), (1.3) *has a solution* $u$ *satisfying*

$$\sigma_1(t) \le u(t) \le \sigma_2(t) \quad \textit{for } t \in [0,T]. \tag{5.3}$$

**5.2. Theorem.** *Let us suppose that* $f \in \mathrm{Car}([0,T] \times \mathbb{R}^2)$, *that the operators* $\mathrm{J}_i$ $i = 1, 2, \dots, p$, *are increasing and continuous mappings of* $\mathbb{R}$ *into* $\mathbb{R}$ *and that the operators* $\mathrm{M}_i$, $i = 1, 2, \dots, p$, *are nondecreasing and continuous mappings of* $\mathbb{R}$ *into* $\mathbb{R}$. *Let* $\sigma_1$ *and* $\sigma_2$ *be respectively lower and upper functions of* (1.1), (5.1), (1.3) *such that* (5.2) *is not fulfilled. Let* $m \in \mathbb{L}[0,T]$ *satisfy* (4.11). *Then the problem* (1.1), (5.1), (1.3) *has a solution* $u$ *satisfying*

$$\sigma_2(t_u) \le u(t_u) \quad \textit{and} \quad u(s_u) \le \sigma_1(s_u) \quad \textit{for some } t_u, s_u \in [0,T]. \tag{5.4}$$

Finally, let us mention that the conditions (3.6) and (4.11) can be weakened by using the method of a priori estimates and the above results can be extended to some classes of unbounded functions $f$. Moreover, as indicated in Remarks 2.4, 2.6 and 2.9, the monotonicity assumptions on $\mathrm{J}_i$, $\mathrm{M}_i$ can be replaced by less restrictive conditions. For these results we refer to our forthcoming papers.

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