Singularities and Laplacians in Boundary Value Problems for Nonlinear Ordinary Differential Equations

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Abstract

In this text we investigate solvability of various nonlinear singular boundary value problems for ordinary differential equations on the compact interval $[0, T]$. The nonlinearities in differential equations may be singular both in the time and space variables. Location of all singular points in $[0, T]$ need not be known.

The work is divided into 6 sections. Sections 1 and 2 are devoted to singular higher order boundary value problems. The remaining ones deal with the second order case. Motivated by various applications in physics we admit here the left hand sides of the equations under consideration containing the ϕ-Laplacian or p-Laplacian operator. The special attention is paid to Dirichlet and periodic problems.

Usually, the main ideas of the proofs of the results mentioned are described. More detailed proofs are included in the cases where no proofs are available in literature or where the details are needed later.

0. Notation

Let $J \subset \mathbb{R}$, $k \in \mathbb{N}$, $p \in (1, \infty)$. Then we will write:

- $L_\infty(J)$ for the set of functions essentially bounded and (Lebesgue) measurable on J; the corresponding norm is $\|u\|_\infty = \sup \operatorname{ess}\{|u(t)|\colon t \in J\}$.
- $L_1(J)$ for the set of functions (Lebesgue) integrable on J; the corresponding norm is $\|u\|_1 = \int_J |u(t)|\, \mathrm{d}t$.
- $L_p(J)$ for the set of functions whose p-th powers of modulus are integrable on J; the corresponding norm is $\|u\|_p = \left(\int_J |u(t)|^p\, \mathrm{d}t\right)^{\frac{1}{p}}$.
- $C(J)$ and $C^k(J)$ for the sets of functions continuous on J and having continuous k-th derivatives on J, respectively.
- $AC(J)$ and $AC^k(J)$ for the sets of functions absolutely continuous on J and having absolutely continuous k-th derivatives on J, respectively.
- $AC_{loc}(J)$ and $AC^k_{loc}(J)$ for the sets of functions absolutely continuous on each compact interval $I \subset J$ and having absolutely continuous k-th derivatives on each compact interval $I \subset J$, respectively.
- If $J = [a, b]$, we will simply write $C[a, b]$ instead of $C([a, b])$ and similarly for other types of intervals and other functional sets defined above.

Further, we use the following notation:

- If $u \in L_\infty[a, b]$ is continuous on $[a, b]$, then $\max\{|u(t)|\colon t \in [a, b]\} = \sup \operatorname{ess}\{|u(t)|\colon t \in [a, b]\}$. Therefore the norm in $C[a, b]$ will be denoted by $\|u\|_\infty = \max\{|u(t)|\colon t \in [a, b]\}$ and the norm in $C^k[a, b]$ by $\|u\|_{C^k} = \sum_{i=0}^k \|u^{(i)}\|_\infty$.
- Let $n \in \mathbb{N}$ and $\mathcal{M} \subset \mathbb{R}^n$. Then $\overline{\mathcal{M}}$ will denote the closure of $\mathcal{M}$, $\partial\mathcal{M}$ the boundary of $\mathcal{M}$ and $\operatorname{meas}(\mathcal{M})$ the Lebesgue measure of $\mathcal{M}$.
- $\deg(\mathcal{I} - \mathcal{F}, \Omega)$ stands for the Leray-Schauder degree of $\mathcal{I} - \mathcal{F}$ with respect to Ω, where $\mathcal{I}$ denotes the identity operator.

We say that a function f satisfies *the Carathéodory conditions* on the set $[a, b] \times \mathcal{M}$ if

$$f(\cdot, x_0, \dots, x_{n-1})\colon [a, b] \to \mathbb{R} \text{ is measurable for all } (x_0, x_1, \dots, x_{n-1}) \in \mathcal{M}; \tag{0.1}$$

$$f(t, \cdot, \dots, \cdot)\colon \mathcal{M} \to \mathbb{R} \text{ is continuous for a.e. } t \in [a, b]; \tag{0.2}$$

$$\left\{\begin{array}{l} \text{for each compact set } \mathcal{K} \subset \mathcal{M} \text{ there is a function } m_{\mathcal{K}} \in L_1[a, b] \text{ such that} \\ \qquad\qquad |f(t, x_0, \dots, x_{n-1})| \le m_{\mathcal{K}}(t) \\ \text{for a.e. } t \in [a, b] \text{ and all } (x_0, x_1, \dots, x_{n-1}) \in \mathcal{K}. \end{array}\right. \tag{0.3}$$

In this case we will write $f \in Car([a,b] \times \mathcal{M})$. If $J \subset [a,b]$ and $J \neq \overline{J}$, then $f \in Car(J \times \mathcal{M})$ will mean that $f \in Car(I \times \mathcal{M})$ for each compact interval $I \subset J$.

1. Principles of solvability of singular higher order BVPs

1.1. *Regular and singular BVPs*

For $n \in \mathbb{N}$, $[0,T] \subset \mathbb{R}$, $i \in \{0,1,\dots,n-1\}$ and a closed set $\mathcal{B} \subset C^i[0,T]$ consider the boundary value problem

$$u^{(n)} = f(t,u,\dots,u^{(n-1)}), \tag{1.1}$$

$$u \in \mathcal{B}. \tag{1.2}$$

In what follows, we will investigate the solvability of problem (1.1), (1.2) on the set $[0,T] \times \mathcal{A}$, where $\mathcal{A}$ is a closed subset of $\mathbb{R}^n$ or $\mathcal{A} = \mathbb{R}^n$. The classical existence results are based on the assumption $f \in Car([0,T] \times \mathcal{A})$. In this case we will say that problem (1.1), (1.2) is *regular on* $[0,T] \times \mathcal{A}$. If $f \notin Car([0,T] \times \mathcal{A})$ we will say that problem (1.1), (1.2) is *singular on* $[0,T] \times \mathcal{A}$.

Motivated by the following applications we will mainly address singular problems.

Example 1. In certain problems in fluid dynamics and boundary layer theory (see e.g. Callegari, Friedman, Nachman [45], [46], [47]) the second order differential equation

$$u'' + \frac{\psi(t)}{u^\lambda} = 0 \tag{1.3}$$

arose. Here $\lambda \in (0,\infty)$ and $\psi \in C(0,1)$, $\psi \notin L_1[0,1]$. Equation (1.3) is known as the generalized Emden-Fowler equation. Its solvability with the Dirichlet boundary conditions

$$u(0) = u(1) = 0 \tag{1.4}$$

was investigated by Taliaferro [149] in 1979 and then by many other authors. Problem (1.3), (1.4) has been studied on the set $[0,1] \times [0,\infty)$ because positive solutions have been sought. We can see that $f(t,x) = \psi(t)\, x^{-\lambda}$ does not fulfil conditions (0.2) and (0.3) with $[a,b] = [0,1]$ and $\mathcal{M} = [0,\infty)$. Hence problem (1.3), (1.4) is singular on $[0,1] \times [0,\infty)$.

Example 2. Consider the fourth order degenerate parabolic equation

$$U_t + (|U|^\mu\, U_{yyy})_y = 0,$$

which arises in droplets and thin viscous flows models (see e.g. [33], [34]). The source-type solutions of this equation have the form

$$U(y,t) = t^{-b}\, u(y\, t^{-b}), \quad b = \frac{1}{\mu+4},$$

which leads to the study of the third order ordinary differential equation on $[-1, 1]$,

$$u''' = b\,t\,u^{1-\mu}.$$

We see that $f(t, x) = b\,t\,x^{1-\mu}$ is singular on $[-1, 1] \times [0, \infty)$ if $\mu > 1$.

Example 3. Similarly to Example 2, the sixth order degenerate equation

$$U_t - (|U|^{\mu}\, U_{yyyyy})_y = 0$$

which arises in semiconductor models (Bernis [31], [32]) leads to the fifth order ordinary differential equation

$$-u^{(5)} = \frac{t}{u^{\lambda}}$$

which is singular for $\lambda > 0$.

A solvability decision for singular boundary value problems requires an exact definition of a solution to such problems. Here, we will work with the same definition of a solution both for regular problems and for singular ones.

Definition 1.1. A function $u \in AC^{n-1}[0, T] \cap \mathcal{B}$ is said to be a *solution of problem* (1.1), (1.2), if it satisfies the equality $u^{(n)}(t) = f(t, u(t), \ldots, u^{(n-1)}(t))$ a.e. on $[0, T]$. If we investigate problem (1.1), (1.2) on $\mathcal{A} \neq \mathbb{R}^n$, we moreover require $(u(t), \ldots, u^{(n-1)}(t)) \in \mathcal{A}$ for $t \in [0, T]$.

In literature, *an alternative approach to solvability* of singular problems can be found. In this approach, solutions are defined as functions whose $(n-1)$-st derivatives can have discontinuities at some points in $[0, T]$. Here we will call them the w-solutions. According to Kiguradze [97] or Agarwal and O'Regan [3] we define them as follows. In contrast to our starting setting, to define w-solutions we assume that $i \in \{0, 1, \ldots, n-2\}$ and $\mathcal{B}$ is a closed subset in $C^i[0, T]$.

Definition 1.2. We say that u is *a w-solution of problem* (1.1), (1.2) if there exists a finite number of points $t_\nu \in [0, T]$, $\nu = 1, 2, \ldots, r$, such that if we denote $J = [0, T] \setminus \{t_\nu\}_{\nu=1}^{r}$, then $u \in C^{n-2}[0, T] \cap AC_{loc}^{n-1}(J) \cap \mathcal{B}$ satisfies $u^{(n)}(t) = f(t, u(t), \ldots, u^{(n-1)}(t))$ a.e. on $[0, T]$. If $\mathcal{A} \neq \mathbb{R}^n$ we require $(u(t), \ldots, u^{(n-1)}(t)) \in \mathcal{A}$ for $t \in J$.

Clearly each solution is a w-solution and each w-solution which belongs to $AC^{n-1}[0, T]$ is a solution.

In the study of singular problem (1.1), (1.2) we will focus our attention on two types of singularities of the function f :

Let $J \subset [0, T]$. We say that $f\colon J \times \mathcal{A} \to \mathbb{R}$ has singularities in its *time variable* t, if $J \neq \overline{J} = [0, T]$ and

$$f \in Car(J \times \mathcal{A}) \quad \text{and} \quad f \notin Car([0, T] \times \mathcal{A}). \tag{1.5}$$

Let $\mathcal{D} \subset \mathcal{A}$. We say that $f\colon [0,T] \times \mathcal{D} \to \mathbb{R}$ has singularities in its *space variables* $x_0, x_1, \ldots, x_{n-1}$, if $\mathcal{D} \neq \overline{\mathcal{D}} = \mathcal{A}$ and

$$f \in Car([0,T] \times \mathcal{D}) \quad \text{and} \quad f \notin Car([0,T] \times \mathcal{A}). \tag{1.6}$$

We will study particular cases of (1.5) and (1.6) which will be described in Section 1.2 and Section 1.3, respectively.

1.2. *Singularities in time variable*

According to (0.3) and (1.5) a function f has a singularity in its *time variable* t, if f is not integrable on $[0,T]$. Let us define it more precisely. Let $k \in \mathbb{N}$, $t_i \in [0,T]$, $i = 1, \ldots, k$, $J = [0,T] \setminus \{t_1, t_2 \ldots, t_k\}$ and let $f \in Car(J \times \mathcal{A})$. Assume that for each $i \in \{1, \ldots, k\}$ there exists $(x_0, \ldots, x_{n-1}) \in \mathcal{A}$ such that

$$\int_{t_i}^{t_i+\varepsilon} |f(t, x_0, \ldots, x_{n-1})| \, \mathrm{d}t = \infty \quad \text{or} \quad \int_{t_i-\varepsilon}^{t_i} |f(t, x_0, \ldots, x_{n-1})| \, \mathrm{d}t = \infty \tag{1.7}$$

for any sufficiently small $\varepsilon > 0$. Then f does not fulfil (0.3) with $\mathcal{M} = \mathcal{A}$ and, according to (1.5), function f has singularities in its time variable t, namely at the values $t_1, \ldots, t_k$. We will call these values the *singular points* of f.

Example. Let $f_i\colon \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2, \ldots, k$, be continuous. Then the function

$$f(t, x_0, \ldots, x_{n-1}) = \sum_{i=1}^{k} \frac{1}{t - t_i} f_i(x_0, \ldots, x_{n-1}),$$

has singular points $t_1, t_2, \ldots, t_k$.

1.3. *Singularities in space variables*

By virtue of (0.2) and (1.6) we see that if f has a singularity in some of its *space variables* then f is not continuous in this variable on a region where f is studied. Motivated by equation (1.3) we will consider the following case. Let $\mathcal{A}_i \subset \mathbb{R}$ be a closed interval and let $c_i \in \mathcal{A}_i$, $\mathcal{D}_i = \mathcal{A}_i \setminus \{c_i\}$, $i = 0, 1, \ldots, n-1$. Assume that there exists $j \in \{0, 1, \ldots, n-1\}$ such that

$$\left\{ \begin{array}{l} \limsup\limits_{x_j \to c_j,\, x_j \in \mathcal{D}_j} |f(t, x_0, \ldots, x_j, \ldots, x_{n-1})| = \infty \\ \text{for a.e. } \; t \in [0,T] \text{ and for some } \; x_i \in \mathcal{D}_i, \; i = 0, 1, \ldots, n-1, \; i \neq j. \end{array} \right. \tag{1.8}$$

If we put $\mathcal{A} = \mathcal{A}_0 \times \cdots \times \mathcal{A}_{n-1}$, we see that f does not fulfil (0.2) with $\mathcal{M} = \mathcal{A}$ and, according to (1.6), the function f has a singularity in its space variable x_j, namely at the value c_j.

Let u be a solution of (1.1), (1.2) and let a point $t_u \in [0,T]$ be such that $u^{(j)}(t_u) = c_j$ for some $j \in \{0, \ldots, n-1\}$. Then t_u is called *a singular point corresponding to the solution* u.

Now, let u be a w-solution of (1.1), (1.2). Assume that a point $t_u \in [0,T]$ is such that $u^{(n-1)}(t_u)$ does not exist or there is a $j \in \{0,\dots,n-1\}$ such that $u^{(j)}(t_u) = c_j$. Then t_u is called *a singular point corresponding to the w-solution u*.

Example. Let h_1, h_2, $h_3 \in L_1[0,T]$, $h_2 \neq 0$, $h_3 \neq 0$ a.e. on $[0,T]$. Consider the Dirichlet problem

$$u'' + h_1(t) + \frac{h_2(t)}{u} + \frac{h_3(t)}{u'} = 0, \quad u(0) = u(T) = 0. \tag{1.9}$$

Let u be a solution of (1.9). Then 0 and T are singular points corresponding to u. Moreover, there exists at least one point $t_u \in (0,T)$ satisfying $u'(t_u) = 0$, which means that t_u is also a singular point corresponding to u. Note that (in contrast to the points 0 and T) we do not know the location of t_u in $(0,T)$.

In accordance with this example, we will distinguish two types of singular points corresponding to solutions or to w-solutions: *singular points of type I*, where we know their location in $[0,T]$ and *singular points of type II* whose location is not known.

1.4. *Existence principles for BVPs with time singularities*

Singular problems are usually investigated by means of auxiliary regular problems. To establish the existence of a solution of a singular problem we introduce a sequence of approximating regular problems which are solvable. Then we pass to the limit for the sequence of approximate solutions to get a solution of the original singular problem. Here we provide existence principles which contain the main rules for a construction of such sequences to get either w-solutions or solutions.

Consider problem (1.1), (1.2) on $[0,T] \times \mathcal{A}$. For the sake of simplicity assume that f has only one time singularity at $t = t_0$, $t_0 \in [0,T]$. It means that

$$\left\{\begin{array}{l} J = [0,T] \setminus \{t_0\},\ f \in Car(J \times \mathcal{A}) \text{ satisfies at least one of the conditions} \\ (i) \quad \displaystyle\int_{t_0-\varepsilon}^{t_0} |f(t,x_0,\dots,x_{n-1})|\,\mathrm{d}t = \infty, \quad t_0 \in (0,T], \\ (ii) \quad \displaystyle\int_{t_0}^{t_0+\varepsilon} |f(t,x_0,\dots,x_{n-1})|\,\mathrm{d}t = \infty, \quad t_0 \in [0,T), \\ \text{for some } (x_0,x_1,\dots,x_{n-1}) \in \mathcal{A} \text{ and each sufficiently small } \varepsilon > 0. \end{array}\right. \tag{1.10}$$

Let us have a sequence of regular problems

$$u^{(n)} = f_k(t,u,\dots,u^{(n-1)}), \quad u \in \mathcal{B}, \tag{1.11}$$

where $f_k \in Car([0,T] \times \mathbb{R}^n)$, $k \in \mathbb{N}$.

Theorem 1.3. **(First existence principle for w-solutions of** (1.1), (1.2)**)**
Let (1.10) *hold and let* $\mathcal{B}$ *be a closed subset in* $C^{n-2}[0,T]$. *Assume that the conditions*

$$\begin{cases} \text{for each } k \in \mathbb{N} \text{ and each } (x_0, \dots, x_{n-1}) \in \mathcal{A}, \\ f_k(t, x_0, \dots, x_{n-1}) = f(t, x_0, \dots, x_{n-1}) \text{ a.e. on } [0,T] \setminus \triangle_k, \\ \text{where } \triangle_k = (t_0 - \frac{1}{k}, t_0 + \frac{1}{k}) \cap [0,T], \end{cases} \tag{1.12}$$

and

$$\begin{cases} \text{there exists a bounded set } \Omega \subset C^{n-1}[0,T] \text{ such that} \\ \text{for each } k \in \mathbb{N} \text{ the regular problem (1.11) has a solution } u_k \in \Omega \\ \text{and } (u_k(t), \dots, u_k^{(n-1)}(t)) \in \mathcal{A} \text{ for } t \in [0,T] \end{cases} \tag{1.13}$$

are fulfilled.

Then

$$\begin{cases} \text{there exist a function } u \in C^{n-2}[0,T] \text{ and a subsequence} \\ \{u_{k_\ell}\} \subset \{u_k\} \text{ such that } \lim_{\ell \to \infty} \|u_{k_\ell} - u\|_{C^{n-2}} = 0; \end{cases} \tag{1.14}$$

$$\begin{cases} \lim_{\ell \to \infty} u_{k_\ell}^{(n-1)}(t) = u^{(n-1)}(t) \text{ locally uniformly on } J \\ \text{and } (u(t), \dots, u^{(n-1)}(t)) \in \mathcal{A} \text{ for } t \in J; \end{cases} \tag{1.15}$$

$$\text{the function } u \in AC_{loc}^{n-1}(J) \text{ is a w-solution of problem (1.1), (1.2).} \tag{1.16}$$

Sketch of the proof. *Step* 1. *Convergence of the sequence of approximate solutions.*
Condition (1.13) implies that the sequences $\{u_k^{(i)}\}$, $0 \le i \le n-2$, are bounded and equicontinuous on $[0,T]$. By the Arzelà–Ascoli theorem the assertion (1.14) is true and $u \in \mathcal{B} \subset C^{n-2}[0,T]$. Since $\{u_k^{(n-1)}\}$ is bounded on $[0,T]$, we get, due to (1.11) and (1.12), that for each $t \in [0, t_0)$ the sequence $\{u_k^{(n-1)}\}$ is equicontinuous on $[0,t]$ and so the same holds on $[t,T]$ if $t \in (t_0, T]$. The Arzelà–Ascoli theorem and the diagonalization principle yield (1.15).

Step 2. *Properties of the limit* u.
By virtue of (1.12), (1.14) and (1.15) we have

$$\lim_{k_\ell \to \infty} f_{k_\ell}(t, u_{k_\ell}(t), \dots, u_{k_\ell}^{(n-1)}(t) = f(t, u(t), \dots, u^{(n-1)}(t)) \text{ a.e. on } [0,T]. \tag{1.17}$$

Hence, using the Lebesgue convergence theorem, we can deduce that if $t_0 \neq 0$ the limit u solves the equation

$$u^{(n-1)}(t) = u^{(n-1)}(0) + \int_0^t f(s, u(s), \dots, u^{(n-1)}(s))\, \mathrm{d}s \quad \text{for } t \in [0, t_0) \tag{1.18}$$

and if $t_0 \neq T$ the limit u solves the equation

$$u^{(n-1)}(t) = u^{(n-1)}(T) - \int_t^T f(s, u(s), \dots, u^{(n-1)}(s))\, \mathrm{d}s \quad \text{for } t \in (t_0, T], \tag{1.19}$$

which immediately yields (1.16). □

For the existence of a solution $u \in AC^{n-1}[0,T]$ of problem (1.1), (1.2) we will impose additional conditions on f on some neighbourhood of t_0.

THEOREM 1.4. **(First existence principle for solutions of** (1.1), (1.2)**)** *Let all assumptions of Theorem* 1.3 *be fulfilled. Further assume that*

$$\left\{\begin{array}{l} \text{there exist } \psi \in L_1[0,T],\ \eta > 0 \text{ and } \lambda_1, \lambda_2 \in \{-1, 1\} \text{ such that} \\ \qquad \lambda_1 f_{k_\ell}(t, u_{k_\ell}(t), \dots, u_{k_\ell}^{(n-1)}(t)) \geq \psi(t) \\ \text{for all } \ell \in \mathbb{N} \text{ and for a.e. } t \in [t_0 - \eta, t_0) \cap [0,T] \text{ provided } (1.10)\,(i) \text{ holds} \\ \text{and} \\ \qquad \lambda_2 f_{k_\ell}(t, u_{k_\ell}(t), \dots, u_{k_\ell}^{(n-1)}(t)) \geq \psi(t) \\ \text{for all } \ell \in \mathbb{N} \text{ and for a.e. } t \in (t_0, t_0 + \eta] \cap [0,T] \text{ provided } (1.10)\,(ii) \text{ is true.} \end{array}\right. \tag{1.20}$$

Then the assertions (1.14) *and* (1.15) *are valid and* $u \in AC^{n-1}[0,T]$ *is a solution of problem* (1.1), (1.2).

SKETCH OF THE PROOF. *Step* 1. As in the proof of Theorem 1.3 we get that (1.14), (1.15) and (1.16) hold.

Step 2. Since u is a w-solution of problem (1.1), (1.2), it remains to prove that $u \in AC^{n-1}[0,T]$. Assume that condition (1.10) (i) holds. Since

$$u_{k_\ell}^{(n-1)}(t) - u_{k_\ell}^{(n-1)}(t_0 - \eta) = \int_{t_0-\eta}^{t} f_{k_\ell}(s, u_{k_\ell}(s), \dots, u_{k_\ell}^{(n-1)}(s))\, \mathrm{d}s \tag{1.21}$$

for $t \in (0, t_0)$, we get due to (1.13) that there is a $c \in (0, \infty)$ such that

$$\lambda_1 \int_{t_0-\eta}^{t_0} f_{k_\ell}(s, u_{k_\ell}(s), \dots, u_{k_\ell}^{(n-1)}(s))\, \mathrm{d}s \leq c \tag{1.22}$$

for each $\ell \in \mathbb{N}$. By the Fatou lemma, having in mind conditions (1.17), (1.20) and (1.22), we deduce that $f(t, u(t), \dots, u^{(n-1)}(t)) \in L_1[0, t_0]$. Similarly, if condition (1.10) (ii) holds, we deduce that $f(t, u(t), \dots, u^{(n-1)}(t)) \in L_1[t_0, T]$. Therefore $f(t, u(t), \dots, u^{(n-1)}(t)) \in L_1[0,T]$ and due to (1.18) and (1.19) we have that $u \in AC^{n-1}[0,T]$ is a solution of problem (1.1), (1.2). □

In the sequel we will need the following definition:

DEFINITION 1.5. Let $[a,b] \subset \mathbb{R}$ and $\{g_k\} \subset L_1[a,b]$. We say that the sequence $\{g_k\}$ is *uniformly integrable on* $[a,b]$ if

$$\left\{\begin{array}{l} \text{for each } \varepsilon > 0 \text{ there exists } \delta > 0 \text{ such that} \\ \qquad \displaystyle\sum_{j=1}^{\infty} (b_j - a_j) < \delta \Longrightarrow \sum_{j=1}^{\infty} \int_{a_j}^{b_j} |g_k(t)|\, \mathrm{d}t < \varepsilon \\ \text{for each } k \in \mathbb{N} \text{ and each sequence of intervals } \{(a_j, b_j)\} \text{ in } [a,b]. \end{array}\right. \tag{1.23}$$

Note that condition (1.23) is satisfied for example if there exists $\psi \in L_1[a,b]$ such that $|g_k(t)| \le \psi(t)$ for a.e. $t \in [a,b]$ and all $k \in \mathbb{N}$.

Theorem 1.6. **(Second existence principle for solutions of** (1.1), (1.2)**)** *Let all assumptions of Theorem* 1.3 *be fulfilled and assume in addition that $\mathcal{B}$ is a closed subset in $C^{n-1}[0,T]$ and that*

$$\left\{\begin{array}{l} \textit{there exists } \eta > 0 \textit{ such that the sequence } \{f_k(t,u_k(t),\dots,u_k^{(n-1)}(t))\} \\ \textit{is uniformly integrable on } [t_0-\eta,t_0+\eta]\cap[0,T]. \end{array}\right. \tag{1.24}$$

Then

$$\left\{\begin{array}{l} \textit{there exist a function } u \in \overline{\Omega} \textit{ and a subsequence } \{u_{k_\ell}\} \subset \{u_k\} \textit{ such that} \\ \quad \lim_{\ell\to\infty} \|u_{k_\ell} - u\|_{C^{n-1}} = 0 \textit{ and } \quad (u(t),\dots,u^{(n-1)}(t)) \in \mathcal{A} \textit{ for } t \in [0,T] \end{array}\right. \tag{1.25}$$

and $u \in AC^{n-1}[0,T]$ is a solution of problem (1.1), (1.2).

Sketch of the proof. *Step* 1. By (1.13) we get that the sequences $\{u_k^{(i)}\}$, $0 \le i \le n-2$, are bounded in $C[0,T]$ and equicontinuous on $[0,T]$ and $\{u_k^{(n-1)}\}$ is bounded in $C[0,T]$. Using (1.24) one can show that $\{u_k^{(n-1)}\}$ is also equicontinuous on $[0,T]$. The Arzelà–Ascoli theorem yields (1.25) and $u \in \mathcal{B} \subset C^{n-1}[0,T]$.

Step 2. As in Step 2 of the proof of Theorem 1.3 we get that u is a w-solution of problem (1.1), (1.2).

Step 3. It remains to prove that $u \in AC^{n-1}[0,T]$. Since $u \in AC^{n-1}_{loc}(J)$, it is sufficient to prove

$$u^{(n-1)} \in AC([t_0-\eta,t_0+\eta]\cap[0,T]). \tag{1.26}$$

Assume that (1.10) (i) holds and $[t_0-\eta,t_0] \subset [0,T]$. By virtue of (1.17) and (1.24), applying Vitali's convergence theorem we obtain $f(t,u(t),\dots,u^{(n-1)}(t)) \in L_1[t_0-\eta,t_0]$. If (1.10) (ii) holds, we can assume $[t_0,t_0+\eta] \subset [0,T]$ and deduce similarly that

$$f(t,u(t),\dots,u^{(n-1)}(t)) \in L_1[t_0,t_0+\eta].$$

Hence, we get (1.26). $\square$

1.5. *Existence principles for BVPs with space singularities*

Similarly to Section 1.4 we will establish sufficient properties for an approximate sequence of regular problems and of their solutions to pass to a limit and yield a solution of the original singular problem (1.1), (1.2). Let $\mathcal{A}_i \subset \mathbb{R}$, $i = 0,\dots,n-1$, be closed intervals and let $\mathcal{A} = \mathcal{A}_0 \times \dots \times \mathcal{A}_{n-1}$. Consider problem (1.1), (1.2) on $[0,T]\times\mathcal{A}$ and assume that f has only one singularity at each x_i, namely at the values $c_i \in \mathcal{A}_i$, $i = 0,\dots,n-1$. Denoting $\mathcal{D} = \mathcal{D}_0 \times \dots \times \mathcal{D}_{n-1}$, $\mathcal{D}_i = \mathcal{A}_i \setminus \{c_i\}$, $i = 0,\dots,n-1$, we will assume that

$$f \in Car([0,T]\times\mathcal{D}) \text{ satisfies (1.8)} \quad \text{for } j = 0,\dots,n-1. \tag{1.27}$$

Consider a sequence of regular problems (1.11) where $f_k \in Car([0,T]\times\mathbb{R}^n)$, $k\in\mathbb{N}$. We will use the approach used by Rachůnková and Staněk in [127] and [128].

THEOREM 1.7. **(Second existence principle for w-solutions of** (1.1), (1.2)**)** *Let* (1.13), (1.27) *hold and let $\mathcal{B}$ be a closed subset in $C^{n-2}[0,T]$. Assume that*

$$\begin{cases} \text{for each } k\in\mathbb{N}, \text{ for a.e. } t\in[0,T] \text{ and each } (x_0,\dots,x_{n-1})\in\mathcal{D} \text{ we have} \\ f_k(t,x_0,\dots,x_{n-1}) = f(t,x_0,\dots,x_{n-1}) \text{ if } |x_i-c_i|\ge \frac{1}{k},\ 0\le i\le n-1. \end{cases} \tag{1.28}$$

Then the assertion (1.14) *is valid.*

If, moreover, the set

$$\Sigma = \Big\{ s\in[0,T] : u^{(i)}(s)=c_i \quad \text{for some } i\in\{0,\dots,n-1\} \\ \text{or} \quad u^{(n-1)}(s) \quad \text{does not exist} \Big\}$$

is finite, then the assertion (1.15) *is valid for $J=[0,T]\setminus\Sigma$. If, in addition,*

$$\begin{cases} \text{the sequence } \quad \{f_{k_\ell}(t,u_{k_\ell}(t),\dots,u_{k_\ell}^{(n-1)}(t))\} \\ \text{is uniformly integrable on each interval } [a,b]\subset J \end{cases} \tag{1.29}$$

then $u\in AC^{n-1}_{loc}(J)$ is a w-solution of problem (1.1), (1.2).

SKETCH OF THE PROOF. *Step* 1. *Convergence of the sequence of approximate solutions.*

As in Step 1 of the proof of Theorem 1.3 we get (1.14)and $u\in\mathcal{B}\subset C^{n-2}[0,T]$. Assume that Σ is finite and choose an arbitrary $[a,b]\subset J$. According to (1.27) and (1.28) we can prove that the sequence $\{u_{k_\ell}^{(n-1)}\}$ is equicontinuous on $[a,b]$. Using the Arzelà-Ascoli theorem and the diagonalization principle we deduce that the subsequence $\{u_{k_\ell}\}$ can be chosen so that it fulfils (1.15).

Step 2. *Convergence of the sequence of regular right-hand sides.*

Consider sets

$$\mathcal{V}_1 = \{t\in[0,T] :\ f(t,\cdot,\dots,\cdot): \mathcal{D}\to\mathbb{R} \text{ is not continuous}\},$$
$$\mathcal{V}_2 = \{t\in[0,T] :\ \text{the equality in } (1.28) \text{ is not valid}\}.$$

We can see that $\operatorname{meas}(\mathcal{V}_1) = \operatorname{meas}(\mathcal{V}_2) = 0$. Denote $\mathcal{U}=\Sigma\cup\mathcal{V}_1\cup\mathcal{V}_2$ and choose an arbitrary $t\in[0,T]\setminus\mathcal{U}$. By (1.14) and (1.15) there exists $\ell_0\in\mathbb{N}$ such that for each $\ell\in\mathbb{N}$, $\ell\ge\ell_0$,

$$|u^{(i)}(t)-c_i| > \frac{1}{k_\ell}, \quad |u_{k_\ell}^{(i)}(t)-c_i| \ge \frac{1}{k_\ell} \quad \text{for } i\in\{0,\dots,n-1\}.$$

According to (1.28) we have

$$f_{k_\ell}(t,u_{k_\ell}(t),\dots,u_{k_\ell}^{(n-1)}(t)) = f(t,u_{k_\ell}(t),\dots,u_{k_\ell}^{(n-1)}(t))$$

and, by (1.14), (1.15),

$$\lim_{\ell\to\infty} f_{k_\ell}(t, u_{k_\ell}(t), \dots, u_{k_\ell}^{(n-1)}(t)) = f(t, u(t), \dots, u^{(n-1)}(t)). \tag{1.30}$$

Since meas $(\mathcal{U}) = 0$, (1.30) holds for a.e. $t \in [0, T]$.

Step 3. *Existence of a w-solution.*

Choose an arbitrary interval $[a, b] \subset J$. By (1.29) and (1.30) we can use Vitali's convergence theorem [24] to show that $f(t, u(t), \dots, u^{(n-1)}(t)) \in L_1[a, b]$ and if we pass to the limit in the sequence

$$u_{k_\ell}^{(n-1)}(t) = u_{k_\ell}^{(n-1)}(a) + \int_a^t f_{k_\ell}(s, u_{k_\ell}(s), \dots, u_{k_\ell}^{(n-1)}(s))\,\mathrm{d}s, \quad t \in [a, b],$$

we get

$$u^{(n-1)}(t) = u^{(n-1)}(a) + \int_a^t f(s, u(s), \dots, u^{(n-1)}(s))\,\mathrm{d}s, \qquad t \in [a, b].$$

Since $[a, b] \subset J$ is an arbitrary interval, we conclude that $u \in AC_{loc}^{n-1}(J)$ satisfies equation (1.1) for a.e. $t \in [0, T]$. $\square$

Theorem 1.8. **(Third existence principle for solutions of** (1.1), (1.2)**)** *Let* (1.13), (1.27), (1.28) *hold and let* $\mathcal{B}$ *be a closed subset of* $C^{n-1}[0, T]$. *Further, assume that the sequence*

$$\{f_k(t, u_k(t), \dots, u_k^{(n-1)}(t))\} \quad \textit{is uniformly integrable on } [0, T]. \tag{1.31}$$

Then the assertion (1.25) *is valid. If, moreover, the functions* $u^{(i)} - c_i$, $0 \le i \le n-1$, *have at most a finite number of zeros in* $[0, T]$, *then* $u \in AC^{n-1}[0, T]$ *is a solution of* (1.1), (1.2).

Sketch of the proof. *Step* 1. As in Step 1 in the proof of Theorem 1.6 we get that (1.25) is valid and $u \in \mathcal{B} \subset C^{n-1}[0, T]$.

Step 2. As in Step 2 in the proof of Theorem 1.7 we get that (1.30) is valid.

Step 3. We can argue as in Step 3 in the proof of Theorem 1.7 if we take $[0, T]$ instead of $[a, b]$ and (1.31) instead of (1.29). $\square$

2. Existence results for singular two-point higher order BVPs

In this section we are interested in problems for higher order differential equations having singularities in their space variables only (see Section 1.3). We consider the focal, conjugate, (n, p), Sturm-Liouville and Lidstone boundary conditions which appear most frequently in literature. Boundary conditions considered are two-point, linear and homogeneous.

Existence results for the above singular problems are proved by regularization and sequential techniques which consist in the construction of a proper sequence of auxiliary regular problems and in limit processes (see Section 1.5). To prove solvability of the auxiliary regular problems we use the Nonlinear Fredholm Alternative (see, e.g., [101], Theorem 4 or [154], p. 25) which we formulate in the form convenient for our problems. In particular, we consider the differential equation

$$u^{(n)} + \sum_{i=0}^{n-1} a_i(t)\, u^{(i)} = g(t, u, \dots, u^{(n-1)}) \tag{2.1}$$

and the corresponding linear homogeneous differential equation

$$u^{(n)} + \sum_{i=0}^{n-1} a_i(t)\, u^{(i)} = 0 \tag{2.2}$$

where $a_i \in L_1[0,T]$, $0 \le i \le n-1$, $g \in Car([0,T] \times \mathbb{R}^n)$. Further, we introduce boundary conditions

$$\mathcal{L}_j(u) = r_j, \quad 1 \le j \le n, \tag{2.3}$$

and the corresponding homogeneous boundary conditions

$$\mathcal{L}_j(u) = 0, \quad 1 \le j \le n, \tag{2.4}$$

where $\mathcal{L}_j \colon C^{n-1}[0,T] \to \mathbb{R}$ are linear and continuous functionals and $r_j \in \mathbb{R}$, $1 \le j \le n$.

THEOREM 2.1. (Nonlinear Fredholm Alternative) *Let the linear homogeneous problem* (2.2), (2.4) *have only the trivial solution and let there exist a function $\psi \in L_1[0,T]$ such that*

$$|g(t, x_0, \dots, x_{n-1})| \le \psi(t) \quad \text{for a.e. } t \in [0,T] \text{ and all } x_0, \dots, x_{n-1} \in \mathbb{R}.$$

Then the nonlinear problem (2.1), (2.3) *has a solution $u \in AC^{n-1}[0,T]$.*

2.1 . *Focal conditions*

We discuss the singular $(p, n-p)$ right focal problem

$$(-1)^{n-p}\, u^{(n)} = f(t, u, \dots, u^{(n-1)}), \tag{2.5}$$

$$u^{(i)}(0) = 0,\ 0 \le i \le p-1, \quad u^{(j)}(T) = 0,\ p \le j \le n-1, \tag{2.6}$$

where $n \ge 2$, $p \in \mathbb{N}$ is fixed, $1 \le p \le n-1$, $f \in Car([0,T] \times \mathcal{D})$ with

$$\mathcal{D} = \begin{cases} \underbrace{\mathbb{R}_+^{p+1} \times \mathbb{R}_- \times \mathbb{R}_+ \times \mathbb{R}_- \times \cdots \times \mathbb{R}_+}_{n} & \text{if } n-p \text{ is odd} \\ \underbrace{\mathbb{R}_+^{p+1} \times \mathbb{R}_- \times \mathbb{R}_+ \times \mathbb{R}_- \times \cdots \times \mathbb{R}_-}_{n} & \text{if } n-p \text{ is even} \end{cases}$$

and f may be singular at the value 0 of all its space variables. Here $\mathbb{R}_- = (-\infty, 0)$ and $\mathbb{R}_+ = (0, \infty)$. Notice that if $f > 0$ then the singular points corresponding to the solutions of problem (2.5), (2.6) are only of type I. The Green function of problem $u^{(n)} = 0$, (2.6) is presented in [19] and [20].

The existence result for the singular problem (2.5), (2.6) is given in the following theorem.

Theorem 2.2. ([126, Theorem 4.3]) *Let $f \in Car([0,T] \times \mathcal{D})$ and let there exist positive constants ε and r such that*

$$\varepsilon\,(T-t)^r \le f(t, x_0, \dots, x_{n-1}) \quad \textit{for a.e. } t \in [0,T] \textit{ and each } (x_0, \dots, x_{n-1}) \in \mathcal{D}.$$

Also assume that for a.e. $t \in [0,T]$ and each $(x_0, \dots, x_{n-1}) \in \mathcal{D}$ we have

$$f(t, x_0, \dots x_{n-1}) \le \varphi(t) + \sum_{i=0}^{n-1} \omega_i(|x_i|) + \sum_{i=0}^{n-1} h_i(t)\,|x_i|^{\alpha_i},$$

where $\alpha_i \in (0,1)$, φ, $h_i \in L_1[0,T]$ are nonnegative, $\omega_i: \mathbb{R}_+ \to \mathbb{R}_+$ are nonincreasing, $0 \le i \le n-1$, and

$$\int_0^T \omega_i(t^{r+n-i})\,\mathrm{d}t < \infty \quad \textit{for } 0 \le i \le n-1.$$

Then there exists a solution $u \in AC^{n-1}[0,T]$ of problem (2.5), (2.6) *with*

$$\begin{cases} u^{(i)} > 0 & \textit{on } (0,T] \quad \textit{for } 0 \le i \le p-1, \\ (-1)^{j-p}\,u^{(j)} > 0 & \textit{on } [0,T) \quad \textit{for } p \le j \le n-1. \end{cases} \tag{2.7}$$

Sketch of proof. *Step* 1. *Construction of a sequence of regular differential equations related to equation* (2.5).

Put

$$\varphi^*(t) = \varphi(t) + \sum_{i=0}^{n-1} \omega_i(1) + \sum_{i=0}^{n-1} h_i(t) \quad \text{for a.e. } t \in [0,T].$$

Then $\varphi^* \in L_1[0,T]$ and there exists $r^* > 0$ such that the estimate $\|u^{(n-1)}\|_\infty < r^*$ is valid for any function $u \in AC^{n-1}[0,T]$ satisfying (2.6), $(-1)^{n-p}\,u^{(n)}(t) \ge \varepsilon\,(T-t)^r$ and

$$(-1)^{n-p}\,u^{(n)}(t) \le \varphi^*(t) + \sum_{i=0}^{n-1} \omega_i(|u^{(i)}(t)|) + \sum_{i=0}^{n-1} h_i(t)\,|u^{(i)}(t)|^{\alpha_i}$$

for a.e. $t \in [0,T]$. Now for $m \in \mathbb{N}$, $0 \le i \le n-1$ and $x \in \mathbb{R}$, put $\varrho_i = 1 + r^* T^{n-i-1}$ and

$$\sigma_i(\tfrac{1}{m}, x) = \begin{cases} \frac{1}{m}\,\text{sign}\,x & \text{for } |x| < \frac{1}{m}, \\ x & \text{for } \frac{1}{m} \le |x| \le \varrho_i, \\ \varrho_i\,\text{sign}\,x & \text{for } \varrho_i < |x|. \end{cases}$$

Extend f onto $[0,T] \times (\mathbb{R} \setminus \{0\})^n$ as an even function in each of its space variables x_i, $0 \le i \le n-1$, and for a.e. $t \in [0,T]$ and each $(x_0, \dots, x_{n-1}) \in \mathbb{R}^n$, define auxiliary functions

$$f_m(t, x_0, \dots, x_{n-1}) = f(t, \sigma_0(\frac{1}{m}, x_0), \dots, \sigma_{n-1}(\frac{1}{m}, x_{n-1})), \quad m \in \mathbb{N}.$$

In this way we get the family of regular differential equations

$$(-1)^{n-p}\, u^{(n)} = f_m(t, u, \dots, u^{(n-1)}) \tag{2.8}$$

depending on $m \in \mathbb{N}$.

Step 2. *Properties of solutions to problems* (2.8), (2.6).

By Theorem 2.1 we show that for any $m \in \mathbb{N}$, problem (2.8), (2.6) has a solution $u_m \in AC^{n-1}[0,T]$ satisfying (for $t \in [0,T]$)

$$\begin{cases} u_m^{(i)}(t) & \geq c\, t^{r+n-i} & \text{if } \ 0 \leq i \leq p-1, \\ (-1)^{i-p}\, u_m^{(i)}(t) & \geq c\,(T-t)^{r+n-i} & \text{if } \ p \leq i \leq n-1, \end{cases} \tag{2.9}$$

where c is a positive constant and $\|u_m^{(n-1)}\|_\infty < r^*$. Moreover, the sequence $\{u_m^{(n-1)}\}$ is equicontinuous on $[0,T]$. By virtue of the Arzelà-Ascoli theorem, a convergent subsequence $\{u_{k_m}\}$ exists and let $\lim_{m\to\infty} u_{k_m} = u$. Then $u \in C^{n-1}[0,T]$, u satisfies (2.6) and, because of (2.9),

$$u^{(i)}(t) \geq c\, t^{r+n-i} \qquad \text{for } \ t \in [0,T] \quad \text{and} \quad 1 \leq i \leq p-1,$$

$$(-1)^{i-p}\, u^{(i)}(t) \geq c\,(T-t)^{r+n-i} \quad \text{for } \ t \in [0,T] \quad \text{and} \quad p \leq i \leq n-1.$$

Also,

$$|f_{k_m}(t, u_{k_m}(t), \dots, u_{k_m}^{(n-1)}(t))| \leq \varrho(t) \quad \text{for a.e. } \ t \in [0,T] \text{ and all } \ m \in \mathbb{N},$$

where $\varrho \in L_1[0,T]$ and

$$\lim_{m\to\infty} f_{k_m}(t, u_{k_m}(t), \dots, u_{k_m}^{(n-1)}(t)) = f(t, u(t), \dots, u^{(n-1)}(t)) \qquad \text{for a.e. } t \in [0,T].$$

Now, letting $m \to \infty$ in

$$u_{k_m}^{(n-1)}(t) = u_{k_m}^{(n-1)}(0) + (-1)^{n-p} \int_0^t f_{k_m}(s, u_{k_m}(s), \dots, u_{k_m}^{(n-1)}(s))\, \mathrm{d}s$$

we conclude

$$u^{(n-1)}(t) = u^{(n-1)}(0) + (-1)^{n-p} \int_0^t f(s, u(s), \dots, u^{(n-1)}(s))\, \mathrm{d}s, \quad t \in [0,T].$$

Hence $u \in AC^{n-1}[0,T]$ and u is a solution of problem (2.5), (2.6). □

Example. Let $\gamma \in (0,1)$, $\alpha_i \in [0,1)$, $c_i \in (0,\infty)$, $\beta_i \in \left(0, \frac{1}{n+\gamma-i}\right)$ and let $h_i \in L_1[0,T]$ be nonnegative for $0 \leq i \leq n-1$. Then, by Theorem 2.2, there exists a solution $u \in AC^{n-1}[0,T]$ of the differential equation

$$(-1)^{n-p}\, u^{(n)} = \left(\frac{T-t}{t}\right)^{\gamma} + \sum_{i=0}^{n-1} \frac{c_i}{|u^{(i)}|^{\beta_i}} + \sum_{i=0}^{n-1} h_i(t)\, |u^{(i)}|^{\alpha_i}$$

satisfying the $(p, n-p)$ right focal boundary conditions (2.6) and (2.7).

Remark 2.3. Substituting $t = T - s$ in (2.5), (2.6) and using Theorem 2.2 we can also give results for the existence of solutions to singular differential equations satisfying $(n-p,p)$ left focal boundary conditions

$$u^{(i)}(0) = 0,\ p \le i \le n-1, \quad u^{(j)}(T) = 0,\ 0 \le j \le p-1$$

(see [126, Theorem 4.4]).

The singular problem (2.5), (2.6) was also considered on the interval $[0,1]$ by Agarwal, O'Regan and Lakshmikantham in [13] where f is assumed to be continuous and independent of space variables $x_p, \dots, x_{n-1}$ and may be singular at $x_i = 0$, $0 \le i \le p-1$. They examined the problem

$$\begin{cases} (-1)^{n-p}\, u^{(n)} = \varphi(t)\, h(t, u, \dots, u^{(p-1)}), \\ u^{(i)}(0) = 0,\ 0 \le i \le p-1, \quad u^{(j)}(1) = 0,\ p \le j \le n-1, \end{cases} \tag{2.10}$$

under the assumptions

$$\varphi \in C^0(0,1) \quad \text{with} \ \ \varphi > 0 \text{ on } (0,1) \text{ and } \ \varphi \in L_1[0,1], \tag{2.11}$$

$$h\colon\ [0,1] \times \mathbb{R}_+^p \to \mathbb{R}_+ \quad \text{is continuous,} \tag{2.12}$$

$$\begin{cases} \qquad h(t, x_0, \dots, x_{p-1}) \le \sum_{i=0}^{p-1} g_i(x_i) + r\,(\max\{x_0, \dots, x_{p-1}\}) \text{ on } \ [0,1] \times \mathbb{R}_+^p \\ \text{with } \ g_i > 0 \text{ continuous and nonincreasing on } \ \mathbb{R}_+ \text{ for each } \ i = 0, \dots, p-1 \\ \text{and } \ r \ge 0 \text{ continuous and nondecreasing on } \ [0,\infty), \end{cases} \tag{2.13}$$

$$\begin{cases} \qquad h(t, x_0, \dots, x_{p-1}) \ge \displaystyle\sum_{i=0}^{p-1} h_i(x_i) \quad \text{on } [0,1] \times \mathbb{R}_+^p \\ \text{with } \ h_i > 0 \text{ continuous and nonincreasing on } \mathbb{R}_+ \text{ for each } \ i = 0, \dots, p-1, \end{cases} \tag{2.14}$$

$$\begin{cases} \qquad \displaystyle\int_0^1 \varphi(t)\, g_i(k_i\, t^{p-i})\, \mathrm{d}t < \infty \quad \text{for each } i = 0, \dots, p-1, \\ \text{where } \ k_i > 0 \ (i = 0, \dots, p-1) \text{ are constants} \end{cases} \tag{2.15}$$

and

$$\begin{cases} \text{if } \ z > 0 \text{ satisfies } \ z \le a_0 + b_0\, r(z) \text{ for constants } \ a_0 \ge 0 \text{ and } \ b_0 \ge 0, \\ \text{then there exists a constant } \ K \text{ (which may depend only on } a_0 \text{ and } \ b_0) \\ \text{such that } \ z \le K. \end{cases} \tag{2.16}$$

The next result was proved by sequential technique and a nonlinear alternative of Leray-Schauder type ([81, Theorem 2.3]).

Theorem 2.4. ([13, Theorem 2.1]) *Suppose* (2.11) – (2.16) *hold.*

Then problem (2.10) *has a solution* $u \in C^{n-1}[0,1] \cap C^n(0,1)$ *with* $u^{(i)} > 0$ *on* $(0,1]$ *for* $0 \le i \le p-1$.

Example. Consider the problem

$$\begin{cases} (-1)^{n-p}\, u^{(n)} = \displaystyle\sum_{i=0}^{p-1}\Big(\frac{1}{(u^{(i)})^{\beta_i}} + \mu_i\,(u^{(i)})^{\alpha_i} + \tau_i\Big), \\ u^{(i)}(0) = 0,\ 0 \le i \le p-1, \quad u^{(j)}(1) = 0,\ p \le j \le n-1 \end{cases} \tag{2.17}$$

with $\beta_i \in (0,\infty)$, $\mu_i,\ \tau_i \in [0,\infty)$, $\alpha_i \in [0,1)$ for $0 \le i \le p-1$. In addition, assume $\beta_i\,(p-i) < 1$ for $i = 0,\dots,p-1$. Theorem 2.4 guarantees that problem (2.17) has a solution $u \in C^{n-1}[0,1] \cap C^n(0,1)$ with $u^{(i)} > 0$ on $(0,1]$ for $0 \le i \le p-1$.

2.2 . *Conjugate conditions*

Let $1 \le p \le n-1$ be a fixed natural number. Consider the $(p, n-p)$ conjugate problem

$$(-1)^p\, u^{(n)} = f(t, u, \dots, u^{(n-1)}), \tag{2.18}$$

$$u^{(i)}(0) = 0,\ 0 \le i \le n-p-1, \quad u^{(j)}(T) = 0,\ 0 \le j \le p-1 \tag{2.19}$$

where $n \ge 2$, $f \in Car([0,T] \times \mathcal{D})$, $\mathcal{D} = (0,\infty) \times (\mathbb{R} \setminus \{0\})^{n-1}$ and f may be singular at the value 0 of all its space variables.

Replacing t by $T-t$, if necessary, we may assume that $p \in \{1,\dots,\frac{n}{2}\}$ for n even and $p \in \{1,\dots,\frac{n+1}{2}\}$ for n odd. We observe that the larger p is chosen, the more complicated structure of the set of all singular points of a solution to (2.18), (2.19) and its derivatives is obtained. We note that solutions of problem (2.18), (2.19) have singular points of type I at $t = 0$ and/or $t = T$ and also singular points of type II. Since the singular problem (2.18), (2.19) for $n = 2$ is the Dirichlet problem discussed in Section 4, we assume that $n > 2$.

Theorem 2.5. ([127, Theorems 2.1 and 2.7] and [129]) *Let $n > 2$ and $1 \le p \le n-1$ be fixed natural numbers. Suppose that the following conditions are satisfied:*

$$\begin{cases} f \in Car([0,T] \times \mathcal{D}) \quad \textit{and there exists} \quad c > 0 \ \textit{such that} \\ \qquad c \le f(t, x_0, \dots, x_{n-1}) \quad \textit{for a.e. } t \in [0,T] \textit{ and all } (x_0,\dots,x_{n-1}) \in \mathcal{D}, \end{cases} \tag{2.20}$$

$$\begin{cases} h \in Car([0,T] \times [0,\infty)) \quad \textit{is nondecreasing in its second variable and} \\ \qquad \limsup\limits_{z\to\infty} \dfrac{1}{z}\displaystyle\int_0^T h(t,z)\,\mathrm{d}t < \Big(1 + \sum_{i=0}^{n-2} \frac{T^{n-i-1}}{(n-i-2)!}\Big)^{-1}, \end{cases} \tag{2.21}$$

$$\begin{cases} \omega_i\colon (0,\infty) \to (0,\infty) \quad \textit{are nonincreasing and} \\ \qquad \displaystyle\int_0^T \omega_i(t^{n-i})\,\mathrm{d}t < \infty \quad \textit{for} \ \ 0 \le i \le n-1, \end{cases} \tag{2.22}$$

$$\begin{cases} f(t, x_0, \dots, x_{n-1}) \le h\Big(t, \displaystyle\sum_{i=0}^{n-1} |x_i|\Big) + \sum_{i=0}^{n-1} \omega_i(|x_i|) \\ \qquad \textit{for a.e. } t \in [0,T] \textit{ and all } \ (x_0,\dots,x_{n-1}) \in \mathcal{D}. \end{cases} \tag{2.23}$$

Then the $(p,n-p)$ *conjugate problem* (2.18), (2.19) *has a solution* $u \in AC^{n-1}[0,T]$ *and* $u > 0$ *on* $(0,T)$.

Sketch of proof. *Step* 1. *Uniform integrability.*

Put

$$\mathcal{B} = \Big\{ u \in AC^{n-1}[0,T]\colon\ u \text{ satisfies } \ (2.19) \text{ and } \ (-1)^p\, u^{(n)}(t) \ge c \text{ for a.e. } t \in [0,T] \Big\},$$

where $c > 0$ is taken from (2.20). Conditions (2.20) and (2.22) guarantee that there exists a positive constant A such that

$$\int_0^T \omega_i(|u^{(i)}(t)|)\,\mathrm{d}t \le A \quad \text{for each } \ u \in \mathcal{B} \text{ and } \ 0 \le i \le n-1$$

and that the set of functions $\{\omega_i(|u^{(i)}(t)|)\colon\ u \in \mathcal{B},\ 0 \le i \le n-1\}$ is uniformly integrable on $[0,T]$. Also, $u > 0$ on $(0,T)$ for each $u \in \mathcal{B}$.

Step 2. *Estimates of functions belonging to* $\mathcal{B}$.

By virtue of (2.21), there exists $r^* > 1$ such that the estimate $\|u\|_{C^{n-1}} < r^*$ holds for each function $u \in \mathcal{B}$ satisfying

$$u^{(n)}(t) \le h\Big(t, n + \sum_{i=0}^{n-1} |u^{(i)}(t)|\Big) + \sum_{i=0}^{n-1} [\omega_i(|u^{(i)}(t)|) + \omega_i(1)] \quad \text{for a.e. } \ t \in [0,T].$$

Step 3. *Construction of regular problems to* (2.18), (2.19) *and properties of their solutions.*

For $m \in \mathbb{N}$, let $h_m \in Car([0,T] \times ([0,\infty) \times \mathbb{R}^{n-1}))$ be such that $h_m(t,x_0,\dots,x_{n-1}) = f(t,x_0,\dots,x_{n-1})$ for a.e. $t \in [0,T]$ and any $x_0 \ge \frac{1}{m}$, $|x_j| \ge \frac{1}{m}$, $1 \le j \le n-1$. Put

$$f_m(t,x_0,x_1,\dots,x_{n-1}) = h_m(t,\sigma_0(x_0),\sigma(x_1),\dots,\sigma(x_{n-1}))$$

for a.e. $t \in [0,T]$ and each $(x_0,x_1,\dots,x_{n-1}) \in \mathbb{R}^n$, where

$$\sigma_0(x) = \begin{cases} |x| & \text{if } \ |x| \le r^*, \\ r^* & \text{if } \ |x| > r^*, \end{cases} \qquad \sigma(x) = \begin{cases} x & \text{if } \ |x| \le r^*, \\ r^* \operatorname{sign} x & \text{if } \ |x| > r^*. \end{cases}$$

Now, the sequence of regular differential equations

$$(-1)^p\, u^{(n)} = f_m(t,u,\dots,u^{(n-1)}) \tag{2.24}$$

is considered. It follows from Theorem 2.1 that for each $m \in \mathbb{N}$ there exists a solution u_m of problem (2.24), (2.19) and $\|u_m\|_{C^{n-1}} < r^*$. Moreover, the sequence of functions $\{f_m(t,u_m(t),\dots,u_m^{(n-1)}(t))\}$ is uniformly integrable on $[0,T]$. By the Arzelà-Ascoli theorem there exists a subsequence $\{u_{k_m}\}$ converging in $C^{n-1}[0,T]$, $\lim_{m\to\infty} u_{k_m} = u$. Then $u \in C^{n-1}[0,T]$ satisfies (2.19) and for each $i \in \{1,\dots,n-1\}$, the function $u^{(n-i)}$ has a finite number of zeros $0 \le a_{i1} < \dots < a_{i,p_i} \le T$ and satisfies

$$|u^{(n-i)}(t)| \ge \frac{c}{i!}\,(t-a_{ik})^i \qquad \text{for } \ t \in [a_{ik}, a_{i,k+1}]$$

or

$$|u^{(n-i)}(t)| \geq \frac{c}{i!}\,(a_{i,k+1}-t)^i \quad \text{for} \ \ t \in [a_{ik}, a_{i,k+1}]$$

(see [129]). Therefore $u \in AC^{n-1}[0,T]$ and u is a solution of problem (2.18), (2.19) due to Theorem 1.8. From Step 1 and (2.20) it follows that $u > 0$ on $(0,T)$. $\square$

Example. Let $p \in \{1,\dots,n-1\}$. Let $\alpha_i \in (0,1)$, $\beta_i \in (0, \frac{1}{n-i})$ and $b_i \in L_1[0,T]$, $c_i \in L_\infty[0,T]$ be nonnegative for $0 \leq i \leq n-1$. Also, let $\varphi \in L_1[0,T]$ and $\varphi(t) \geq c$ for a.e. $t \in [0,T]$ with $c > 0$. Then the differential equation

$$(-1)^p\, u^{(n)} = \varphi(t) + \sum_{i=0}^{n-1}\Big(b_i(t)\,|u^{(i)}|^{\alpha_i} + \frac{c_i(t)}{|u^{(i)}|^{\beta_i}}\Big)$$

has a solution $u \in AC^{n-1}[0,T]$ satisfying (2.19) and $u > 0$ on $(0,T)$.

2.3 . (n,p) *boundary conditions*

Here we are concerned with the singular (n,p) problem

$$-u^{(n)} = f(t,u,\dots,u^{(n-1)}), \tag{2.25}$$

$$u^{(i)}(0) = 0, \ 0 \leq i \leq n-2, \quad u^{(p)}(T) = 0, \ p \text{ fixed}, \ 0 \leq p \leq n-1, \tag{2.26}$$

where $n \geq 2$, $f \in Car([0,T] \times \mathcal{D})$, $\mathcal{D} = (0,\infty) \times (\mathbb{R} \setminus \{0\})^{n-2} \times \mathbb{R}$ and f may be singular at the value 0 of its space variables $x_0,\dots,x_{n-2}$. Notice that the $(n,0)$ problem is simultaneously also the $(1,n-1)$ conjugate problem. For $f > 0$, solutions of problem (2.25), (2.26) have singular points of type I at $t = 0$, $t = T$ and also singular points of type II.

Theorem 2.6. ([15, Theorem 4.2]) *Suppose*

$$\left\{\begin{array}{l} f \in Car([0,T] \times \mathcal{D}) \ \textit{and there exist positive} \ \ \psi \in L_1[0,T] \ \textit{and} \\ K \in (0,\infty) \ \textit{such that} \ \ \psi(t) \leq f(t,x_0,\dots,x_{n-1}) \ \textit{for a.e.}\ t \in [0,T] \\ \textit{and each}\ (x_0,\dots,x_{n-1}) \in (0,K] \times (\mathbb{R}\setminus\{0\})^{n-2} \times \mathbb{R}, \end{array}\right. \tag{2.27}$$

$$\left\{\begin{array}{l} h_j \in L_1[0,T] \quad \textit{is nonnegative,} \quad \omega_i\colon (0,\infty) \to (0,\infty) \ \textit{is nonincreasing and} \\ \qquad \displaystyle\sum_{k=0}^{n-1} \frac{1}{(n-k-1)!} \int_0^T h_k(t) t^{n-k-1}\,\mathrm{d}t < 1, \quad \int_0^T \omega_i(t^{n-i-1})\,\mathrm{d}t < \infty \\ \textit{for} \ \ 0 \leq j \leq n-1 \ \textit{and} \ \ 0 \leq i \leq n-2, \end{array}\right. \tag{2.28}$$

$$\left\{\begin{array}{l} \textit{for a.e.} \ \ t \in [0,T] \ \textit{and each} \ \ (x_0,\dots,x_{n-1}) \in \mathcal{D} \quad \textit{we have} \\ \qquad \displaystyle f(t,x_0,\dots,x_{n-1}) \leq \varphi(t) + \sum_{i=0}^{n-2} \omega_i(|x_i|) + \sum_{j=0}^{n-1} h_j(t)\,|x_j| \\ \textit{where} \ \ \varphi \in L_1[0,T] \ \textit{is nonnegative.} \end{array}\right. \tag{2.29}$$

Then there exists a solution $u \in AC^{n-1}[0,T]$ *of problem* (2.25), (2.26) *with* $u^{(i)} > 0$ *on* $(0,T]$ *for* $0 \le i \le p-1$ *and* $u^{(p)} > 0$ *on* $(0,T)$.

Sketch of proof. *Step* 1. *A priori bounds for solutions of problem* (2.25), (2.26).

Upper and lower bounds for solutions of problem (2.25), (2.26) and their derivatives are given by means of the Green function of the problem $-u^{(n)} = 0$, (2.26) (see e.g. [1]).

Step 2. *Construction of auxiliary regular problems and properties of their solutions.*

Using Step 1, a sequence of regular differential equations

$$-u^{(n)} = f_m(t, u, \dots, u^{(n-1)}), \quad m \in \mathbb{N},\ m \ge m_0 \ge \frac{1}{K}, \tag{2.30}$$

with $f_m \in Car([0,T] \times \mathbb{R}^n)$ is constructed. By the Leray-Schauder degree theory, we prove that for any $m \ge m_0$ problem (2.30), (2.26) has a solution u_m. The sequence $\{u_m\}_{m=m_0}^{\infty}$ is bounded in $C^{n-1}[0,T]$ and $\{u_m^{(n-1)}\}_{m=m_0}^{\infty}$ is equicontinuous on $[0,T]$. The Arzelà-Ascoli theorem guarantees the existence of a subsequence $\{u_k\}_{k=1}^{\infty}$ of $\{u_m\}_{m=m_0}^{\infty}$ converging in $C^{n-1}[0,T]$ to a function u. Then $u \in C^{n-1}[0,T]$ satisfies (2.26) and $u^{(i)} > 0$ on $(0,T]$ for $0 \le i \le p-1$ (if $p \ge 1$) and $u^{(p)} > 0$ on $(0,T)$. Moreover, for each $i \in \{p+1, \dots, n-2\}$, the function $u^{(i)}$ has a unique zero ξ_i in $(0,T)$ ($0 < \xi_{n-2} < \xi_{n-1} < \dots < \xi_{p+1} < T$) and satisfies

$$u^{(i)}(t) \ge \begin{cases} c\,t^{n-i-1} & \text{for } t \in [0, \xi_{i+1}], \\ c\,(\xi_i - t) & \text{for } t \in [\xi_{i+1}, \xi_i], \end{cases} \qquad u^{(i)}(t) \le c\,(\xi_i - t) \quad \text{for } t \in [\xi_i, T]$$

where c is a positive constant. Since $\{f_k(t, u_k(t), \dots, u_k^{(n-1)}(t))\}$ is uniformly integrable on $[0,T]$, we can use Theorem 1.8 concluding that $u \in AC^{n-1}[0,T]$ and u is a solution of problem (2.25), (2.26). □

A related existence result for the singular (n,p) problem

$$\begin{cases} -u^{(n)} = \varphi(t)\, h(t, u, \dots, u^{(p-1)}), \\ u^{(i)}(0) = 0,\ 0 \le i \le n-2, \quad u^{(p)}(1) = 0,\ p \text{ fixed},\ 1 \le p \le n-1 \end{cases} \tag{2.31}$$

was presented in [13] with h continuous and positive on $[0,1] \times (0,\infty)^p$ and $\varphi \in C^0(0,1) \cap L_1[0,1]$ positive on $(0,1)$. In this setting solutions of problem (2.31) cannot have singular points of type II. The result is the following.

Theorem 2.7. ([13, Theorem 3.1]) *Suppose that* (2.11) – (2.14) *and* (2.16) *hold and*

$$\begin{cases} \displaystyle\int_0^1 \varphi(t)\, g_i(k_i\, t^{n-1-i})\, \mathrm{d}t < \infty \quad \text{for each } i = 0, \dots, p-1, \\ \text{where} \quad k_i > 0,\ i = 0, \dots, p-1,\ \text{are constants.} \end{cases}$$

Then problem (2.31) *has a solution* $u \in C^{n-1}[0,1] \cap C^n(0,1)$ *with* $u^{(j)} > 0$ *on* $(0,1]$ *for* $0 \le j \le p-1$.

2.4. *Sturm-Liouville conditions*

We are now concerned with the Sturm-Liouville problem for the n-order differential equation (2.25), $n \geq 3$, and the boundary conditions

$$\begin{cases} u^{(i)}(0) = 0, \quad 0 \leq i \leq n-3, \\ \alpha\, u^{(n-2)}(0) - \beta\, u^{(n-1)}(0) = 0, \\ \gamma\, u^{(n-2)}(T) + \delta\, u^{(n-1)}(T) = 0, \end{cases} \tag{2.32}$$

where $\alpha,\ \gamma > 0$ and $\beta,\ \delta \geq 0$. Notice that the function f in equation (2.25) may be singular at the value 0 of its space variables $x_0, \ldots, x_{n-1}$. If $f > 0$, solutions of problem (2.25), (2.32) have singular points of type I at the end points of the interval $[0, T]$ and also singular points of type II.

We will impose the following conditions on the function f in (2.25):

$$\begin{cases} f \in Car([0,T] \times \mathcal{D}) \quad \text{where} \ \ \mathcal{D} = (0,\infty)^{n-1} \times (\mathbb{R} \setminus \{0\}) \\ \text{and there exist positive constants } \varepsilon \text{ and } \ r \text{ such that} \\ \qquad \varepsilon\, t^r \leq f(t, x_0, \ldots, x_{n-1}) \\ \text{for a.e. } t \in [0,T] \text{ and each } \ (x_0, \ldots, x_{n-1}) \in \mathcal{D}; \end{cases} \tag{2.33}$$

$$\begin{cases} \text{for a.e. } \ t \in [0,T] \text{ and each } \ (x_0, \ldots, x_{n-1}) \in \mathcal{D}, \\ \qquad f(t, x_0, \ldots, x_{n-1}) \leq \varphi(t) + \displaystyle\sum_{i=0}^{n-1} \omega_i(|x_i|) + \sum_{i=0}^{n-1} h_i(t)\, |x_i|^{\alpha_i} \\ \text{with } \ \alpha_i \in (0,1), \ \ \varphi, h_i \in L_1[0,T] \text{ nonnegative,} \\ \omega_i \colon \ (0,\infty) \to (0,\infty) \text{ nonincreasing, } 0 \leq i \leq n-1, \text{ and} \\ \qquad \displaystyle\int_0^T \omega_{n-1}(t^{r+1})\, \mathrm{d}t < \infty, \quad \int_0^T \omega_i(t^{n-i-1})\, \mathrm{d}t < \infty \quad \text{for } \ 0 \leq i \leq n-2; \end{cases} \tag{2.34}$$

$$\begin{cases} \text{for a.e. } t \in [0,T] \text{ and each } \ (x_0, \ldots, x_{n-1}) \in \mathcal{D}, \\ f(t, x_0, \ldots, x_{n-1}) \leq \varphi(t) + \displaystyle\sum_{i=0, i \neq n-2}^{n-1} \omega_i(|x_i|) + q(t)\, \omega_{n-2}\,(|x_{n-2}|) + \sum_{i=0}^{n-1} h_i(t)\, |x_i|^{\alpha_i} \\ \text{with } \alpha_i \in (0,1), \ \ \varphi, \ q, \ h_i \in L_1[0,T] \text{ nonnegative,} \\ \omega_i \colon \ (0,\infty) \to (0,\infty) \text{ nonincreasing, } 0 \leq i \leq n-1, \text{ and} \\ \qquad \displaystyle\int_0^T \omega_{n-1}(t^{r+1})\, \mathrm{d}t < \infty, \quad \int_0^T \omega_j(t^{n-j-2})\, \mathrm{d}t < \infty \quad \text{for } \ 0 \leq j \leq n-3. \end{cases} \tag{2.35}$$

The next two theorems show that our sufficient conditions for the solvability of problem (2.25), (2.32) with $\min\{\beta, \delta\} > 0$ are weaker then those for this problem with $\min\{\beta, \delta\} = 0$.

Theorem 2.8. ([126, Theorem 4.1]) *Let conditions* (2.33) *and* (2.34) *be satisfied and let* $\min\{\beta,\delta\}=0$.

Then problem (2.25), (2.32) *has a solution* $u\in AC^{n-1}[0,T]$, $u^{(n-2)}>0$ *on* $(0,T)$ *and* $u^{(j)}>0$ *on* $(0,T]$ *for* $0\le j\le n-3$.

Sketch of proof. *Step* 1. *A priori bounds for the solution of* (2.25), (2.32).

By (2.33) and by the properties of the Green function to problem

$$-u''=0,\quad \alpha\,u(0)-\beta\,u'(0)=0,\ \gamma\,u(T)+\delta\,u'(T)=0,$$

the existence of a positive constant A is proved such that for any function $u\in AC^{n-1}[0,T]$ satisfying (2.32) and $-u^{(n)}(t)\ge\varepsilon\,t^r$ for a.e. $t\in[0,T]$ we have

$$u^{(n-2)}(t)\ge\begin{cases} A\,t & \text{for }\ t\in[0,\frac{T}{2}],\\ A\,(T-t) & \text{for }\ t\in(\frac{T}{2},T],\end{cases}\tag{2.36}$$

$$u^{(j)}(t)\ge\frac{A}{4\,(n-j-1)!}\,t^{n-j-1}\quad\text{for }\ t\in[0,T]\text{ and }\ 0\le j\le n-3\tag{2.37}$$

and

$$u^{(n-1)}(t)\begin{cases}\ge\dfrac{\varepsilon}{r+1}\,(\xi-t)^{r+1} & \text{for }\ t\in[0,\xi],\\ <-\dfrac{\varepsilon}{r+1}\,(t-\xi)^{r+1} & \text{for }\ t\in(\xi,T],\end{cases}\tag{2.38}$$

where $\xi\in(0,T)$ (depending on the solution u) is the unique zero of $u^{(n-1)}$. Condition (2.34) guarantees the existence of a positive constant S such that $\|u\|_{C^{n-1}}\le S$ for any solution u to (2.25), (2.32).

Step 2. *Construction of regular differential equations.*

Using Step 1, a sequence of regular differential equations (2.30) is constructed where $f_m\in Car([0,T]\times\mathbb{R}^n)$,

$$f_m(t,x_0,\dots,x_{n-1})=f(t,x_0,\dots,x_{n-1})\quad\text{for a.e. }t\in[0,T]\text{ and all }(x_0,\dots,x_{n-1})\in\mathbb{R}^n$$

such that

$$\frac{1}{m}\le x_j\le S+1\quad\text{if}\ \ 0\le j\le n-2,\qquad \frac{1}{m}\le|x_{n-1}|\le S+1$$

and

$$\sup\{f_m(t,x_0,\dots,x_{n-1})\colon\,(x_0,\dots,x_{n-1})\in\mathbb{R}^n\}\in L_1[0,T]\quad\text{for all }m\in\mathbb{N}.$$

Then Theorem 2.1 guarantees that the regular problem (2.30), (2.32) has a solution u_m which satisfies (2.36)-(2.38) (with u_m instead of u).

Step 3. *Properties of solutions to regular problems* (2.30, (2.32).

The sequence $\{u_m\}$ is considered. It is proved that $\{u_m\}$ is bounded in $C^{n-1}[0,T]$ and, by (2.34), the sequence of functions $\{f_m(t,u_m(t),\dots,u_m^{(n-1)}(t))\}$ is uniformly integrable on $[0,T]$, which implies that $\{u_m^{(n-1)}\}$ is equicontinuous on $[0,T]$. Hence a subsequence $\{u_{k_m}\}$ converging in $C^{n-1}[0,T]$ exists and let $\lim_{m\to\infty} u_{k_m} = u$. Since u satisfies (2.36)-(2.38), the functions $u^{(i)}$, $0 \le i \le n-1$, have a finite number of zeros in $[0,T]$. Therefore, by Theorem 1.8, $u \in AC^{n-1}[0,T]$ and u is a solution of problem (2.25), (2.32) such that $u^{(j)} > 0$ on $(0,T]$ for $0 \le j \le n-3$ and $u^{(n-2)} > 0$ on $(0,T)$. □

THEOREM 2.9. ([126, Theorem 4.2]) *Let conditions* (2.33) *and* (2.35) *be satisfied and let* $\min\{\beta,\delta\} > 0$.

Then there exists a solution $u \in AC^{n-1}[0,T]$ *of problem* (2.25), (2.32) *such that* $u^{(n-2)} > 0$ *on* $[0,T]$ *and* $u^{(j)} > 0$ *on* $(0,T]$ *for* $0 \le j \le n-3$.

SKETCH OF PROOF. Since $\min\{\beta,\delta\} > 0$, there is a positive constant B such that $u^{(n-2)} \ge B$ on $[0,T]$ for any solution u of problem (2.25), (2.32). Further, the inequalities (2.37) with B instead of A and (2.38) hold. Next we argue as in the sketch of proof to Theorem 2.8. □

2.5 . *Lidstone conditions*

Let $\mathbb{R}_- = (-\infty, 0)$, $\mathbb{R}_+ = (0,\infty)$ and $\mathbb{R}_0 = \mathbb{R} \setminus \{0\}$. Here we will consider the singular problem

$$(-1)^n\, u^{(2n)} = f(t, u, \dots, u^{(2n-2)}), \tag{2.39}$$

$$u^{(2j)}(0) = u^{(2j)}(T) = 0, \quad 0 \le j \le n-1, \tag{2.40}$$

where $n \ge 1$ and $f \in Car([0,T] \times \mathcal{D})$ with

$$\mathcal{D} = \begin{cases} \underbrace{\mathbb{R}_+ \times \mathbb{R}_0 \times \mathbb{R}_- \times \mathbb{R}_0 \times \cdots \times \mathbb{R}_+}_{4k-3} & \text{if } \; n = 2k-1 \\ \underbrace{\mathbb{R}_+ \times \mathbb{R}_0 \times \mathbb{R}_- \times \mathbb{R}_0 \times \cdots \times \mathbb{R}_-}_{4k-1} & \text{if } \; n = 2k \end{cases}$$

(for $n = 1, 2$ and 3 we have $\mathcal{D} = \mathbb{R}_+$, $\mathcal{D} = \mathbb{R}_+ \times \mathbb{R}_0 \times \mathbb{R}_-$ and $\mathcal{D} = \mathbb{R}_+ \times \mathbb{R}_0 \times \mathbb{R}_- \times \mathbb{R}_0 \times \mathbb{R}_+$, respectively). The function f may be singular at the value 0 of its space variables $x_0, \dots, x_{2n-2}$. If f is positive on $[0,T] \times \mathcal{D}$, solutions of problem (2.39), (2.40) have singular points of type I at $t = 0$ and $t = T$ as well as singular points of type II.

THEOREM 2.10. ([15, Theorem 4.1]) *Let the following conditions be satisfied:*

$$\begin{cases} f \in Car([0,T] \times \mathcal{D}) \text{ and there exists } \varphi \in L_1[0,T] \text{ such that} \\ \qquad 0 < \varphi(t) \le f(t, x_0, \dots, x_{2n-2}) \\ \text{for a.e. } t \in [0,T] \text{ and each } (x_0, \dots, x_{2n-2}) \in \mathcal{D}; \end{cases} \tag{2.41}$$

$$\left\{\begin{array}{l} \textit{for}\ \ 0 \le j \le 2n-2,\ \ h_j \in L_1[0,T]\ \ \textit{are nonnegative and} \\ \qquad \displaystyle\sum_{j=0}^{n-1} \frac{T^{2(n-j)-3}}{6^{n-j-1}} \int_0^T t\,(T-t)\,h_{2j}(t)\,\mathrm{d}t \\ \qquad\qquad \displaystyle +\sum_{j=0}^{n-2} \frac{T^{2(n-j-2)}}{6^{n-j-2}} \int_0^T t\,(T-t)\,h_{2j+1}(t)\,\mathrm{d}t < 1 \\ (\textit{here}\ \ \sum_{j=0}^{n-2} = 0 \quad \textit{if}\ \ n=1); \end{array}\right. \tag{2.42}$$

$$\left\{\begin{array}{l} \omega_j\colon \mathbb{R}_+ \to \mathbb{R}_+ \quad \textit{are nonincreasing,}\ \Lambda > 0\ \ \textit{and} \\ \qquad \displaystyle\int_0^T \omega_j(s)\,\mathrm{d}s < \infty, \quad \omega_j(u\,v) \le \Lambda\,\omega_j(u)\,\omega_j(v) \\ \textit{for}\ \ 0 \le j \le 2n-2\ \ \textit{and}\ \ u,v \in \mathbb{R}_+; \end{array}\right. \tag{2.43}$$

$$\left\{\begin{array}{l} \qquad \displaystyle f(t,x_0,\dots,x_{2n-2}) \le \psi(t) + \sum_{j=0}^{2n-2} \omega_j(|x_j|) + \sum_{j=0}^{2n-2} h_j(t)\,|x_j| \\ \textit{for a.e.}\ t \in [0,T]\ \textit{and each}\ (x_0,\dots,x_{2n-2}) \in \mathcal{D}, \\ \textit{where}\ \ \psi \in L_1[0,T]\ \ \textit{is nonnegative.} \end{array}\right. \tag{2.44}$$

Then problem (2.39), (2.40) *has a solution* $u \in AC^{2n-1}[0,T]$ *and*

$$(-1)^j\,u^{(2j)} > 0 \quad \textit{on}\ \ (0,T) \quad \textit{for}\ \ 0 \le j \le n-1.$$

Sketch of proof. *Step* 1. *A priori bounds for solutions of problem* (2.39), (2.40).

Using (2.41) and the properties of the Green functions to problems $u^{(2j)} = 0$, $u^{(2i)}(0) = u^{(2i)}(T) = 0$, $0 \le i \le j-1$, it is proved that for any solution u of problem (2.39), (2.40) and for each j, $0 \le j \le n-1$, the inequality $(-1)^j\,u^{(2j)} > 0$ holds on $(0,T)$ and the function $(-1)^j\,u^{(2j+1)}$ is decreasing on $[0,T]$ and vanishes at a unique $\xi_j \in (0,T)$ (depending on u). Moreover,

$$|u^{(2j)}(t)| \ge A\,\frac{T^{2(n-j)-5}}{30^{n-j-1}}\,t\,(T-t), \quad t \in [0,T],\ 0 \le j \le n-1,$$

and (if $n > 1$)

$$|u^{(2j+1)}(t)| \ge A\,\frac{T^{2(n-j)-7}}{30^{n-j-2}}\,\Big|\int_{\xi_j}^t s\,(T-s)\,\mathrm{d}s\Big|, \quad t \in [0,T],\ 0 \le j \le n-2,$$

where $A = \displaystyle\int_0^T t\,(T-t)\,\varphi(t)\,\mathrm{d}t.$

Step 2. *Construction of a sequence of regular problems.*

For each $m \in \mathbb{N}$, define $f_m \in Car([0,T] \times \mathbb{R}^{2n-1})$ satisfying

$$f_m(t,x_0,\dots,x_{2n-2}) = f(t,x_0,\dots,x_{2n-2})$$

for a.e. $t \in [0,T]$ and each $(x_0,\dots,x_{2n-2}) \in \mathcal{D}$, $|x_j| \ge \frac{1}{m}$, $0 \le j \le 2n-2$. By virtue of (2.44) and a fixed point theorem of Leray-Schauder type (see e.g. [54, Corollary 8.1]), for each $m \in \mathbb{N}$ there exists a solution u_m of the regular differential equation $(-1)^n u^{(2n)} = f_m(t,u,\dots,u^{(2n-2)})$ satisfying (2.40). Further, $\|u_m\|_{C^{2n-1}} \le B$ for each $m \in \mathbb{N}$ where B is a positive constant and the sequence $\{f_m(t,u_m(t),\dots,u_m^{(2n-2)}(t))\}$ is uniformly integrable on $[0,T]$ due to (2.43) and (2.44).

Step 3. *Limit processes.*

From Step 2 it follows that $\{u_m\}$ is bounded in $C^{2n-1}[0,T]$. Hence, by the Arzelà-Ascoli theorem and a compactness principle, there exists its subsequence $\{u_{k_m}\}$ which converges in $C^{2n-2}[0,T]$ and $\{u_{k_m}^{(2n-1)}(0)\}$ converges in $\mathbb{R}$. Let $\lim_{m\to\infty} u_{k_m} = u$, $\lim_{m\to\infty} u_{k_m}^{(2n-1)}(0) = C$. Then $u \in C^{2n-2}[0,T]$ satisfies (2.40) and, by Step 1, the functions $u^{(i)}$, $0 \le i \le 2n-1$, have a finite number of zeros on $[0,T]$. Therefore, by Theorem 1.8, $u \in AC^{2n-1}[0,T]$ and u is a solution of (2.39). Moreover, $(-1)^j u^{(2j)} > 0$ on $(0,T)$ for $0 \le j \le n-1$. $\square$

2.6 . *Historical and bibliographical notes*

Higher order boundary value problems with space singularities have been mostly studied by Agarwal, Eloe, Henderson, Lakshmikantham, O'Regan, Rachůnková and Staněk.

Positive solutions in the set $C^{n-1}[0,1] \cap C^n(0,1)$ were obtained in [7] for the singular $(p,n-p)$ right focal problem $(-1)^{n-p} u^{(n)} = \varphi(t)\, f(t,u)$, (2.6) on the interval $[0,1]$ where $f \in C^0([0,1]\times(0,\infty))$ is positive and may be singular at $u=0$. In [7] the authors also discussed applications in fluid theory and boundary layer theory.

Singular $(p,n-p)$ conjugate problems were studied in [5], [62] (with $p=1$) and [63] for the differential equation $(-1)^{n-p} u^{(n)} = f(t,u)$ where $f \in C^0((0,1)\times(0,\infty))$ and may be singular at $u=0$. Here positive solutions on $(0,1)$ belong to the class $C^{n-1}[0,1]\cap C^n(0,1)$. Existence results in [62] and [63] are proved by a fixed point theorem for operators that are decreasing with respect to a cone and those in [5] by a nonlinear alternative of Leray-Schauder.

The existence of positive solutions on $(0,1)$ to singular Sturm-Liouville problems for the differential equation $-u^{(n)} = f(t,u,\dots,u^{(n-2)})$ can be found in [21]. There $f \in C((0,1)\times(0,\infty)^{n-1})$ is positive and may be singular at the value 0 of all its space variables. The results are proved by a fixed point theorem for mappings that are decreasing with respect to a cone in a Banach space.

Existence results for positive solutions to singular $(p,n-p)$ focal, conjugate and (n,p) problems are given in [8] and [9] for differential equations with the right-hand side $\varphi(t)\, f(t,u)$ where $f \in C([0,1]\times(0,\infty))$ and may be singular at $u=0$. The paper [8] is the first to establish conditions for the existence of two solutions to singular $(p,n-p)$ focal and (n,p) problems. Further multiple solutions for singular $(p,n-p)$ focal, conjugate and (n,p) problems are established in [9]. The technique presented in [8] and [9] to guarantee the existence of twin solutions to the singular problems combines (i) a nonlinear alternative of Leray-Schauder, (ii) the Krasnoselskii fixed point theorem in a cone, and (iii) lower type inequalities.

Notice that in all cited papers singular points corresponding to solutions of the singular problems under discussion are only of type I.

3. Principles of solvability of singular second order BVPs with ϕ-Laplacian

In the theory of partial differential equations, the p-Laplace equation

$$\operatorname{div}\left(|\nabla v|^{p-2}\,\nabla v\right) = h(|x|, v) \tag{3.1}$$

is considered. Here ∇ is the gradient, $p > 1$ and $|x|$ is the Euclidean norm in $\mathbb{R}^n$ of $x = (x_1, \ldots, x_n)$, $n > 1$. Radially symmetric solutions of (3.1) (i.e., solutions that depend only on the variable $r = |x|$) satisfy the ordinary differential equation

$$r^{1-n}\,(r^{n-1}\,|v'|^{p-2}\,v')' = h(r, v), \quad ' = \frac{\mathrm{d}}{\mathrm{d}r}. \tag{3.2}$$

If $p = n$, the change of variables $t = \ln r$ transforms (3.2) into the equation

$$(|u'|^{p-2}\,u')' = e^{n\,t}\,h(e^t, u), \quad ' = \frac{\mathrm{d}}{\mathrm{d}t}$$

and for $p \neq n$, the change of variables $t = r^{(p-n)/(p-1)}$ transforms (3.2) into the equation

$$(|u'|^{p-2}\,u')' = \left|\frac{p-1}{p-n}\right|^p t^{\frac{p-n}{p\,(1-n)}}\,h(t^{\frac{p-1}{p-n}}, u), \quad ' = \frac{\mathrm{d}}{\mathrm{d}t}.$$

The operator $u \to (|u'|^{p-2}\,u')'$ is called the (one-dimensional) *p-Laplacian*. Its natural generalization is the *ϕ-Laplacian*

$$u \to (\phi(u'))', \text{ where } \phi\colon \mathbb{R} \to \mathbb{R} \text{ is an increasing homeomorphism and } \phi(\mathbb{R}) = \mathbb{R}. \tag{3.3}$$

Therefore the equation (3.1) was a motivation for discussing the solutions to the differential equations

$$(|u'|^{p-2}\,u')' = f(t, u, u')$$

and

$$(\phi(u'))' = f(t, u, u')$$

with the p-Laplacian and the ϕ-Laplacian, respectively.

In the next part of this section, we treat problems for second order differential equations with the ϕ-Laplacian on the left hand side and with nonlinearities on the right hand sides which can have singularities in their space variables. Boundary conditions under discussion are generally nonlinear and nonlocal. Using regularization and sequential techniques we present general existence principles for solvability of regular and singular problems.

3.1 . *Regularization of singular problems with ϕ-Laplacian*

We discuss singular differential equations of the form

$$(\phi(u'))' = f(t, u, u') \tag{3.4}$$

with the ϕ-Laplacian. Here $f \in Car([0,T] \times \mathcal{D})$, the set $\mathcal{D} = \mathcal{D}_1 \times \mathcal{D}_2 \subset \mathbb{R}^2$ is not necessarily closed, $\mathcal{D}_1$, $\mathcal{D}_2$ are intervals and f may have singularities in its space variables on the boundary $\partial\mathcal{D}_j$ of $\mathcal{D}_j$ $(j = 1, 2)$. We note that f has a singularity on $\partial\mathcal{D}_j$ in its space variable x_j if there is an $a_j \in \partial\mathcal{D}_j$ such that for a.e. $t \in [0, T]$ and some $x_{3-j} \in \mathcal{D}_{3-j}$,

$$\limsup_{x_j \to a_j,\, x_j \in \mathcal{D}_j} |f(t, x_1, x_2)| = \infty.$$

Let $\mathcal{A}$ denote the set of functionals $\alpha\colon C^1[0,T] \to \mathbb{R}$ which are

(a) continuous and

(b) bounded, that is $\alpha(\Omega)$ is bounded (in $\mathbb{R}$) for any bounded $\Omega \subset C^1[0,T]$.

For α, $\beta \in \mathcal{A}$, consider the (generally nonlinear and nonlocal) boundary conditions

$$\alpha(u) = 0, \quad \beta(u) = 0. \tag{3.5}$$

Definition 3.1. A function $u : [0,T] \to \mathbb{R}$ is said to be a *solution of problem* (3.4), (3.5) if $\phi(u') \in AC[0,T]$, u satisfies the boundary conditions (3.5) and $(\phi(u'(t)))' = f(t, u(t), u'(t))$ holds for a.e. $t \in [0,T]$.

Special cases of the boundary conditions (3.5) are the Dirichlet (Neumann; mixed; periodic and Sturm-Liouville type) boundary conditions which we get setting $\alpha(x) = x(0)$, $\beta(x) = x(T)$ $(\alpha(x) = x'(0)$, $\beta(x) = x'(T)$; $\alpha(x) = x(0)$, $\beta(x) = x'(T)$; $\alpha(x) = x(0) - x(T)$, $\beta(x) = x'(0) - x'(T)$ and $\alpha(x) = a_0\, x(0) + a_1\, x'(0)$, $\beta(x) = b_0\, x(T) + b_1\, x'(T))$.

In order to obtain an existence result for problem (3.4), (3.5), we use regularization and sequential techniques. For this purpose consider a sequence of regular differential equations

$$(\phi(u'))' = f_n(t, u, u') \tag{3.6}$$

where $f_n \in Car([0,T] \times \mathbb{R}^2)$. The function f_n is constructed in such a way that

$$f_n(t, x, y) = f(t, x, y) \quad \text{for a.e. } t \in [0,T] \text{ and all } (x, y) \in \mathcal{Q}_n$$

where $\mathcal{Q}_n \subset \mathcal{D}$ and roughly speaking $\mathcal{Q}_n$ converges to $\mathcal{D}$ as $n \to \infty$.

Let $h \in Car([0,T] \times \mathbb{R}^2)$ and consider the regular differential equation

$$(\phi(u'))' = h(t, u, u'). \tag{3.7}$$

A function $u\colon [0,T] \to \mathbb{R}$ is called a *solution of the regular problem* (3.7), (3.5) if $\phi(u') \in AC[0,T]$, u satisfies (3.5) and $(\phi(u'(t)))' = h(t, u(t), u'(t))$ for a.e. $t \in [0,T]$.

The next general existence principle can be used for solving the regular problem (3.7), (3.5).

Theorem 3.2. (**General existence principle for regular problems**) *Assume* (3.3), $h \in Car([0,T]\times\mathbb{R}^2)$ *and* $\alpha,\ \beta \in \mathcal{A}$. *Suppose there exist positive constants* S_0 *and* S_1 *such that*

$$\|u\|_\infty < S_0, \quad \|u'\|_\infty < S_1$$

for all solutions u *to the problem*

$$(\phi(u'))' = \lambda\, h(t,u,u'), \quad \alpha(u) = 0,\ \beta(u) = 0 \tag{3.8}$$

and each $\lambda \in [0,1]$. *Also assume there exist positive constants* Λ_0 *and* Λ_1 *such that*

$$|A| < \Lambda_0, \quad |B| < \Lambda_1 \tag{3.9}$$

for all solutions $(A,B) \in \mathbb{R}^2$ *of the system*

$$\begin{cases} \alpha\,(A + B\,t) - \mu\,\alpha\,(-A - B\,t) = 0, \\ \beta\,(A + B\,t) - \mu\,\beta(-A - B\,t) = 0 \end{cases} \tag{3.10}$$

and each $\mu \in [0,1]$.

Then problem (3.7), (3.5) *has a solution.*

Proof. Set

$$\Omega = \Big\{x \in C^1[0,T]\colon\ \|x\|_\infty < \max\{S_0, \Lambda_0 + \Lambda_1\,T\},\ \|x'\|_\infty < \max\{S_1,\Lambda_1\}\Big\}.$$

Then Ω is an open, bounded and symmetric with respect to $0 \in C^1[0,T]$ subset of the Banach space $C^1[0,T]$. Define an operator $\mathcal{P}\colon\ [0,1]\times\overline{\Omega} \to C^1[0,T]$ by the formula

$$\begin{cases} \mathcal{P}(\lambda,x)(t) = x(0) + \alpha(x) \\ \qquad\qquad + \displaystyle\int_0^t \phi^{-1}\Big(\phi(x'(0)+\beta(x)) + \lambda\int_0^s h(v,x(v),x'(v))\,\mathrm{d}v\Big)\,\mathrm{d}s. \end{cases} \tag{3.11}$$

A standard argument shows that $\mathcal{P}$ is a continuous operator. We claim that $\mathcal{P}([0,1]\times\overline{\Omega})$ is compact in $C^1[0,T]$. Indeed, since $\overline{\Omega}$ is bounded in $C^1[0,T]$, we have

$$|\alpha(x)| \le r, \quad |\beta(x)| \le r, \quad |h(t,x(t),x'(t))| \le \varrho(t)$$

for a.e. $t \in [0,T]$ and $x \in \overline{\Omega}$, where r is a positive constant and $\varrho \in L_1[0,T]$. Set $K = \max\{S_1,\Lambda_1\} + r$ and $V = \max\{|\phi(-K)|, |\phi(K)|\}$. Then

$$|\mathcal{P}(\lambda,x)(t)| \le \max\{S_0,\Lambda_0+\Lambda_1\,T\} + r + T\,\max\{|\phi^{-1}(-V - \|\varrho\|_1)|, |\phi^{-1}(V + \|\varrho\|_1)|\}$$

$$|\mathcal{P}(\lambda,x)'(t)| \le \max\{|\phi^{-1}(-V - \|\varrho\|_1)|, |\phi^{-1}(V + \|\varrho\|_1)|\}$$

and

$$\big|\phi\big(\mathcal{P}(\lambda,x)'(t_2)\big) - \phi\big(\mathcal{P}(\lambda,x)'(t_1)\big)\big| \le \Big|\int_{t_1}^{t_2} \varrho(t)\,\mathrm{d}t\Big|$$

for t, t_1, $t_2 \in [0,T]$ and $(\lambda, x) \in [0,1] \times \overline{\Omega}$. Hence $\mathcal{P}([0,1] \times \overline{\Omega})$ is bounded in $C^1[0,T]$ and $\{\phi[\mathcal{P}(\lambda,x)']\}$ is equicontinuous on $[0,T]$. The mapping ϕ^{-1} being an increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$, we deduce from

$$\big|\mathcal{P}(\lambda,x)'(t_2) - \mathcal{P}(\lambda,x)'(t_1)\big| = \big|\phi^{-1}\big(\phi(\mathcal{P}(\lambda,x)'(t_2))\big) - \phi^{-1}\big(\phi(\mathcal{P}(\lambda,x)'(t_1))\big)\big|$$

that $\{\mathcal{P}(\lambda,x)'\}$ is also equicontinuous on $[0,T]$. Now the Arzelà-Ascoli theorem shows that $\mathcal{P}([0,1] \times \overline{\Omega})$ is compact in $C^1[0,T]$. Thus $\mathcal{P}$ is a compact operator.

Suppose that x_0 is a fixed point of the operator $\mathcal{P}(1,\cdot)$. Then

$$x_0(t) = x_0(0) + \alpha(x_0) + \int_0^t \phi^{-1}\Big(\phi(x_0'(0) + \beta(x_0)) + \int_0^s h(v, x_0(v), x_0'(v))\,\mathrm{d}v\Big)\,\mathrm{d}s.$$

Hence $\alpha(x_0) = 0$, $\beta(x_0) = 0$ and x_0 is a solution of the differential equation (3.7). Therefore x_0 is a solution of problem (3.7), (3.5) and to prove our theorem, it suffices to show that

$$\deg(\mathcal{I} - \mathcal{P}(1,\cdot), \Omega) \neq 0 \tag{3.12}$$

where "deg" stands for the Leray-Schauder degree and $\mathcal{I}$ is the identity operator on $C^1[0,T]$. To see this let a compact operator $\mathcal{K}\colon [0,1] \times \overline{\Omega} \to C^1[0,T]$ be given by

$$\mathcal{K}(\mu,x)(t) = x(0) + \alpha(x) - \mu\,\alpha(-x) + [x'(0) + \beta(x) - \mu\,\beta(-x)]\,t.$$

Then $\mathcal{K}(1,\cdot)$ is odd (i.e. $\mathcal{K}(1,-x) = -\mathcal{K}(1,x)$ for $x \in \overline{\Omega}$) and

$$\mathcal{K}(0,\cdot) = \mathcal{P}(0,\cdot). \tag{3.13}$$

If $\mathcal{K}(\mu_0, x_0) = x_0$ for some $\mu_0 \in [0,1]$ and $x_0 \in \overline{\Omega}$, then

$$x_0(t) = x_0(0) + \alpha(x_0) - \mu_0\,\alpha(-x_0) + [x_0'(0) + \beta(x_0) - \mu_0\beta(-x_0)]\,t, \quad t \in [0,T].$$

Thus $x_0(t) = A_0 + B_0\,t$ where $A_0 = x_0(0) + \alpha(x_0) - \mu_0\,\alpha(-x_0)$ and $B_0 = x_0'(0) + \beta(x_0) - \mu_0\,\beta(-x_0)$, so $\alpha(x_0) - \mu_0\,\alpha(-x_0) = 0$, $\beta(x_0) - \mu_0\,\beta(-x_0) = 0$. Hence

$$\alpha(A_0 + B_0\,t) - \mu_0\,\alpha(-A_0 - B_0 t) = 0,$$

$$\beta(A_0 + B_0\,t) - \mu_0\,\beta(-A_0 - B_0 t) = 0.$$

Therefore $|A_0| < \Lambda_0$, $|B_0| < \Lambda_1$ and $\|x_0\|_\infty < \Lambda_0 + \Lambda_1 T$, $\|x_0'\|_\infty < \Lambda_1$, which gives $x_0 \notin \partial\Omega$. Now, by the Borsuk antipodal theorem and a homotopy property,

$$\deg(\mathcal{I} - \mathcal{K}(0,\cdot), \Omega) = \deg(\mathcal{I} - \mathcal{K}(1,\cdot), \Omega) \neq 0. \tag{3.14}$$

Finally, assume that $\mathcal{P}(\lambda_*, x_*) = x_*$ for some $\lambda_* \in [0,1]$ and $x_* \in \overline{\Omega}$. Then x_* is a solution of problem (3.8) with $\lambda = \lambda_*$ and, by our assumptions, $\|x_*\|_\infty < S_0$ and $\|x_*'\|_\infty < S_1$. Hence $x_* \notin \partial\Omega$ and the homotopy property yields

$$\deg(\mathcal{I} - \mathcal{P}(0,\cdot), \Omega) = \deg(\mathcal{I} - \mathcal{P}(1,\cdot), \Omega).$$

This, with (3.13) and (3.14), implies (3.12). Therefore, problem (3.7), (3.5) has a solution. $\square$

Remark 3.3. If functionals $\alpha, \beta \in \mathcal{A}$ are linear, then system (3.10) has the form

$$A\,\alpha(1) + B\,\alpha(t) = 0,$$
$$A\,\beta(1) + B\,\beta(t) = 0.$$

All of its solutions (A, B) are bounded if and only if $\alpha(1)\,\beta(t) - \alpha(t)\,\beta(1) \neq 0$ (and then $(A, B) = (0, 0)$). This is satisfied for example for the Dirichlet conditions but not for the periodic ones.

3.2 . *General existence principle for singular BVPs with ϕ-Laplacian*

Let us consider the singular problem (3.4), (3.5). By regularization and sequential techniques, we construct an approximating sequence of regular problems (3.6), (3.5) for whose solvability Theorem 3.2 can be used. Existence results for the singular problem (3.4), (3.5) can be proved by the following two general existence principles. The first principle uses the Vitali convergence theorem, the other is based on a combination of the Lebesgue dominated convergence theorem and the Fatou theorem.

Theorem 3.4. (**General existence principle for singular problems I**) *Assume* (3.3). *Let there exist a bounded set $\Omega \subset C^1[0, T]$ such that*

(i) for each $n \in \mathbb{N}$, the regular problem (3.6), (3.5) *has a solution $u_n \in \Omega$,*

(ii) the sequence $\{f_n(t, u_n(t), u_n'(t))\}$ is uniformly integrable on $[0, T]$.

Then

(a) there exist $u \in \overline{\Omega}$ and a subsequence $\{u_{k_n}\}$ of $\{u_n\}$ such that $\lim_{n\to\infty} u_{k_n} = u$ *in* $C^1[0, T]$,

(b) u is a solution of problem (3.4), (3.5) *if*

$$\lim_{n\to\infty} f_{k_n}(t, u_{k_n}(t), u_{k_n}'(t)) = f(t, u(t), u'(t))$$

for almost all $t \in [0, T]$.

Proof. Since Ω is bounded in $C^1[0, T]$ and $\{u_n\} \subset \Omega$, we have

$$\|u_n\|_\infty \le r, \quad \|u_n'\|_\infty \le r, \quad n \in \mathbb{N}, \tag{3.15}$$

where r is a positive constant. Now (ii) guarantees that for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|\phi(u_n'(t_2)) - \phi(u_n'(t_1))| \le \Big| \int_{t_1}^{t_2} |f_n(t, u_n(t), u_n'(t))| \,\mathrm{d}t \Big| < \varepsilon$$

for each t_1, $t_2 \in [0, T]$, $|t_1 - t_2| < \delta$ and $n \in \mathbb{N}$. Therefore $\{\phi(u_n')\}$ is equicontinuous on $[0, T]$, and by virtue of (3.15) and the fact that ϕ is continuous and increasing on $\mathbb{R}$, $\{u_n'\}$ is equicontinuous on $[0, T]$ as well. The Arzelà-Ascoli theorem guarantees the existence of a subsequence $\{u_{k_n}\}$ of $\{u_n\}$ converging in $C^1[0, T]$ to some $u \in \overline{\Omega}$.

Suppose that $\lim_{n\to\infty} f_{k_n}(t,u_{k_n}(t),u'_{k_n}(t)) = f(t,u(t),u'(t))$ for a.e. $t\in[0,T]$. By (ii), $\{f_{k_n}(t,u_{k_n}(t),u'_{k_n}(t))\}$ is uniformly integrable on $[0,T]$. Therefore, by Vitali's convergence theorem, $f(t,u(t),u'(t))\in L_1[0,T]$ and letting $n\to\infty$ in

$$\phi(u'_{k_n}(t)) = \phi(u'_{k_n}(0)) + \int_0^t f_{k_n}(s,u_{k_n}(s),u'_{k_n}(s))\,\mathrm{d}s, \quad t\in[0,T],\ n\in\mathbb{N},$$

we arrive at

$$\phi(u'(t)) = \phi(u'(0)) + \int_0^t f(s,u(s),u'(s))\,\mathrm{d}s, \quad t\in[0,T].$$

Consequently, $\phi(u')\in AC[0,T]$ and u is a solution of (3.4). In addition, since

$$\lim_{n\to\infty} u_{k_n} = u \quad \text{in} \quad C^1[0,T]$$

and α and β are continuous in $C^1[0,T]$, it follows that $\alpha(u)=0$, $\beta(u)=0$. Hence u is a solution of problem (3.4), (3.5). □

Remark 3.5. Let f in (3.4) have singularities only at the value 0 of its space variables and let f_n in (3.6) satisfy $f_n(t,x,y) = f(t,x,y)$ for a.e. $t\in[0,T]$ and all $(x,y)\in\mathcal{D}$, $n\in\mathbb{N}$, $|x|\ge\frac{1}{n}$ and $|y|\ge\frac{1}{n}$. Then the condition

$$\lim_{n\to\infty} f_{k_n}(t,u_{k_n}(t),u'_{k_n}(t)) = f(t,u(t),u'(t))$$

for a.e. $t\in[0,T]$ is satisfied if u and u' have a finite number of zeros.

Theorem 3.6. (**General existence principle for singular problems II**) *Assume* (3.3). *Let f have singularities only at the value* 0 *of its space variables. Let f_n in equation* (3.6) *satisfy*

$$\begin{cases} \text{for a.e. } \ t\in[0,T] \text{ and each } \ x,y\in\mathbb{R}\setminus\{0\}, \\ 0\le f_n(t,x,y)\le p(|x|,|y|) \quad \text{where} \ \ p\in C((0,\infty)\times(0,\infty)). \end{cases} \tag{3.16}$$

Suppose that for each $n\in\mathbb{N}$, the regular problem (3.6), (3.5) *has a solution u_n and there exists a subsequence $\{u_{k_n}\}$ of $\{u_n\}$ converging in $C^1[0,T]$ to some u. Then u is a solution of the singular problem* (3.4), (3.5) *if u and u' have a finite number of zeros and*

$$\lim_{n\to\infty} f_{k_n}(t,u_{k_n}(t),u'_{k_n}(t)) = f(t,u(t),u'(t)) \quad \text{for a.e. } t\in[0,T]. \tag{3.17}$$

Proof. Assume that (3.17) is true and $0\le\xi_1<\xi_2<\cdots<\xi_m\le T$ are all the zeros of u and u'. Since $\|u_{k_n}\|_\infty\le L$ and $\|u'_{k_n}\|_\infty\le L$ for each $n\in\mathbb{N}$ where L is a positive constant, it follows from (3.16), (3.17),

$$\phi(u'_{k_n}(T)) - \phi(u'_{k_n}(0)) = \int_0^T f_{k_n}(t,u_{k_n}(t),u'_{k_n}(t))\,\mathrm{d}t, \quad n\in\mathbb{N},$$

and the Fatou theorem that

$$\int_0^T f(t,u(t),u'(t))\,\mathrm{d}t \le \phi(L) - \phi(-L).$$

Hence $f(t,u(t),u'(t)) \in L_1[0,T]$. Set $\xi_0 = 0$ and $\xi_{m+1} = T$. We claim that for all $j \in \{0,1,\dots,m\}$, $\xi_j < \xi_{j+1}$, the equality

$$\phi(u'(t)) = \phi\Big(u'(\tfrac{\xi_j+\xi_{j+1}}{2})\Big) + \int_{\frac{\xi_j+\xi_{j+1}}{2}}^{t} f(s,u(s),u'(s))\,\mathrm{d}s \tag{3.18}$$

is satisfied for $t \in [\xi_j,\xi_{j+1}]$. Indeed, let $j \in \{0,1,\dots,m\}$ and $\xi_j < \xi_{j+1}$. Let us look at the interval $[\xi_j+\delta, \xi_{j+1}-\delta]$ where $\delta \in (0, \frac{\xi_j+\xi_{j+1}}{2})$. We know that $|u| > 0$ and $|u'| > 0$ on (ξ_j,ξ_{j+1}) and therefore $|u(t)| \ge \varepsilon$, $|u'(t)| \ge \varepsilon$ for $t \in [\xi_j+\delta, \xi_{j+1}-\delta]$ with a positive constant ε. Hence there exists $n_0 \in \mathbb{N}$ such that $|u_{k_n}(t)| \ge \frac{\varepsilon}{2}$, $|u'_{k_n}(t)| \ge \frac{\varepsilon}{2}$ for $t \in [\xi_j+\delta,\xi_{j+1}-\delta]$ and $n \ge n_0$. This yields (see (3.16))

$$0 \le f_{k_n}(t,u_{k_n}(t),u'_{k_n}(t)) \le \max\Big\{p(u,v)\colon\ u,v \in [\frac{\varepsilon}{2}, L]\Big\}$$

for a.e. $t \in [\xi_j+\delta,\xi_{j+1}-\delta]$ and all $n \ge n_0$. Letting $n \to \infty$ in

$$\phi(u'_{k_n}(t)) = \phi\Big(u'_{k_n}(\tfrac{\xi_j+\xi_{j+1}}{2})\Big) + \int_{\frac{\xi_j+\xi_{j+1}}{2}}^{t} f_{k_n}(s,u_{k_n}(s),u'_{k_n}(s))\,\mathrm{d}s$$

gives (3.18) for $t \in [\xi_j+\delta,\xi_{j+1}-\delta]$ by the Lebesgue dominated convergence theorem. Since $\delta \in (0, \frac{\xi_j+\xi_{j+1}}{2})$ is arbitrary, (3.18) is true on the interval (ξ_j,ξ_{j+1}) and using the fact that $f(t,u(t),u'(t)) \in L_1[0,T]$, we conclude that (3.18) holds also at $t = \xi_j$ and ξ_{j+1}. From (3.18) it follows that $\phi(u') \in AC[0,T]$ and equation (3.4) is satisfied for a.e. $t \in [0,T]$. Finally, $\alpha(u_{k_n}) = 0$ and $\beta(u_{k_n}) = 0$ and the continuity of α and β yields $\alpha(u) = 0$ and $\beta(u) = 0$. Hence u is a solution of problem (3.4), (3.5). $\square$

3.3. *Nonlocal singular BVPs*

We consider differential equations of the type

$$(\phi(u'))' = f(t,u,u') \tag{3.19}$$

where ϕ is an increasing and odd homeomorphism, $\phi(\mathbb{R}) = \mathbb{R}$, f satisfies the Carathéodory conditions on a subset of $[0,T]\times\mathbb{R}^2$ and f may be singular in its space variables.

We also discuss nonlinear nonlocal boundary conditions

$$u(0) = u(T), \quad \max\{u(t)\colon\ 0 \le t \le T\} = c, \quad c \in \mathbb{R}, \tag{3.20}$$

$$u(0) = u(T) = -\gamma\,\min\{u(t)\colon\ 0 \le t \le T\}, \quad \gamma \in (0,\infty), \tag{3.21}$$

$$\min\{u(t)\colon\ 0 \le t \le T\} = 0, \quad \delta(u') = 0, \quad \delta \in \mathcal{B}, \tag{3.22}$$

where $\mathcal{B}$ denotes the set of functionals $\delta\colon C[0,T] \to \mathbb{R}$ which are

(a) continuous, $\delta(0) = 0$ and

(b) increasing, that is $x, y \in C[0,T]$, $x < y$ on $(0,T) \Rightarrow \delta(x) < \delta(y)$.

Example. Let $n \in \mathbb{N}$ and $0 \le a < b \le T$. Then the functionals $\delta_1(x) = x(\frac{T}{2}) + \max\{x(t)\colon 0 \le t \le T\}$, $\delta_2(x) = \int_a^b x^{2n+1}(t)\,\mathrm{d}t$, $\delta_3(x) = \int_0^T e^{x(t)}\,\mathrm{d}t - T$ belong to the set $\mathcal{B}$. The functionals $\delta_4(x) = x(0)$ and $\delta_5(x) = x(T)$ satisfy condition (a) of $\mathcal{B}$ but do not satisfy condition (b). Hence $\delta_4, \delta_5 \notin \mathcal{B}$.

The boundary conditions (3.20)-(3.22) are special cases of (3.5) where

$$\begin{aligned} &\alpha(x) = x(0) - x(T), \quad \beta(x) = \max\{x(t)\colon 0 \le t \le T\} - c && \text{for } (3.20),\\ &\alpha(x) = x(0) - x(T), \quad \beta(x) = x(0) + \gamma \min\{x(t)\colon 0 \le t \le T\} && \text{for } (3.21), \end{aligned}$$

and

$$\alpha(x) = \min\{x(t)\colon 0 \le t \le T\}, \quad \beta(x) = \delta(x') \qquad \text{for } (3.22).$$

The next theorems give sufficient conditions for solvability of the three nonlocal singular problems given above. Their proofs are based on applying general existence principles presented in Theorems 3.2, 3.4 and 3.6. Notice that if $f < 0$ in equation (3.19) then the singular points corresponding to the solutions of problem (3.19), (3.20) are of type II and, if $f > 0$, the solutions of problems (3.19), (3.21) and (3.19), (3.22) have singular points of type II.

Theorem 3.7. ([145, Theorem 2.1]) *Suppose $f \in Car([0,T] \times \mathbb{R} \times (\mathbb{R} \setminus \{0\})$,*

$$\left\{ \begin{array}{l} \qquad -q(x)\big(\omega_1(|y|) + \omega_2(|y|)\big) \le f(t,x,y) \le -a \\ \textit{for a.e. } t \in [0,T] \textit{ and each } (x,y) \in \mathbb{R} \times (\mathbb{R} \setminus \{0\}), \\ \textit{where } a > 0,\ q \in C(\mathbb{R}) \textit{ is positive, } \omega_1 \in C[0,\infty) \textit{ is nonnegative,} \\ \omega_2 \in C(0,\infty) \quad \textit{is positive and nonincreasing and} \\ \qquad \displaystyle\int_0^1 \omega_2(\phi^{-1}(s))\,\mathrm{d}s < \infty \end{array} \right. \tag{3.23}$$

and

$$\int_{d-1}^{d} \frac{\mathrm{d}s}{H^{-1}\Big(\int_s^d q(v)\,\mathrm{d}v\Big)} < \infty \quad \textit{for any } d \in \mathbb{R},$$

where

$$H(x) = \int_0^{\phi(x)} \frac{\phi^{-1}(s)\,\mathrm{d}s}{\omega_1(1 + \phi^{-1}(s)) + \omega_2(\phi^{-1}(s))} \quad \textit{for } x \in [0,\infty).$$

Let

$$\mathcal{S} = \Big\{c \in \mathbb{R}\colon \lim_{x \to -\infty} \int_x^c \frac{\mathrm{d}s}{H^{-1}\big(\int_s^c q(v)\mathrm{d}v\big)} > \frac{T}{2}\Big\}.$$

Then problem (3.19), (3.20) *has a solution for each $c \in \mathcal{S}$.*

Sketch of proof. *Step* 1. *Regularization.*

A sequence of auxiliary regular differential equations $(\phi(u'))' = f_n(t,u,u')$ is constructed where $f_n \in Car([0,T] \times \mathbb{R}^2)$ and

$$f_n(t,x,y) = f(t,x,y) \quad \text{for a.e. } t \in [0,T] \text{ and each } \ x \in \mathbb{R},\ |y| \ge \frac{1}{n},\ n \in \mathbb{N}.$$

Step 2. *Existence of solutions of regular problems* (3.6, (3.20).

Let $c \in \mathcal{S}$. By (3.23), the existence of positive constants S_0 and S_1 (independent of n and λ) is proved such that $\|u\|_\infty < S_0$ and $\|u'\|_\infty < S_1$ for any $\lambda \in [0,1]$, $n \in \mathbb{N}$ and each solution u of the differential equation

$$(\phi(u'))' = \lambda\, f_n(t,u,u') \tag{3.24}$$

satisfying the conditions (3.20). Put $\alpha(x) = x(0) - x(T)$ and $\beta(x) = \max\{x(t) : 0 \le t \le T\} - c$ for $x \in C^1[0,T]$. Then system (3.10) has a unique solution $(A,B) = (c\,\frac{1-\mu}{1+\mu}, 0)$ for each $\mu \in [0,1]$ and therefore all solutions of this system are bounded in $\mathbb{R}^2$. Hence Theorem 3.2 guarantees that for each $n \in \mathbb{N}$, problem (3.6), (3.20) has a solution u_n and $\|u_n\|_\infty < S_0$, $\|u_n'\|_\infty < S_1$.

Step 3. *Properties of solutions of regular problems* (3.6), (3.20).

The sequence $\{u_n\}$ is considered. It is proved that

$$u_n'(t) \ge \phi^{-1}(a\,(\xi_n - t)) \quad \text{for} \ \ t \in [0,\xi_n] \quad \text{and} \quad |u_n'(t)| \ge \phi^{-1}(a\,(t-\xi_n)) \quad \text{for} \ \ t \in [\xi_n, T],$$

where ξ_n is the unique zero of u_n' and $a > 0$ appears in (3.23). Next, it is shown that the sequence $\{f_n(t,u_n(t),u_n'(t))\}$ is uniformly integrable on $[0,T]$. Hence $\{u_n\}$ is bounded in $C^1[0,T]$ and $\{u_n'\}$ is equicontinuous on $[0,T]$ and, by the Arzelà-Ascoli theorem and the compactness principle, we can assume without loss of generality that $\{u_n\}$ converges in $C^1[0,T]$ and $\{\xi_n\}$ converges in $\mathbb{R}$. Let

$$\lim_{n\to\infty} u_n = u \quad \text{and} \quad \lim_{n\to\infty} \xi_n = \xi.$$

Then $u \in C^1[0,T]$ satisfies (3.20),

$$u'(t) \ge \phi^{-1}(a\,(\xi - t)) \quad \text{for} \ \ t \in [0,\xi], \quad |u'(t)| \ge \phi^{-1}(a\,(t-\xi)) \quad \text{for} \ \ t \in [\xi, T]$$

and

$$\lim_{n\to\infty} f_n(t,u_n(t),u_n'(t)) = f(t,u(t),u'(t)) \quad \text{for a.e. } t \in [0,T].$$

Theorem 3.4 now guarantees that u is a solution of problem (3.19), (3.20). □

Example. Let $p > 2$, $\alpha \in [0, p-2)$ and $\beta \in (0, p-1)$. Then for any $c \in \mathbb{R}$ there exists a solution of the differential equation

$$(|u'|^{p-2}\,u')' + \big(2 + \sin(t\,u) + |u|\big)\left(|u'|^\alpha + \frac{1}{|u'|^\beta}\right) = 0$$

satisfying boundary conditions (3.20).

THEOREM 3.8. ([146, Theorem 4.1]) *Let* $f \in Car([0,T] \times (\mathbb{R} \setminus \{0\})^2)$. *Let*

$$\begin{cases} \qquad a \le f(t,x,y) \le \big(h_1(|x|) + h_2(|x|)\big)\big(\omega_1(\phi(|y|)) + \omega_2(\phi(|y|))\big) \\ \textit{for a.e.} \quad t \in [0,T] \ \textit{and all} \ \ (x,y) \in (\mathbb{R} \setminus \{0\})^2, \\ \textit{where } a > 0, \ h_1, \omega_1 \in C[0,\infty) \ \textit{are nonnegative and nondecreasing,} \\ h_2, \ \omega_2 \in C(0,\infty) \quad \textit{are positive and nonincreasing,} \\ \quad \displaystyle\int_0^1 h_2(s)\,\mathrm{d}s < \infty, \quad \int_0^1 \omega_2(s)\,\mathrm{d}s < \infty, \quad \int_0^\infty \frac{\mathrm{d}s}{\omega_2(s)} = \infty, \end{cases} \tag{3.25}$$

and let

$$\liminf_{x\to\infty} \int_0^x \frac{\mathrm{d}s}{K^{-1}\left(\frac{T}{2}\ (h_1(x) + h_2(s))\right)} > \frac{T}{2}, \tag{3.26}$$

where

$$K(x) = \int_0^{\phi(x)} \frac{\mathrm{d}s}{\omega_1(\phi(1)+s) + \omega_2(s)}, \quad x \in [0,\infty). \tag{3.27}$$

Then there exists a solution of problem (3.19), (3.21) *for each* $\gamma > 0$.

SKETCH OF PROOF. *Step* 1. *Regularization.*

A sequence of approximating differential equations

$$(\phi(u'))' = f_n(t,u,u')$$

is introduced where $f_n \in Car([0,T] \times \mathbb{R}^2)$ and

$$f_n(t,x,y) = f(t,x,y) \quad \text{for a.e. } t \in [0,T] \text{ and each } \ |x| \ge \frac{1}{n}, \ |y| \ge \frac{1}{n}, \ n \in \mathbb{N}.$$

Step 2. *Existence of solutions of regular problems* (3.6), (3.21).

Let $\gamma > 0$ in (3.21). Using (3.25) and (3.26), the existence of a positive constant P (depending on γ) is proved such that $\|u\|_\infty < PT$ and $\|u'\|_\infty < P$ for each solution u of problem (3.24), (3.21) with $\lambda \in [0,1]$ and $n \in \mathbb{N}$. Put $\alpha(x) = x(0) - x(T)$ and

$$\beta(x) = x(0) + \gamma \min\{x(t)\colon 0 \le t \le T\} \quad \text{for} \ \ x \in C^1[0,T].$$

The system (3.10) has a unique solution $(A,B) = (0,0)$ for each $\mu \in [0,1]$. Hence, by Theorem 3.2 for each $n \in \mathbb{N}$, there exists a solution u_n of problem (3.6), (3.21) and $\|u_n\|_\infty < PT$, $\|u_n'\|_\infty < P$.

Step 3. *Properties of solutions of regular problems* (3.6), (3.21).

The sequence $\{u_n\}$ is considered. From (3.25) it follows that u_n' is increasing on $[0,T]$ and has a unique zero $\xi_n \in (0,T)$ and u_n vanishes exactly at two points t_{1n}, t_{2n},

$0 < t_{1n} < \xi_n < t_{2n} < T$, $u_n > 0$ on $[0, t_{1n}) \cup (t_{2n}, T]$ and $u_n < 0$ on (t_{1n}, t_{2n}). Further, u_n satisfies the inequality

$$|u_n(t)| \geq \begin{cases} \dfrac{S\,|t - t_{1n}|}{\xi_n - t_{1n}} & \text{for } t \in [0, \xi_n], \\[2mm] \dfrac{S\,|t - t_{2n}|}{t_{2n} - \xi_n} & \text{for } t \in [\xi_n, T], \end{cases}$$

where S is a positive constant and the sequence $\{f_n(t, u_n(t), u_n'(t))\}$ is uniformly integrable on $[0,T]$, which implies that $\{u_n'\}$ is equicontinuous on $[0,T]$. Moreover, there exists a positive constant Δ such that

$$t_{1n} \geq \gamma\,\Delta, \quad \xi_n - t_{1n} > \Delta, \quad t_{2n} - \xi_n > \Delta, \quad T - t_{2n} > \gamma\,\Delta \quad \text{for } n \in \mathbb{N}.$$

Hence, by the Arzelà-Ascoli theorem, there exists a subsequence $\{u_{k_n}\}$ which converges in $C^1[0,T]$ and let $u = \lim_{n\to\infty} u_{k_n}$. Then u vanishes exactly at two points in $[0,T]$, u' has a unique zero and $\lim_{n\to\infty} f_{k_n}(t, u_{k_n}(t), u_{k_n}'(t)) = f(t, u(t), u'(t))$ for a.e. $t \in [0,T]$. Now Theorem 3.4 guarantees that u is a solution of problem (3.19), (3.21). □

Remark 3.9. If $\lim_{x\to\infty} h_2(x) < A$ for some $A > 0$ and

$$\liminf_{x\to\infty} \frac{x}{K^{-1}\big(\frac{T}{2}\,(h_1(x) + A)\big)} > \frac{T}{2},$$

then condition (3.26) is satisfied.

Example. Let $q_j \in L_\infty[0,T]$ be nonnegative $(1 \leq j \leq 6)$, $q_1(t) \geq a > 0$ for a.e. $t \in [0,T]$, $p > 1$, β_1, β_2, $\beta_3 \in (0, p-1)$, $\alpha_1 \in (0, p-1+\beta_2)$, α_2, $\alpha_3 \in (0,1)$. Then for each $\gamma > 0$, there exists a solution of the differential equation

$$(|u'|^{p-2}\,u')' = q_1(t) + q_2(t)\,|u|^{\alpha_1} + \frac{q_3(t)}{|u|^{\alpha_2}} + \frac{q_4(t)}{|u|^{\alpha_3}\,|u'|^{\beta_1}} + q_5(t)\,|u'|^{\beta_2} + \frac{q_6(t)}{|u'|^{\beta_3}}$$

satisfying boundary conditions (3.21).

Theorem 3.10. *Suppose $f \in Car([0,T]\times(0,\infty)\times(\mathbb{R}\setminus\{0\}))$ and the following conditions are satisfied:*

$$\begin{cases} \quad \varphi(t) \leq f(t,x,y) \leq (h_1(x) + h_2(x))\,[\omega_1(\phi(|y|)) + \omega_2(\phi(|y|))] \\ \textit{for a.e. } t \in [0,T] \quad \textit{and each } (x,y) \in (0,\infty) \times (\mathbb{R}\setminus\{0\}), \\ \textit{where } \ \varphi \in L_\infty[0,T] \ \textit{is positive}, \\ h_1, \ \omega_1 \in C[0,\infty) \ \textit{are positive and nondecreasing}, \\ h_2, \ \omega_2 \in C(0,\infty) \ \textit{are positive and nonincreasing}, \\ \qquad\qquad \displaystyle\int_0^1 h_2(s)\,\mathrm{d}s < \infty \end{cases} \tag{3.28}$$

and

$$\liminf_{x\to\infty} \frac{V(x)}{H(T\,x)} > 1 \tag{3.29}$$

where

$$V(x) = \int_0^{\phi(x)} \frac{\phi^{-1}(s)\,\mathrm{d}s}{\omega_1(s+1)+\omega_2(s)}, \quad H(x) = \int_0^x (h_1(s+1)+h_2(s))\,\mathrm{d}s \ \textit{for}\ x \in [0,\infty).$$

Then for each $\delta \in \mathcal{B}$, problem (3.19), (3.22) *has a solution.*

Sketch of proof. *Step* 1. *Regularization.*

A sequence of auxiliary regular differential equations $(\phi(u'))' = f_n(t,u,u')$ is constructed with $f_n \in Car([0,T]\times\mathbb{R}^2)$ satisfying

$$f_n(t,x,y) = f(t,x,y) \quad \text{for a.e. } t\in[0,T] \text{ and each } \ x \ge \frac{1}{n},\ |y| \ge \frac{1}{n},\ n\in\mathbb{N}.$$

Step 2. *Existence of solutions of regular problems* (3.6), (3.22).

Fix $\delta \in \mathcal{B}$. From (3.28), (3.29) and from the properties of δ we obtain the existence of positive constants M_0 and M_1 such that $\|u\|_\infty < M_0$, $\|u'\|_\infty < M_1$ for each solution of problem (3.24), (3.22) with $\lambda \in [0,1]$ and $n \in \mathbb{N}$. Set $\alpha(x) = \min\{x(t)\colon\ 0 \le t \le T\}$ and $\beta(x) = \delta(x')$ for $x \in C^1[0,T]$. Then system (3.10) has a unique solution $(A,B) = (0,0)$ for each $\mu \in [0,1]$. Therefore, by Theorem 3.2, for each $n \in \mathbb{N}$ there exists a solution u_n of problem (3.6), (3.22) and $\|u_n\|_\infty < M_0$, $\|u'\|_\infty < M_1$.

Step 3. *Properties of solutions of regular problems* (3.6), (3.22).

By Step 2, $\{u_n\}$ is bounded in $C^1[0,T]$ and from (3.28) it follows that $\{u_n'\}$ is equicontinuous on $[0,T]$. The assumption (3.28) and the properties of δ show that u_n has a unique zero ξ_n, $\xi_n \in (0,T)$, u_n' is increasing on $[0,T]$, $u_n'(\xi_n) = 0$ and

$$|u_n'(t)| \ge \left|\int_{\xi_n}^t \varphi(s)\,\mathrm{d}s\right|, \quad u_n(t) \ge \int_{\xi_n}^t (t-s)\,\varphi(s)\,\mathrm{d}s \tag{3.30}$$

for $t\in[0,T]$. According to the Arzelà-Ascoli theorem, there exists a subsequence $\{u_{k_n}\}$ converging in $C^1[0,T]$ to some u and from (3.30) we see that u and u' vanish at a unique point. Since

$$\lim_{n\to\infty} f_{k_n}(t,u_{k_n}(t),u_{k_n}'(t)) = f(t,u(t),u'(t)) \qquad \text{for a.e. } t\in[0,T],$$

Theorem 3.6 gives that u is a solution of problem (3.19), (3.22). □

Remark 3.11. Problem (3.19), (3.22) was investigated in [148]. The conditions for the solvability of this problem are stronger there than those in Theorem 3.10. This is due to the fact that [148] uses the Vitali convergence theorem in limit processes whereas Theorem 3.10 is proved by Theorem 3.6.

Example. Let $\varphi \in L_\infty[0,T]$ be positive, $p>1$, $c_j > 0$ $(1\le j\le 4)$, $\beta\in(0,1)$, α, γ, δ, $\lambda \in (0,\infty)$ and $\alpha+\gamma < p-1$. Then for each $\delta\in\mathcal{B}$, the differential equation

$$(|u'|^{p-2}\,u')' = \varphi(t)\left(1 + c_1\,u^\alpha + \frac{c_2}{u^\beta}\right)\left(1 + c_3\,|u'|^\gamma + \frac{c_4}{|u'|^\lambda}\right)$$

has a solution u satisfying boundary conditions (3.22).

3.4. *Historical and bibliographical notes*

The general existence principles presented in Theorems 3.2 and 3.4 are special cases of the principles stated by Agarwal, O'Regan and Staněk in [17] for a class of second-order functional differential equations. Some general existence principles for second-order regular differential equations with the ϕ-Laplacian and Dirichlet or mixed boundary data have been established using the nonlinear alternative of Leray-Schauder type by O'Regan [115].

Second-order differential equations with the p-Laplacian and the ϕ-Laplacian occur in the study of the p-Laplace equations ([95]), general diffusion theory ([23], [41]), non-Newtonian fluid theory ([86]) and the turbulent flow of a polytropic gas in a porous medium ([64], [37]).

In recent years problems for $p(t)$-Laplacian equations have been studied (e.g. [67], [68]). The $p(t)$-Laplacian is defined by $u \to (|u'|^{p(t)-2}\, u')'$ where $p \in C[0,T]$ and $p > 1$ on $[0,T]$. The $p(t)$-Laplacian is a generalization of the p-Laplacian.

4. Singular Dirichlet BVPs with ϕ-Laplacian

Motivated by various significant applications to non-Newtonian fluid theory, diffusion of flows in porous media, nonlinear elasticity and theory of capillary surfaces (see [23], [64], [118], and Section 3.4), several authors have proposed the study of equations $(\phi_p(u'))' + f(t,u,u') = 0$ with the p-Laplacian $(\phi_p(u'))'$, where $p \in (1,\infty)$ and $\phi_p(y) = |y|^{p-2}\, y$ for $y \in \mathbb{R}$. Usually the p-Laplacian is replaced by its abstract and more general version, which leads to clearer exposition and better understanding of the methods that are employed to derive existence results. Therefore, similarly to Section 3, we will work with a ϕ-Laplacian which satisfies (3.3), i.e. ϕ is an increasing homeomorphism with $\phi(\mathbb{R}) = \mathbb{R}$.

We will consider a singular Dirichlet problem of the form

$$(\phi(u'))' + f(t,u,u') = 0, \quad u(0) = u(T) = 0 \tag{4.1}$$

and its special cases, in particular, a problem of the form

$$u'' + f(t,u,u') = 0, \quad u(0) = u(T) = 0, \tag{4.2}$$

where $\phi(y) = y$ on $\mathbb{R}$.

We will investigate problems (4.1) and (4.2) on the set $[0,T] \times \mathcal{A}$. In general, the function f depends on a time variable $t \in [0,T]$ and on two space variables x and y, where $(x,y) \in \mathcal{A}$ and $\mathcal{A}$ is a closed subset of $\mathbb{R}^2$ or $\mathcal{A} = \mathbb{R}^2$.

We assume that problems (4.1) and (4.2) are singular, which means, by Section 1, that f does not satisfy the Carathéodory conditions on $[0,T] \times \mathcal{A}$. In what follows, the types of singularities of f will be exactly specified for each problem under consideration.

In accordance with Section 1 we define:

Definition 4.1. A function $u\colon [0,T] \to \mathbb{R}$ with $\phi(u') \in AC[0,T]$ is *a solution of problem* (4.1) if u satisfies $(\phi(u'(t)))' + f(t,u(t),u'(t)) = 0$ a.e. on $[0,T]$ and fulfils the boundary conditions $u(0) = u(T) = 0$.

A function $u \in C[0,T]$ is *a w-solution of problem* (4.1) if there exists a finite number of singular points $t_\nu \in [0,T]$, $\nu = 1,\dots,r$, such that if we denote $J = [0,T] \setminus \{t_\nu\}_{\nu=1}^r$, then $\phi(u') \in AC_{loc}(J)$, u satisfies $(\phi(u'(t)))' + f(t,u(t),u'(t)) = 0$ a.e. on $[0,T]$ and fulfils the boundary conditions $u(0) = u(T) = 0$.

Note that the condition $\phi(u') \in AC[0,T]$ implies $u \in C^1[0,T]$ and the condition $\phi(u') \in AC_{loc}(J)$ implies $u \in C^1(J)$. We will mention some papers where f is supposed to be continuous on $(0,T) \times \mathbb{R}^2$ and can have only time singularities at $t = 0$ and $t = T$. Then any solution (any w-solution) u of (4.1) moreover satisfies $\phi(u') \in C^1(0,T)$. If we investigate the solvability of problem (4.1) or (4.2) on the set $[0,T] \times \mathcal{A}$ and $\mathcal{A} \neq \mathbb{R}^2$, we impose on its solution u in addition the condition

$$(u(t), u'(t)) \in \mathcal{A} \quad \text{for } t \in [0,T]. \tag{4.3}$$

If u is a w-solution, then one requires it to satisfy (4.3) for $t \in J$ only.

In some cases, see e.g. (4.9), f does not depend on y. Then we work with a set $\mathcal{A}$ which is a closed subset of $\mathbb{R}$ or $\mathcal{A} = \mathbb{R}$ and condition (4.3) has the form $u(t) \in \mathcal{A}$ for $t \in [0,T]$.

Remark 4.2. We will carry out the investigation of the singular problem (4.1) in the spirit of the existence principles presented in Sections 1 and 3:

- the singular problem is approximated by a sequence of solvable regular problems;
- a sequence $\{u_n\}$ of approximate solutions is generated;
- a convergence of a suitable subsequence $\{u_{k_n}\}$ is investigated;
- the type of this convergence determines the properties of its limit u and, among others, determines whether u is a w-solution or a solution of the original singular problem.

There are more possibilities how to construct an approximating sequence of regular problems. Their choice depends on the type of singularities of the nonlinearity f in (4.1) (time, space), on the type of singular points corresponding to a solution (w-solution) of (4.1) (type I, type II), on the type of results desired (existence of a solution, a positive solution, a w-solution, uniqueness), and so on. A common idea is that approximate functions f_n have no singularities, $f_n \neq f$ on neighbourhoods U_n of singular points of f, $f_n = f$ elsewhere, and $\lim_{n\to\infty} \operatorname{meas}(U_n) = 0$.

Having such a sequence of $\{f_n\}$ we study problems

$$(\phi(u'))' + f_n(t,u,u') = 0, \quad u(0) = u(T) = \varepsilon_n, \ n \in \mathbb{N}, \tag{4.4}$$

where $\varepsilon_n \in \mathbb{R}$, $\lim_{n\to\infty} \varepsilon_n = 0$. In some proofs, one simply puts $\varepsilon_n = 0$ for $n \in \mathbb{N}$.

Solvability of (4.4) can be investigated by means of various methods which have been developed for regular Dirichlet problems (fixed point theorems, topological degree arguments, the topological transversality method, variational methods, lower and upper functions, the Fredholm nonlinear alternative, etc.). See also Section 3. Using one of the above methods we generate a sequence of approximate solutions $\{u_n\}$ of (4.4). The crucial information which enables us to realize the limit process concerns a priori estimates of the approximate solutions u_n.

4.1 . *Method of lower and upper functions*

It is well known that for regular second order boundary value problems *the lower and upper functions method* is a profitable instrument for proofs of their solvability and for a priori estimates of their solutions. See e.g. [51], [52], [98], [100], [130] or [154]. Hence, it seems to be a good idea to extend this method to the singular problem (4.1). In literature there are several definitions of lower and upper functions for regular boundary value problems. (Note that in some papers they are called lower and upper solutions.) Here we will use the following definition which is the same both for regular problems with $f \in Car([0,T] \times \mathbb{R}^2)$ and for singular ones with $f \in Car((0,T) \times \mathbb{R}^2)$ having time singularities at $t = 0$ and $t = T$.

Definition 4.3. A function $\sigma : [0,T] \to \mathbb{R}$ with $\phi(\sigma') \in AC[0,T]$ is called *a lower function of* (4.1) if σ satisfies

$$(\phi(\sigma'(t))' + f(t, \sigma(t), \sigma'(t)) \geq 0 \quad \text{for a.e. } t \in [0,T] \tag{4.5}$$

and

$$\sigma(0) \leq 0, \quad \sigma(T) \leq 0. \tag{4.6}$$

If the inequalities in (4.5) and (4.6) are reversed, then σ is called *an upper function of* (4.1).

For the special case (4.2) we admit a more general definition.

Definition 4.4. A function $\sigma \in C[0,T]$ is called *a lower function of* (4.2) if there exists a finite set $\Sigma \subset (0,T)$ such that $\sigma \in AC^1_{loc}([0,T] \setminus \Sigma)$, $\sigma'(\tau+), \sigma'(\tau-) \in \mathbb{R}$ for each $\tau \in \Sigma$,

$$\sigma''(t) + f(t, \sigma(t), \sigma'(t)) \geq 0 \quad \text{for a.e. } t \in [0,T], \tag{4.7}$$

$$\sigma(0) \leq 0, \ \sigma(T) \leq 0, \quad \sigma'(\tau-) < \sigma'(\tau+) \text{ for each } \tau \in \Sigma. \tag{4.8}$$

If the inequalities in (4.7) and (4.8) are reversed, then σ is called *an upper function of* (4.2).

Remark 4.5. (i) If, moreover, f is continuous on $(0,T) \times \mathbb{R}^2$, then a lower (upper) function σ of (4.1) is supposed to satisfy $\phi(\sigma') \in C^1(0,T)$ and a lower (upper) function of (4.2) belongs also to $C^2(0,T)$.

(ii) If the boundary conditions in (4.1) or in (4.2) are replaced by inhomogeneous ones, i.e. they have the form

$$u(0) = a, \quad u(T) = b$$

for some $a, b \in \mathbb{R}$, then the corresponding boundary inequalities in (4.6) or in (4.8) are modified to

$$\sigma(0) \le a, \quad \sigma(T) \le b.$$

We present straightforward extensions of the classical lower and upper functions method to a singular problem with the p-Laplacian

$$(\phi_p(u'))' + f(t,u) = 0, \quad u(0) = a, \quad u(T) = b, \tag{4.9}$$

where

$$\begin{cases} \phi_p(y) = |y|^{p-2}\, y, \ p > 1, \ a, b \in \mathbb{R}, \quad f \in Car((0,T) \times \mathbb{R}), \\ f \text{ can have time singularities at } t = 0 \text{ and } t = T. \end{cases} \tag{4.10}$$

Recall that f has time singularities at $t = 0$ and $t = T$ if there exist $x, y \in \mathbb{R}$ such that

$$\int_0^{\varepsilon} |f(t,x,y)| \, \mathrm{d}t = \infty, \qquad \int_{T-\varepsilon}^{T} |f(t,x,y)| \, \mathrm{d}t = \infty$$

for each sufficiently small $\varepsilon > 0$.

Making use of ideas of the papers [106] by Lomtatidze and Torres and [85] by Habets and Zanolin, one can prove the following result for the special case of (4.9) with $p = 2$.

Theorem 4.6. *Let $p = 2$ and (4.10) hold. Let σ_1 and σ_2 be a lower and an upper function for problem (4.9) and $\sigma_1 \le \sigma_2$ on $[0,T]$. Assume also that there is a function $h \in L_1(I)$ on each compact interval $I \subset (0,T)$ such that*

$$|f(t,x)| \le h(t) \quad \text{for a.e. } t \in (0,T) \text{ and each } x \in [\sigma_1(t), \sigma_2(t)],$$

and

$$\int_0^T t\,(T-t)\,h(t)\,\mathrm{d}t < \infty.$$

Then problem (4.9) has a w-solution $u \in C[0,T] \cap AC^1_{loc}(0,T)$ such that

$$\sigma_1(t) \le u(t) \le \sigma_2(t) \quad \text{for } t \in [0,T]. \tag{4.11}$$

If for a.e. $t \in (0,T)$ the function $f(t,x)$ is nonincreasing in x, then the w-solution is unique. If $h \in L_1[0,T]$, then u belongs to $AC^1[0,T]$, i.e. u is a solution of (4.9).

Theorem 4.6 can be proved by means of the Schauder fixed point theorem which is applied to the operator $\mathcal{T} \colon C[0,T] \to C[0,T]$, where

$$(\mathcal{T}\,u)(t) = a + \frac{t}{T}\,(b-a) + \int_0^T G(t,s)\, f^*(s, u(s))\,\mathrm{d}s.$$

Here G is the Green function of the problem $-u'' = 0$, $u(0) = u(T) = 0$ and f^* is given by

$$f^*(t,x) = \begin{cases} f(t,\sigma_1(t)) & \text{if} \quad x < \sigma_1(t), \\ f(t,x) & \text{if} \quad \sigma_1(t) \le x \le \sigma_2(t), \\ f(t,\sigma_2(t)) & \text{if} \quad x > \sigma_2(t) \end{cases}$$

for a.e. $t \in [0,T]$ and each $x \in \mathbb{R}$.

Remark 4.7. By virtue of (4.11) we can investigate problem (4.9) on $[0,T] \times \mathcal{A}_t$, where $\mathcal{A}_t = [\sigma_1(t), \sigma_2(t)]$ for $t \in [0,T]$. Therefore, in Theorem 4.6, instead of $f \in Car((0,T) \times \mathbb{R})$ it is sufficient to assume $f \in Car((0,T) \times A_t)$.

Jiang in [91] dealt with problem (4.9) under the assumption (4.10) and, in addition,

$$f \in C((0,T) \times \mathbb{R}). \tag{4.12}$$

He modified Theorem 4.6 for $p \neq 2$.

Theorem 4.8. *Let* (4.10) *and* (4.12) *hold. Let σ_1 and σ_2 be a lower and an upper function for problem* (4.9) *and $\sigma_1 \le \sigma_2$ on $[0,T]$. Assume also that there is a function $h \in C(0,T)$ such that*

$$|f(t,x)| \le h(t) \quad \text{for } t \in (0,T),\ x \in [\sigma_1(t), \sigma_2(t)],$$

and that there exist μ, $\nu \in [0, p-1)$ such that

$$\int_0^T t^\mu \,(T-t)^\nu \, h(t)\, \mathrm{d}t < \infty. \tag{4.13}$$

Then problem (4.9) *has a w-solution $u \in C[0,T]$ satisfying $\phi_p(u') \in C^1(0,T)$ and* (4.11).

In contrast to $p = 2$, there is no Green function for $p \neq 2$, which makes the proof of Theorem 4.8 more difficult and complicated than that for $p = 2$.

Remark 4.9. Motivated by physical and technical problems, there is a lot of papers studying problems *with both time and space singularities.* If such a problem has a singularity at $x = 0$, one often searches for solutions (w-solutions) which are positive on $(0,T)$. Although they vanish at 0 and T, they are still called *positive solutions (positive w-solutions)* in literature. In this case, problems (4.1) and (4.2) are investigated on the set $[0,T] \times \mathcal{A}$, where $\mathcal{A} = [0,\infty) \times \mathbb{R}$, and f in (4.1) or (4.2) is supposed to satisfy $f \in Car((0,T) \times \mathcal{D})$, where $\mathcal{D} = (0,\infty) \times \mathbb{R}$ (or more specifically f is supposed to be continuous on $(0,T) \times \mathcal{D}$). In this case lower and upper functions have to be positive on $(0,T)$ and consequently a lower function σ_1 has to satisfy $\sigma_1(0) = \sigma_1(T) = 0$.

Having in mind Remarks 4.7 and 4.9 we will search for positive w-solutions of a singular problem

$$u'' + f(t,u) = 0, \quad u(0) = u(T) = 0, \tag{4.14}$$

where

$$\begin{cases} f \in Car((0,T)\times(0,\infty)) \quad \text{can have} \\ \text{time singularities at } \ t=0,\ t=T \text{ and a space singularity at } x=0. \end{cases} \tag{4.15}$$

Recall that f has a space singularity at $x=0$ if

$$\limsup_{x\to 0+} |f(t,x)| = \infty \qquad \text{for a.e. } t \in [0,T].$$

Let us present a simple application of Theorem 4.6 in the spirit of Habets and Zanolin [85].

THEOREM 4.10. *Let a function f be positive and satisfy* (4.15). *Assume that for a.e. $t \in (0,T)$ the function $f(t,x)$ is nonincreasing in x. Suppose that there exists a lower function σ_1 of problem* (4.14) *such that $\sigma_1 > 0$ on $(0,T)$ and*

$$\int_0^T f(s,\sigma_1(s))\,\mathrm{d}s < \infty.$$

Then problem (4.14) *has a unique positive solution $u \in AC^1[0,T]$ such that $\sigma_1 \le u$ on $[0,T]$.*

Theorem 4.10 follows from Theorem 4.6 and Remark 4.7 if we put $h(t) = f(t,\sigma_1(t))$ and

$$\sigma_2(t) = \int_0^T G(t,s)\,f(s,k)\,\mathrm{d}s + k,$$

where $k = \max\{\sigma_1(t)\colon\ t \in [0,T]\}$ and G is the Green function of the problem

$$-u'' = 0, \quad u(0) = u(T) = 0.$$

The next result can be viewed as a corollary of [106, Theorem 1.1], where Lomtatidze and Torres studied an equation including an additional term $g(t,u)\,u'$.

THEOREM 4.11. *Let* (4.15) *hold. Let σ_1 and σ_2 be a lower and an upper function of problem* (4.14) *and*

$$\sigma_2(0) > 0, \quad \sigma_2(T) > 0, \quad 0 < \sigma_1 \le \sigma_2 \quad \textit{on} \ \ (0,T).$$

Let, moreover, for every $0 < \eta < \min\{\sigma_2(t)\colon\ t \in [0,T]\}$ there exist $h_\eta \in C(0,T)$ such that

$$|f(t,x)| \le h_\eta(t) \quad \textit{for} \ \ t \in (0,T) \ \textit{and all} \ \ x \in [\sigma_{1\eta}(t), \sigma_2(t)],$$

where $\sigma_{1\eta}(t) = \max\{\eta, \sigma_1(t)\}$ and

$$\int_0^T t\,(T-t)\,h_\eta(t)\,\mathrm{d}t < \infty.$$

Then problem (4.14) *has a positive w-solution $u \in C[0,T] \cap AC^1_{loc}(0,T)$ such that*

$$\sigma_1(t) \le u(t) \le \sigma_2(t) \quad \textit{for } t \in [0,T].$$

Sketch of the proof. *Step* 1. *Construction of auxiliary intervals.*

A decreasing sequence $\{a_n\} \subset (0,T)$ and an increasing sequence $\{b_n\} \subset (0,T)$ are constructed such that, among other, $\lim_{n\to\infty} a_n = 0$, $\lim_{n\to\infty} b_n = T$.

Step 2. *Construction of auxiliary regular problems.*

For $t \in [0,T]$, $x \in \mathbb{R}$, $n \in \mathbb{N}$, functions χ_n, α_n, β, f_n are given by

$$\chi_n(x) = \begin{cases} \sigma_1(a_n) & \text{if} \quad x < \sigma_1(a_n), \\ x & \text{if} \quad x \ge \sigma_1(a_n), \end{cases}$$

$$\alpha_n(t) = \begin{cases} \sigma_1(a_n) & \text{if} \quad 0 \le t \le a_n, \\ \sigma_1(t) & \text{if} \quad a_n \le t \le b_n, \\ \sigma_1(b_n) & \text{if} \quad b_n \le t \le T, \end{cases} \qquad \beta(t) = \begin{cases} v_1(t) & \text{if} \quad 0 \le t \le a_1, \\ \sigma_2(t) & \text{if} \quad a_1 \le t \le b_1, \\ v_2(t) & \text{if} \quad b_1 \le t \le T, \end{cases}$$

$$f_n(t,x) = \begin{cases} \frac{1}{2}\left[f(t,\chi_n(x)) + |f(t,\chi_n(x))|\right] & \text{if} \quad t \in (0,a_n] \cup [b_n,T), \\ f(t,\chi_n(x)) & \text{if} \quad t \in (a_n,b_n), \end{cases}$$

where v_1 and v_2 are solutions of some auxiliary linear Dirichlet problems.

Step 3. *Convergence of the sequence of approximating solutions.*

Solvability of a sequence of regular problems

$$u'' + f_n(t,u) = 0, \quad u(0) = \sigma_1(a_n), \quad u(T) = \sigma_1(b_n), \quad n \in \mathbb{N}, \tag{4.16}$$

is investigated. The functions α_1 and β are a lower and an upper function of (4.16) with $n = 1$, and hence, by Theorem 4.6, there is a w-solution u_1 of (4.16) with $n = 1$ such that $\alpha_1 \le u_1 \le \beta$ on $[0,T]$. Further, α_2 and u_1 are a lower and an upper function of (4.16) with $n = 2$, and so Theorem 4.6 guarantees the existence of a w-solution u_2 of (4.16) with $n = 2$ such that $\alpha_2 \le u_2 \le u_1$ on $[0,T]$. In this way a sequence of w-solutions is obtained and then a limit process is applied. □

At the end of this subsection we will show another existence assertion in terms of the lower and upper functions for a problem with the p-Laplacian of the form

$$(\phi_p(u'))' + \psi(t)\, g(t,u) = 0, \quad u(0) = u(1) = 0, \tag{4.17}$$

where $\phi_p(y) = |y|^{p-2}y$, $p > 1$. Here we assume that

$$\begin{cases} \psi\colon\ (0,1) \to (0,\infty) \text{ is continuous} \\ \text{and can have time singularities at } t = 0 \text{ and } t = 1, \end{cases} \tag{4.18}$$

$$\begin{cases} g\colon\ [0,1] \times (0,\infty) \to \mathbb{R} \text{ is continuous} \\ \text{and can have a space singularity at } x = 0. \end{cases} \tag{4.19}$$

In this setting, using the paper by Agarwal, Lü and O'Regan [2], we offer the following result about the existence of positive w-solutions of (4.17).

THEOREM 4.12. *Let* (4.18) *and* (4.19) *hold. Assume that the following conditions are satisfied:*

$$\begin{cases} \{\rho_n\} \text{ is a nonincreasing sequence of real numbers,} \\ \lim_{n\to\infty} \rho_n = 0 \text{ and } n_0 \in \mathbb{N},\ n_0 \ge 3 \text{ being fixed,} \end{cases} \tag{4.20}$$

$$\max\left\{ \int_0^{\frac{1}{2}} \phi_p^{-1}\left(\int_s^{\frac{1}{2}} \psi(t)\,\mathrm{d}t\right)\mathrm{d}s, \int_{\frac{1}{2}}^1 \phi_p^{-1}\left(\int_{\frac{1}{2}}^s \psi(t)\,\mathrm{d}t\right)\mathrm{d}s \right\} = b_0 < \infty \tag{4.21}$$

and

$$\psi(t)\,g(t,\rho_n) \ge 0 \quad \text{for } t \in [\frac{1}{2^{n+1}}, 1), \quad n \ge n_0.$$

Further assume that σ_1 *and* σ_2 *are a lower and an upper function of problem* (4.17) *with* $\sigma_1 > 0$ *on* $(0,1)$, $\max\{\rho_{n_0}, \sigma_1(t)\} \le \sigma_2(t)$ *for* $t \in [0,1]$ *and*

$$(\phi_p(\sigma_2'(t)))' + \psi(t)\,g(\frac{1}{2^{n_0+1}}, \sigma_2(t)) \le 0 \quad \text{for } t \in (0, \frac{1}{2^{n_0+1}}).$$

Then problem (4.17) *has a positive w-solution* $u \in C[0,1]$ *with* $\phi_p(u') \in C^1(0,1)$ *and*

$$\sigma_1(t) \le u(t) \le \sigma_2(t) \quad \text{for } t \in [0,1].$$

In Section 4.3 we will show how lower and upper functions for regular problems can be applied to get not only a w-solution but also a solution of a given singular problem (see Theorem 4.18).

4.2. *Positive nonlinearities*

Many papers studying problem (4.1) or (4.2) with a space singularity at $x = 0$ concern the case that the nonlinearity f is positive. Such problems are refered to as *positone* ones in literature, see [11], [12] or [144]. The positivity of f implies that each solution is concave and hence positive on $(0,T)$, and if, moreover, f has a space singularity at $x = 0$ but not at y, then each solution has only two corresponding singular points 0, T which are of type I. This makes the study of such problems easier than of those having sign-changing f or space singularities at y.

First we will discuss mixed singularities at t *and* x. In Section 1.1 we have presented problem (1.3), (1.4) the solvability of which was investigated by Taliaferro [149]. This problem has mixed singularities: the time ones at $t = 0$ and $t = 1$ as well as the space one at $x = 0$. Among many papers generalizing Taliaferro's existence results we choose the paper by Tineo [150] devoted to the existence of positive solutions or w-solutions to a singular problem

$$u'' + f(t,u,u') = 0, \quad u(0) = u(1) = 0, \tag{4.22}$$

where

$$\begin{cases} f\colon (0,1)\times(0,\infty)\times\mathbb{R} \to (0,\infty) \text{ is continuous and can have} \\ \text{time singularities at } t = 0,\ t = 1 \text{ and a space singularity at } x = 0. \end{cases} \tag{4.23}$$

Theorem 4.13. ([150, Theorem 0.1]) *Let* (4.23) *hold. Suppose that there are continuous functions* $\varphi\colon (0,1) \to (0,\infty)$, $\psi\colon (0,1) \to [0,\infty)$ *and* $g\colon (0,\infty) \to (0,\infty)$ *such that* g *is decreasing and*

$$f(t,x,y) \le \varphi(t)\, g(x) + \psi(t)\, |y| \quad \textit{for } t \in (0,1),\ x \in (0,\infty),\ y \in \mathbb{R},$$

$$\int_0^1 t\,(1-t)\,\varphi(t)\,\mathrm{d}t < \infty, \qquad \int_0^1 \psi(t)\,\mathrm{d}t < \infty.$$

Assume further that for each constant $M > 0$ *there exists a continuous function* $\varepsilon_M :$ $(0,1) \to (0,\infty)$ *such that*

$$\varepsilon_M(t) \le f(t,x,y) \quad \textit{for } t \in (0,1),\ x \in (0,M],\ y \in \mathbb{R}.$$

Then problem (4.22) *has a positive w-solution* $u \in C[0,1] \cap C^2(0,1)$. *If, moreover,*

$$\int_0^1 g(k\,t\,(1-t))\,\varphi(t)\,\mathrm{d}t < \infty \quad \textit{for all } k > 0,$$

then u *belongs to* $AC^1[0,1]$, *which means that* u *is a solution of* (4.22).

The proof of Theorem 4.13 proceeds according to Remark 4.2. Solvability of auxiliary regular problems is obtained by the Leray-Schauder degree argument and the limit process is guaranteed by means of a priori estimates of the approximate solutions.

Example. Let α, $\beta \in (0,2)$, k, $\lambda \in (0,\infty)$, $\varepsilon \in C(0,1)$, $\varepsilon > 0$ on $(0,1)$. By Theorem 4.13 the problem

$$u'' + \frac{1}{t^\alpha\,(1-t)^\beta\, u^\lambda} + t^k\,|u'| + \varepsilon(t) = 0, \quad u(0) = u(1) = 0$$

has a positive w-solution $u \in C[0,1] \cap C^2(0,1)$. If $\alpha + \lambda$, $\beta + \lambda \in (0,1)$ then, moreover, $u \in C^1[0,1]$. Hence u is a solution.

Let us turn back to problem (4.14) with f satisfying (4.15). Wang in [156] considered f which can have at most linear growth in x. He illustrated his result by functions

$$f(t,x) = \frac{1}{t^\alpha\, x^\beta} \quad \text{with } \ \alpha \in (1,2),\ \beta > 0$$

or

$$f(t,x) = \delta\, x\, \exp\left(\frac{1}{x}\right) \quad \text{with a sufficiently small } \ \delta > 0.$$

For f which is moreover continuous on $(0,T) \times (0,\infty)$, Agarwal and O'Regan [3], [4], [12] proved the existence of a positive w-solution of (4.14) with f increasing in x for large x. An example of such f is

$$f(t,x) = \delta\left(\frac{1}{x^\alpha} + x^\beta + 1\right), \quad \alpha,\ \beta,\ \delta \in (0,\infty).$$

If f has sublinear growth in x (i.e. $\beta \in (0,1)$), then (4.14) has a positive w-solution for each $\delta > 0$. If f has linear or superlinear growth in x (i.e. $\beta = 1$ or $\beta \in (1,\infty)$), then (4.14) has a positive w-solution for any sufficiently small $\delta > 0$. A formula for an upper bound of δ is also given.

Now let us consider space singularities at x and y. We will present conditions ensuring solvability of problems with singularities in space variables x and y and with singular points both of type I and of type II. The main difficulty in the study of singular points of type II is the fact that their location in $[0,T]$ is not known. This is the reason why in mathematical literature there are only few papers concerning solvability of such problems and no results about w-solutions are known.

The first existence result in this direction was reached by Staněk [144] in 2001. The following theorem can be viewed as a corollary of [144, Theorem 1].

For a fixed $A > 0$ we consider a singular problem

$$u'' + \mu\, f(t,u,u') = 0, \quad u(0) = u(T) = 0, \tag{4.24}$$

with a positive real parameter μ, where

$$\begin{cases} f \text{ is continuous on } \mathcal{D}_A = [0,T] \times (0,A) \times \big[-\frac{2A}{T}, 0\big) \cup \big(0, \frac{2A}{T}\big] \\ \text{and can have space singularities at } x = 0,\ x = A,\ y = 0. \end{cases} \tag{4.25}$$

Sufficient conditions on μ and f for the solvability of (4.24) in the set $[0,T] \times \mathcal{A}$, where $\mathcal{A} = [0,A] \times [-\frac{2A}{T}, \frac{2A}{T}]$, are given in the next theorem.

Theorem 4.14. *Let* (4.25) *hold. Suppose that there exists* $\delta > 0$ *such that* f *satisfies*

$$\delta \le f(t,x,y) \le g(x)\,\omega(|y|) \quad \textit{on } \mathcal{D}_A,$$

where $\omega \ge \delta$ *is continuous on* $\big(0, \frac{2A}{T}\big]$ *and* $g \in C(0,A) \cap L_1[0,A]$. *Let*

$$\mu_T = \left(\int_0^{\frac{2A}{T}} \frac{y}{\omega(y)}\, \mathrm{d}y \right) \left(\int_0^A g(x)\, \mathrm{d}x \right)^{-1}.$$

Then for any $\mu \in (0, \mu_T]$ *problem* (4.24) *has a solution* $u_\mu \in AC^1[0,T]$ *satisfying*

$$0 < u_\mu(t) < A \quad \textit{for} \ \ t \in (0,T). \tag{4.26}$$

Take notice of the fact that for any solution u of problem (4.24) there is a point $t_u \in (0,T)$ with $u'(t_u) = 0$. Since f has a singularity at $y = 0$ and we do not know the position of $t_u \in (0,T)$, t_u is a singular point of type II.

To prove Theorem 4.14 a two-parameter family of regular problems is constructed and their solvability is established by the topological transversality method. Then, a priori bounds for approximate solutions of regular problems are derived. Using these bounds and the Arzelà-Ascoli theorem, a solution of (4.24) is obtained by a limiting process.

Example. Let $A > 0$, a, b, c, d, $\gamma \in [0,\infty)$, $a+b+d > 0$, α, $\beta \in (0,1)$. Then for a sufficiently small $\mu > 0$ the problem

$$u'' + \mu\left(1 + \frac{a}{u^\alpha\,(A-u)^\beta} + b\,u^\gamma\right)\left(1 + c\,u'^2\right)\left(1 + \frac{d}{u'^2}\right) = 0, \quad u(0) = u(T) = 0$$

has a solution u_μ satisfying (4.26). The upper bound μ_T is explicitly expressed in [144].

The next existence result is in the spirit of Staněk [147], where a more general state-dependent functional differential equation was studied. Here we consider problem (4.1) with the ϕ-Laplacian and a function f satisfying

$$\begin{cases} f \in Car([0,T]\times\mathcal{D}),\ \mathcal{D} = (0,\infty)\times(\mathbb{R}\setminus\{0\}), \\ \text{and } f \text{ can have space singularities at } x = 0,\ y = 0. \end{cases} \tag{4.27}$$

THEOREM 4.15. *Let* (4.27) *hold and let* ϕ *be odd. Suppose that there exists* $\delta \in (0,\infty)$ *such that* f *satisfies*

$$\delta \le f(t,x,y) \le \big(h_1(x) + h_2(x)\big)\big(\omega_1(\phi(|y|)) + \omega_2(\phi(|y|))\big)$$

for a.e. $t \in [0,T]$ *and all* $(x,y) \in \mathcal{D}$, *where* h_1, $\omega_1 \in C[0,\infty)$ *are positive and nondecreasing,* h_2, $\omega_2 \in C(0,\infty)\cap L_1[0,1]$ *are positive and nonincreasing and*

$$\int_0^\infty \frac{\mathrm{d}s}{\omega_1(s)} = \infty.$$

Let

$$\frac{T}{2} < \liminf_{u\to\infty} \frac{u}{K^{-1}\big(T\,(h_1(u) + h_2(u))\big)}\,,$$

where K^{-1} *denotes the inverse function to* $K\colon [0,\infty) \to [0,\infty)$,

$$K(u) = \int_0^{\phi(u)} \frac{\mathrm{d}s}{\omega_1(s) + \omega_2(s)}\,.$$

Then problem (4.1) *has a positive solution* $u \in AC^1[0,T]$.

In the proof of this result the solvability of a sequence of regular problems is obtained by the Leray-Schauder degree theory. Limit processes are guaranteed by Vitali's convergence theorem.

Example. Let $c \in L_\infty[0,T]$, $p \in (1,\infty)$, $\beta \in (0,1)$, γ, $\eta \in (0,p)$, $\delta \in (0,\infty)$ and $\alpha \in (0,p-\gamma)$. Further, let $c(t) \ge \delta$ a.e. on $[0,T]$. Then the problem

$$(|u'|^{p-2}\,u')' + c(t)\left(1 + u^\alpha + \frac{1}{u^\beta}\right)\left(1 + |u'|^\gamma + \frac{1}{|u'|^\eta}\right) = 0, \quad u(0) = u(T) = 0$$

has a positive solution.

4.3 . *Sign-changing nonlinearities*

Results about the solvability of singular Dirichlet problems with sign-changing nonlinearities mostly concern w-solutions. Making use of the arguments of Section 1 we can show a new existence principle giving positive solutions to singular Dirichlet problems of the form

$$u'' + f(t, u, u') = 0, \quad u(0) = u(T) = 0, \tag{4.28}$$

where

$$\begin{cases} f \in Car((0,T) \times \mathcal{D}) \text{ can change its sign, } \mathcal{D} = (0,\infty) \times \mathbb{R}, \\ \text{and } f \text{ can have mixed singularities at } t = 0,\ t = T,\ x = 0. \end{cases} \tag{4.29}$$

For $k \in \mathbb{N}$, $k \ge \frac{3}{T}$, $t \in [0,T]$, $x \in \mathbb{R}$, put $\Delta_k = [0, \frac{1}{k}) \cup (T - \frac{1}{k}, T]$,

$$\gamma_k(t) = \begin{cases} \frac{1}{k} & \text{if } t < \frac{1}{k}, \\ t & \text{if } t \in [0,T] \setminus \Delta_k, \\ T - \frac{1}{k} & \text{if } t > T - \frac{1}{k}, \end{cases} \qquad \delta_k(x) = \begin{cases} |x| & \text{if } |x| \ge \frac{1}{k}, \\ \frac{1}{k} & \text{if } |x| < \frac{1}{k}. \end{cases}$$

Then we construct the sequence of regular functions

$$f_k(t, x, y) = f(\gamma_k(t), \delta_k(x), y) \quad \text{for a.e. } t \in [0,T], \text{ each } x,\ y \in \mathbb{R} \tag{4.30}$$

and the sequence of regular problems

$$u'' + f_k(t, u, u') = 0, \quad u(0) = u(T) = \frac{1}{k}, \tag{4.31}$$

where $f_k \in Car([0,T] \times \mathbb{R}^2)$, $k \in \mathbb{N}$, $k \ge \frac{3}{T}$.

Theorem 4.16. **(Existence principle for solutions of (4.28))** *Let* (4.29) *hold. Assume that*

$$\begin{cases} \textit{there exists a bounded set } \Omega \subset C^1[0,T] \textit{ such that} \\ \textit{problem } (4.31) \textit{ has a solution } u_k \in \Omega \textit{ for each } k \in \mathbb{N},\ k \ge \frac{3}{T}, \end{cases} \tag{4.32}$$

$$\begin{cases} \textit{there exists a function } \varepsilon \in C[0,T],\ \varepsilon(0) = \varepsilon(T) = 0, \\ \textit{such that } u_k(t) \ge \varepsilon(t) > 0 \quad \textit{for } t \in (0,T) \textit{ and each } k \in \mathbb{N},\ k \ge \frac{3}{T} \end{cases} \tag{4.33}$$

and

$$\textit{the sequence } \{f_k(t, u_k(t), u_k'(t))\} \textit{ is uniformly integrable on } [0,T]. \tag{4.34}$$

Then

$$\begin{cases} \textit{there exist a function } u \in \overline{\Omega} \textit{ and a subsequence } \{u_{k_n}\} \subset \{u_k\} \\ \textit{such that } \lim_{n\to\infty} \|u_{k_n} - u\|_{C^1} = 0, \end{cases} \tag{4.35}$$

and

$$u \in AC^1[0,T] \textit{ is a positive solution of problem } (4.28). \tag{4.36}$$

Proof. Since f can have singularities both at t and at x, we cannot obtain Theorem 4.16 as a direct consequence of some of the theorems in Section 1. Nevertheless, we can use the ideas of their proofs and argue as follows.

Step 1. *Convergence of the sequence of approximating solutions.*

By (4.32) we get that $\{u_k\}$ and $\{u_k'\}$ are bounded in $C[0,T]$. The boundedness of $\{u_k'\}$ implies the equicontinuity of $\{u_k\}$ on $[0,T]$. Condition (4.34) yields the equicontinuity of $\{u_k'\}$ on $[0,T]$ and hence the Arzelà-Ascoli theorem gives the assertion (4.35).

Step 2. *Convergence of the sequence of regular right-hand sides.*

The conditions $u_{k_n}(0) = u_{k_n}(T) = \frac{1}{k_n}$ imply $u(0) = u(T) = 0$. By (4.33) we get $u(t) > 0$ on $(0,T)$. Choose $\xi \in (0,T)$ such that $f(\xi,\cdot,\cdot)\colon \mathcal{D} \to \mathbb{R}$ is continuous. Then, by virtue of (4.30) and (4.33), we have

$$u_{k_n}(\xi) \ge \varepsilon(\xi) > \frac{1}{k_n}, \quad \xi \in [0,T] \setminus \Delta_{k_n}, \quad f_{k_n}(\xi, u_{k_n}(\xi), u_{k_n}'(\xi)) = f(\xi, u_{k_n}(\xi), u_{k_n}'(\xi))$$

for a sufficiently large k_n. Therefore

$$\lim_{n\to\infty} f_{k_n}(t, u_{k_n}(t), u_{k_n}'(t)) = f(t, u(t), u'(t)) \quad \text{for a.e. } t \in [0,T]. \tag{4.37}$$

Step 3. *Properties of the limit u.*

By (4.34), (4.37) and Vitali's convergence theorem, we get that $f(t,u(t),u'(t))$ belongs to $L_1[0,T]$ and we can pass to the limit in the sequence

$$u_{k_n}'(t) = u_{k_n}'(0) - \int_0^t f_{k_n}(s, u_{k_n}(s), u_{k_n}'(s))\,\mathrm{d}s, \ \ t \in [0,T],$$

thus obtaining

$$u'(t) = u'(0) - \int_0^t f(s, u(s), u'(s))\,\mathrm{d}s, \quad t \in [0,T].$$

Hence (4.36) is true. □

Remark 4.17. Having in mind the absolute continuity of the Lebesgue integral we see that if there exists $\varphi \in L_1[0,T]$ such that

$$|f_k(t, u_k(t), u_k'(t))| \le \varphi(t) \quad \text{for a.e. } t \in [0,T] \text{ and each } k \in \mathbb{N},\ k \ge \tfrac{3}{T}, \tag{4.38}$$

then condition (4.34) is valid.

In Section 4.1, the classical lower and upper functions method has been extended to singular problems (see Theorems 4.6, 4.8, 4.11 and 4.12). Motivated by Agarwal, O'Regan, Lakshmikantham and Leela [14], we will show another approach which consists in the employment of a sequence of lower and upper functions of approximating regular problems.

THEOREM 4.18. *Let* (4.29) *and* (4.30) *hold. Assume that there exists* $k_0 \in \mathbb{N}$ *such that for each* $k \in \mathbb{N}$, $k \geq k_0$, *the following conditions are satisfied:*

$$\begin{cases} \alpha_k \text{ and } \beta \text{ are a lower and an upper function of (4.31) and} \\ \alpha_k',\ \beta' \in L_\infty[0,T], \quad \frac{1}{k} \leq \alpha_k(t) \leq \beta(t) \quad \text{for } t \in [0,T], \end{cases} \tag{4.39}$$

$$\begin{cases} |f_k(t,x,y)| \leq \psi(t)\, g(x)\, \omega(|y|) \quad \text{for a.e. } t \in [0,T] \\ \text{and for all } x \in (0, \|\beta\|_\infty),\ y \in \mathbb{R}, \end{cases} \tag{4.40}$$

with

$$\begin{cases} \text{positive functions } \psi \in L_1[\frac{1}{k}, T - \frac{1}{k}],\ \omega \in C[0,\infty) \text{ and} \\ \text{a positive nonincreasing function } \ g \in C(0,\infty). \end{cases} \tag{4.41}$$

Further assume that there is a function $\alpha \in C[0,T]$ *with* $\alpha(0) = \alpha(T) = 0$, $\alpha > 0$ *on* $(0,T)$, $\alpha_k \geq \alpha$ *on* $[0,T]$, *such that*

$$\int_0^T \psi(t)\, g(\alpha(t))\, \mathrm{d}t < \int_0^\infty \frac{\mathrm{d}s}{\omega(s)}\,. \tag{4.42}$$

Then problem (4.28) *has a positive solution* $u \in AC^1[0,T]$ *satisfying*

$$\alpha(t) \leq u(t) \leq \beta(t) \quad \text{for } t \in [0,T].$$

SKETCH OF THE PROOF. Theorem 4.18 can be proved by means of Theorem 4.16 in the following way.

Step 1. *Construction of the sequence of regular problems.*

Condition (4.42) implies that there exists $\rho > 0$ such that

$$\int_0^T \psi(t) g(\alpha(t))\, \mathrm{d}t < \int_0^\rho \frac{\mathrm{d}s}{\omega(s)}\,. \tag{4.43}$$

For $k \in \mathbb{N}$, $k \geq k_0$, put $\rho_k = \max\{\rho, \|\alpha_k'\|_\infty, \|\beta'\|_\infty\}$ and consider a sequence of regular problems

$$u'' + \widetilde{f}_k(t,u,u') = 0, \quad u(0) = u(T) = \frac{1}{k}\,, \tag{4.44}$$

where for a.e. $t \in [0,T]$ and all $(x,y) \in \mathbb{R}^2$ we set

$$\widetilde{f}_k(t,x,y) = \chi_k(y)\, f_k(t,x,y)$$

and

$$\chi_k(y) = \begin{cases} 1 & \text{if} \quad |y| \leq \rho_k, \\ 2 - \dfrac{|y|}{\rho_k} & \text{if} \quad \rho_k < |y| < 2\,\rho_k, \\ 0 & \text{if} \quad |y| \geq 2\,\rho_k. \end{cases}$$

Then α_k and β are respectively a lower and an upper function of (4.44), $\widetilde{f}_k$ satisfies (4.40) and, moreover, there exists $\widetilde{h}_k \in L_1[0,T]$ such that

$$|\widetilde{f}_k(t,x,y)| \le \widetilde{h}_k(t) \quad \text{for a.e. } t \in [0,T],\ \text{all } x \in [\alpha_k(t),\beta(t)],\ y \in \mathbb{R}.$$

The classical existence result based on the lower and upper functions method for regular problems (see e.g. [154]) guarantees that for each $k \ge k_0$ problem (4.44) has a solution u_k with

$$\alpha \le \alpha_k \le u_k \le \beta \quad \text{on } [0,T]. \tag{4.45}$$

Step 2. A priori estimates of approximate solutions.

Using (4.40), (4.41) and (4.43) we deduce that $\|u_k'\|_\infty \le \rho$, which implies that u_k is also a solution of (4.31) for $k \ge k_0$. Hence, if we put $\varepsilon(t) = \alpha(t)$ for $t \in [0,T]$ and

$$\Omega = \{x \in C^1[0,T]\colon\ \alpha \le x \le \beta \text{ on } [0,T],\ \|x'\|_\infty \le \rho\},$$

we get that (4.32), (4.33) are fulfilled.

Step 3. Uniform integrability of regular right-hand sides.

By (4.40), (4.41) and (4.45) we get

$$|f_k(t,u_k(t),u_k'(t))| \le \psi(t)\, g(u_k(t))\, \omega(|u_k'(t)|) \le M\, \psi(t)\, g(\alpha(t)) = \varphi(t)$$

for a.e. $t \in [0,T]$, where $M = \max\{\omega(|s|)\colon\ s \in [-\rho,\rho]\}$. By virtue of (4.43), $\varphi \in L_1[0,T]$ and we conclude by Remark 4.17 that condition (4.34) is valid. Therefore the assertion follows from Theorem 4.16 and condition (4.45). □

Remark 4.19. If f does not depend on y, then (4.42) takes the form

$$\int_0^T \psi(t)\, g(\alpha(t))\, \mathrm{d}t < \infty.$$

In the rest of this section we present a selection of existence results about w-solutions which can be obtained by theorems from Section 4.1. Consider a singular Dirichlet problem

$$u'' + f(t,u) = 0, \quad u(0) = u(T) = 0, \tag{4.46}$$

where

$$\left\{ \begin{array}{l} f \in Car((0,T)\times(0,\infty)) \text{ can change its sign and} \\ \text{can have singularities at } t=0,\ t=T \text{ and at } x=0. \end{array} \right. \tag{4.47}$$

The first result is due to Lomtatidze [105].

Theorem 4.20. *Let (4.47) hold. Assume that f is nonincreasing as a function of its second argument and that there is $\varepsilon > 0$ for which*

$$f(t,\varepsilon) \ge 0 \quad \textit{for a.e. } t \in [0,T],\ \text{meas}\,\{t \in (0,T)\colon\ f(t,\varepsilon) > 0\} > 0.$$

Then the condition

$$\int_0^T t\,(T-t)\,|f(t,\delta)|\,\mathrm{d}t < \infty \quad \textit{for any } \delta \in (0,\varepsilon]$$

is necessary and sufficient for problem (4.46) *to have a unique positive w-solution* $u \in C[0,T] \cap AC^1_{loc}(0,T)$.

The proof of Theorem 4.20 is based on the lower and upper functions method via Theorem 4.11. More general results were obtained by Lomtatidze and Torres in [106], where a differential equation having moreover the term $g(t,u)\,u'$ was investigated.

A very similar result for a simpler case of problem (4.46) with f satisfying (4.49) was proved by Habets and Zanolin in [85]. The analogue of their results was proved by Jiang [91] for a singular Dirichlet problem with the p-Laplacian

$$(\phi_p(u'))' + f(t,u) = 0, \quad u(0) = u(T) = 0, \tag{4.48}$$

where $\phi(y) = |y|^{p-2}\,y,\;\; p > 1$ and

$$\begin{cases} f \in C((0,T)\times(0,\infty)) \text{ can change its sign and} \\ \text{can have singularities at } t=0,\; t=T \text{ and at } x=0. \end{cases} \tag{4.49}$$

Theorem 4.21. ([91, Theorem 3]) *Let* (4.49) *hold. Assume that*

(*i*) *there exists a constant* $L > 0$ *such that for any compact set* $K \subset (0,T)$ *there is* $\varepsilon = \varepsilon_K > 0$ *such that*

$$f(t,x) > L \quad \textit{for all } t \in K,\; x \in (0,\varepsilon],$$

(*ii*) *for any* $\delta > 0$ *there are* $h_\delta \in C(0,T)$ *and* $\mu,\, \nu \in [0,p-1)$ *such that*

$$|f(t,x)| \le h_\delta(t) \quad \textit{for all } t \in (0,T),\; x \ge \delta$$

and

$$\int_0^T t^\mu\,(T-t)^\nu\,h_\delta(t)\,\mathrm{d}t < \infty.$$

Then problem (4.48) *has a positive w-solution* $u \in C[0,T]$ *with* $\phi_p(u') \in C^1(0,T)$. *If, moreover, for each* $t \in (0,T)$ *the function* $f(t,x)$ *is nonincreasing in* x, *then* u *is a unique w-solution.*

Another existence result for differential equation where the nonlinearity f can depend on u' is due to Jiang in [92]. He studied the singular Dirichlet problem of the form

$$u'' + f(t,u,u') = 0, \quad u(0) = u(1) = 0, \tag{4.50}$$

where

$$\begin{cases} f \in C((0,1)\times(0,\infty)\times\mathbb{R}) \text{ can change its sign} \\ \text{and can have singularities at } t=0,\ t=1 \text{ and } x=0. \end{cases} \tag{4.51}$$

Motivated by the example

$$f(t,x,y) = \frac{\delta}{t^m\,(1-t)^n}\left(\frac{1}{x^\alpha} + x^\beta + \sin(8\,\pi\,t)\right)\left(1 + t\,(1-t)\,|y|^{\frac{1}{\gamma}}\right) \tag{4.52}$$

with real numbers $\alpha > 0$, $\beta \geq 0$, $\gamma > 1$, $\delta > 0$, $0 \leq m,\ n < 2$, he proved existence of a positive w-solution of (4.50) for f satisfying (4.51) and having superlinear growth in x for large x and sublinear growth in y for large $|y|$. Particularly, if f in (4.50) has the form (4.52), then an upper bound for δ is found guaranteeing that (4.50) has a positive w-solution. The proof is based on the papers [85] and [4].

Let us turn back to problem (4.17). Using the lower and upper functions method established in Theorem 4.12 we can get sufficient conditions for the existence of positive w-solutions. Specifically, we report the result motivated by Agarwal, Lü and O'Regan [2].

Theorem 4.22. *Let* (4.18) – (4.21) *hold. Assume that there exist* $n_0 \in \mathbb{N}$, $n_0 \geq 3$ *and* $c_0 \in (0,\infty)$ *such that*

$$\psi(t)\,g(t,x) \geq c_0 \quad \textit{for } t \in [\frac{1}{2^{n+1}}, 1-\frac{1}{2^{n+1}}],\ x \in (0,\rho_n],\ n \geq n_0$$

and

$$|g(t,x)| \leq \eta(x) + h(x) \quad \textit{for} \ \ (t,x) \in [0,1]\times(0,\infty),$$

where $\eta \in AC_{loc}(0,\infty)$ *is positive,* $h \in C[0,\infty)$ *is nonnegative and* $\frac{h}{\eta}$ *is nondecreasing on* $(0,\infty)$. *Further assume that*

$$\eta' < 0 \quad \textit{a.e. on} \ \ (0,R) \quad \textit{and} \quad \frac{\eta'}{\eta^2} \in L_1[0,R] \quad \textit{for any } R > 0$$

and that there exists $r \in (0,\infty)$ *such that*

$$b_0 < \left(\phi_p^{-1}\left(1 + \frac{h(r)}{\eta(r)}\right)\right)^{-1} \int_0^r \frac{\mathrm{d}u}{\phi_p^{-1}\,(\eta(u))}\,.$$

Then problem (4.17) *has a positive w-solution* $u \in C[0,1]$ *with* $\phi_p(u') \in C^1(0,1)$.

4.4 . *Sign-changing solutions and w-solutions*

If we consider a singular differential equation with a space singularity at $x=0$, a question about the existence of sign-changing solutions of such an equation can arise. A single result in this direction was proved by Rachůnková and Staněk in [124], where an equation of the form $(r(u)\,u')' = \mu\,q(t)\,f(t,u)$ has been studied.

Here we will present this result for a simplified equation

$$u'' + \mu f(t,u) = 0 \tag{4.53}$$

with a positive real parameter μ and with boundary conditions

$$u(0) = u(T) = 0, \quad \max\{u(t)\colon t \in [0,T]\} \min\{u(t)\colon t \in [0,T]\} < 0. \tag{4.54}$$

We assume that

$$\begin{cases} f\colon [0,T] \times (\mathbb{R} \setminus \{0\}) \to \mathbb{R} \quad \text{is continuous} \\ \text{and can have a space singularity at } \ x = 0. \end{cases} \tag{4.55}$$

By *a solution of problem* (4.53), (4.54) we mean a function $u \in C^1[0,T]$ having precisely one zero $t_u \in (0,T)$. Moreover, $u \in C^2((0,T) \setminus \{t_u\})$ fulfils (4.54) and there exists $\mu_u > 0$ such that u satisfies (4.53) for $\mu = \mu_u$ and $t \in (0,T) \setminus \{t_u\}$.

Theorem 4.23. *Let* (4.55) *hold. Assume that for each $t \in [0,T]$ the function $f(t,x)$ is nondecreasing with respect to x on $(-\infty,0)$ and nonincreasing on $(0,\infty)$ and*

$$\varepsilon \le f(t,x) \operatorname{sign} x \le g(x) \quad \textit{for} \ \ (t,x) \in [0,T] \times (\mathbb{R} \setminus \{0\}),$$

where $\varepsilon \in (0,\infty)$ and $g \in C(\mathbb{R} \setminus \{0\}) \cap L_1[-1,1]$.

Then for each $A \in (0,\infty)$ and $B \in (-\infty,0)$ there exist solutions u and v of problem (4.53), (4.54) *satisfying*

$$\max\{u(t)\colon t \in [0,T]\} = A \quad \textit{and} \quad \min\{v(t)\colon t \in [0,T]\} = B. \tag{4.56}$$

By virtue of Theorem 4.23, any solution u of problem (4.53), (4.54) vanishes at some point $t_u \in (0,T)$. Since f has a singularity at $x = 0$ and we do not know the position of t_u, we see that t_u is a singular point of type II.

The proof of Theorem 4.23 is based on a combination of four main theorems in [124], where a new method of proofs was developed. It is based on "gluing" the positive and negative parts of solutions and smoothing them.

In accordance with the paper [125] by Rachůnková and Staněk we define *a w-solution of problem* (4.53), (4.54) as a function $u \in C[0,T]$ having precisely one zero $t_u \in (0,T)$. Further, $u \in C^2((0,T) \setminus \{t_u\})$ fulfils (4.54), there exist finite limits

$$\lim_{t \to t_u -} u'(t), \quad \lim_{t \to t_u +} u'(t)$$

and there exists $\mu_u > 0$ such that u satisfies (4.53) for $\mu = \mu_u$ and $t \in (0,T) \setminus \{t_u\}$.

In [125], the following existence result for w-solutions is proved.

Theorem 4.24. *Let all assumptions of Theorem* 4.23 *be satisfied.*

Then for each $t_0 \in (0,T)$ and each $A \in (0,\infty)$, $B \in (-\infty,0)$ problem (4.53), (4.54) *has just two different w-solutions vanishing at t_0 and having their maximum value on $[0,T]$ equal to A, and just two different w-solutions vanishing at t_0 and having their minimum value on $[0,T]$ equal to B.*

In the proof of Theorem 4.24, w-solutions are constructed by means of solutions of auxiliary Dirichlet problems on $[0, t_0]$ and $[t_0, T]$.

Example. Let α, $\beta \in (0, 1)$, $a \in (0, \infty)$, $b \in (-\infty, 0)$ and

$$f(x) = \begin{cases} \dfrac{a}{x^\alpha} & \text{for } x > 0, \\[2ex] \dfrac{b}{(-x)^\beta} & \text{for } x < 0. \end{cases}$$

Consider the differential equation

$$u'' + \mu f(u) = 0. \tag{4.57}$$

By Theorem 4.23, for each $A > 0$ and $B < 0$ there exist solutions u and v of problem (4.57), (4.54) satisfying (4.56). Moreover, by Theorem 4.24, for each $t_0 \in (0, T)$ and for each $A > 0$ there exist just two different w-solutions u_1 and u_2 of problem (4.57), (4.54) satisfying

$$\max\{u_1(t) \colon t \in [0, T]\} = \max\{u_2(t) \colon t \in [0, T]\} = A.$$

Further, for each $t_0 \in (0, T)$ and for each $B < 0$ there exist just two different w-solutions v_1 and v_2 of problem (4.57), (4.54) satisfying

$$\min\{v_1(t) \colon t \in [0, T]\} = \min\{v_2(t) \colon t \in [0, T]\} = B.$$

4.5 . *Historical and bibliographical notes*

A systematic study of solvability of Dirichlet problems having both time and space singularities was initiated in 1979 by Taliaferro [149], who found necessary and sufficient conditions for the existence of solutions (w-solutions) of problem (1.3), (1.4). A contribution to the more general problem (4.22) was published by Bobisud, O'Regan and Royalty [38] in 1988. In 1989, in contrast to the shooting method used in [149] and the topological transversality method applied in [38], Gatica, Oliker and Waltman [76] proved a fixed point theorem for decreasing maps on cones and applying it they obtained solvability of (4.14). However, in these works the nonlinearity f had to be bounded in its space variables x and y for large x and large $|y|$.

An extension of the above results permitting linear growth of f in its third variable y for large $|y|$ was treated by Baxley [25] in 1991 and by Tineo [150] in 1992. The condition of boundedness of f in its second variable x for large x was overcome by Agarwal and O'Regan in [3] (1996), where the existence of a positive w-solution was proved even for f having superlinear growth for large x. In 1999, the first multiplicity result for Dirichlet problems with time and space singularities was reached. Particularly, Agarwal and O'Regan [6] proved the existence of two different positive w-solutions. All these results rely on the fact that nonlinearities in the equations considered are positive.

In 1987, this assumption was removed by Lomtatidze [105] for problem (4.14). We can also refer to papers by Janus and Myjak [88] for a nonhomogeneous equation (1.3) and by Habets and Zanolin [85] for the continuous case of (4.14) which appeared in 1994. From papers providing more general existence results for problems with sign-changing nonlinearities we mention the recent papers by Jiang [92] (2002), by Agarwal, Staněk [18] (2003) or by Lomtatidze, Torres [106] (2003). These papers deal with problems of the type (4.2).

Existence results for problems of the type (4.1) with the ϕ-Laplacian and sign-changing nonlinearities were presented by Wang and Gao in [158] (1996), where Taliaferro's results were extended. Existence results in the spirit of Habets and Zanolin which are applicable to problems with the p-Laplacian of the form (4.48) were given by Jiang in [91] (2001). In 2003, Agarwal, Lü and O'Regan [2] published the existence result for the problem with the p-Laplacian of the form (4.17).

Further results and references for positive and for sign-changing nonlinearities can be found in the monographs by Kiguradze [97] (1975), by Kiguradze and Shekhter [98] (1987), by O'Regan [116] (1994), by Agarwal and O'Regan [11] (2003) and in [12] (2004). Note that there exists a large group of papers investigating Dirichlet boundary value problems having only time singularities. These results are not discussed here but some of them can be found in the above cited monographs.

In the study of Dirichlet problems with space singularities and singular points both of type I and of type II the first existence result was reached by Staněk [144] in 2001, and the existence of sign-changing solutions was proved by Rachůnková and Staněk [124] in 2003. Numerical algorithms and computation of solutions and w-solutions of singular Dirichlet problems were given by Baxley [26] (1995) and by Baxley and Thompson [29] (2000).

5. Singular periodic BVPs with ϕ-Laplacian

The aim of this section is to present existence results for singular periodic problems of the form

$$(\phi(u'))' = f(t, u, u'), \tag{5.1}$$

$$u(0) = u(T), \quad u'(0) = u'(T), \tag{5.2}$$

where $0 < T < \infty$, $\phi\colon \mathbb{R} \to \mathbb{R}$ is an increasing and odd homeomorphism such that $\phi(\mathbb{R}) = \mathbb{R}$ and

$$f \in Car([0,T] \times ((0,\infty) \times \mathbb{R})) \quad \text{and} \quad f \text{ can have a space singularity at } x = 0. \tag{5.3}$$

In accordance with Section 1.3, this means that

$$\limsup_{x \to 0+} |f(t,x,y)| = \infty \qquad \text{for a.e. } t \in [0,T] \text{ and some } y \in \mathbb{R}$$

may happen.

Remark 5.1. Physicists say that f has an *attractive* singularity at $x = 0$ if

$$\liminf_{x\to 0+} f(t,x,y) = -\infty \qquad \text{for a.e. } t \in [0,T] \text{ and some } y \in \mathbb{R}$$

since near the origin the force is directed inward. Alternatively, f is said to have a *repulsive* singularity at $x = 0$ if

$$\limsup_{x\to 0+} f(t,x,y) = \infty \qquad \text{for a.e. } t \in [0,T] \text{ and some } y \in \mathbb{R}$$

holds.

In the setting of Section 1.3, problem (5.1), (5.2) is investigated on the set $[0,T] \times \mathcal{A}$, where $\mathcal{A} = [0,\infty) \times \mathbb{R}$. In contrast to the Dirichlet problem (4.1), where each solution vanishes at $t = 0$ and $t = T$ and hence enters the space singularity $x = 0$ of f, all known existence results for the periodic problem (5.1), (5.2) under the assumption (5.3) concern positive solutions which do not touch the space singularity $x = 0$ of the function f.

Definition 5.2. A function $u\colon [0,T] \to \mathbb{R}$ is a *positive solution* to problem (5.1), (5.2) if $\phi(u') \in AC[0,T]$, $u > 0$ on $[0,T]$, $(\phi(u'(t)))' = f(t,u(t),u'(t))$ for a.e. $t \in [0,T]$ and (5.2) is satisfied.

The restriction to positive solutions causes that the general existence principle in Theorem 1.8 about a limit of a sequence of approximate solutions need not be employed here. On the other hand, the singular problem (5.1), (5.2) will be also investigated through regular approximating periodic problems having differential equations of the form

$$(\phi(u'))' = h(t,u,u'), \tag{5.4}$$

where $h \in Car([0,T] \times \mathbb{R}^2)$. As usual, by a *solution of the regular problem* (5.4), (5.2) we understand a function u such that $\phi(u') \in AC[0,T]$, (5.2) is true and $(\phi(u'(t)))' = h(t,u(t),u'(t))$ for a.e. $t \in [0,T]$.

Notice that the requirement $\phi(u') \in AC[0,T]$ implies that $u \in C^1[0,T]$.

We will also discuss various special cases of (5.1) including the classical one with $\phi(y) \equiv y$ or those with f not depending on u' or with f depending on u' linearly.

Let us notice that the assumption that ϕ is an odd function is only technical and it is sufficient to assume (3.3) as in Section 3 in most cases. We employ it just to simplify some formulas occurring in this section.

5.1 . *Method of lower and upper functions for regular problems*

First, we will consider problem (5.4), (5.2), where $h \in Car([0,T] \times \mathbb{R}^2)$. We bring some results which will be exploited in the investigation of the singular problem (5.1), (5.2). The lower and upper functions method combined with the topological degree argument is an important tool for proofs of solvability of regular periodic problems. Several rather general definitions of lower and upper functions are available (see e.g. [51], [52], [66], [98], [130], [155]). However, for our purposes the following one seems to be optimal.

DEFINITION 5.3. We say that a function $\sigma \in C[0,T]$ is a *lower function* of problem (5.4), (5.2) if there is a finite set $\Sigma \subset (0,T)$ such that $\phi(\sigma') \in AC_{loc}([0,T] \setminus \Sigma)$, $\sigma'(\tau+)$, $\sigma'(\tau-) \in \mathbb{R}$ for each $\tau \in \Sigma$ and

$$(\phi(\sigma'(t)))' \geq h(t, \sigma(t), \sigma'(t)) \quad \text{for a.e. } t \in [0,T], \tag{5.5}$$

$$\sigma(0) = \sigma(T), \quad \sigma'(0) \geq \sigma'(T), \tag{5.6}$$

$$\sigma'(\tau+) > \sigma'(\tau-) \quad \text{for all } \tau \in \Sigma. \tag{5.7}$$

If the inequalities in (5.5)–(5.7) are reversed, σ is called an *upper function* of problem (5.4), (5.2).

The role of lower and upper functions is demonstrated by the following "maximum principle":

LEMMA 5.4. *Let σ_1 and σ_2 be a lower and an upper function of* (5.4), (5.2) *and $\sigma_1 \leq \sigma_2$ on $[0,T]$.*

Then for each $d \in [\sigma_1(0), \sigma_2(0)]$ and each $\widetilde{f} \in Car([0,T] \times \mathbb{R}^2)$ such that

$$\begin{cases} \widetilde{f}(t,x,y) < h(t, \sigma_1(t), \sigma_1'(t)) \text{ for a.e. } t \in [0,T], \text{ all } x \in (-\infty, \sigma_1(t)) \\ \qquad \text{and all } y \in \mathbb{R} \text{ such that } |y - \sigma_1'(t)| \leq \dfrac{\sigma_1(t) - x}{\sigma_1(t) - x + 1}, \\ \widetilde{f}(t,x,y) > h(t, \sigma_2(t), \sigma_2'(t)) \text{ for a.e. } t \in [0,T], \text{ all } x \in (\sigma_2(t), \infty) \\ \qquad \text{and all } y \in \mathbb{R} \text{ such that } |y - \sigma_2'(t)| \leq \dfrac{x - \sigma_2(t)}{x - \sigma_2(t) + 1}, \end{cases} \tag{5.8}$$

any solution u of the problem

$$(\phi(u'))' = \widetilde{f}(t, u, u'), \quad u(0) = u(T) = d$$

satisfies $\sigma_1 \leq u \leq \sigma_2$ on $[0,T]$.

PROOF. Denote $v = u - \sigma_1$ and assume that $v(\alpha) = \min\{v(t) : t \in [0,T]\} < 0$. Since $d \in [\sigma_1(0), \sigma_2(0)]$ and thanks to (5.6) and (5.7), we may assume that $\alpha \in (0,T) \setminus \Sigma$, $v'(\alpha) = 0$, and there is $\beta \in (\alpha, T]$ such that $(\alpha, \beta] \cap \Sigma = \emptyset$ and

$$v(t) < 0 \quad \text{and} \quad |v'(t)| < \frac{-v(t)}{1 - v(t)} \text{ for all } t \in [\alpha, \beta].$$

Using (5.5) (where $\sigma = \sigma_1$) and (5.8), we obtain

$$(\phi(u'(t)) - \phi(\sigma_1'(t)))' < h(t, \sigma_1(t), \sigma_1'(t)) - (\phi(\sigma_1'(t)))' \leq 0 \text{ for a.e. } t \in [\alpha, \beta].$$

Hence

$$0 > \int_\alpha^t (\phi(u'(s)) - \phi(\sigma_1'(s)))' \, \mathrm{d}s = \phi(u'(t)) - \phi(\sigma_1'(t)) \quad \text{for all } t \in (\alpha, \beta],$$

which leads to a contradiction with the definition of α, i.e. $u \ge \sigma_1$ on $[0,T]$. Similarly we can show that $u \le \sigma_2$ on $[0,T]$. □

Problem (5.4), (5.2) is often transformed to a fixed point problem (see e.g. [42], [107], [110], [159]). Here we present one possibility how to find an operator representation of (5.4), (5.2) in the space $C^1[0,T]$. Having in mind that the periodic conditions (5.2) can be equivalently rewritten as

$$u(0) = u(T) = u(0) + u'(0) - u'(T),$$

let us consider the quasilinear Dirichlet problem

$$(\phi(x'))' = b(t) \quad \text{a.e. on } [0,T], \quad x(0) = x(T) = d \tag{5.9}$$

with $b \in L_1[0,T]$ and $d \in \mathbb{R}$. A function $x \in C^1[0,T]$ is a solution of (5.9) if and only if there is $a \in \mathbb{R}$ such that

$$x(t) = d + \int_0^t \phi^{-1}\Big(a + \int_0^s b(\tau)\,\mathrm{d}\tau\Big)\,\mathrm{d}s \quad \text{for } t \in [0,T]$$

and

$$\int_0^T \phi^{-1}\Big(a + \int_0^s b(\tau)\,\mathrm{d}\tau\Big)\,\mathrm{d}s = 0. \tag{5.10}$$

Since ϕ is increasing on $\mathbb{R}$ and $\phi(\mathbb{R}) = \mathbb{R}$, equation (5.10) has exactly one solution $a = a(b) \in \mathbb{R}$ for each $b \in L_1[0,T]$. So, we can define an operator $\mathcal{K}\colon L_1[0,T] \to C^1[0,T]$ by

$$(\mathcal{K}(b))(t) = \int_0^t \phi^{-1}\Big(a(b) + \int_0^s b(\tau)\,\mathrm{d}\tau\Big)\,\mathrm{d}s \quad \text{for } \ t \in [0,T]. \tag{5.11}$$

Let $\mathcal{N}: C^1[0,T] \to L_1[0,T]$ and $\mathcal{F}: C^1[0,T] \to C^1[0,T]$ have the form $(\mathcal{N}(u))(t) = h(t,u(t),u'(t))$ for a.e. $t \in [0,T]$ and

$$(\mathcal{F}(u))(t) = u(0) + u'(0) - u'(T) + (\mathcal{K}(\mathcal{N}(u)))(t) \quad \text{for } t \in [0,T]. \tag{5.12}$$

In view of the definition of $\mathcal{K}$, a function $x \in C^1[0,T]$ is a solution to (5.9) if and only if $x = d + \mathcal{K}(b)$. Therefore, $u \in C^1[0,T]$ is a solution to (5.4), (5.2) if and only if it is a fixed point of $\mathcal{F}$.

An alternative representation of the operator $\mathcal{F}$ can be obtained by inserting $\alpha(u) = u(0) - d$, $\beta(u) = d - u(T)$ and $d = u(0) + u'(0) - u'(T)$ into the operator $\mathcal{P}(1,u)$ defined in the proof of Theorem 3.2. In this way we get

$$\begin{aligned}(\mathcal{F}(u))(t) &= u(0) + u'(0) - u'(T)\\ &\quad + \int_0^t \phi^{-1}\Big(\phi(u'(T) + u(T) - u(0)) + \int_0^s (\mathcal{N}(u))(\tau)\,\mathrm{d}\tau\Big)\,\mathrm{d}s\end{aligned}$$

for $t \in [0,T]$ and $u \in C^1[0,T]$.

Taking into account [107, Proposition 2.2] or the proof of Theorem 3.2, we can summarize:

Lemma 5.5. *Let $\mathcal{F}\colon C^1[0,T] \to C^1[0,T]$ be defined by* (5.12).

Then $\mathcal{F}$ is completely continuous and $u \in C^1[0,T]$ is a solution to (5.4), (5.2) *if and only if $\mathcal{F}(u) = u$.*

The next lemma describes the relationship between lower and upper functions and the Leray-Schauder topological degree. We will consider the class of auxiliary problems

$$(\phi(v'))' = \eta(v')\, h(t,v,v'), \quad v(0) = v(T), \quad v'(0) = v'(T), \tag{5.13}$$

where $\eta\colon \mathbb{R} \to [0,1]$ may be an arbitrary continuous function.

Lemma 5.6. *Let σ_1 and σ_2 be a lower and an upper function of* (5.4), (5.2) *and $\sigma_1 < \sigma_2$ on $[0,T]$. Furthermore, let there exist $r^* > 0$ such that*

$$\begin{cases} \|v'\|_\infty < r^* \quad \textit{for each continuous } \eta\colon \mathbb{R} \to [0,1] \textit{ and for} \\ \textit{each solution } \ v \ \textit{ of } \ (5.13) \textit{ such that } \quad \sigma_1 \le v \le \sigma_2 \textit{ on } [0,T]. \end{cases} \tag{5.14}$$

Finally, assume that $\mathcal{F}\colon C^1[0,T] \to C^1[0,T]$ is defined by (5.12) *and, for $\rho > 0$, denote*

$$\Omega_\rho = \big\{ u \in C^1[0,T]\colon \sigma_1 < u < \sigma_2 \textit{ on } [0,T] \textit{ and } \ \|u'\|_\infty < \rho \big\}. \tag{5.15}$$

Then

$$\deg(\mathcal{I} - \mathcal{F}, \Omega_\rho) = 1 \quad \textit{for each } \ \rho \ge r^* \textit{ such that } \ \mathcal{F}(u) \ne u \textit{ on } \partial\Omega_\rho.$$

Proof. Put $\Omega = \Omega_{r^*}$, $\ R^* = r^* + \|\sigma_1'\|_\infty + \|\sigma_2'\|_\infty$,

$$\eta(y) = \begin{cases} 1 & \text{if } \ |y| \le R^*, \\ 2 - \frac{|y|}{R^*} & \text{if } \ R^* < |y| \le 2\,R^*, \\ 0 & \text{if } \ |y| > 2\,R^* \end{cases}$$

and assume that $\mathcal{F}(x) \ne x$ for all $x \in \partial\Omega$. Then σ_1 and σ_2 are a lower and an upper function for the modified problem (5.13) and there is a $\psi \in L_1[0,T]$ satisfying

$$|\eta(y)\, h(t,x,y)| \le \psi(t) \text{ for a.e. } t \in [0,T] \text{ and all } (x,y) \in [\sigma_1(t), \sigma_2(t)] \times \mathbb{R}.$$

We can construct (see the proof of Theorem 2.1 in [135]) the function $\widetilde{f} \in Car([0,T] \times \mathbb{R}^2)$ so that

$$\widetilde{f}(t,x,y) = \eta(y)\, h(t,x,y) \text{ for a.e. } t \in [0,T] \text{ and all } (x,y) \in [\sigma_1(t), \sigma_2(t)] \times \mathbb{R},$$
$$|\widetilde{f}(t,x,y)| \le \widetilde{\psi}(t) \text{ for a.e. } t \in [0,T], \text{ all } (x,y) \in \mathbb{R}^2 \text{ and some } \ \widetilde{\psi} \in L_1[0,T]$$

and $\widetilde{f}$ satisfies the assumptions of Lemma 5.4 with $\eta(y)\, h(t,x,y)$ in place of $h(t,x,y)$. Define $\widetilde{\mathcal{F}}\colon C^1[0,T] \to C^1[0,T]$ by

$$\widetilde{\mathcal{F}}(u) = \alpha(u(0) + u'(0) - u'(T)) + \mathcal{K}(\widetilde{\mathcal{N}}(u)),$$

where

$$(\widetilde{\mathcal{N}}(u))(t) = \widetilde{f}(t, u(t), u'(t)) \quad \text{for } u \in C^1[0,T] \text{ and a.e. } \ t \in [0,T],$$

$$\alpha(x) = \begin{cases} \sigma_1(0) & \text{if } \ x < \sigma_1(0), \\ x & \text{if } \ \sigma_1(0) \le x \le \sigma_2(0), \\ \sigma_2(0) & \text{if } \ x > \sigma_2(0) \end{cases}$$

and let $\mathcal{K}: \ L_1[0,T] \to C^1[0,T]$ be given in (5.11). By Lemma 5.5, the operator $\widetilde{\mathcal{F}}$ is completely continuous. Moreover, it follows from the definition of the operator $\mathcal{K}$ that the problem

$$(\phi(u'))' = \widetilde{f}(t, u, u'), \quad u(0) = u(T) = \alpha(u(0) + u'(0) - u'(T))$$

is equivalent to the operator equation $\widetilde{\mathcal{F}}(u) = u$. We can find $r_0 \in (0, \infty)$ such that for any $\lambda \in [0,1]$, each fixed point u of the operator $\lambda\widetilde{\mathcal{F}}$ belongs to $\mathcal{B}(r_0) = \{x \in C^1[0,T]: \|x\|_\infty + \|x'\|_\infty < r_0\}$. So, $\mathcal{I} - \lambda\widetilde{\mathcal{F}}$ is a homotopy on $\overline{\mathcal{B}}(r_0) \times [0,1]$ and

$$\deg(\mathcal{I} - \widetilde{\mathcal{F}}, \mathcal{B}(r_0)) = \deg(\mathcal{I}, \mathcal{B}(r_0)) = 1.$$

Put $\Omega_1 = \big\{u \in \Omega: \ \sigma_1(0) < u(0) + u'(0) - u'(T) < \sigma_2(0)\big\}$. Clearly, $\widetilde{\mathcal{F}} = \mathcal{F}$ on $\overline{\Omega}_1$ and $u \in \Omega_1$ whenever $\mathcal{F}(u) = u$ and $u \in \Omega$. Using Lemma 5.4, we can prove that

$$(\widetilde{\mathcal{F}}(u) = u) \implies u \in \Omega_1$$

which, by the excision property of the degree, yields

$$\deg(\mathcal{I} - \mathcal{F}, \Omega) = \deg(\mathcal{I} - \mathcal{F}, \Omega_1) = \deg(\mathcal{I} - \widetilde{\mathcal{F}}, \Omega_1) = \deg(\mathcal{I} - \widetilde{\mathcal{F}}, \mathcal{B}(r_0)) = 1.$$

Finally, according to (5.14) all fixed points u of $\mathcal{F}$ such that $\sigma_1 < u < \sigma_2$ on $[0,T]$ belong to Ω. Thus

$$\deg(\mathcal{I} - \mathcal{F}, \Omega_\rho) = \deg(\mathcal{I} - \mathcal{F}, \Omega) = 1$$

for each $\rho \ge r^*$ such that $\mathcal{F}(x) \ne x$ on $\partial\Omega_\rho$. □

Lemma 5.6 offers a possibility to get existence results for problems having a pair of lower and upper functions σ_1 and σ_2 satisfying

$$\sigma_1 \le \sigma_2 \quad \text{on } [0,T]. \tag{5.16}$$

In such a case we say that σ_1 and σ_2 are *well-ordered* and the existence of an a priori estimate r^* with the property (5.14) is usually ensured by conditions of Nagumo type. The most general known version of such conditions is provided by the next lemma which is a modified version of the result by Staněk [142, Lemma 1].

LEMMA 5.7. *Let σ_1, $\sigma_2 \in C[0,T]$ satisfy* (5.16) *and assume that*

$$\begin{cases} \psi \in L_1[0,T] \text{ is nonnegative, } \varepsilon_1,\ \varepsilon_2 \in \{-1,1\}, \\ \omega \in C(\mathbb{R}) \text{ is positive and } \displaystyle\int_{-\infty}^{0} \frac{\mathrm{d}s}{\omega(s)} = \int_{0}^{\infty} \frac{\mathrm{d}s}{\omega(s)} = \infty. \end{cases} \tag{5.17}$$

Then there is an $r^* > 0$ such that

$$\|v'\|_\infty < r^* \tag{5.18}$$

holds for each $v \in C^1[0,T]$ such that $\phi(v') \in AC[0,T]$, $v(0) = v(T)$, $v'(0) = v'(T)$, $\sigma_1 \le v \le \sigma_2$ on $[0,T]$ and, for a.e. $t \in [0,T]$,

$$\begin{cases} \varepsilon_1\ (\phi(v'(t)))' \le (\psi(t) + v'(t))\,\omega(\phi(v'(t))) & \text{if } v'(t) > 0, \\ \varepsilon_2\ (\phi(v'(t)))' \le (\psi(t) - v'(t))\,\omega(\phi(v'(t))) & \text{if } v'(t) < 0. \end{cases}$$

□

Lemma 5.6 provides also a crucial argument for the proof of existence of a solution even in the case that the given problem possesses lower and upper functions σ_1 and σ_2 which do not satisfy (5.16), i.e. if

$$\sigma_1(\tau) > \sigma_2(\tau) \quad \text{for some} \quad \tau \in [0,T]. \tag{5.19}$$

In such a case, the following a priori estimate is available.

LEMMA 5.8. *Let $\psi \in L_1[0,T]$. Then there is $r^* > 0$ such that* (5.18) *holds for each $v \in C^1[0,T]$ fulfilling $\phi(v') \in AC[0,T]$, $v(0) = v(T)$, $v'(0) = v'(T)$ and $(\phi(v'(t)))' > \psi(t)$ (or $(\phi(v'(t)))' < \psi(t)$) for a.e. $t \in [0,T]$.*

PROOF. We will restrict ourselves to the case that $(\phi(v'(t)))' > \psi(t)$ for a.e. $t \in [0,T]$. (The other case can be proved by a similar argument.) By the proof of [137, Lemma 1.1], we can see that $\|w\|_\infty < \|\psi\|_1$ holds for each $w \in AC[0,T]$ such that $w(0) = w(T)$, $w(t_w) = 0$ for some $t_w \in (0,T)$ and $w'(t) > \psi(t)$ for a.e. $t \in [0,T]$. The assertion of the lemma follows by setting $w = \phi(v')$ and

$$r^* = \phi^{-1}(\|\psi\|_1). \tag{5.20}$$

□

The next lemma provides an existence principle which will be helpful later:

LEMMA 5.9. *Let σ_1 and σ_2 be a lower and an upper function of* (5.4), (5.2) *and let* (5.19) *be true. Furthermore, let there be $m \in L_1[0,T]$ such that*

$$h(t,x,y) > m(t) \ (\text{or } h(t,x,y) < m(t)) \ \text{ for a.e. } t \in [0,T] \text{ and all } x, y \in \mathbb{R}$$

and let $r^* > 0$ *be given by* (5.20), *where* $\psi = |m| + 2$.

Then problem (5.4), (5.2) *has a solution* u *satisfying*

$$\|u'\|_\infty < r^* \tag{5.21}$$

and

$$\min\{\sigma_1(\tau_u), \sigma_2(\tau_u)\} \le u(\tau_u) \le \max\{\sigma_1(\tau_u), \sigma_2(\tau_u)\} \quad \textit{for some } \tau_u \in [0, T]. \tag{5.22}$$

Sketch of the proof. We follow the ideas of the proof of Theorem 3.2 in [135]. Assume e.g. that $h(t, x, y) > m(t)$ for a.e. $t \in [0, T]$ and all $x, y \in \mathbb{R}$.

Step 1. *Construction of an auxiliary problem and the operator representation.*

Define $\psi(t) := -(|m(t)| + 2)$ for a.e. $t \in [0, T]$, find $r^* > 0$ as in Lemma 5.8 and set $c^* = \|\sigma_1\|_\infty + \|\sigma_2\|_\infty + T\, r^*$. Consider the auxiliary problem

$$(\phi(u'))' = \widetilde{f}(t, u, u'), \quad u(0) = u(T), \quad u'(0) = u'(T), \tag{5.23}$$

where

$$\widetilde{f}(t, x, y) = \begin{cases} -(|m(t)| + 1) & \text{if } x \le -(c^* + 1), \\ h(t, x, y) + (x + c^*)\,(|m(t)| + 1 + h(t, x, y)) & \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \text{if } -(c^* + 1) < x < -c^*, & \\ h(t, x, y) & \text{if } -c^* \le x \le c^*, \\ h(t, x, y) + (x - c^*)\,|m(t)| & \text{if } c^* < x < c^* + 1, \\ h(t, x, y) + |m(t)| & \text{if } x \ge c^* + 1. \end{cases}$$

We have

$$\begin{cases} \widetilde{f}(t, x, y) < 0 & \text{if } x \le -(c^* + 1), \\ \widetilde{f}(t, x, y) > 0 & \text{if } x \ge c^* + 1, \\ \widetilde{f}(t, x, y) = h(t, x, y) & \text{if } x \in [-c^*, c^*], \end{cases} \tag{5.24}$$

and

$$\widetilde{f}(t, x, y) > \psi(t) \quad \text{for a.e. } t \in [0, T] \text{ and all } x, y \in \mathbb{R} \tag{5.25}$$

and σ_1 and σ_2 are a lower and an upper function of (5.23). Moreover, $\sigma_3(t) \equiv -c^* - 2$ and $\sigma_4(t) \equiv c^* + 2$ form another pair of a lower and an upper function for (5.23) and

$$\sigma_3 < \min\{\sigma_1, \sigma_2\} \le \max\{\sigma_1, \sigma_2\} < \sigma_4 \text{ on } [0, T].$$

Denote $\Omega_0 = \{u \in C^1[0, T] \colon \sigma_3 < u < \sigma_4 \text{ on } [0, T],\ \|u'\|_\infty < r^*\}$,

$$\Omega_1 = \{u \in \Omega_0 \colon \sigma_3 < u < \sigma_2 \text{ on } [0, T]\}, \quad \Omega_2 = \{u \in \Omega_0 \colon \sigma_1 < u < \sigma_4 \text{ on } [0, T]\}$$

and $\Omega = \Omega_0 \setminus \overline{\Omega_1 \cup \Omega_2}$. By Lemma 5.5, problem (5.23) is equivalent to the operator equation $\widetilde{\mathcal{F}}(u) = u$ in $C^1[0, T]$, where $\widetilde{\mathcal{F}}(u) = u(0) + u'(0) - u'(T) + \mathcal{K}(\widetilde{\mathcal{N}}(u))$, $(\widetilde{\mathcal{N}}(u))(t) =$

$\widetilde{f}(t, u(t), u'(t))$ and $\mathcal{K}\colon L_1[0,T] \to C^1[0,T]$ is given by (5.11). Clearly, $\widetilde{\mathcal{F}}(u) = \mathcal{F}(u)$ for $u \in C^1[0,T]$ such that $\|u\|_\infty \le c^*$.

Step 2. *A priori estimates.* We show that

$$\|u'\|_\infty < r^* \quad \text{and} \quad \|u\|_\infty < c^*$$

is true for all $u \in \overline{\Omega}$ such that $\widetilde{\mathcal{F}}(u) = u$.

Step 3. *Existence of a solution to* (5.4), (5.2).

Let $\widetilde{\mathcal{F}}(u) = u$ and $u \in \partial\Omega$. By Step 3, we have $\mathcal{F}(u) = \widetilde{\mathcal{F}}(u) = u$ and u solves (5.4), (5.2). Let $\widetilde{\mathcal{F}}(u) \ne u$ on $\partial\Omega$. Then using Lemma 5.6 we get

$$\deg(\mathcal{I} - \widetilde{\mathcal{F}}, \Omega_0) = \deg(\mathcal{I} - \widetilde{\mathcal{F}}, \Omega_1) = \deg(\mathcal{I} - \widetilde{\mathcal{F}}, \Omega_2) = 1.$$

Furthermore, by (5.19), we have $\Omega_1 \cap \Omega_2 = \emptyset$. Therefore, due to the additive property of the degree,

$$\deg(\mathcal{I} - \widetilde{\mathcal{F}}, \Omega) = \deg(\mathcal{I} - \widetilde{\mathcal{F}}, \Omega_0) - \deg(\mathcal{I} - \widetilde{\mathcal{F}}, \Omega_1) - \deg(\mathcal{I} - \widetilde{\mathcal{F}}, \Omega_2) = -1$$

which implies that $\widetilde{\mathcal{F}}$ has a fixed point $u \in \Omega$. It follows by Step 3 that $\|u\|_\infty < c^*$ which, by virtue of (5.24), means that u solves (5.4), (5.2).

We can proceed analogously when $h(t,x,y) < m(t)$ for a.e. $t \in [0,T]$ and all $x, y \in \mathbb{R}$. $\square$

5.2. *Method of lower and upper functions for singular problems*

Now, we consider problem (5.1), (5.2) where f satisfies (5.3). We present sufficient conditions in terms of lower and upper functions for the existence of positive solutions to (5.1), (5.2). Lower and upper functions σ_1 and σ_2 are defined in the same way as for the regular problem (5.4), (5.2) (see Definition 5.3). However, since problem (5.1), (5.2) is investigated on $[0,T] \times \mathcal{A}$ where $\mathcal{A} = [0,\infty) \times \mathbb{R}$, only such σ_1 and σ_2 which are positive a.e. on $[0,T]$ make sense.

The first existence result concerns problem (5.1), (5.2) having well-ordered lower and upper functions.

Theorem 5.10. *Let there exist lower and upper functions σ_1 and σ_2 of problem* (5.1), (5.2) *such that* (5.16) *is true and $\sigma_1 > 0$ on $[0,T]$. Furthermore, let for a.e. $t \in [0,T]$ and each $(x,y) \in [\sigma_1(t), \sigma_2(t)] \times \mathbb{R}$ the inequalities*

$$\begin{cases} \varepsilon_1 f(t,x,y) \le (\psi(t) + y)\,\omega(\phi(y)) & \text{if } \; y > 0, \\ \varepsilon_2 f(t,x,y) \le (\psi(t) - y)\,\omega(\phi(y)) & \text{if } \; y < 0 \end{cases} \tag{5.26}$$

hold with ε_1, ε_2, ω and ψ satisfying (5.17).

Then problem (5.1), (5.2) *has a positive solution u such that*

$$\sigma_1 \le u \le \sigma_2 \qquad \text{on } [0,T]. \tag{5.27}$$

PROOF. *Step* 1. *The case* $\sigma_1 < \sigma_2$.

Assume that $\sigma_1 < \sigma_2$ on $[0,T]$. Consider the auxiliary problem (5.4), (5.2) with $h(t,x,y) = f(t, \max\{\sigma_1(t), \min\{x, \sigma_2(t)\}, y)$ for a.e. $t \in [0,T]$ and $(x,y) \in \mathbb{R}^2$. Clearly, $h \in Car([0,T] \times \mathbb{R}^2)$ and $h(t,x,y) = f(t,x,y)$ if $x \in [\sigma_1(t), \sigma_2(t)]$. Further, σ_1 and σ_2 are a lower and an upper function of (5.4), (5.2). Choose an arbitrary continuous $\eta\colon \mathbb{R} \to [0,1]$ and let v be an arbitrary solution of (5.13) fulfilling $\sigma_1 \le v \le \sigma_2$ on $[0,T]$. Since (5.26) is satisfied with h in place of f, we have for a.e. $t \in [0,T]$

$$\begin{aligned}\varepsilon_1\,(\phi(v'(t)))' &= \varepsilon_1\,\eta(v'(t))\,h(t,v(t),v'(t)) \le \eta(v'(t))\,(\psi(t)+v'(t))\,\omega(\phi(v'(t)))\\ &\le (\psi(t)+v'(t))\,\omega(\phi(v'(t))) \qquad \text{if } \ v'(t) > 0\end{aligned}$$

and

$$\varepsilon_2\,(\phi(v'(t)))' \le (\psi(t)-v'(t))\,\omega(\phi(v'(t))) \qquad \text{if } \ v'(t) < 0.$$

Hence we can apply Lemma 5.7 to deduce that (5.14) is satisfied. Let $\mathcal{F}:\ C^1[0,T] \to C^1[0,T]$ and $\Omega = \Omega_{r^*}$ be defined by (5.12) and (5.15), respectively. Then there are two possibilities: either $\mathcal{F}$ has a fixed point $u \in \partial\Omega$ or $\mathcal{F}(u) \ne u$ on $\partial\Omega$.
(a) Let $\mathcal{F}(u) = u$ for some $u \in \partial\Omega$. In view of Lemma 5.5 and of the definition of h, it follows that u is a solution to (5.1), (5.2) fulfilling (5.27).
(b) If $\mathcal{F}(u) \ne u$ on $\partial\Omega$, then by Lemma 5.6 we have $\deg(\mathcal{I} - \mathcal{F}, \Omega) = 1$, which implies that $\mathcal{F}$ has a fixed point $u \in \Omega$. As in (a), this fixed point is a solution to (5.1), (5.2) fulfilling (5.27).

Step 2. The case $\sigma_1 \le \sigma_2$.

For each $k \in \mathbb{N}$, the function $\widetilde{\sigma}_k = \sigma_2 + \frac{1}{k}$ is also an upper function of (5.4), (5.2) and $\sigma_1 < \widetilde{\sigma}_k$ on $[0,T]$. Hence, in the general case, when the strict inequality between σ_1 and σ_2 need not hold, we can use Step 1 to show that for each $k \in \mathbb{N}$ there exists a solution u_k to (5.4), (5.2) such that

$$u_k(t) \in [\sigma_1(t), \sigma_2(t) + \tfrac{1}{k}] \quad \text{ for } \ t \in [0,T] \quad \text{ and } \quad \|u_k'\|_\infty < \rho^*,$$

where $\rho^* > 0$ is the a priori estimate given by Lemma 5.7 with $\sigma_2 + 1$ in place of σ_2. Using the Arzelà-Ascoli theorem and the Lebesgue dominated convergence theorem for the sequence $\{u_k\}$ we get a solution u of (5.1), (5.2) as the C^1-limit of a subsequence of $\{u_k\}$. □

REMARK 5.11. Theorem 5.10 provides the existence of a positive solution to problem (5.1), (5.2) with $f(t,x,y) = -h(x)\,y + g(t,x)$, if $h \in C[0,\infty)$, $g \in Car([0,T] \times (0,\infty))$ and if the existence of well-ordered and positive lower and upper functions is ensured. Indeed, for a.e. $t \in [0,T]$ and each $(x,y) \in [\sigma_1(t), \sigma_2(t)] \times \mathbb{R}$, we have

$$|f(t,x,y)| \le |h(x)|\,|y| + |g(t,x)| \le K\,(\psi(t) + |y|)$$

where $K = 1 + \max\{|h(x)|:\ x \in [\delta, \|\sigma_2\|_\infty]\}$, $\psi(t) = \sup\{|g(t,x)|:\ x \in [\delta, \|\sigma_2\|_\infty]\}$ and $\delta = \min\{\sigma_1(t)\colon t \in [0,T]\}$. (By assumption, we have $\delta > 0$.)

Now, we will consider problem (5.1), (5.2) which has lower and upper functions, but no pair of them is well-ordered. We will deal with the periodic problem for the equation

$$(\phi(u'))' = g(u) + p(t,u,u'). \tag{5.28}$$

THEOREM 5.12. *Assume $g \in C(0,\infty)$, $p \in Car([0,T]\times\mathbb{R}^2)$ and*

$$\lim_{x\to 0+}\int_x^1 g(\xi)\,\mathrm{d}\xi = +\infty. \tag{5.29}$$

Let there exist lower and upper functions σ_1 and σ_2 of problem (5.28), (5.2) *such that* (5.19) *is true and $\sigma_2 > 0$ on $[0,T]$. Furthermore, let there exist an $m \in L_1[0,T]$ such that*

$$g(x) + p(t,x,y) > m(t) \quad \text{for a.e. } t\in[0,T] \text{ and all } x>0, y\in\mathbb{R} \tag{5.30}$$

holds and let r^* be given by (5.20) *with $\psi = |m| + 2$.*

Then problem (5.28), (5.2) *has a positive solution u satisfying* (5.21) *and* (5.22).

PROOF. We will use the ideas of [138]. Similarly to [138, Lemma 2.5] we can deduce from (5.29) and (5.30) that σ_1 is positive on $[0,T]$. Thus, $\delta := \min\{\{\sigma_1(t),\sigma_2(t)\}\colon t\in[0,T]\} > 0$. Put $R = \|\sigma_1\|_\infty + \|\sigma_2\|_\infty$ and $B = R + r^*T$. Furthermore, as $p \in Car([0,T]\times\mathbb{R}^2)$, there is $\widetilde{p} \in L_1[0,T]$ such that $|p(t,x,y)| \le \widetilde{p}(t)$ for a.e. $t\in[0,T]$ and all $(x,y)\in[0,B]\times[-r^*,r^*]$. Put

$$K = \|\widetilde{p}\|_1\, r^* + \int_\delta^B |g(\xi)|\,\mathrm{d}\xi.$$

By (5.29) there exists $\varepsilon \in (0,\delta)$ such that $g(\varepsilon) > 0$ and

$$\int_\varepsilon^\delta g(\xi)\,\mathrm{d}\xi > K. \tag{5.31}$$

For a.e. $t\in[0,T]$ and $(x,y)\in\mathbb{R}^2$, define

$$h(t,x,y) = \widetilde{g}(x) + p(t,x,y), \quad \text{where } \widetilde{g}(x) = \begin{cases} g(\varepsilon) & \text{if } x<\varepsilon,\\ g(x) & \text{if } x\ge\varepsilon. \end{cases}$$

Then $h \in Car([0,T]\times\mathbb{R}^2)$, σ_1 and σ_2 are lower and upper functions of (5.4), (5.2) and, by (5.30), $h(t,x,y) > m(t)$ for a.e. $t\in[0,T]$ and all $x>0$, $y\in\mathbb{R}$. By Lemma 5.9, problem (5.4), (5.2) has a solution u satisfying (5.21) and $\delta \le u(t_u) \le R$ for some $t_u\in[0,T]$. In particular, $u \le B$ for all $t\in[0,T]$. It remains to show that $u \ge \varepsilon$ on $[0,T]$.

Let t_0, $t_1 \in [0,T]$ be such that $u(t_0) = \min\{u(t)\colon t\in[0,T]\}$ and $u(t_1) = \max\{u(t)\colon t\in[0,T]\}$. We have $u'(t_0) = u'(t_1) = 0$ and $u(t_1)\in[\delta,B]$. Put $v(t) = \phi(u'(t))$ for $t\in[0,T]$. Then $u'(t) = \phi^{-1}(v(t))$ on $[0,T]$, $v(t_0) = v(t_1) = \phi(0)$ and

$$\int_{t_0}^{t_1} (\phi(u'(s)))'\,u'(s)\,\mathrm{d}s = \int_{t_0}^{t_1} v'(s)\,\phi^{-1}(v(s))\,\mathrm{d}s = \int_{v(t_0)}^{v(t_1)} \phi^{-1}(\xi)\,\mathrm{d}\xi = 0.$$

Thus, multiplying both sides of the equality $(\phi(u'(t)))' = h(t,u(t),u'(t))$ by $u'(t)$ and integrating from t_0 to t_1, we get

$$\int_{u(t_0)}^{u(t_1)} \widetilde{g}(\xi)\,\mathrm{d}\xi \le \int_{t_0}^{t_1} |p(t,u(t),u'(t))|\,|u'(t)|\,\mathrm{d}t \le \|\widetilde{p}\|_1\, r^*.$$

Therefore

$$g(\varepsilon)\,(\varepsilon - u(t_0)) + \int_{\varepsilon}^{\delta} g(\xi)\,\mathrm{d}\xi = \int_{u(t_0)}^{\delta} \widetilde{g}(\xi)\,\mathrm{d}\xi$$

$$\le \int_{u(t_0)}^{u(t_1)} \widetilde{g}(\xi)\,\mathrm{d}\xi + \int_{\delta}^{B} |g(\xi)|\,\mathrm{d}\xi \le \|\widetilde{p}\|_1\, r^* + \int_{\delta}^{B} |g(\xi)|\,\mathrm{d}\xi = K.$$

Since $g(\varepsilon) > 0$, this contradicts (5.31) whenever $u(t_0) = \min\{u(t)\colon t \in [0,T]\} < \varepsilon$. Hence, $u(t) \ge \varepsilon$ on $[0,T]$ which means that u is a solution to (5.28), (5.2). ☐

Remark 5.13. Let g and p fulfil the assumptions of Theorem 5.12 and $f(t,x,y) = g(x) + p(t,x,y)$. Then the condition (5.29) implies that

$$\limsup_{x\to 0+} g(x) = +\infty, \tag{5.32}$$

which means by Remark 5.1 that f and g have a space repulsive singularity at $x = 0$. Each repulsive singularity having the property (5.29) is called a *strong singularity* of f and the corresponding function g is usually called a *strong repulsive singular force.* On the contrary, if (5.32) holds together with

$$\lim_{x\to 0+} \int_{x}^{1} g(\xi)\,\mathrm{d}\xi \in \mathbb{R}, \tag{5.33}$$

then the singularity of f at $x = 0$ is called a *weak singularity* and g is called a *weak repulsive singular force.*

5.3. *Attractive singular forces*

This section is devoted to singular problem (5.1), (5.2) where f can have an attractive singularity at $x = 0$. (See Remark 5.1.)

In what follows we use the standard notation for mean values of integrable functions: for $y \in L_1[0,T]$, the symbol $\overline{y}$ stands for

$$\overline{y} := \frac{1}{T}\int_0^T y(t)\,\mathrm{d}t.$$

Theorem 5.14. *Let there exist $r > 0$, $A > r$ and $b \in L_1[0,T]$ such that $\overline{b} \ge 0$,*

$$f(t,r,0) \le 0 \quad \text{for a.e. } t \in [0,T], \tag{5.34}$$

$$\begin{cases} f(t,x,y) \ge b(t) \\ \qquad \text{for a.e. } t \in [0,T] \text{ and all } x \in [A,B] \text{ and } |y| \le \phi^{-1}(\|b\|_1), \end{cases} \tag{5.35}$$

where

$$B - A \ge 2\,T\,\phi^{-1}\big(\|b\|_1\big). \tag{5.36}$$

Furthermore, let for a.e. $t \in [0,T]$ *and each* $(x,y) \in [r,B] \times \mathbb{R}$ *the inequalities* (5.26) *hold with* ε_1, ε_2, ω *and* ψ *satisfying* (5.17).

Then problem (5.1), (5.2) *has a positive solution* u *such that*

$$r \le u \le B \ \text{ on } [0,T]. \tag{5.37}$$

Proof. For a given $d \in \mathbb{R}$, let x_d be a solution of (5.9). Then

$$\phi(x_d'(t)) = \phi(x_d'(t_0)) + \int_{t_0}^{t} b(s)\,\mathrm{d}s \quad \text{ for all } t,\ t_0 \in [0,T].$$

Since $\overline{b} \ge 0$, it follows that $x_d'(T) \ge x_d'(0)$. Since $x_d(0) = x_d(T)$, there is a $t_d \in (0,T)$ such that $x_d'(t_d) = 0$. Thus

$$\phi(x_d'(t)) = \int_{t_d}^{t} b(s)\,\mathrm{d}s \quad \text{for } t \in [0,T]$$

and so $\|x_d'\|_\infty \le \phi^{-1}(\|b\|_1)$ for each $d \in \mathbb{R}$ and $\|x_0\|_\infty \le T\,\phi^{-1}(\|b\|_1)$. Put $\sigma_2 = A + T\,\phi^{-1}(\|b\|_1) + x_0$. Then

$$A \le \sigma_2 \le A + 2\,T\,\phi^{-1}(\|b\|_1) \le B \quad \text{ on } [0,T]. \tag{5.38}$$

Having in mind (5.35) and (5.9), we can see that σ_2 is an upper function of (5.1), (5.2). Furthermore, $\sigma_1 = r$ is a lower function of problem (5.1), (5.2) and $0 < \sigma_1 < \sigma_2$ on $[0,T]$. By Theorem 5.10, problem (5.1), (5.2) has a positive solution u satisfying (5.37). □

Now, let us consider the Liénard periodic problem

$$(\phi(u'))' + h(u)\,u' = g(t,u) + e(t), \quad u(0) = u(T), \quad u'(0) = u'(T), \tag{5.39}$$

where g can have an attractive space singularity at $x = 0$.

Theorem 5.15. *Assume*

$$h \in C[0,\infty),\ e \in L_1[0,T],\ g \in Car([0,T] \times (0,\infty)), \tag{5.40}$$

$$\textit{there exists } \alpha > 0 \textit{ such that } \liminf_{|y|\to\infty} \frac{|\phi(y)|}{|y|^\alpha} > 0, \tag{5.41}$$

$$\left\{ \begin{array}{l} \textit{there exists } r > 0 \textit{ such that} \\ g(t,r) + e(t) \le 0 \quad \textit{for a.e. } t \in [0,T], \end{array} \right. \tag{5.42}$$

$$\left\{ \begin{array}{l} \textit{there exist } A > r \textit{ and } g_0 \in L_1[0,T] \textit{ such that} \\ g(t,x) \ge g_0(t) \quad \textit{for a.e. } t \in [0,T] \textit{ and all } x \ge A \end{array} \right. \tag{5.43}$$

and

$$\overline{g}_0 + \overline{e} \ge 0. \tag{5.44}$$

Then problem (5.39) *has a positive solution* u *such that* $u \ge r$ *on* $[0,T]$.

Sketch of the proof. We follow the ideas of the paper [136]. Define

$$f(t,x,y) = -h(x)\,y + g(t,x) + e(t) \quad \text{for a.e. } t \in [0,T] \text{ and all } x \in (0,\infty),\ y \in \mathbb{R}.$$

Step 1. First, notice that, due to (5.42), $\sigma_1(t) \equiv r$ is a lower function of (5.39).

Step 2. Thanks to (5.41), (5.43) and (5.44), we can construct an upper function σ_2 of (5.39). To this aim, take an arbitrary $C \in \mathbb{R}$ and consider a parameter auxiliary problem

$$(\phi(v'))' + \lambda\, h(v+C)\, v' = \lambda\, b(t), \quad v(0) = v(T) = 0, \quad \lambda \in [0,1], \tag{5.45}$$

where $b(t) = g_0(t) + e(t)$ for a.e. $t \in [0,T]$. By (5.41), there are $k > 0$ and $y_0 > 0$ such that

$$|\phi(y)| > \frac{k}{2}\,|y|^{\alpha} \quad \text{for } |y| \ge y_0. \tag{5.46}$$

Multiplying (5.45) by $v(t)$ and integrating over $[0,T]$, we obtain

$$-\int_0^T \phi(v'(t))\, v'(t)\, \mathrm{d}t = \lambda \int_0^T b(t)\, v(t)\, \mathrm{d}t. \tag{5.47}$$

Using (5.46), (5.47) and the Hölder inequality, we can find $\rho \in (0,\infty)$, independent of $C \in \mathbb{R}$, such that $v \in \mathcal{B}(\rho) = \{x \in C^1[0,T] :\ \|x\|_\infty + \|x'\|_\infty < \rho\}$ holds for each $\lambda \in [0,1]$ and each solution v of (5.45). Thus, choosing a proper operator representation of problem (5.45) and using a standard homotopy and topological degree argument we can show that, for each $C \in \mathbb{R}$, problem (5.45) with $\lambda = 1$ has a solution $v_C \in \mathcal{B}(\rho)$. Now, it is already easy to see that if $C > A + \rho$, then $\sigma_2 = v_C + C$ is an upper function of (5.39). Indeed, we have $\sigma_2(0) = \sigma_2(T) = C$ and, due to (5.44),

$$\phi(\sigma_2'(T)) - \phi(\sigma_2'(0)) = T\,\overline{b} = T\,[\overline{g}_0 + \overline{e}] \ge 0.$$

Moreover, $\sigma_2(t) \ge C - \rho > A > r$ on $[0,T]$. Hence, by (5.43), we have

$$\begin{aligned} (\phi(\sigma_2'(t)))' &= -h(\sigma_2(t))\,\sigma_2'(t) + g_0(t) + e(t) \\ &\le -h(\sigma_2(t))\,\sigma_2'(t) + g(t,\sigma_2(t)) + e(t) = f(t,\sigma_2(t),\sigma_2'(t)) \quad \text{for a.e. } t \in [0,T]. \end{aligned}$$

Step 3. Finally, similarly as in Remark 5.11, we show that f satisfies (5.26) with $\omega(s) \equiv 1 + \max\{|h(x)| :\ x \in [r, \|\sigma_2\|_\infty]\}$, $\psi(t) = |e(t)| + \sup\{|g(t,x)| :\ x \in [r, \|\sigma_2\|_\infty]\}$ and $\varepsilon_1 = \varepsilon_2 = 1$. Therefore, by Theorem 5.10, problem (5.39) has a positive solution u such that $u \ge r$ on $[0,T]$. □

Remark 5.16. If g does not depend on t, i.e. $g(t,x) \equiv g(x)$ for a.e. $t \in [0,T]$ and all $x \in (0,\infty)$, then the condition (5.42) is satisfied if $\liminf_{x\to 0+} (g(x) + \|e\|_\infty) < 0$ which is true e.g. if $\liminf_{x\to 0+} g(x) = -\infty$ and $\sup\operatorname{ess}\{e(t) :\ t \in [0,T]\} < \infty$. Similarly, the conditions (5.43) and (5.44) are in such a case satisfied if $\liminf_{x\to\infty} (g(x) + \overline{e}) > 0$. In particular, Theorem 5.15 applies to problem (5.39) if $\phi = \phi_p$, $p > 1$, $\sup\operatorname{ess}\{e(t) :\ t \in [0,T]\} < \infty$, $\overline{e} > 0$, $g(t,x) = -\beta(t)\,x^{-\lambda}$, where $\beta \in L_1[0,T]$, $\beta \ge \varepsilon > 0$ a.e. on $[0,T]$ and $\lambda \ge 1$. Notice that the condition (5.41) is satisfied e.g. by $\phi(y) = (|y|\,y + y)\,\ln(1 + \frac{1}{|y|})$ or $\phi(y) = y\,(\exp(y^2) - 1)$.

5.4 . *Repulsive singular forces*

In this section we study the singular problem (5.1), (5.2) with f having a repulsive singularity at $x = 0$. Recall (see Remark 5.1) that this means that the relation

$$\limsup_{x\to 0+} f(t,x,y) = \infty \quad \text{for a.e. } t \in [0,T] \text{ and some } y \in \mathbb{R}$$

is true. In general, for the case of a repulsive singularity, the existence of a pair of associated lower and upper functions having opposite order is typical. This causes that such a case is more difficult and more interesting than that of an attractive singularity. The next assertion deals with equation (5.28) and is a direct corollary of Theorem 5.12.

THEOREM 5.17. *Assume* $g \in C(0,\infty)$, $p \in Car([0,T]\times\mathbb{R}^2)$, (5.29) *and* (5.30) *with some* $m \in L_1[0,T]$. *Furthermore, let there be* $r > 0$, $A > r$, $B \ge A$ *and* $b \in L_1[0,T]$ *such that* $\overline{b} \le 0$, (5.36),

$$g(r) + p(t,r,0) \ge 0 \quad \textit{for a.e. } t \in [0,T], \tag{5.48}$$

and

$$g(x) + p(t,x,y) \le b(t) \textit{ for a.e. } t \in [0,T] \textit{ and all } x \in [A,B] \textit{ and } |y| \le \phi^{-1}(\|b\|_1) \tag{5.49}$$

hold.

Then problem (5.28), (5.2) *has a positive solution* u *such that* $u(t_u) \in [r,B]$ *for some* $t_u \in [0,T]$.

PROOF. By (5.48), $\sigma_2(t) \equiv r$ is an upper function of (5.28), (5.2). Furthermore, let x_0 be a solution of

$$(\phi(x'))' = b(t), \quad u(0) = u(T) = 0.$$

Using (5.49) and having in mind that $\overline{b} \le 0$, we can show by a reasoning analogous to that applied in the proof of Theorem 5.14 to construct an upper function that the function $\sigma_1 = A + T\,\phi^{-1}(\|b\|_1) + x_0$ is a lower function of (5.28), (5.2). Using Theorem 5.12 we complete the proof. □

In particular, when restricted to the Duffing equation with the ϕ-Laplacian

$$(\phi(u'))' = g(u) + e(t), \tag{5.50}$$

Theorem 5.17 has the following corollary.

COROLLARY 5.18. *Let* $e \in L_1[0,T]$ *with* $\inf \operatorname{ess}\{e(t)\colon t \in [0,T]\} > -\infty$ *and let* $g \in C(0,\infty)$ *have a strong repulsive singularity* (5.29). *Further, let*

$$g_* := \inf\{g(x)\colon x \in (0,\infty)\} > -\infty \tag{5.51}$$

and let there be $A > 0$ *such that*

$$g(x) + \overline{e} \le 0 \quad \textit{for } x \in [A,B], \quad \textit{where } B - A \ge 2T\,\phi^{-1}(\|e - \overline{e}\|_1).$$

Then problem (5.50), (5.2) *has a positive solution* u *such that* $u(t_u) \le B$ *for some* $t_u \in [0,T]$.

Proof. By (5.29) we have (5.32) and consequently there is an $r \in (0, A)$ such that $g(r) + e(t) \geq 0$ for a.e. $t \in [0, T]$. The assertion follows from Theorem 5.17 if we put $b(t) = e(t) - \overline{e}$ and $m(t) = g_* + e(t)$ a.e. on $[0, T]$. $\square$

Consider the periodic problem for the Liénard equation

$$(\phi_p(u'))' + h(u)\, u' = g(u) + e(t) \tag{5.52}$$

with the p-Laplacian $\phi_p(y) = |y|^{p-2}\, y$. To this end, the following easy corollary of the continuation type principle due to Manásevich and Mawhin turned out to be essential.

Lemma 5.19. ([107, Theorem 3.1] and [89, Lemma 3]) *Let $p > 1$, $h \in C[0, \infty)$, $g \in C(0, \infty)$ and $e \in L_1[0, T]$. Furthermore, assume there exist $r > 0$, $R > r$ and $R' > 0$ such that*

(*i*) *the inequalities $r < v < R$ on $[0, T]$ and $\|v'\|_\infty < R'$ hold for each $\lambda \in (0, 1]$ and for each positive solution v of the problem*

$$(\phi_p(v'))' = \lambda\, (-h(v)\, v' + g(v) + e(t)), \quad v(0) = v(T), \quad v'(0) = v'(T), \tag{5.53}$$

(*ii*) $(g(x) + \overline{e} = 0) \implies r < x < R,$

(*iii*) $(g(r) + \overline{e})\,(g(R) + \overline{e}) < 0.$

Then problem (5.52), (5.2) *has at least one solution u such that $r < u < R$ on $[0, T]$.*

Under the assumptions ensuring that g is bounded below on $(0, \infty)$, the following result was delivered by Jebelean and Mawhin.

Theorem 5.20. ([89, Theorem 2]) *Let $p > 1$, $h \in C[0, \infty)$, $e \in L_1[0, T]$ and let $g \in C(0, \infty)$ have a strong repulsive singularity* (5.29). *Furthermore, assume*

$$\liminf_{x\to\infty}\, g(x) > -\infty \tag{5.54}$$

and

$$\liminf_{x\to 0+}\, [g(x) + \overline{e}] > 0 > \limsup_{x\to\infty}\, [g(x) + \overline{e}]. \tag{5.55}$$

Then problem (5.52), (5.2) *has a positive solution.*

Proof. We will verify that the assumptions of Lemma 5.19 are satisfied.

Step 1. First, we will show that

$$\begin{cases} \text{there are } \ R_0 > 0 \text{ and } \ R_1 > R_0 \text{ such that} \\ v(t_v) \in (R_0, R_1) \quad \text{for some } \ t_v \in [0, T] \\ \text{holds for each } \ \lambda \in (0, 1] \text{ and each positive solution } \ v \text{ of (5.53).} \end{cases} \tag{5.56}$$

To this aim, assume that $\lambda \in (0,1]$ and that v is a positive solution to (5.53). Integrating the differential equation in (5.53) over $[0,T]$ and having in mind the periodicity of v, we get:

$$\int_0^T (g(v(t)) + e(t))\,\mathrm{d}t = 0. \tag{5.57}$$

By the first inequality in (5.55), there is an $R_0 > 0$ such that

$$g(x) + \overline{e} > 0 \quad \text{whenever} \quad x \in (0, R_0). \tag{5.58}$$

If $g(v(t)) + \overline{e} > 0$ were valid on $[0,T]$, we would have

$$\int_0^T (g(v(t)) + e(t))\,\mathrm{d}t = \int_0^T (g(v(t)) + \overline{e})\,\mathrm{d}t > 0.$$

Since this contradicts (5.57), we see that $\max\{v(t)\colon\, t \in [0,T]\} > R_0$. Similarly, by the second inequality in (5.55), there is an $R_1 > R_0$ such that $g(x) + \overline{e} < 0$ for $x \geq R_1$ and $v(t_1) < R_1$ for some $t_1 \in [0,T]$. Therefore (5.56) is true.

Step 2. Now we show that

$$\begin{cases} \text{there is } \ R > 0 \text{ such that } \ v < R \text{ on } \ [0,T] \\ \text{for each } \ \lambda \in (0,1] \text{ and each positive solution } \ v \text{ of (5.53).} \end{cases} \tag{5.59}$$

Notice that, due to (5.54) and (5.58), we have $g_* = \inf\{g(x)\colon\, x \in (0,\infty)\} > -\infty$. Thus, multiplying (5.53) by v and integrating over $[0,T]$, we get

$$\|v'\|_p^p \leq \int_0^T (|g_*| + |e(t)|)\, v(t)\,\mathrm{d}t.$$

Furthermore, for R_1 given as in Step 1 and $\frac{1}{p} + \frac{1}{q} = 1$, we deduce that

$$\|v'\|_p^p \leq \left(\int_0^T (|g_*| + |e(t)|)\,\mathrm{d}t\right)\big(R_1 + T^{\frac{1}{q}}\|v'\|_p\big).$$

The right-hand side being a linear function of $\|v'\|_p$, this is possible only if there is $C_1 > 0$, independent of v and λ and such that $\|v'\|_p < C_1$. Therefore

$$v(t) = v(t_1) + \int_{t_1}^t v'(s)\,\mathrm{d}s < R_1 + T^{\frac{1}{q}}\, C_1$$

for all $\lambda \in (0,1]$ and all positive periodic solutions v of (5.53), i.e. the assertion (5.59) is true with $R := R_1 + T^{\frac{1}{q}}\, C_1$.

Step 3. Next we show that

$$\begin{cases} \text{there is } \ R_2 > 0 \text{ such that } \ |v'| < \lambda^{\frac{1}{p-1}} R_2 \text{ on } \ [0,T] \\ \text{for each } \ \lambda \in (0,1] \text{ and each positive solution } \ v \text{ of (5.53).} \end{cases} \tag{5.60}$$

Having in mind that v satisfies the periodic conditions, we can see that there is $t' \in [0,T]$ such that $v'(t') = 0$. Integrating the differential equation in (5.53) over $[t',t]$ and taking into account (5.59), we get

$$|v'(t)|^{p-1} \le \lambda \left(\int_0^R |h(x)|\,\mathrm{d}x + \|e\|_1 + \left| \int_{t'}^t |g(v(s))|\,\mathrm{d}s \right| \right) \quad \text{for } t \in [0,T]. \tag{5.61}$$

By (5.58), there is $b > 0$ such that $g(x) \ge -b$ for all $x \in (0,R]$. So, by (5.59), $g(v(t)) \ge -b$ on $[0,T]$ holds for each possible positive solution v of (5.53). Therefore, $|g(v(t))| \le g(v(t)) + 2\,b$ for all $t \in [0,T]$ wherefrom, using (5.57), we deduce

$$\left| \int_{t'}^t |g(v(s))|\,\mathrm{d}s \right| \le 2\,b\,T + \|e\|_1,$$

which inserted into (5.61) yields (5.60) with

$$R_2^{p-1} = \int_0^R |h(x)|\,\mathrm{d}x + 2\,(b + \|e\|_1) > 0.$$

Step 4. We show that

$$\left\{ \begin{array}{l} \text{there is } r \in (0,R_0) \text{ such that } v > r \text{ on } [0,T] \\ \text{for each } \lambda \in (0,1] \text{ and each positive solution } v \text{ of (5.53).} \end{array} \right. \tag{5.62}$$

Put $h_R := \max\{|h(x)|\colon x \in [0,R]\}$, $R^* = \frac{R_2^p}{q} + R_2\,(h_R\,R_2\,T + \|e\|_1)$ and

$$K^* = R^* + \int_{R_0}^R |g(x)|\,\mathrm{d}x. \tag{5.63}$$

By (5.29), there is $r > 0$ such that

$$\int_r^{R_0} g(x)\,\mathrm{d}x > K^*. \tag{5.64}$$

Put $w(t) = \phi_p(v'(t))$ for $t \in [0,T]$. Then $|w(t)|^q = |v'(t)|^p$ for $t \in [0,T]$,

$$v'(t) = |w(t)|^{q-2}\,w(t) \quad \text{for } t \in [0,T] \tag{5.65}$$

and

$$w'(t) = \lambda\,\big(-h(v(t))\,v'(t) + g(v(t)) + e(t)\big) \quad \text{for a.e. } t \in [0,T]. \tag{5.66}$$

Multiplying (5.65) by $w'(t)$ and (5.66) by $v'(t)$ and subtracting we get

$$\tfrac{1}{q}\,(|v'(t)|^p)' = \lambda \Big(-h(v(t))\,(v'(t))^2 + g(v(t))\,v'(t) + e(t)\,v'(t) \Big) \quad \text{for a.e. } t \in [0,T]. \tag{5.67}$$

Now, suppose that $\min\{v(t)\colon t\in[0,T]\}<r$. Let us extend v to a T-periodic function on $\mathbb{R}$ and choose $t'\in[0,T]$ and $t^*\in(t',t'+T]$ so that $v(t')=r$, $v(t^*)=\max\{v(t)\colon t\in[0,T]\}$ and $v(t)\ge r$ on $[t',t^*]$. Integrating (5.67) from t' to t^*, we get

$$\lambda\int_r^{v(t^*)} g(x)\,\mathrm{d}x=-\frac{1}{q}\,|v'(t')|^p-\lambda\left(\int_{t'}^{t^*} h(v(t))\,(v'(t))^2\,\mathrm{d}t+\int_{t'}^{t^*} e(t)\,v'(t)\,\mathrm{d}t\right).$$

Consequently, by (5.59) and (5.60),

$$\begin{aligned}\int_r^{v(t^*)} g(x)\,\mathrm{d}x &\le \frac{\lambda^{\frac{p}{p-1}}}{q}\,R_2^p+h_R\,\lambda^{\frac{2}{p-1}}\,R_2^2\,T+\|e\|_1\,\lambda^{\frac{1}{p-1}}\,R_2\\ &\le \frac{R_2^p}{q}+R_2\left(h_R\,R_2\,T+\|e\|_1\right)=R^*,\end{aligned}$$

which, by virtue of (5.63), finally gives

$$\int_r^{R_0} g(x)\,\mathrm{d}x=\int_r^{v(t^*)} g(x)\,\mathrm{d}x-\int_{R_0}^{v(t^*)} g(x)\,\mathrm{d}x\le R^*+\int_{R_0}^{R}|g(x)|\,\mathrm{d}x=K^*.$$

This being contradictory to (5.64) implies that $v>r$ holds on $[0,T]$, i.e. (5.62) is true.

Step 5. To summarize, there are r, R and R' such that the assumption (i) from Lemma 5.19 is satisfied. Furthermore, since by Step 1 we have

$$g(x)+\overline{e}>0\quad\text{if}\ \ 0<x<R_0\quad\text{and}\quad g(x)+\overline{e}<0\quad\text{if}\ \ x>R_1$$

and $0<r<R_0<R_1<R$, it is easy to see that also the assumptions (ii) and (iii) of Lemma 5.19 are satisfied. □

Assume that the dissipativity condition

$$h(x)\ge h_*>0\quad\text{or}\quad h(x)\le -h_*<0\qquad\text{for all } x\in[0,\infty) \tag{5.68}$$

is fulfilled instead of (5.54) and $e\in L_2[0,T]$. Then the existence of a positive solution to problem (5.52) is ensured by Jebelean and Mawhin.

Theorem 5.21. ([90, Theorem 3]) *Let $p>1$, $h\in C[0,\infty)$, $e\in L_2[0,T]$ and let $g\in C(0,\infty)$ have a strong repulsive singularity* (5.29). *Furthermore, assume* (5.55) *and* (5.68).

Then problem (5.52), (5.2) *has a positive solution.*

Proof. The proof is analogous to that of Theorem 5.20, just the estimate (5.59) is, thanks to (5.68), obtained more easily. Indeed: let $\lambda\in(0,1]$ and let v be a positive solution of (5.53). Let R_0, R_1 and t_1 be found as in Step 1 of the proof of Theorem 5.20, i.e. R_0 is such that (5.58) is true, $R_1>R_0$, $g(x)+\overline{e}<0$ for $x\ge R_1$ and $v(t_1)<R_1$. Integrating equality (5.67) over $[0,T]$, we get $h_*\,\|v'\|_2\le\|e\|_2$ and, consequently,

$$v(t)=v(t_1)+\int_{t_1}^{t} v'(s)\,\mathrm{d}s<R_1+\sqrt{T}\,\frac{\|e\|_2}{h_*}+1\ \text{for all } t\in[0,T].$$

Thus, (5.59) is true with $R = R_1 + \sqrt{T}\,\frac{\|e\|_2}{h_*} + 1$. Now, we can repeat Steps 3–5 of the proof of Theorem 5.20. ☐

Lemma 5.19 enables us to prove also the following result concerning both the non-dissipative case and the case where g need not be bounded below on $(0,\infty)$. Recall that the symbol π_p is defined for $p > 1$ by

$$\pi_p = \frac{2\,\pi\,(p-1)^{\frac{1}{p}}}{p\,\sin\left(\frac{\pi}{p}\right)}$$

and $(\frac{\pi_p}{T})^p$ is the first eigenvalue of the Dirichlet problem

$$(\phi_p(u'))' + \lambda\,\phi_p(u) = 0, \quad u(0) = u(T) = 0$$

(see [61]).

Theorem 5.22. *Let $p > 1$, $h \in C[0,\infty)$, $e \in L_1[0,T]$ and let $g \in C(0,\infty)$ have a strong repulsive singularity* (5.29). *Furthermore, assume* (5.55) *and*

$$\left\{\begin{array}{l} \text{there exist } a,\ 0 \le a < \left(\frac{\pi_p}{T}\right)^p, \text{ and } \gamma \ge 0 \text{ such that} \\ g(x)\,x \ge -(a\,x^p + \gamma) \quad \text{for all } x > 0. \end{array}\right. \tag{5.69}$$

Then problem (5.52), (5.2) *has a positive solution.*

Proof. Similarly to the proof of Theorem 5.21, it suffices to verify (5.59). Assume that $\lambda \in (0,1]$, v is a positive solution to (5.53) and let R_1 and t_1 have the same meaning as in Step 1 of the proof of Theorem 5.20. Multiplying (5.53) by $v(t)$ and integrating over $[0,T]$, we get

$$\|v'\|_p^p \le a\,\|v\|_p^p + \|e\|_1\,\|v\|_\infty + \gamma\,T. \tag{5.70}$$

Since $v(t_1) \le R_1$, we have

$$0 < v(t) < R_1 + T^{\frac{1}{q}}\,\|v'\|_p \quad \text{for} \quad t \in [0,T], \tag{5.71}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Now put

$$y(t) = \left\{\begin{array}{ll} v(t+t_1) - v(t_1) & \text{if } 0 \le t \le T - t_1, \\ v(t+t_1-T)) - v(t_1) & \text{if } T - t_1 \le t \le T. \end{array}\right.$$

We have $y \in C^1[0,T]$, $y(0) = y(T) = 0$ and $\|y + v(t_1)\|_p^p = \|v\|_p^p$. Therefore, by the generalized Poincaré-Wirtinger inequality (see e.g. [162, Lemma 3]),

$$\|y\|_p \le \frac{T}{\pi_p}\,\|y'\|_p = \frac{T}{\pi_p}\,\|v'\|_p\,.$$

Hence, for an arbitrary $\varepsilon > 0$, there is a $C_1 > 0$ such that

$$\|v\|_p^p \le \left(\|y\|_p + v(t_1)\, T^{\frac{1}{p}}\right)^p \le (1+\varepsilon)\left(\frac{T}{\pi_p}\right)^p \|v'\|_p^p + C_1.$$

Inserting this into (5.70), choosing $\varepsilon \in \left(0, \frac{1}{a}\,(\frac{\pi_p}{T})^p - 1\right)$ and having in mind (5.71), we deduce that

$$\alpha\, \|v'\|_p^p \le T^{\frac{1}{q}}\, \|e\|_1\, \|v'\|_p + C_2$$

for some $C_2 > 0$, where $\alpha = \left(1 - a\,(1+\varepsilon)\left(\frac{T}{\pi_p}\right)^p\right) > 0$. However, this is possible only if there is $R_p \in (0,\infty)$, independent of λ and v, such that $\|v'\|_p < R_p$. Therefore $0 < v(t) < R_1 + T^{\frac{1}{q}}\, R_p + 1$ on $[0,T]$ for all $\lambda \in (0,1]$ and all positive T-periodic solutions v of (5.53), i.e. the assertion (5.59) is true with $R = R_1 + T^{\frac{1}{q}}\, R_p + 1$.

By virtue of (5.55), we can choose $b > 0$ so that $\inf\{g(x)\colon\ x \in (0,R]\} \ge -b$. Thus, we can continue by Steps 3–5 of the proof of Theorem 5.20 to verify that the assumptions of Lemma 5.19 are satisfied. □

Remark 5.23. Theorem 5.22 is a slightly modified scalar version of the result by Liu [104, Theorem 1].

In the undamped case of the Duffing type equation

$$(\phi_p(u'))' = g(u) + e(t), \tag{5.72}$$

condition (5.69) can be replaced by a related asymptotic condition. It is shown in the next theorem which has been proved for $p = 2$ by Rachůnková and Tvrdý in [131, Theorem 3.1].

Theorem 5.24. *Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $g \in C(0,\infty)$ and $e \in L_q[0,T]$. Furthermore, assume* (5.29),

$$\liminf_{x\to 0+}\ g(x) > -\infty, \tag{5.73}$$

$$\liminf_{x\to\infty}\ \frac{g(x)}{x^{p-1}} > -\big(\frac{\pi_p}{T}\big)^p. \tag{5.74}$$

Further, assume that there exist $r > 0$ and $A > r$ such that the conditions

$$g(r) + e(t) \ge 0 \quad \text{for a.e. } t \in [0,T] \tag{5.75}$$

and

$$g(x) + \overline{e} \le 0 \quad \text{for}\ \ x \in [A,B], \tag{5.76}$$

where

$$B - A \le 2\,T\, \|e - \overline{e}\|_1^{q-1}, \tag{5.77}$$

are satisfied.

Then problem (5.72), (5.2) *has a positive solution u such that $u(t_u) \in [r,B]$ for some $t_u \in [0,T]$.*

Proof. *Step* 1. *Lower and upper functions.*

By (5.75), $\sigma_2 \equiv r$ is an upper function of (5.72), (5.2). Let v be a solution of the quasilinear Dirichlet problem (5.9) with $b(t) = e(t) - \overline{e}$ a.e. on $[0,T]$ and $d = 0$ and let $\sigma_1 = A + T\,\phi_p^{-1}(\|e-\overline{e}\|_1) + v$ on $[0,T]$. Let us recall that $\phi_p^{-1} = \phi_q$. Hence $\phi_p^{-1}(\|e-\overline{e}\|_1) = \|e-\overline{e}\|_1^{q-1}$. Having in mind assumption (5.76), we can see, similarly to the proof of Theorem 5.17 (see also the proof of Theorem 5.14), that σ_1 is a lower function of (5.72), (5.2) and $\sigma_1(t) \in [A,B]$ for $t \in [0,T]$.

Step 2. *Construction of an auxiliary problem having a right hand side bounded below.*

By (5.29) we have (5.32) and hence there is a sequence $\{\varepsilon_n\} \subset (0,r)$ such that

$$g(\varepsilon_n) > 0 \quad \text{for } n \in \mathbb{N}, \quad \lim_{n\to\infty} \varepsilon_n = 0 \quad \text{and} \quad \lim_{n\to\infty} g(\varepsilon_n) = \infty. \tag{5.78}$$

For $n \in \mathbb{N}$ and $M \in \mathbb{R}$, $M > r$, define

$$g_{n,M}(x) = \begin{cases} 0 & \text{if } x < 0, \\ \dfrac{g(\varepsilon_n)}{\varepsilon_n^{p-1}}\, x^{p-1} & \text{if } x \in [0,\varepsilon_n], \\ g(x) & \text{if } x \in [\varepsilon_n, M], \\ g(M) & \text{if } x > M. \end{cases} \tag{5.79}$$

By (5.74), there are $\eta \in \bigl(0, \bigl(\frac{\pi_p}{T}\bigr)^p\bigr)$ and $x_0 > 1$ such that

$$\frac{g(x)}{x^{p-1}} \ge -\left(\left(\frac{\pi_p}{T}\right)^p - \eta\right) \quad \text{for all} \quad x \ge x_0.$$

Put

$$p(x) = \begin{cases} 0 & \text{if } x \le 0, \\ \dfrac{g(x_0)}{x_0^{p-1}}\, x^{p-1} & \text{if } x \in (0, x_0), \\ g(x) & \text{if } x \ge x_0 \end{cases}$$

and $q_{n,M}(x) = g_{n,M}(x) - p(x)$ for $x \in \mathbb{R}$. By virtue of (5.73), there is $\gamma \ge 0$ such that $q_{n,M}(x) \ge -\gamma$ for all $x \in \mathbb{R}$, $n \in \mathbb{N}$ and $M > r$. Consequently, each function $\widetilde{g}(x) = g_{n,M}(x)$, $n \in \mathbb{N}$, $M > r$, satisfies the estimate

$$\widetilde{g}(x)\, x \ge -\left(\left(\frac{\pi_p}{T}\right)^p - \eta\right) |x|^p - \gamma\, |x| \quad \text{for all } x \in \mathbb{R}. \tag{5.80}$$

Step 3. *A priori estimates.*

Now, we will give uniform a priori estimates for solutions of periodic problems associated to the equations

$$(\phi_p(u'))' = \widetilde{g}(u) + e(t), \tag{5.81}$$

where $\widetilde{g}$ may be an arbitrary function satisfying the estimate (5.80). To this aim, we will prove the following assertion.

Claim. *Let* $\gamma \geq 0$ *and* $\eta \in \left(0, (\frac{\pi_p}{T})^p\right)$. *Then for any* $\delta > 0$, *there are* $R \geq \delta$ *and* $R' > 0$ *such that the estimates*

$$u \leq R \quad \text{on} \quad [0,T] \quad \text{and} \quad \|u'\|_p \leq R' \tag{5.82}$$

hold whenever

$$\begin{cases} \widetilde{g} \in C(0,\infty) \quad \textit{fulfils} \ \ (5.80) \text{ and } \ u \text{ is a solution of } \ (5.81),(5.2) \\ \textit{such that} \quad \min\{u(t)\colon t \in [0,T]\} \leq \delta. \end{cases} \tag{5.83}$$

Proof of Claim. We will follow ideas from the proof of [138, Lemma 2.4]. Suppose that for each $k \in \mathbb{N}$ there are $g_k \in C(0,\infty)$ and a solution u_k of

$$(\phi_p(u'))' = g_k(u) + e(t), \quad u(0) = u(T), \ u'(0) = u'(T) \tag{5.84}$$

such that

$$g_k(x)\, x \geq -\left(\left(\frac{\pi_p}{T}\right)^p - \eta\right) |x|^p - \gamma\, |x| \tag{5.85}$$

and

$$u_k(t_k) = \delta \quad \text{for some} \ \ t_k \in [0,T] \quad \text{and} \quad \max\{u_k(t)\colon t \in [0,T]\}) > k. \tag{5.86}$$

In particular, we have

$$\lim_{k\to\infty} \max\{u_k(t)\colon t \in [0,T]\} = \infty. \tag{5.87}$$

Let us extend u_k and e to functions T-periodic on $\mathbb{R}$. We have

$$(\phi_p(u_k'(t)))' = g_k(u_k(t)) + e(t) \quad \text{for a.e. } t \in \mathbb{R}.$$

Multiplying this equality by u_k, integrating from t_k to $t_k + T$ and making use of (5.85), we obtain

$$\begin{aligned} \|u_k'\|_p^p &= -\int_{t_k}^{t_k+T} g_k(u_k(s))\, u_k(s)\, \mathrm{d}s - \int_{t_k}^{t_k+T} e(s)\, u_k(s)\, \mathrm{d}s \\ &\leq \left(\left(\frac{\pi_p}{T}\right)^p - \eta\right) \|u_k\|_p^p + \gamma\, T^{\frac{1}{q}}\, \|u_k\|_p + \|e\|_q\, \|u_k\|_p. \end{aligned}$$

Let us set $v_k = u_k - \delta$. By (5.87) we have $\lim_{k\to\infty} \|v_k\|_\infty = \infty$. Therefore, applying the Hölder inequality we can conclude that

$$\lim_{k\to\infty} \|v_k'\|_p = \infty. \tag{5.88}$$

Furthermore, it is easy to verify that

$$\|v_k'\|_p^p \leq \left(\left(\frac{\pi_p}{T}\right)^p - \epsilon\right) \|v_k\|_p^p + a\, \|v_k\|_p + b \tag{5.89}$$

holds with some $\epsilon \in \big(0, (\frac{\pi_p}{T})^p\big)$ and $a,\ b \geq 0$ not depending on v_k. This, together with (5.88), gives

$$\lim_{k\to\infty} \|v_k\|_p = \infty. \tag{5.90}$$

Moreover, as $v_k(t_k) = v_k(t_k + T) = 0$, we can apply the generalized Poincaré-Wirtinger inequality (see e.g. [162, Lemma 3]) to get

$$\|v_k\|_p^p \leq \left(\frac{T}{\pi_p}\right)^p \|v_k'\|_p^p \quad \text{for each} \quad k \in \mathbb{N}.$$

Hence the inequality (5.89) can be rewritten as

$$\left(\frac{\pi_p}{T}\right)^p \leq \frac{\|v_k'\|_p^p}{\|v_k\|_p^p} \leq \left(\frac{\pi_p}{T}\right)^p - \epsilon + \frac{a}{\|v_k\|_p^{p-1}} + \frac{b}{\|v_k\|_p^p}, \tag{5.91}$$

which, in view of (5.90), leads to a contradiction

$$\left(\frac{\pi_p}{T}\right)^p \leq \left(\frac{\pi_p}{T}\right)^p - \epsilon.$$

As a consequence, we can conclude that the sequences $\{\|v_k\|_\infty\}$ and $\{\|v_k\|_p\}$ are bounded. By (5.89), this implies that also the sequence $\{\|v_k'\|_p\}$ is bounded. In particular, there are $R \in [\delta, \infty)$ and $R' \in (0, \infty)$ such that $u \leq R$ and $\|u'\|_p \leq R'$ hold whenever (5.83) is true. This completes the proof of Claim.

Now, let $R > B$ and $R' > 0$ be constants given by Claim for $\delta = B$. Put

$$K = \int_A^R |g(x)| \,\mathrm{d}x + \|e\|_q R'.$$

It follows from (5.29) and (5.78) that we can choose $\varepsilon = \varepsilon_{n^*} \in \{\varepsilon_n\}$ such that

$$\int_\varepsilon^A g(x)\,\mathrm{d}x > K \quad \text{and} \quad g(\varepsilon) > 0. \tag{5.92}$$

By (5.73), there is $g_R \in \mathbb{R}$ such that

$$g(x) \geq g_R \quad \text{for} \quad x \in (0, R]. \tag{5.93}$$

Define

$$\widetilde{g}(x) = g_{n^*,R}(x) \quad \text{for} \quad x \in \mathbb{R} \tag{5.94}$$

and consider the regular periodic problem for the auxiliary equation

$$(\phi_p(u'))' = \widetilde{g}(u) + e(t). \tag{5.95}$$

Clearly, σ_1 and σ_2 are lower and upper functions of (5.95), (5.2) and $\widetilde{g}(x) + e(t) \geq g_R + e(t)$ for a.e. $t \in [0, T]$ and all $x \in \mathbb{R}$. Thus, by Lemma 5.9, problem (5.95), (5.2) possesses a solution u such that $u(t_u) \in [r, B]$ for some $t_u \in [0, T]$. In particular, $\min\{u(t) : t \in [0, T]\} \leq B$. Furthermore, by Step 2 it is easy to see that $\widetilde{g}$ satisfies the estimate (5.80). Thus, by Claim and by the definitions of R and R', the estimates (5.82) are true.

Step 4. *Existence of a solution to* (5.72), (5.2).

It remains to show that $u \ge \varepsilon$ on $[0,T]$. Let $t_0,\ t_1 \in [0,T]$ be such that

$$u(t_0) = \min\{u(t) \colon t \in [0,T]\} \quad \text{and} \quad u(t_1) = \max\{u(t) \colon t \in [0,T]\}.$$

Due to the periodicity of u, we have $u'(t_0) = u'(t_1) = 0$. Multiplying the equality $(\phi_p(u'(t)))' = \widetilde{g}(u(t)) + e(t)$ by $u'(t)$ and integrating, we get

$$\int_{u(t_0)}^{u(t_1)} \widetilde{g}(x)\,\mathrm{d}x \le \|e\|_q\, R'.$$

Therefore,

$$\int_{u(t_0)}^{A} \widetilde{g}(x)\,\mathrm{d}x \le \int_{A}^{R} |g(x)|\,\mathrm{d}x + \|e\|_q\, R' = K.$$

Let $u(t_0) < \varepsilon$. Then, by (5.92),

$$\begin{aligned}\int_{u(t_0)}^{A} \widetilde{g}(x)\,\mathrm{d}x &= \int_{u(t_0)}^{\varepsilon} \widetilde{g}(x)\,\mathrm{d}x + \int_{\varepsilon}^{A} g(x)\,\mathrm{d}x \\ &= g(\varepsilon)\,(\varepsilon - u(t_0)) + \int_{\varepsilon}^{A} g(x)\,\mathrm{d}x > \int_{\varepsilon}^{A} g(x)\,\mathrm{d}x > K,\end{aligned}$$

a contradiction. So $u(t) \ge \varepsilon$ on $[0,T]$, which together with (5.79), (5.82) and (5.94) yields that u is a solution of (5.72), (5.2). $\square$

Examples. (i) Let $p > 1$, $h \in C[0,\infty)$, $\beta > 0$, $\alpha \ge 1$, $e \in L_1[0,T]$. Then, by Theorem 5.20, the problem

$$(|u'|^{p-2}u')' + h(u)\,u' = \frac{\beta}{u^\alpha} + e(t), \quad u(0) = u(T), \quad u'(0) = u'(T) \tag{5.96}$$

has a positive solution if $\overline{e} < 0$. Integrating both sides of the differential equation in (5.96) over $[0,T]$ and taking into account the positivity of $g(x) = \beta\, x^{-\alpha}$ on $(0,\infty)$, we can see that the condition $\overline{e} < 0$ is also necessary for the existence of a positive solution to (5.96).

(ii) Let $p > 1$, $c \ne 0$, $a > 1$, $\beta > 0$, $\alpha \ge 1$. Then, by Theorem 5.21, the problem

$$(|u'|^{p-2}u')' + c\,u' = \frac{\beta}{u^\alpha} - a\,\exp(u) + e(t), \quad u(0) = u(T), \quad u'(0) = u'(T)$$

has a solution for each $e \in L_2[0,T]$.

(iii) Let $p > 1$, $h \in C[0,\infty)$, $0 < a < (\frac{\pi_p}{T})^p$, $\beta > 0$ and $\alpha \ge 1$. Then, by Theorem 5.22, the problem

$$(|u'|^{p-2}u')' + h(u)\,u' = -a\,u^{p-1} + \frac{\beta}{u^\alpha} + e(t), \quad u(0) = u(T), \quad u'(0) = u'(T)$$

has a positive solution for each $e \in L_1[0,T]$.

For the classical case $p = 2$, the following result due to Omari and Ye is known. Its proof combines the lower and upper functions method, the degree theory and connectedness arguments for some properly chosen truncated equations and a posteriori estimates.

THEOREM 5.25. ([114, Theorem 1.2]) *Assume* $e \in L_\infty[0,T]$, (5.29), $\lim\limits_{x\to 0+} g(x) = \infty$ *and*

$$\liminf_{x\to\infty} \frac{g(x)}{x} \geq -\left(\frac{\pi}{T}\right)^2 \quad \textit{and} \quad \liminf_{x\to\infty} \frac{2\,G(x)}{x^2} > -\left(\frac{\pi}{T}\right)^2,$$

where

$$G(x) = \int_x^1 g(\xi)\,\mathrm{d}\xi \quad \textit{for}\ \ x \in (0,\infty).$$

Then the problem $u'' + h(u)\,u' = g(u) + e(t)$, (5.2) *has a solution if and only if it possesses a lower function* $\sigma_1 \in AC^1[0,T]$.

Hitherto we have assumed the strong singularity condition (5.29). The next existence principle enables us to treat also problems with weak repulsive singularities. We shall restrict ourselves to the case that $\phi(y) \equiv y$ and f does not depend on u', i.e. we consider the equation

$$u'' = f(t,u), \tag{5.97}$$

where $f\colon\ [0,T] \times (0,\infty) \to \mathbb{R}$.

THEOREM 5.26. *Let* $f \in Car([0,T] \times (0,\infty))$, $r > 0$, $A \geq r$ *and let* $\mu \in L_1[0,T]$ *and* $\beta \in L_1[0,T]$ *be such that* $\mu(t) \geq 0$ *on* $[0,T]$,

$$\overline{\beta} \leq 0 \quad \textit{and} \quad f(t,x) \leq \beta(t) \quad \textit{for a.e. } t \in [0,T] \textit{ and all } x \in [A,B] \tag{5.98}$$

and

$$f(t,x) \geq -\mu(t)\,(x-r) \quad \textit{for a.e. } t \in [0,T] \textit{ and all } x \in [r,B], \tag{5.99}$$

where

$$B - A \geq \frac{T}{2}\,\overline{m},\ \ m(t) = \max\big\{\sup\{f(t,x)\colon\ x \in [r,A]\}, \beta(t), 0\big\} \ \ \textit{for a.e. } t \in [0,T]$$

and

$$\left\{\begin{array}{l} v \geq 0 \textit{ on } [0,T] \textit{ holds for each } \ v \in AC^1[0,T] \textit{ such that} \\ v''(t) + \mu(t)\,v(t) \geq 0 \textit{ for a.e. } t \in [0,T], \quad v(0) = v(T),\ v'(0) = v'(T). \end{array}\right. \tag{5.100}$$

Then problem (5.97), (5.2) *has a positive solution* u *such that* $r \leq u \leq B$ *on* $[0,T]$.

PROOF. The proof follows the ideas of the proof of [137, Theorem 2.5]. First, assume that $\overline{\beta} < 0$.

Step 1. Existence of a solution u *to a certain auxiliary problem.*

Put

$$\widetilde{f}(t,x) = \begin{cases} f(t,r) - \mu(t)\,(x-r) & \text{if } x \le r, \\ f(t,x) & \text{if } x \in [r,B], \\ f(t,B) & \text{if } x \ge B \end{cases} \tag{5.101}$$

and consider the problem

$$u'' = \widetilde{f}(t,u), \quad u(0) = u(T), \quad u'(0) = u'(T). \tag{5.102}$$

We have $\widetilde{f} \in Car([0,T] \times \mathbb{R})$. By (5.98), (5.99) and (5.101), the inequalities

$$\widetilde{f}(t,x) \le \beta(t) \quad \text{if } x \ge A \tag{5.103}$$

and

$$\widetilde{f}(t,x) \ge -\mu(t)\, \min\{x-r, B-r\} \tag{5.104}$$

are valid for a.e. $t \in [0,T]$ and all $x \in \mathbb{R}$. In particular, $\widetilde{f}(t,x) \ge -\mu(t)\,(B-r)$. By (5.104), $\sigma_2 \equiv r$ is an upper function of (5.102). Further, let σ_0 be the solution of the Dirichlet problem $v'' = b$, $v(0) = v(T) = 0$, where $b(t) = \beta(t) - \overline{\beta}$ for a.e. $t \in [0,T]$, and let $\sigma_c(t) = c + \sigma_0(t)$ for $t \in [0,T]$ and $c \in \mathbb{R}$. Then $\sigma_c'' = b$ a.e. on $t \in [0,T]$ and $\sigma_c(0) = \sigma_c(T) = c$. Moreover, $\sigma_c'(T)) - \sigma_c'(0) = T\,\overline{b} = 0$. Let us choose $c^* > 0$ so that $\sigma_1 = \sigma_{c^*} \ge A$ on $[0,T]$. Due to (5.103), where $\beta < b$ a.e. on $[0,T]$, we can see that σ_1 is a lower function of (5.102). Therefore, by Lemma 5.9, the regular problem (5.102) has a solution u such that $u(t_u) \ge r$ for some $t_u \in [0,T]$.

Step 2. *Lower estimate for u.*

We shall show that

$$u \ge r \text{ on } [0,T]. \tag{5.105}$$

Set $z = u - r$. By virtue of (5.99) and (5.101), we have

$$z''(t) + \mu(t)\,z(t) = u''(t) + \mu(t)\,z(t) = \widetilde{f}(t,u(t)) + \mu(t)\,(u(t)-r) \ge 0$$

for a.e. $t \in [0,T]$. By (5.100), it follows that $z(t) \ge 0$ on $[0,T]$, i.e. (5.105) is true.

Step 3. *Upper estimate for u.*

We shall show that

$$u \le B \text{ on } [0,T]. \tag{5.106}$$

By the definition of m and by (5.101) and (5.103) we have

$$\widetilde{f}(t,x) \le m(t) \quad \text{for a.e. } t \in [0,T] \text{ and all } x \ge r.$$

Hence, we can use Lemma 5.8 (see (5.20)) to get

$$\|u'\|_\infty \le \|m\|_1 = \overline{m}. \tag{5.107}$$

If $u \ge A$ were valid on $[0,T]$, then taking into account the periodicity of u' and (5.103), we would get

$$0 = \int_0^T \widetilde{f}(t,u(t))\,\mathrm{d}t \le \int_0^T \beta(t)\,\mathrm{d}t = T\,\overline{\beta} < 0,$$

a contradiction. Thus, there is $\tau \in [0,T]$ such that $u(\tau) < A$. Now, assume that $u(s) > A$ for some $s \in [0,T]$ and extend u to the function T-periodic on $\mathbb{R}$. There are s_1, s_2 and $s^* \in \mathbb{R}$ such that $s_1 < s^* < s_2$, $s_2 - s_1 < T$, $u(s_1) = u(s_2) = A$ and $u(s^*) = \max\{u(s)\colon s \in [0,T]\} > A$. In particular, due to (5.107),

$$2\,(u(s^*) - A) = \int_{s_1}^{s^*} u'(s)\,\mathrm{d}s + \int_{s_2}^{s^*} u'(s)\,\mathrm{d}s \le T\,\overline{m},$$

wherefrom the estimate

$$u(t) - A \le \frac{T}{2}\,\overline{m} \le B - A \text{ on } [0,T]$$

follows. Consequently, (5.106) is true.
Step 4. *Conclusion: u is a solution to* (5.97), (5.2).

The estimates (5.105) and (5.106) mean that $r \le u \le B$ holds on $[0,T]$. By (5.101), we conclude that u is a solution to (5.97), (5.2).

If $\overline{\beta} = 0$, we can approximate the solution to (5.1), (5.2) by solutions of the problems

$$u'' = \widetilde{f}_n(t,u), \quad u(0) = u(T), \quad u'(0) = u'(T),$$

where

$$\widetilde{f}_n(t,x) = \begin{cases} f(t,r) & \text{if } x \le r, \\ f(t,x) & \text{if } x \in [r,A], \\ f(t,x) - \frac{1}{n}\,\frac{x-A}{x-A+1} & \text{if } x \in [A,B], \\ f(t,B) - \frac{1}{n}\,\frac{B-A}{B-A+1} & \text{if } x \ge B. \end{cases}$$

□

Recently, using the Krasnoselskii fixed point theorem, Torres proved for the Hill equation

$$u'' + \mu(t)\,u = g(t,u) \tag{5.108}$$

the existence result which is related to Theorem 5.26.

Theorem 5.27. ([152, Theorem 4.5]) *Let $\mu \in L_1[0,T]$ be such that the problem*

$$v'' + \mu(t)\,v = 0, \quad v(0) = v(T),\ v'(0) = v'(T) \tag{5.109}$$

possesses the Green function $G(t,s)$ which is positive on $[0,T] \times [0,T]$. Moreover, assume that there is an $R > 0$ such that

$$g(t,x) \ge 0 \textit{ for all } x \in (0, \tfrac{M}{m}\,R] \quad \textit{and} \quad g(t,x) \le \tfrac{1}{T\,M}\,x \textit{ for all } x \in [R, \tfrac{M}{m}\,R]$$

for a.e. $t \in [0,T]$, *where*

$$m = \min\{G(t,s)\colon\ t,s \in [0,T]\} \quad \textit{and} \quad M = \max\{G(t,s)\colon\ t,s \in [0,T]\}.$$

Then problem (5.108), (5.2) *has a positive solution.*

It is easy to check that the function $G(t,s) = \sin\left(\frac{\pi}{T}|t-s|\right)$, $t,s \in [0,T]$, is the Green function for $v'' + (\frac{\pi}{T})^2 v = 0$, $v(0) = v(T)$, $v'(0) = v'(T)$ and $G(t,s) \ge 0$ on $[0,T] \times [0,T]$. Hence, the statement (5.100) holds if $\mu(t) \equiv \mu_1 = \left(\frac{\pi}{T}\right)^2$. Notice that $\mu_1 = \left(\frac{\pi}{T}\right)^2$ is the first eigenvalue of the related Dirichlet problem and it is optimal in the sense that for $\mu(t) = \mu$ a.e. on $[0,T]$ and $\mu \in (\mu_1, 4\,\mu_1)$ the corresponding Green function of $v'' + (\frac{\pi}{T})^2 v = 0$, $v(0) = v(T)$, $v'(0) = v'(T)$ is not nonnegative on $[0,T] \times [0,T]$.

In particular, when restricted to the Duffing equation

$$u'' = g(u) + e(t), \tag{5.110}$$

Theorem 5.26 has the following corollary.

Corollary 5.28. ([137, Corollary 3.7]) *Suppose that* $g \in C(0,\infty)$, $e \in L_1[0,T]$,

$$\overline{e} + \limsup_{x\to\infty}\, g(x) < 0$$

and there is $r > 0$ *such that*

$$e(t) + g(x) + \left(\frac{\pi}{T}\right)^2 x \ge \left(\frac{\pi}{T}\right)^2 r \quad \textit{for a.e. } t \in [0,T] \textit{ and all } x > r.$$

Then problem (5.110), (5.2) *has a positive solution* u *such that* $u \ge r$ *on* $[0,T]$.

More detailed information on the sign properties of the associated Green functions is provided by the next proposition which is due to Torres [152] (see also [163, Lemma 2.5]). Before formulating it, let us define the function $K\colon\ [0,\infty] \to (0,\infty)$ by

$$K(z) = \begin{cases} \dfrac{2\,\pi}{z\,T^{1+\frac{2}{z}}} \left(\dfrac{2}{2+z}\right)^{1-\frac{2}{z}} \left(\dfrac{\Gamma(\frac{1}{z})}{\Gamma(\frac{1}{2}+\frac{1}{z})}\right)^2 & \text{if } 1 \le z < \infty, \\[2ex] \dfrac{4}{T} & \text{if } z = \infty. \end{cases} \tag{5.111}$$

Let us recall that for a given z, $1 \le z \le \infty$, $K(z)$ is the best Sobolev constant for the inequality $C\,\|u\|_z^2 \le \|u'\|_2^2$, i.e.

$$K(z) = \inf\Big\{\frac{\|u'\|_2^2}{\|u\|_z^2}\colon\ u \in H_0^1[0,T] \setminus \{0\}\Big\},$$

where $H_0^1[0,T] = \{u \in AC[0,T]\colon\ u' \in L_2[0,T],\ u(0) = u(T) = 0\}$.

Proposition 5.29. ([152, Corollary 2.3]) *Let* $1 \le q \le \infty$ *and let* $\mu \in L_q[0,T]$. *Then* (5.100) *is true provided*

$$\mu(t) \ge 0 \quad a.e.\ on \quad [0,T], \quad \overline{\mu} > 0 \quad and \quad \|\mu\|_q \le K(2\,q^*), \tag{5.112}$$

where

$$\frac{1}{q} + \frac{1}{q^*} = 1 \quad if \quad 1 < q < \infty, \quad q^* = \infty \quad if \quad q = 1, \quad q^* = 1 \quad if \quad q = \infty \tag{5.113}$$

and the function K *is defined by* (5.111).

Moreover, if $\|\mu\|_q < K(2\,q^*)$, *then the corresponding Green function is positive on* $[0,T] \times [0,T]$.

Notice that if $\mu(t) \equiv \mu \in (0,\infty)$ on $[0,T]$, then we can take $q = \infty$, $q^* = 1$ and so we get $K(2\,q^*) = K(2) = \left(\frac{\pi}{T}\right)^2$, which confirms the above mentioned fact that in such a case (5.100) is satisfied if $\mu \in (0, \left(\frac{\pi}{T}\right)^2]$.

Example. Consider the Brillouin beam focusing equation

$$u'' + a\,(1 + \cos\,t)\,u = \frac{1}{u} \tag{5.114}$$

on the interval $[0, 2\,\pi]$, where $a > 0$ is a parameter. (See [35] for a description of the model.) The problem of existence of a positive $2\,\pi$-periodic solution to (5.114) has been considered by several authors (see e.g. [59], [152], [153], [161] and [163]). Put $A = \frac{1}{\sqrt{a}}$. Then for all $x \ge A$ and $t \in [0, 2\,\pi]$ we have

$$f(t,x) := \frac{1}{x} - a\,(1 + \cos\,t)\,x \le \beta(t) := \frac{1}{A} - a\,(1 + \cos\,t)\,A$$

and $\overline{\beta} = \frac{1}{A} - a\,A = 0$. So, the assumption (5.98) is satisfied with $B = \infty$ and $T = 2\,\pi$. Furthermore, for $r \in (0, r_0]$ and a.e. $t \in [0,T]$ define

$$m_r(t) := \max\big\{\sup\{f(t,x)\colon\ x \in [r, A]\}, \beta(t), 0\big\}.$$

Let $r_0 = \frac{1}{\sqrt{2\,a}}$. Then $\frac{1}{r} - a\,(1 + \cos\,t)\,r \ge \frac{1}{r_0} - 2\,a\,r_0 = 0$ holds for $r \in (0, r_0]$ and $t \in [0,T]$. Consequently,

$$m_r(t) = \frac{1}{r} - a\,(1 + \cos\,t)\,r \quad \text{for a.e. } t \in [0,T] \text{ and all } \ r \in (0, r_0].$$

For a given $r \in (0, r_0]$, put $B_r = A + \pi\,\overline{m_r}$. Now, it is easy to check that it is possible to find $r \in (0, r_0]$ such that the assumption (5.99) is satisfied with $B = B_r$ and $\mu(t) = a\,(1 + \cos\,t)$ whenever $a < \frac{1}{2\,\pi} \approx 0.15915$. Finally, notice that by virtue of Proposition 5.29, the assumption (5.100) is satisfied if

$$a \le K_{\max} := \max\left\{\frac{K(2\,q^*)}{\|1 + \cos\,t\|_q}\colon\ 1 \le q \le \infty\right\} \approx 0.16488.$$

(The maximum is attained at $q \approx 2.1941$, see [153, Corollary 4.8].) By Theorem 5.26, we can conclude that the equation (5.114) has a positive 2π-periodic solution for $a < \frac{1}{2\pi}$. To compare, notice that for $q = \infty$ and $T = 2\pi$, we get $q^* = 1$ and

$$\frac{K(2\,q^*)}{\|1 + \cos t\|_\infty} = \frac{1}{8} = 0.125.$$

Finally, let us note that using more sophisticated and involved techniques, Zhang proved (see [163, Theorem 4.5] that for $a < K_{\max}$, $b > 0$, $\lambda \geq 1$, $e \in C[0,T]$ and $h \in C[0,\infty)$ the problem

$$u'' + h(u)u' + a\,(1 + \cos t)\,u = \frac{b}{u^\lambda} + e(t), \quad u(0) = u(2\,\pi),\ u'(0) = u'(2\,\pi)$$

has a positive solution.

The hitherto mentioned conditions for the existence of a positive solution of problem (5.97), (5.2) concern the case when $f(t,x)$ asymptotically behaves like $-k\,x$ with $k \leq \mu_1$, $\mu_1 = (\frac{\pi}{T})^2$ being the first eigenvalue of the related Dirichlet problem. The next theorem deals with the case corresponding to k lying between two adjacent higher eigenvalues.

Let us denote by $\{\mu_k\}_{k=1}^\infty$ the sequence of eigenvalues of the related linear Dirichlet problem $u'' + \mu\,x = 0$, $u(0) = u(T) = 0$, that is

$$\mu_k = \left(\frac{\pi\,k}{T}\right)^2, \quad k \in \mathbb{N}. \tag{5.115}$$

Furthermore, we set $\mu_0 = 0$.

Theorem 5.30. ([56, Theorem 1.1]) *Assume that $f\colon [0,T] \times (0,\infty) \to \mathbb{R}$ is continuous and there are positive constants c, c', δ and $\nu \geq 1$ such that*

$$\frac{c'}{x^\nu} \leq f(t,x) \leq \frac{c}{x^\nu} \quad \textit{for all} \;\; x \in (0,\delta). \tag{5.116}$$

Moreover, let there exist a nonnegative integer k such that

$$-\mu_{k+1} < \liminf_{x\to\infty} \frac{f(t,x)}{x} \leq \limsup_{x\to\infty} \frac{f(t,x)}{x} < -\mu_k \quad \textit{uniformly in} \;\; t \in [0,T]. \tag{5.117}$$

Then problem (5.97), (5.2) *has a positive solution.*

Sketch of the proof For a given $e \in C[0,T]$ denote by $\mathcal{R}(e)$ the unique solution of the problem

$$u'' + u = e(t), \quad u(0) = u(T), \quad u'(0) = u'(T).$$

It is known that $\mathcal{R}$ defines a compact linear operator on the space $C_T[0,T]$ of continuous T-periodic functions endowed with the sup norm $\|\,.\,\|_\infty$. Problem (5.97), (5.2) is thus

equivalent to finding a positive solution $u \in C_T[0,T]$ of the fixed point problem $u = \mathcal{T}(u)$, where
$$\mathcal{T}(u) = \mathcal{R}(u + f(\cdot, u)) \quad \text{for} \quad u \in C_T[0,T].$$
For $0 < \varepsilon < M$ define $\Omega_{\varepsilon,M} = \{u \in C_T[0,T]\colon\ \varepsilon < u < M \text{ on } [0,T]\}$. Then $\mathcal{T}\colon\ \overline{\Omega}_{\varepsilon,M} \to C_T[0,T]$ is a completely continuous operator.

The proof of the theorem consists in showing that there are ε, M such that $\deg(\mathcal{I} - \mathcal{T}, \Omega_{\varepsilon,M}) \neq 0$:

Let k be from (5.117) and choose an arbitrary $\gamma \in (\mu_k, \mu_{k+1})$. Further, for $\lambda \in [0,1]$ and $u \in C_T[0,T]$, define

$$\widetilde{\mathcal{T}}(u) = \mathcal{R}\big(\gamma\, u - \frac{1}{u^\nu} + u\big) \quad \text{and} \quad \mathcal{T}_\lambda(u) = \lambda\, \widetilde{\mathcal{T}}(u) + (1-\lambda)\, \mathcal{T}(u)$$

Notice that
$$\mathcal{T}_\lambda(u) = \mathcal{R}\left(\lambda\left(\gamma\, u - \frac{1}{u^\nu}\right) + (1-\lambda)\, f(.,u) + u\right).$$
Furthermore, $\mathcal{T}_0 = \mathcal{T}$ and $\mathcal{T}_1 = \widetilde{\mathcal{T}}$. It can be shown that there are ε, M such that $0 < \varepsilon < M$ and $u \neq \mathcal{T}_\lambda(u)$ for all $u \in \partial\,\Omega_{\varepsilon,M}$. By the homotopy property of the degree, it follows that $\deg(\mathcal{I} - \mathcal{T}, \Omega_{\varepsilon,M}) = \deg(\mathcal{I} - \widetilde{\mathcal{T}}, \Omega_{\varepsilon,M})$.

Define

$$\widetilde{\mathcal{S}}(u) = \mathcal{R}\left(\gamma\, u - \frac{\lambda}{u^3} + u\right) \quad \text{and} \quad \mathcal{S}_\lambda(u) = \lambda\, \widetilde{\mathcal{T}}(u) + (1-\lambda)\, \widetilde{\mathcal{S}}(u)$$

for $u \in C_T[0,T]$ and $\lambda \in [0,1]$. We have

$$\mathcal{S}_\lambda(u) = \mathcal{R}\left(\gamma\, u + u - (1-\lambda)\,\frac{1}{u^\nu} - \lambda\,\frac{1}{u^3}\right),$$

$\mathcal{S}_0 = \widetilde{\mathcal{S}}$ and $\mathcal{S}_1 = \widetilde{\mathcal{T}}$. Now, we prove that $u \neq \mathcal{S}_\lambda(u)$ on $\partial\,\Omega_{\varepsilon,M}$ for each $\lambda \in [0,1]$ and for some suitable ε and M. Similarly to Step 1, this yields that $\deg(\mathcal{I} - \widetilde{\mathcal{T}}, \Omega_{\varepsilon,M}) = \deg(\mathcal{I} - \widetilde{\mathcal{S}}, \Omega_{\varepsilon,M})$.

The proof is completed by proving that $\deg(\mathcal{I} - \widetilde{\mathcal{S}}, \Omega_{\varepsilon,M}) \neq 0$. □

Remark 5.31. Consider the problem

$$u'' + k\, u = \frac{\beta}{u^\lambda} + e(t), \quad u(0) = u(T),\ u'(0) = u'(T) \tag{5.118}$$

with $\lambda > 0$, $\beta > 0$ and $k \geq 0$. Denote

$$g(x) = \frac{\beta}{x^\lambda} - k\, x \quad \text{for} \quad x > 0.$$

If $e \in L_1[0,T]$, $k = 0$ and $\lambda \geq 1$, i.e. the function g has a strong singularity at $x = 0$, then by [102, Theorem 3.12] problem (5.118) has a positive solution if and only if $\overline{e} < 0$ and, in the case $\lambda \in (0,1)$, this condition need not ensure the existence of a positive solution to (5.118) (cf. [102, Theorem 4.1]). Further, if $e \in C[0,T]$ and $\lambda \geq 1$, then by Theorem 5.30, problem (5.118) has a positive solution whenever the condition

$$k \neq (n\,\frac{\pi}{T})^2 \quad \text{for all} \quad n \in \mathbb{N}$$

is satisfied. It is worth mentioning that the resonance case of $k = (\frac{\pi}{T})^2$ is covered neither by Theorem 5.30 nor by Theorem 5.25 even for the strong singularity $\lambda \geq 1$.

In comparison to these results, it should be pointed out that using Corollary 5.28, we can obtain existence results also for the cases $\lambda \in (0,1)$ and $k = (\frac{\pi}{T})^2$. In particular, for problem (5.118), with $e \in L_1[0,T]$ we get the existence of a positive solution in the following cases:

$$k = 0, \quad \overline{e} < 0 \quad \text{and} \quad \inf \operatorname{ess}\{e(t)\colon\, t \in [0,T]\} > -\Big(\frac{\pi^2}{T^2\,\lambda\,\beta}\Big)^{\frac{\lambda}{\lambda+1}}\,(\lambda+1)\,\beta$$

or

$$0 < k < \Big(\frac{\pi}{T}\Big)^2 \quad \text{and} \quad \inf \operatorname{ess}\{e(t)\colon\, t \in [0,T]\} > -\Big(\frac{\pi^2 - T^2\,k}{T^2\,\lambda\,\beta}\Big)^{\frac{\lambda}{\lambda+1}}\,(\lambda+1)\,\beta$$

or

$$k = \Big(\frac{\pi}{T}\Big)^2 \qquad \text{and} \quad \inf \operatorname{ess}\{e(t)\colon\, t \in [0,T]\} > 0.$$

Notice that for the case $0 < k < (\frac{\pi}{T})^2$, Theorem 5.27 provides a complementary existence condition. For details, see [152, Corollary 4.6].

We close this section by mentioning some results concerning the case when the nonlinearity can have both a space singularity at $x = 0$ and superlinear descent for large x. The first is due to del Pino and Manásevich. It was motivated by [165], where an equation governing the nonlinear vibrations of a radially forced thickwalled and incompressible material was derived. The proof makes use of a version of the Poincaré-Birkhoff theorem due to Ding [60] together with an analysis of some oscillatory properties of solutions to the related initial value problems.

Theorem 5.32. ([55, Theorem 2.1]) *Let $f : \mathbb{R} \times (0,\infty) \to \mathbb{R}$ be continuous, locally Lipschitz in x, T-periodic in t and such that for s, $\beta \in \mathbb{R}$ and $\alpha > 0$, the solution $u(t)$ of the local initial value problem*

$$u'' = f(t,u), \quad u(s) = \alpha,\; u'(s) = \beta$$

is continuable to the whole real line $\mathbb{R}$ and $u > 0$ on $\mathbb{R}$. Furthermore, assume that

$$0 < \liminf_{x \to 0+}\, x\,f(t,x) \leq \infty \quad \text{and} \quad \lim_{x \to \infty} \frac{f(t,x)}{x} = -\infty$$

uniformly in t.

Then there is $n_0 \in \mathbb{N}$ *such that for every* $n \geq n_0$ *there exist two distinct* T*-periodic positive solutions* u_{n1}, u_{n2} *of* (5.97), (5.2) *such that both* $u_{n1} - 1$ *and* $u_{n2} - 1$ *have exactly* $2\,n$ *zeros in* $[0, T)$. *In particular, problem* (5.97), (5.2) *possesses infinitely many* T*-periodic positive solutions.*

The other result is due to Ge and Mawhin. It deals with the equation

$$u'' = g(u) + p(t, u, u'). \tag{5.119}$$

Its proof was obtained by use of some continuation theorems valid in absence of a priori bounds and given by Capietto, Mawhin and Zanolin e.g. in [48] and [109].

Theorem 5.33. ([77, Theorem 1]) *Let* $g \in C(0, \infty)$ *and* $p \in Car([0, T] \times \mathbb{R}^2)$. *Furthermore, assume that there are constants* α, $\beta \geq 1$, $M \geq 0$, $L \geq 0$ *such that*

$$\lim_{x \to 0+} g(x)\, x^{\alpha} = \infty, \quad \lim_{x \to \infty} \frac{g(x)}{x^{\beta}} = -\infty, \tag{5.120}$$

$$|p(t, x, y)| \leq \begin{cases} M \left(|x|^{\frac{1-\alpha}{2}} + |y| \right) + L & \text{if } \ 0 < x < 1, \\ M \left(|x|^{\frac{1+\beta}{2}} + |y| \right) + L & \text{if } \ x \geq 1. \end{cases} \tag{5.121}$$

Then problem (5.119), (5.2) *has a positive solution.*

5.5 . *Historical and bibliographical notes*

In 1958 Bevc, Palmer and Süsskind [35] searched for positive $2\,\pi$-periodic solutions of the Brillouin electron beam focusing system (5.114) which is a singular perturbation of the Mathieu equation. Before, in 1950, Pinney [119] considered the so-called Ermakov-Pinney equation $r'' + a(t)\, r = \frac{K}{r^3}$, where $a(t)$ is T-periodic and $K > 0$. Another example of the singular problem mentioned in literature are the parametric resonances of certain nonlinear Schrödinger systems (see [75]). As mentioned by Mawhin and Jebelean in their exhaustive historical introduction to the paper [89], second order nonlinear differential equations or systems with singularities appear naturally in the description of particles submitted to Newtonian type forces or to forces caused by compressed gases. Their mathematical study started in the sixties by Forbat and Huaux [73], Huaux [87], Derwidué [58] and Faure [69], who considered positive solutions of equations describing e.g. the motion of a piston in a cylinder closed at one extremity and submitted to a T-periodic exterior force, to the restoring force of a perfect gas and to a viscosity friction. The equations under their study may be after suitable substitutions transformed to

$$u'' + c\, u' = \frac{\beta}{u} + e(t),$$

where $c \neq 0$ and $\beta \in \mathbb{R}$ can be either positive or negative. Equations of this form are usually called Forbat's equations and their Liénard type generalizations like

$$u'' + h(u)\, u' = g(t, u) + e(t) \tag{5.122}$$

are sometimes also referred to as the generalized Forbat's equations. It is worth mentioning that, while Forbat and others relied on the dissipativeness properties, Faure made use of the Leray-Schauder topological method.

Later, in the seventies, techniques of critical point theory were applied for the first time by Gordon [80], who also introduced the *strong force condition* of the type (5.29).

In 1986, Gaete and Manásevich [77], using variational methods, proved the existence of at least two different positive T-periodic solutions of the equation $u'' = p(t)\, u^2 - u + u^{-5}$ which governs the radial oscillations of an elastic spherical membrane made up of a neo-Hookean material, and subjected to an internal pressure $p: \mathbb{R} \to (0, \infty)$ continuous, T-periodic and non-constant.

In 1987, keeping in mind the model equation $u'' = \beta\, u^{-\lambda} + e(t)$ with $\lambda > 0$, $\beta \neq 0$ and $e \in L_1[0, T]$, Lazer and Solimini [102] employed topological arguments and the lower and upper functions method to investigate the existence of positive solutions to the Duffing equation $u'' = g(u) + e(t)$. The restoring force g was allowed to have an attractive singularity or a strong repulsive singularity at origin. Starting with this paper, the interest in periodic singular problems considerably increased. The results by Lazer and Solimini have been generalized or extended e.g. by Habets and Sanchez [83] (1990), Mawhin [108] (1991), del Pino, Manásevich and Montero [56] (1992), del Pino and Manásevich [55] (1993), Fonda [70] (1993), Omari and Ye [114] (1995), Zhang [161] (1996) and [163] (1998) and Ge and Mawhin [77] (1998). Some of these papers (e.g. [83], [114], [161] or [163]) cover also the Liénard equation (5.122) with $g(t, x)$ having at $x = 0$ an essentially autonomous singularity. However, all of them, when dealing with the repulsive singularity, supposed that the strong force condition of the type (5.29) is satisfied. Furthermore, except for [55], [56] and [77], they dealt with restoring forces $g(t, x)$ behaving at ∞ like $-k\, x$ with $0 < k < \mu_1$, $\mu_1 = (\frac{\pi}{T})^2$ being the first Dirichlet eigenvalue of $x'' + \mu\, x = 0$. The paper [56] was concerned with the cases corresponding to k lying between two adjacent higher eigenvalues, while the papers [55] and [77] dealt with the superlinear case. Recently, Yan and Zhang [160] (2003) proved an existence result assuming that the nonlinearity grows semilinearly as $x \to \infty$ and fulfil a certain higher-order non-resonance condition in terms of the periodic and antiperiodic eigenvalues. Let us mention also that Martinez-Amores and Torres [111] considered in 1996 stability of periodic solutions of problems with singularities of attractive type. Furthermore, in 1998, Torres delivered results on the existence of bounded solutions to singular equations of repulsive type.

In 2001, Rachůnková, Tvrdý and Vrkoč [137], motivated by results on the existence of positive solutions to regular periodic problems by Nkashama and Santanilla [113] (1990) and Sanchez [139] (1992), made use of the lower and upper functions method to deliver related results in the form applicable also to singular problems. Unlike the above mentioned papers, their results concern also the resonance case $k = \mu_1$ and do not need any strong force condition. Later, in 2002, further step was done by Bonheure, Fonda and Smets [40] who made use of the properties of forced isochronous oscillators. Their results are also valid in the resonance case $k = \mu_1$ with a weak singularity. It turned out that in the resonance case $k = \mu_1$ problem (5.118) with $\lambda \geq 0$ has a solution whenever there is a $\delta > 0$ such that

$$\min_{t\in[0,T]} \int_t^{t+T} e(t)\,\sin(\pi\,\frac{s-t}{T})\,\mathrm{d}s \geq \delta.$$

Analogous results were derived also by Bonheure and De Coster [39] in 2003 by means of the lower and upper functions method. Simultaneously, Torres [152] noticed that having a thorough analysis of the sign properties of the related Green's functions, solvability of the periodic problem with a weak singularity can be ensured also by the Krasnoselskii fixed point theorem. His results turned out to be complementary to those already known when $0 < k < \mu_1$.

For related multiplicity results we refer to the papers by Fonda, Manásevich and Zanolin [72] (1993), Rachůnková [121] and [122] (2000) and Rachůnková, Tvrdý and Vrkoč [138] (2003). In particular, in [121] an extension of the results of Gaete and Manásevich from [77] applicable to the equation modelling radial oscillations of an elastic spherical membrane can be found.

The regular periodic problem with ϕ- or p-Laplacian on the left hand side was considered by several authors. For example, del Pino, Manásevich and Murúa [57] (1992) and Yan [159] (2003) proved the existence or multiplicity of periodic solutions of the equation $(\phi_p(u'))' = f(t,u)$ under non resonance conditions imposed on f. In 1998, general existence principles for the regular vector problem, based on the homotopy to the averaged nonlinearity, were presented by Manásevich and Mawhin [107] (1998) (see also Mawhin [110]). Multiplicity results of the Ambrosetti-Prodi type were given by Liu in [103] (1998).

The first steps to establish the lower/upper functions method for problems with a ϕ-Laplacian operator on the left hand side were done by Cabada and Pouso in [42] (1997) and by Jiang and Wang in [93] (1997), the latter paper dealing with the p-Laplacian. They assumed the existence of a pair of well-ordered lower and upper functions and both-sided Nagumo conditions. These results were extended by Staněk [142] (2001) to the case when a functional right-hand side fulfils one-sided growth conditions of Nagumo type. The paper by Cabada, Habets and Pouso [43] (1999) was the first to present the lower/upper functions method for periodic problems with a ϕ-Laplacian operator under the assumption that lower/upper functions are in the reverse order. If $\phi = \phi_p$ the authors got the solvability for $1 < p \leq 2$ only. The general existence principle valid also when lower/upper functions are non-ordered was presented by Rachůnková and Tvrdý in [135] (2005) and for the case when impulses are admitted also in [134] (2005).

The singular periodic problem for the Liénard equation $(\phi_p(u'))' + h(u)\,u' = g(u) + e(t)$ with p-Laplacian on the left hand side was treated by Liu [104] (2002) and Jebelean and Mawhin [89] (2002) and [90] (2004). Their main tool was the existence principle due to Manásevich and Mawhin from [107] (1998). Furthermore, in [89], the significance of the lower/upper functions method was shown.

Extensions to vector systems of the second order were not the subject of this text. We can only refer e.g. to the papers by Habets and Sanchez [84] (1990), Solimini [141] (1990), Fonda [71] (1995) and Zhang [164] (1999) for the classical case and by Manásevich and Mawhin [107] (1998) and Liu [104] (2002) for systems with p-Laplacian operators on their left hand sides.

6. Other types of two-point second order BVPs

In Sections 4 and 5, under the assumption that ϕ satisfies (3.3), we have investigated the nonlinear second order differential equation of the form

$$(\phi(u'))' + f(t,u,u') = 0, \tag{6.1}$$

subjected to Dirichlet and periodic boundary conditions, respectively. In this section we will study solvability of equation (6.1) with some other types of two-point boundary conditions on the interval $[0,T] \subset \mathbb{R}$.

We will focus our attention on problems with *space* or with *mixed (time and space)* singularities. According to Section 1, *solutions* and *w-solutions* of the problems are defined in the same way as for the Dirichlet problem (see Section 4.1) just replacing the Dirichlet conditions by the boundary conditions under consideration. We can define *lower and upper functions* of the second order boundary value problem in the same way as in Definition 4.3 replacing inequalities (4.6) with inequalities corresponding to the boundary conditions in question.

In the sequel we consider two-point linear boundary conditions arising in the study of physical, chemical or engineering problems and having the form

$$\begin{cases} a_0\, u(0) - b_0\, u'(0) = 0, \quad a_1\, u(T) + b_1\, u'(T) = 0, \\ a_i,\, b_i \in \mathbb{R}, \quad a_i^2 + b_i^2 > 0,\ i = 0,1. \end{cases} \tag{6.2}$$

Conditions (6.2) include conditions of the Dirichlet type (with $b_0 = b_1 = 0$), of the Neumann type (with $a_0 = a_1 = 0$), of the mixed type (with $a_0 = b_1 = 0$ or $b_0 = a_1 = 0$), of the Robin type (with $a_i > 0$, $b_i > 0$, $i = 0,1$) and of the standard Sturm-Liouville type (with a_i, $b_i \in [0,\infty)$, $i = 0,1$). We will also mention problems involving inhomogeneous form of the above boundary conditions, i.e.

$$\begin{cases} a_0\, u(0) - b_0\, u'(0) = A, \quad a_1\, u(T) + b_1\, u'(T) = B, \\ a_i,\, b_i \in \mathbb{R}, \quad a_i^2 + b_i^2 > 0,\ i = 0,1,\ A, B \in \mathbb{R}. \end{cases} \tag{6.3}$$

However, there is no restriction in assuming just the homogeneous conditions since a change from $u(t)$ to $y(t) = u(t) - q(t)$, where q is a polynomial satisfying (6.3), will reduce (6.3) to (6.2).

Consider a class of nonlinear singular boundary value problems whose importance is derived, in part, from the fact that they arise when searching for *positive, radially symmetric solutions* to the nonlinear elliptic partial differential equation

$$\Delta\, u + g(r,u) = 0 \quad \text{on } \Omega, \quad u\,|_\Gamma = 0,$$

where Δ is the Laplace operator, Ω is the open unit disk in $\mathbb{R}^n$ (centered at the origin), Γ is its boundary, and r is the radial distance from the origin. Radially symmetric solutions to this problem are solutions of the ordinary differential equation with the mixed boundary conditions

$$u'' + \frac{n-1}{t}\, u' + g(t,u) = 0, \quad u'(0) = 0,\ u(1) = 0.$$

See e.g. [30] or [78]. Particularly, Gatica, Oliker and Waltman [76] investigated the singular problem

$$u'' + \frac{n-1}{t}\,u' + \psi(t)\,\frac{1}{u^\alpha} = 0, \quad u'(0) = 0,\ u(1) = 0, \tag{6.4}$$

where

$$\begin{cases} n \ge 2,\ \alpha \in (0,1), \quad \psi \in C[0,1) \text{ is non-negative,} \\ \psi \quad \text{can have a time singularity at } t = 1. \end{cases} \tag{6.5}$$

Theorem 6.1. ([76, Theorem 4.1]) *Let* (6.5) *hold. Assume that*

$$0 < \int_0^1 (1-t)^{-\alpha}\,\psi(t)\,\mathrm{d}t < \infty.$$

Then problem (6.4) *has a solution that is positive on* $[0,1)$.

The technical arguments in the proof involve concavity of solutions and the use of iterative techniques. The main tool is a fixed point theorem for decreasing mappings on cones.

In the theory of diffusion and reaction a class of differential equations

$$u'' - \eta^2\, g_\kappa(u) = 0 \tag{6.6}$$

appears. Here η^2 is the (positive) Thiele modulus, $u \ge 0$ is the concentration of one of the reactants and κ is a positive parameter. The functions g_κ are continuous on $[0,\infty)$,

$$\lim_{\kappa\to 0+} g_\kappa(x) = g(x) \quad \text{for } x \in (0,\infty)$$

and g can have a space singularity at $x = 0$. The model functions are

$$g_\kappa(x) = \frac{x}{\kappa + x^{1+\gamma}}, \tag{6.7}$$

where γ is a positive parameter. Aris [22] proposed such equations as descriptions of the steady state for chemicals reacting and diffusing according to the Langmuir-Hinshelwood kinetics. Bobisud [36] studied a class of equations (6.6) on $[-1,1]$ subjected to the inhomogeneous Robin boundary conditions

$$\alpha\, u(-1) - u'(-1) = A, \quad \alpha\, u(1) + u'(1) = A, \quad \alpha,\ A > 0, \tag{6.8}$$

and with functions g_κ behaving qualitatively very much like the model functions in (6.7). He proved that for η^2 in (6.6) sufficiently small the limit problem with $\kappa = 0$ has a positive solution which can be approximated uniformly on $[-1,1]$ by solutions of (6.6), (6.8) with small κ.

Motivated by problem (6.6), (6.8) as well as by problem (6.4), Baxley and Gersdorff [28] studied a singular equation in which u' can appear nonlinearly,

$$u'' + h(t, u') - \eta^2\, g(t, u) = 0, \quad \eta^2 > 0, \tag{6.9}$$

with inhomogeneous Sturm-Liouville boundary conditions

$$u'(0) = 0, \quad \alpha\, u(T) + \beta\, u'(T) = A, \quad \alpha,\ A > 0,\ \beta \geq 0, \tag{6.10}$$

where

$$\left\{ \begin{array}{l} h \in C((0,T] \times [0,\infty)) \quad \text{is non-negative} \\ \text{and can have a time singularity at } t = 0, \end{array} \right. \tag{6.11}$$

$$\left\{ \begin{array}{l} g \in C([0,T] \times (0, \frac{A}{\alpha}]) \quad \text{is positive} \\ \text{and can have a space singularity at } x = 0. \end{array} \right. \tag{6.12}$$

In contrast to Theorem 6.1 where positive singular nonlinearity $\psi(t)\, x^{-\alpha}$ appears, the next theorem applies to equations involving a negative singular term $-\eta^2\, g(t, x)$.

Theorem 6.2. ([28, Theorem 17]) *Let* (6.11) *and* (6.12) *hold. Assume that*

$$h(t, 0) = 0 \quad \textit{for } t \in (0, T]$$

and that there exists $G \in L[0, \frac{A}{\alpha}]$ *satisfying*

$$g(t, x) \leq G(x) \quad \textit{on } [0,T] \times (0, \tfrac{A}{\alpha}]. \tag{6.13}$$

Then for η^2 *sufficiently small problem* (6.9), (6.10) *has a solution that is positive on* $[0, T]$.

Moreover, if η^2 is sufficiently large, Baxley and Gersdorff guarantee the existence of the so called *dead core* solutions which are defined as functions belonging for some $t_0 \in (0, T)$ to $C^1[0,T] \cap C^2(t_0, T]$, satisfying equation (6.9) on $(t_0, T]$, vanishing on $[0, t_0]$ and fulfilling (6.10). The proof is based on a priori estimates of approximate solutions of auxiliary regular problems and on the Arzelà-Ascoli theorem.

Agarwal, O'Regan and Staněk [16] considered a singular equation with a ϕ-Laplacian generalizing (6.9) and subjected to inhomogeneous mixed conditions

$$(\phi(u'))' - \mu\, f(t, u, u') = 0, \quad u'(0) = 0,\ u(T) = b, \quad b > 0, \tag{6.14}$$

where μ is a real positive parameter and

$$\left\{ \begin{array}{l} \phi(0) = 0, \quad f \in Car([0,T] \times (\mathbb{R} \setminus \{b\}) \times (\mathbb{R} \setminus \{0\})), \\ f \text{ can have space singularities at } \ x = b \text{ and } y = 0. \end{array} \right. \tag{6.15}$$

Theorem 6.3. ([16, Theorem 3.1]) *Let (6.15) hold. Assume that there exist $\varepsilon > 0$, $\nu \in (0,T]$ and a positive non-decreasing function $\rho \in C[0,T]$ such that*

$$f(t,x,\rho(t)) = 0 \quad \text{for a.e. } t \in [0,T] \text{ and all } x \in [0,b),$$

$$\varepsilon \le f(t,x,y) \quad \text{for a.e. } t \in [0,\nu] \text{ and all } x \in [0,b),\ y \in [0,\nu].$$

Further, let for a.e. $t \in [0,T]$ and for all $x \in [0,b)$, $y \in (0,\rho(t)]$

$$0 \le f(t,x,y) \le (h_1(x) + h_2(x))\,(\omega_1(y) + \omega_2(y)),$$

where $h_1 \in C[0,b]$, $\omega_1 \in C[0,\infty)$ are non-negative, $h_2 \in C[0,b)$, $\omega_2 \in C(0,\infty)$ are positive, h_1 and ω_2 are non-increasing, h_2 and ω_1 are non-decreasing and $\omega_1+\omega_2$ is non-increasing on a right neighbourhood of 0. Moreover, let

$$\int_0^b h_2(s)\,\mathrm{d}s < \infty, \qquad \int_0^1 \omega_2(\phi^{-1}(s))\,\mathrm{d}s < \infty.$$

Finally, let there exist $\mu_* > 0$ such that

$$\int_0^b \frac{\mathrm{d}s}{\Omega^{-1}(\mu_* H(s))} = T, \tag{6.16}$$

where

$$H(u) = \int_0^u (h_1(s) + h_2(s))\,\mathrm{d}s, \quad \Omega(u) = \int_0^{\phi(u)} \frac{\phi^{-1}(s)\,\mathrm{d}s}{\omega_1(\phi^{-1}(s)) + \omega_2(\phi^{-1}(s))}.$$

Then for each $\mu \in (0,\mu_*)$ problem (6.14) has a solution u satisfying

$$0 < u(t) \le b, \quad 0 \le u'(t) \le \rho(t) \quad \text{for } t \in [0,T].$$

To prove this existence result the authors used regularization and sequential techniques. First, they defined a family of auxiliary regular differential equations depending on $n \in \mathbb{N}$ and then, using the topological transversality theorem, they obtained a sequence of positive approximate solutions. Applying the Arzelà-Ascoli theorem and the Lebesgue convergence theorem they showed that its limit is a solution of problem (6.14).

Remark 6.4. Note that if there exists $\mu_0 \in (0,\infty)$ such that

$$\int_0^b \frac{\mathrm{d}s}{\Omega^{-1}(\mu_0 H(s))} \in (T,\infty),$$

then $\mu_* \in (0,\mu_0)$ satisfying (6.16) can be always found.

Comparing problem (6.9), (6.10) and problem (6.14), they seem to be in some sense close, because both of them have negative singular nonlinearities in differential equations and the boundary conditions of (6.14) are contained in (6.10). However, there is a large difference between them. For example, positive solutions of (6.9), (6.10) do not touch the space singularity of f at $x=0$. On the other hand, each solution u of (6.14) satisfies $u(T)=b$ and hence enters the space singularity of f at $x=b$. Another difference between them consists in the fact that f in (6.14) can have also a space singularity at $y=0$ and hence Theorem 6.3 can be used in the following example whereas Theorem 6.2 cannot.

Example. Let $\alpha \in (0,\infty)$ and $\beta \in (0,1)$. By Theorem 6.3 there exists a positive number μ_* depending on β only such that for any $\mu \in (0,\mu_*)$ the problem

$$u'' - \mu\,(1-|u|^\alpha)\left(\frac{1}{|u'|^\beta}-1\right)=0, \quad u'(0)=0, \quad u(1)=\frac{1}{2},$$

has a solution u such that $0<u(t)\le \frac{1}{2}$, $0\le u'(t)\le 1$ for $t\in[0,1]$. An explicit formula for μ^* can be found in [16].

Assumption (6.13) in Theorem 6.2 means that the space singularity of g at $x=0$ is a *weak singularity.* See Remark 5.1 for more detail. Note that the assumption (6.13) is not satisfied for the problem

$$u'' + \frac{t^2}{32\,u^2} - \frac{\lambda^2}{8} = 0, \quad u(0)=0, \quad 2\,u'(1)-(1-\nu)\,u(1)=0, \tag{6.17}$$

where $\lambda \in (0,\infty)$, $\nu \in (0,1)$, which models the large deflection membrane response of a spherical cap. This problem has been solved numerically by various techniques in engineering literature [79], [112], [117]. Baxley in [27] proved existence and uniqueness of a solution of this problem, gave qualitative information about the solution, and used this information to suggest an approach to numerical computation. In the proof the maximum principle plays a fundamental role. In contrast to (6.8), (6.11) and (6.14), problem (6.17) has boundary conditions which are not included in the Sturm-Liouville ones, because $\nu<1$ and so the coefficient $-(1-\nu)$ at $u(1)$ is negative.

Existence results for equations whose nonlinearities have a singularity at $x=0$ and can be increasing for $x\to\infty$ are proved in [4], where Agarwal and O'Regan obtained the existence of a w-solution $u>0$ on $(0,1]$ of such equations with mixed boundary conditions. Their theorem can be applied for example to the problem

$$u'' + \left(\frac{1}{u^\alpha}+u^\beta+1\right)(1+(u')^3)=0, \quad u'(0)=0, \quad u(1)=0, \tag{6.18}$$

with $\alpha\in(0,1)$, $\beta\ge 0$. We see that the nonlinearity

$$f(t,x,y)=\left(\frac{1}{x^\alpha}+x^\beta+1\right)(1+y^3) \tag{6.19}$$

has a weak space singularity at $x=0$ and can be increasing for large x. If $\beta\in(0,1)$ the growth of f is sublinear. For $\beta=1$ or $\beta>1$ the growth of f is linear or superlinear, respectively.

In the investigation of singular problems (6.1), (6.2) or (6.1), (6.3), lower and upper functions of the corresponding regular problems can be a fruitful tool. See for example papers by Kannan and O'Regan in [94] or by Agarwal and Staněk in [18]. We will demonstrate the role of lower and upper functions on the following singular problem with mixed boundary conditions

$$u'' + f(t, u, u') = 0, \quad u'(0) = 0, \quad u(T) = 0, \tag{6.20}$$

where

$$\begin{cases} \mathcal{D} = (0, \infty) \times (-\infty, 0), \quad f \in Car((0, T) \times \mathcal{D}), \\ f \text{ can have time singularities at } t = 0,\ t = T \\ \text{and space singularities at } \ x = 0,\ y = 0. \end{cases} \tag{6.21}$$

First, consider an auxiliary regular problem

$$u'' + h(t, u, u') = 0, \quad u'(0) = 0, \quad u(T) = 0, \tag{6.22}$$

where $h \in Car([0, T] \times \mathbb{R}^2)$.

Definition 6.5. A function $\sigma \in C[0, T]$ is called *a lower function of* (6.22) if there exists a finite set $\Sigma \subset (0, T)$ such that $\sigma \in AC^1_{loc}([0, T] \setminus \Sigma)$, $\sigma'(\tau+), \sigma'(\tau-) \in \mathbb{R}$ for each $\tau \in \Sigma$,

$$\sigma''(t) + f(t, \sigma(t), \sigma'(t)) \ge 0 \quad \text{ for a.e. } t \in [0, T], \tag{6.23}$$

$$\sigma'(0) \ge 0, \quad \sigma(T) \le 0, \quad \sigma'(\tau-) < \sigma'(\tau+) \quad \text{for each } \tau \in \Sigma. \tag{6.24}$$

If the inequalities in (6.23) and (6.24) are reversed, then σ is called *an upper function of* (6.22).

In what follows we will need the classical lower and upper functions result for the mixed problem (6.22).

Lemma 6.6. [98, Lemma 3.7] *Let σ_1 and σ_2 be a lower and an upper function for problem* (6.22) *such that $\sigma_1 \le \sigma_2$ on $[0, T]$. Assume also that there is a function $\psi \in L_1[0, T]$ such that*

$$|h(t, x, y| \le \psi(t) \quad \textit{ for a.e. } t \in [0, T],\ \textit{all } \ x \in [\sigma_1(t), \sigma_2(t)],\ y \in \mathbb{R}. \tag{6.25}$$

Then problem (6.22) *has a solution $u \in AC^1[0, T]$ satisfying*

$$\sigma_1(t) \le u(t) \le \sigma_2(t) \quad \textit{for } \ t \in [0, T]. \tag{6.26}$$

We will apply Lemma 6.6 to the singular mixed problem (6.20).

THEOREM 6.7. ([123, Theorem 3.1]) *Let* (6.21) *hold. Assume that there exist* $\varepsilon \in (0,1)$, $\nu \in (0,T)$, $c \in (\nu,\infty)$ *such that*

$$f(t, c\,(T-t), -c) = 0 \quad \text{for a.e. } t \in [0,T], \tag{6.27}$$

$$0 \le f(t,x,y) \quad \text{for a.e. } t \in [0,T], \text{ and all } x \in (0, c\,(T-t)],\ y \in [-c,0), \tag{6.28}$$

$$\varepsilon \le f(t,x,y) \quad \text{for a.e. } t \in [0,\nu], \text{ and all } x \in (0, c\,(T-t)],\ y \in [-\nu,0). \tag{6.29}$$

Then problem (6.20) *has a solution* $u \in AC^1[0,T]$ *satisfying*

$$0 < u(t) \le c\,(T-t), \quad -c \le u'(t) < 0 \text{ for } t \in (0,T). \tag{6.30}$$

PROOF. Let $k \in \mathbb{N}$, $k \ge \frac{3}{T}$.

Step 1. *Approximate solutions.*

For $t \in [\frac{1}{k}, T-\frac{1}{k}]$, $x \in \mathbb{R}$ and $y \in \mathbb{R}$ put

$$\alpha_k(t,x) = \begin{cases} c\,(T-t) & \text{if} \quad x > c\,(T-t), \\ x & \text{if} \quad \frac{c}{k} \le x \le c\,(T-t), \\ \frac{c}{k} & \text{if} \quad x < \frac{c}{k}, \end{cases}$$

$$\beta_k(y) = \begin{cases} -\frac{\varepsilon}{k} & \text{if} \quad y > -\frac{\varepsilon}{k}, \\ y & \text{if} \quad -c \le y \le -\frac{\varepsilon}{k}, \\ -c & \text{if} \quad y < -c, \end{cases} \qquad \gamma(y) = \begin{cases} \varepsilon & \text{if} \quad y \ge -\nu, \\ \varepsilon\,\frac{c+y}{c-\nu} & \text{if} \quad -c < y < -\nu, \\ 0 & \text{if} \quad y \le -c. \end{cases}$$

For a.e. $t \in [0,T]$ and x, $y \in \mathbb{R}$ define

$$f_k(t,x,y) = \begin{cases} \gamma(y) & \text{if} \quad t \in [0,\frac{1}{k}), \\ f(t,\alpha_k(t,x),\beta_k(y)) & \text{if} \quad t \in [\frac{1}{k}, T-\frac{1}{k}], \\ 0 & \text{if} \quad t \in (T-\frac{1}{k}, T]. \end{cases}$$

Then $f_k \in Car([0,T]\times\mathbb{R}^2)$ and there is $\psi_k \in L_1[0,T]$ such that

$$|f_k(t,x,y)| \le \psi_k(t) \quad \text{for a.e. } t \in [0,T], \text{ all } x,y \in \mathbb{R}. \tag{6.31}$$

We have got an auxiliary regular problem

$$u'' + f_k(t,u,u') = 0, \quad u'(0) = 0, \quad u(T) = 0. \tag{6.32}$$

Conditions (6.27) and (6.28) yield

$$f_k(t, c\,(T-t), -c) = 0 \quad \text{and} \quad f_k(t,0,0) \ge 0 \quad \text{for a.e. } t \in [0,T].$$

Put $\sigma_1(t) = 0$, $\sigma_2(t) = c\,(T-t)$ for $t \in [0,T]$. Then σ_1 and σ_2 are a lower and an upper function of (6.32). Hence, by Lemma 6.6, problem (6.32) has a solution u_k and

$$0 \le u_k(t) \le c\,(T-t) \quad \text{on} \quad [0,T]. \tag{6.33}$$

Step 2. *A priori estimates of approximate solutions.*

Since $u_k'(0) = 0$ and $f_k(t, x, y) \geq 0$ for a.e. $t \in [0, T]$ and all x, $y \in \mathbb{R}$, we get $u_k'(t) \leq 0$ on $[0, T]$. Condition (6.33) and $u_k(T) = 0$ give $u_k(T) - u_k(t) \geq -c\,(T - t)$, which yields $u_k'(T) \geq -c$. Since u_k' is nonincreasing on $[0, T]$, we have proved

$$-c \leq u_k'(t) \leq 0 \quad \text{on} \quad [0, T]. \tag{6.34}$$

Due to $u_k'(0) = 0$, there is $t_k \in (0, T]$ such that

$$-\nu \leq u_k'(t) \leq 0 \quad \text{for} \quad t \in [0, t_k].$$

If $t_k \geq \nu$, we get by (6.29)

$$u_k'(t) \leq -\varepsilon\, t \quad \text{for} \quad t \in [0, \nu]. \tag{6.35}$$

Assume that $t_k < \nu$ and $u_k'(t) < -\nu$ for $t \in (t_k, \nu]$. Then

$$u_k'(t) \leq -\varepsilon\, t \quad \text{for } t \in [0, t_k].$$

Since $-\nu < -\varepsilon t$ for $t \in (t_k, \nu]$, we get (6.35) again. Integrating (6.35) over $[0, \nu]$ and using the concavity of u_k on $[0, T]$ we deduce that

$$\frac{\varepsilon\, \nu^2}{2\,T}\,(T - t) \leq u_k(t) \quad \text{on} \quad [0, T]. \tag{6.36}$$

Step 3. *Convergence of a sequence of approximate solutions.*

Consider the sequence $\{u_k\}$. Choose an arbitrary compact interval $J \subset (0, T)$. By virtue of (6.33)–(6.36) there is $k_0 \in \mathbb{N}$ such that for each $k \in \mathbb{N}$, $k \geq k_0$,

$$\frac{c}{k_0} \leq u_k(t) \leq c(T - t), \quad -c \leq u_k'(t) \leq -\frac{\varepsilon}{k_0} \quad \text{on} \quad J, \tag{6.37}$$

and hence there is $\psi \in L_1(J)$ such that

$$|f_k(t, u_k(t), u_k'(t))| \leq \psi(t) \quad \text{for a.e. } t \in J. \tag{6.38}$$

Using conditions (6.33), (6.34), (6.38), the Arzelà-Ascoli theorem and the diagonalization principle, we can choose $u \in C[0, T] \cap C^1(0, T)$ and a subsequence of $\{u_k\}$ which we denote for the sake of simplicity in the same way such that

$$\begin{cases} \lim_{k\to\infty} u_k = u & \text{uniformly on} \quad [0, T], \\ \lim_{k\to\infty} u_k' = u' & \text{locally uniformly on} \quad (0, T). \end{cases} \tag{6.39}$$

Therefore we have $u(T) = 0$.

Step 4. *Convergence of the sequence of approximate problems.*

Choose an arbitrary $\xi \in (0, T)$ such that

$$f(\xi, \cdot, \cdot) \quad \text{is continuous on} \quad (0, \infty) \times (-\infty, 0).$$

By (6.37) there exist a compact interval $J^* \subset (0,T)$ and $k_* \in \mathbb{N}$ such that $\xi \in J^*$ and for each $k \ge k_*$

$$u_k(\xi) > \frac{c}{k_*}, \quad u_k'(\xi) < -\frac{\varepsilon}{k_*}, \quad J^* \subset [\tfrac{1}{k}, T - \tfrac{1}{k}].$$

Therefore

$$f_k(\xi, u_k(\xi), u_k'(\xi)) = f(\xi, u_k(\xi), u_k'(\xi))$$

and, due to (6.39),

$$\lim_{k\to\infty} f_k(t, u_k(t), u_k'(t)) = f(t, u(t), u'(t)) \quad \text{for a.e. } t \in (0,T). \tag{6.40}$$

Choose an arbitrary $t \in (0,T)$. Then there exists a compact interval $J \subset (0,T)$ such that (6.38) holds for all sufficiently large k. By virtue of (6.32) we get

$$u_k'(\tfrac{T}{2}) - u_k'(t) = \int_{\frac{T}{2}}^{t} f_k(s, u_k(s), u_k'(s))\,\mathrm{d}s.$$

Letting $k \to \infty$ and using (6.38), (6.39), (6.40) and the Lebesgue convergence theorem on J, we get

$$u'(\tfrac{T}{2}) - u'(t) = \int_{\frac{T}{2}}^{t} f(s, u(s), u'(s))\,\mathrm{d}s \quad \text{for each } \ t \in (0,T). \tag{6.41}$$

Therefore $u \in AC^1_{loc}(0,T)$ satisfies

$$u''(t) + f(t, u(t), u'(t)) = 0 \quad \text{for a.e. } t \in (0,T). \tag{6.42}$$

Further, according to (6.32) and (6.34) we have for each $k \ge \frac{3}{T}$

$$\int_0^T f_k(s, u_k(s), u_k'(s))\,\mathrm{d}s = -u_k'(T) \le c,$$

which together with (6.28), (6.33), (6.34) and (6.40) yields, by the Fatou lemma, that $f(t, u(t), u'(t)) \in L_1[0,T]$. Therefore, by (6.42), $u \in AC^1[0,T]$. Moreover for each $k \ge \frac{3}{T}$ and $t \in (0,T)$

$$|u_k'(t)| \le \int_0^t |f_k(s, u_k(s), u_k'(s)) - f(s, u(s), u'(s))|\,\mathrm{d}s + \int_0^t |f(s, u(s), u'(s))|\,\mathrm{d}s.$$

Hence by (6.39) and (6.40) for each $\varepsilon > 0$ there exists $\delta > 0$ and for each $t \in (0,\delta)$ there exists $k_0 = k_0(\varepsilon, t) \in \mathbb{N}$ such that

$$|u'(t)| \le |u'(t) - u_{k_0}'(t)| + |u_{k_0}'(t)| < \varepsilon.$$

It means that $u'(0) = \lim_{t\to 0+} u'(t) = 0$. We have proved that u is a solution of problem (6.20). $\square$

Example. Let $\alpha > 0$, $\beta \ge 0$ be arbitrary numbers. By Theorem 6.7 problem (6.18) has a solution $u \in AC^1[0,1]$ satisfying

$$0 < u(t) \le 1 - t, \quad -1 \le u'(t) < 0 \quad \text{for } \ t \in (0,1).$$

Note that Theorem 6.7 guarantees solvability of problem (6.18) even for the nonlinearity (6.19) having a strong space singularity ($\alpha \ge 1$) at $x = 0$.

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