Second Order Periodic Problem with ϕ-Laplacian and Impulses - Part II

Irena Rachůnková* and Milan Tvrdý†

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Abstract. Existence principles for the BVP $(\phi(u'))' = f(t,u,u')$, $u(t_i+) = J_i(u(t_i))$, $u'(t_i+) = M_i(u'(t_i))$, $i = 1,2,\dots,m$, $u(0) = u(T)$, $u'(0) = u'(T)$ are presented. They are based on the method of lower/upper functions which are not well-ordered. We continue our investigations from [16], where existence principles based on well-ordered lower/upper functions have been proved and from [13]–[15], where related results for the case that ϕ is the identity have been delivered.
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1. Introduction

We will consider the problem

$$(\phi(u'(t)))' = f(t,u(t),u'(t)) \quad \text{a.e. on} \quad [0,T], \tag{1.1}$$

$$u(t_i+) = J_i(u(t_i)), \quad u'(t_i+) = M_i(u'(t_i)), \quad i = 1,2,\dots,m, \tag{1.2}$$

$$u(0) = u(T), \quad u'(0) = u'(T), \tag{1.3}$$

where

$$u'(t_i) = u'(t_i-) = \lim_{t\to t_i-} u'(t),\ i = 1,2,\dots,m+1, \quad u'(0) = u'(0+) = \lim_{t\to 0+} u'(t)$$

and

$$\begin{cases} m \in \mathbb{N},\ 0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T < \infty, \\ f \text{ is a Carathéodory function on } [0,T]\times\mathbb{R}^2, \\ J_i \text{ and } M_i \text{ are continuous on } \mathbb{R},\ i = 1,2,\dots,m, \\ \phi \text{ is an increasing homeomorphism } \mathbb{R}\to\mathbb{R},\ \phi(0)=0,\ \phi(\mathbb{R})=\mathbb{R}. \end{cases} \tag{1.4}$$

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Throughout the paper we keep the following notation and conventions:

For a function u defined a.e. on $[0,T]$, we put

$$\|u\|_\infty = \sup_{t\in[0,T]} \text{ess}\, |u(t)| \quad \text{and} \quad \|u\|_1 = \int_0^T |u(s)|\,\mathrm{d}s.$$

For a given interval $J \subset \mathbb{R}$, $\mathbb{C}(J)$ is the set of functions which are continuous on J, $\mathbb{C}^1(J)$ is the set of functions having continuous first derivatives on J and $\mathbb{L}(J)$ is the set of functions which are Lebesgue integrable on J.

Denote $\mathrm{D} = \{t_1, t_2, \dots, t_m\}$ and define $\mathbb{C}_\mathrm{D}$ (or $\mathbb{C}^1_\mathrm{D}$) as the sets of functions $u\colon [0,T] \mapsto \mathbb{R}$,

$$u(t) = \begin{cases} u_{[0]}(t) & \text{if } t \in [0,t_1], \\ u_{[1]}(t) & \text{if } t \in (t_1,t_2], \\ \dots & \quad \dots \\ u_{[m]}(t) & \text{if } t \in (t_m,T], \end{cases}$$

where $u_{[i]}$ is continuous on $[t_i,t_{i+1}]$ (or continuously differentiable on $[t_i,t_{i+1}]$) for $i = 0,1,\dots,m$. If $u \in \mathbb{C}^1_\mathrm{D}$, we define $\|u\|_\mathrm{D} = \|u\|_\infty + \|u'\|_\infty$. $\mathbb{C}_\mathrm{D}$ and $\mathbb{C}^1_\mathrm{D}$ respectively with the norms $\|.\|_\infty$ and $\|.\|_\mathrm{D}$ are Banach spaces. Further, $\mathbb{AC}_\mathrm{D}$ is the set of functions $u \in \mathbb{C}_\mathrm{D}$ which are absolutely continuous on each subinterval (t_i,t_{i+1}), $i = 0,1,\dots,m$. The set of functions satisfying the Carathéodory conditions on $[0,T]\times\mathbb{R}^2$ will be denoted by $\mathrm{Car}([0,T]\times\mathbb{R}^2)$. As usual, χ_M will denote the characteristic function of the set $M \subset \mathbb{R}$. For $\psi \in \mathbb{C}(\mathbb{R})$ increasing on $\mathbb{R}$ and $x \in \mathbb{R}$, we define

$$\Big\{x\Big\}_\psi = \max\{|\psi(-x)|, |\psi(x)|\}.$$

Given a Banach space $\mathbb{X}$ and its subset M, let $\mathrm{cl}(M)$ and ∂M denote the closure and the boundary of M, respectively.

Let Ω be an open bounded subset of $\mathbb{X}$. Assume that the operator $\mathcal{F}\colon \mathrm{cl}(\Omega) \mapsto \mathbb{X}$ is completely continuous and $\mathcal{F}u \neq u$ for all $u \in \partial\,\Omega$. Then $\deg(\mathrm{I}-\mathcal{F},\Omega)$ denotes the *Leray-Schauder topological degree* of $\mathrm{I}-\mathcal{F}$ with respect to Ω, where I is the identity operator on $\mathbb{X}$.

A *solution* of the problem (1.1)–(1.3) is a function $u \in \mathbb{C}^1_\mathrm{D}$ such that $\phi(u') \in \mathbb{AC}_\mathrm{D}$ and (1.1)–(1.3) hold.

A function $\sigma \in \mathbb{C}^1_\mathrm{D}$ is called a *lower function* of (1.1)–(1.3) if $\phi(\sigma') \in \mathbb{AC}_\mathrm{D}$ and

$$\begin{cases} \phi(\sigma'(t))' \geq f(t,\sigma(t),\sigma'(t)) \quad \text{for a.e. } t \in [0,T], \\ \sigma(t_i+) = J_i(\sigma(t_i)),\ \sigma'(t_i+) \geq M_i(\sigma'(t_i)),\ i = 1,2,\dots,m, \\ \sigma(0) = \sigma(T),\ \sigma'(0) \geq \sigma'(T). \end{cases} \tag{1.5}$$

Similarly, a function $\sigma \in \mathbb{C}^1_{\mathrm{D}}$ with $\phi(\sigma') \in \mathbb{AC}_{\mathrm{D}}$ is an *upper function* of (1.1)–(1.3) if it satisfies the relations (1.5) but with reversed inequalities.

Up to now, the only paper dealing with the problems with a ϕ-Laplacian and impulses is our previous paper [16], where we have established existence principles based on the existence of well-ordered lower/upper functions. As concerns problem (1.1), (1.3) (without impulses), there are various results about its solvability, see e.g. [4], [5], [6], [8], [9], [10], [11], [12] and [19]. The papers which are devoted to the lower/upper functions method for the problem (1.1), (1.3) mostly assume well-ordered σ_1/σ_2. We can refer to the papers [1], [3], [7] and [18]. The paper [2] is, to our knowledge, the only one presenting the lower/upper functions method for the problem $(\phi(u')' = f(t,u)$, (1.3) under the assumption that $\sigma_1 \geq \sigma_2$, i.e. lower/upper functions are in the reverse order. If $\phi = \phi_p$ the authors get the existence results for $1 < p \leq 2$, only. Therefore the existence principle (Theorem 3.1) which we state here for the impulsive problem (1.1)–(1.3) and the case (1.6) are new even for the non-impulsive problem (1.1), (1.3).

Our basic assumption is the existence of lower/upper functions:

$$\begin{aligned} &\sigma_1 \quad \text{and} \quad \sigma_2 \text{ are respectively lower and upper functions of (1.1)–(1.3)} \\ &\text{such that } \ \sigma_1(\tau) > \sigma_2(\tau) \quad \text{for some } \ \tau \in [0,T], \end{aligned} \tag{1.6}$$

i.e., in contrast to [16], they *are not well-ordered*. Furthermore, as in [14]–[16], we will assume that the impulse functions J_i, M_i fulfil the following weak monotonicity like conditions

$$\left\{ \begin{aligned} x > \sigma_1(t_i) \ &\Longrightarrow \ J_i(x) > J_i(\sigma_1(t_i)), \\ x < \sigma_2(t_i) \ &\Longrightarrow \ J_i(x) < J_i(\sigma_2(t_i)), \quad i = 1,2,\dots,m, \end{aligned} \right. \tag{1.7}$$

$$\left\{ \begin{aligned} y \leq \sigma_1'(t_i) \ &\Longrightarrow \ M_i(y) \leq M_i(\sigma_1'(t_i)), \\ y \geq \sigma_2'(t_i) \ &\Longrightarrow \ M_i(y) \geq M_i(\sigma_2'(t_i)), \quad i = 1,2,\dots,m, \end{aligned} \right. \tag{1.8}$$

To transfer the given problem (1.1)–(1.3) into a fixed point problem in $\mathbb{C}^1_{\mathrm{D}}$, we will borrow some ideas from [10] and [16]. First, notice that it can be equivalently rewritten as (1.1), (1.2),

$$u(0) = u(T) = u(0) + u'(0) - u'(T). \tag{1.9}$$

Further, for $(\ell, d) \in \mathbb{C}_{\mathrm{D}} \times \mathbb{R}$ denote by $a(\ell,d)$ the unique solution $a \in \mathbb{R}$ of the equation

$$d + \int_0^T \phi^{-1}\big(a + \ell(t)\big)\,\mathrm{d}t = 0 \tag{1.10}$$

(see [16, Lemma 3.2]) and define the operators $\mathcal{N}: \mathbb{C}^1_{\rm D} \mapsto \mathbb{C}_{\rm D}$ and $\mathcal{J}: \mathbb{C}^1_{\rm D} \mapsto \mathbb{C}^1_{\rm D}$ respectively by

$$(1.11)\qquad \left\{ \begin{aligned} &(\mathcal{N}(x))(t) = \int_0^t f(s,x(s),x'(s))\,\mathrm{d}s \\ &\qquad + \sum_{i=1}^m \left[\phi\big(M_i(x'(t_i))\big) - \phi\big(x'(t_i)\big)\right] \chi_{(t_i,T]}(t), \quad t \in [0,T], \end{aligned} \right.$$

and

$$(1.12)\qquad (\mathcal{J}(x))(t) = \sum_{i=1}^m \left[J_i(x(t_i)) - x(t_i)\right] \chi_{(t_i,T]}(t), \quad t \in [0,T].$$

Finally, for $x \in \mathbb{C}^1_{\rm D}$ and $t \in [0,T]$, define

$$(1.13)\qquad \left\{ \begin{aligned} &(\mathcal{F}(x))(t) = \int_0^t \phi^{-1}\Big(a\big(\mathcal{N}(x),(\mathcal{J}(x))(T)\big) + (\mathcal{N}(x))(s)\Big)\,\mathrm{d}s \\ &\qquad\qquad + x(0) + x'(0) - x'(T) + (\mathcal{J}(x))(t). \end{aligned} \right.$$

Then $\mathcal{F}: \mathbb{C}^1_{\rm D} \mapsto \mathbb{C}^1_{\rm D}$ is an absolutely continuous operator and u is a solution of (1.1)–(1.3) if and only if $\mathcal{F}(u) = u$ (see [16, Theorem 3.5]).

In the proof of our main result we will need to evaluate the Leray-Schauder degree of a certain auxiliary operator with respect to sets determined by couples of well-ordered lower/upper functions. This is enabled by the following proposition which follows from [16, Theorem 4.4].

1.1. Proposition. *Assume that* (1.4) *holds and let* α *and* β *be respectively lower and upper functions of* (1.1) – (1.3) *such that*

$$(1.14)\qquad \alpha(t) < \beta(t) \ \ \textit{for } t \in [0,T] \quad \textit{and} \quad \alpha(\tau+) < \beta(\tau+) \ \ \textit{for } \tau \in {\rm D},$$

$$(1.15)\qquad \alpha(t_i) \le x \le \beta(t_i) \implies J_i(\alpha(t_i)) < J_i(x) < J_i(\beta(t_i)), \quad i = 1,2,\dots,m$$

and

$$(1.16)\qquad \left\{ \begin{aligned} &y \le \alpha'(t_i) \implies M_i(y) \le M_i(\alpha'(t_i)), \\ &y \ge \beta'(t_i) \implies M_i(y) \ge M_i(\beta'(t_i)), \quad i = 1,2,\dots,m. \end{aligned} \right.$$

Further, let $h \in \mathbb{L}[0,T]$ *be such that*

$$(1.17)\qquad |f(t,x,y)| \le h(t) \qquad \textit{for a.e. } t \in [0,T] \ \textit{ and all } (x,y) \in [\alpha(t),\beta(t)] \times \mathbb{R}$$

and let the operator $\mathcal{F}$ *be defined by* (1.10) – (1.13). *Finally, for* $\gamma \in (0,\infty)$ *denote*

$$(1.18)\qquad \left\{ \begin{aligned} &\Omega(\alpha,\beta,\gamma) = \{u \in \mathbb{C}_{\rm D} : \alpha(t) < u(t) < \beta(t) \ \ \textit{for } t \in [0,T], \\ &\qquad\qquad \alpha(\tau+) < u(\tau+) < \beta(\tau+) \ \ \textit{for } \tau \in {\rm D},\ \|u'\|_\infty < \gamma\}. \end{aligned} \right.$$

Then $\deg(\mathrm{I}-\mathcal{F},\Omega(\alpha,\beta,\gamma))=1$ *whenever* $\mathcal{F}u\neq u$ *on* $\partial\Omega(\alpha,\beta,\gamma)$ *and*

$$\gamma > \Big\{\|h\|_1\Big\}_{\phi^{-1}} + \frac{\|\alpha\|_\infty+\|\beta\|_\infty}{\Delta}, \quad \textit{where}\ \ \Delta=\min_{i=1,2,\dots,m+1}(t_i-t_{i-1}). \tag{1.19}$$

Proof. Using the Mean Value Theorem, we can show that

$$\|u'\|_\infty \le \Big\{\|h\|_1\Big\}_{\phi^{-1}} + \frac{\|\alpha\|_\infty+\|\beta\|_\infty}{\Delta} \tag{1.20}$$

holds for each $u\in\mathbb{C}_{\mathrm{D}}$ fulfilling $\alpha(t)<u(t)<\beta(t)$ on $[0,T]$ and $\alpha(\tau+)<u(\tau+)<\beta(\tau+)$ on D. Thus, if we denote by c the right-hand side of (1.20), we can follow the proof of [16, Theorem 4.4]. □

2. A priori estimates

Notice that from a priori estimates given by Lemmas 2.1–2.3 in [15] and Lemma 2.4 in [14], only the first one depend on the form of the differential equation (1.1) and requires a modification for the purposes of this paper.

2.1. Lemma. *Let* $\rho_1\in(0,\infty)$, $\widetilde{h}\in\mathbb{L}[0,T]$, $M_i\in\mathbb{C}(\mathbb{R})$, $i=1,2,\dots,m$. *Then there exists* $d\in(\rho_1,\infty)$ *such that the estimate*

$$\|u'\|_\infty<d \tag{2.1}$$

is valid for each $u\in\mathbb{AC}^1_{\mathrm{D}}$ *and each* $\widetilde{M}_i\in\mathbb{C}(\mathbb{R})$, $i=1,2,\dots,m$, *satisfying* (1.3),

$$|\phi(u'(\xi_u))|<\rho_1 \quad \textit{for some}\ \ \xi_u\in[0,T], \tag{2.2}$$

$$u'(t_i+)=\widetilde{M}_i(u'(t_i)), \quad i=1,2,\dots,m, \tag{2.3}$$

$$|(\phi(u'(t)))'|<\widetilde{h}(t) \quad \textit{for a.e.}\ \ t\in[0,T] \tag{2.4}$$

and

$$\sup\{|M_i(y)|:\,|y|<a\}<b \implies \sup\{|\widetilde{M}_i(y)|:\,|y|<a\}<b \qquad \textit{for}\ \ i=1,2,\dots,m,\ a\in(0,\infty),\ b\in(a,\infty). \tag{2.5}$$

Proof. Suppose that $u\in\mathbb{AC}^1_{\mathrm{D}}$ and $\widetilde{M}_i\in\mathbb{C}(\mathbb{R})$, $i=1,2,\dots,m$, satisfy (1.3) and (2.2)–(2.5). Due to (1.3), we can assume that $\xi_u\in(0,T]$, i.e. there is $j\in\{1,2,\dots,m+1\}$ such that $\xi_u\in(t_{j-1},t_j]$. We will distinguish 3 cases: either $j=1$ or $j=m+1$ or $1<j<m+1$.

Let $j=1$. Then, using (2.2) and (2.4), we obtain $|\phi(u'(t))|<\rho_1+\|\widetilde{h}\|_1$ for $t\in[0,t_1]$, i.e.

$$|u'(t)|<a_1 \ \text{ on } [0,t_1], \tag{2.6}$$

where $a_1 = \Big\{\rho_1 + \|\widetilde{h}\|_1\Big\}_{\phi^{-1}}$. Since $M_1 \in \mathbb{C}(\mathbb{R})$, we can find $b_1(a_1) \in (a_1, \infty)$ such that $|M_1(y)| < b_1(a_1)$ for all $y \in (-a_1, a_1)$. Hence, in view of (2.3) and (2.5), we have $|u'(t_1+)| < b_1(a_1)$, wherefrom, using (2.4), we deduce that

$$|u'(t)| < \Big\{\Big\{b_1(a_1)\Big\}_{\phi} + \|\widetilde{h}\|_1\Big\}_{\phi^{-1}} \quad \text{for } t \in (t_1, t_2].$$

Continuing by induction, we get $b_i(a_i) \in (a_i, \infty)$ such that

$$|u'(t)| < a_{i+1} = \Big\{\Big\{b_i(a_i)\Big\}_{\phi} + \|\widetilde{h}\|_1\Big\}_{\phi^{-1}} \quad \text{on } (t_i, t_{i+1}]$$

for $i = 2, \ldots, m$, i.e.

$$\|u'\|_\infty < d := \max\{a_i : i = 1, 2, \ldots, m+1\}. \tag{2.7}$$

Assume that $j = m + 1$. Then, using (2.2) and (2.4), we obtain

$$|u'(t)| < a_{m+1} \quad \text{on } (t_m, T], \tag{2.8}$$

where

$$a_{m+1} = \Big\{\rho_1 + \|\widetilde{h}\|_1\Big\}_{\phi^{-1}}.$$

Furthermore, due to (1.3), we have $|u'(0)| < a_{m+1}$ which together with (2.4) yields that (2.6) is true with

$$a_1 = \Big\{\Big\{a_{m+1}\Big\}_{\phi} + \|\widetilde{h}\|_1\Big\}_{\phi^{-1}}.$$

Now, proceeding as in the case $j = 1$, we show that (2.7) is true also in the case $j = m + 1$.

Assume that $1 < j < m + 1$. Then (2.2) and (2.4) yield

$$|u'(t)| < a_{j+1} = \Big\{\rho_1 + \|\widetilde{h}\|_1\Big\}_{\phi^{-1}} \quad \text{on } (t_j, t_{j+1}].$$

If $j < m$, then

$$|u'(t)| < a_{j+2} = \Big\{\Big\{b_{j+1}(a_{j+1})\Big\}_{\phi} + \|\widetilde{h}\|_1\Big\}_{\phi^{-1}} \quad \text{on } (t_{j+1}, t_{j+2}],$$

where $b_{j+1}(a_{j+1}) > a_{j+1}$. Proceeding by induction we get (2.8) with

$$a_{m+1} = \Big\{\Big\{b_m(a_m)\Big\}_{\phi} + \|\widetilde{h}\|_1\Big\}_{\phi^{-1}}$$

and $b_m(a_m) > a_m$, wherefrom (2.7) again follows as in the previous case. $\square$

Remaining a priori estimates can be taken from [15] and [16] without any change:

2.2. Lemma. ([15, Lemma 2.2].) *Let $\rho_0, d, q \in (0,\infty)$ and $J_i \in \mathbb{C}(\mathbb{R})$, $i = 1,2,\dots,m$. Then there exists $c \in (\rho_0,\infty)$ such that the estimate*

$$\|u\|_\infty < c \tag{2.9}$$

is valid for each $u \in \mathbb{C}_{\mathrm{D}}$ and each $\widetilde{J}_i \in \mathbb{C}(\mathbb{R})$, $i = 1,2,\dots,m$, satisfying (1.3), (2.1),

$$u(t_i+) = \widetilde{J}_i(u(t_i)), \quad i = 1,2,\dots,m, \tag{2.10}$$

$$|u(\tau_u)| < \rho_0 \quad \textit{for some} \;\; \tau_u \in [0,T] \tag{2.11}$$

and

$$\sup\{|J_i(x)| : |x| < a\} < b \implies \sup\{|\widetilde{J}_i(x)| : |x| < a\} < b \quad \textit{for} \;\; i = 1,2,\dots,m, \; a \in (0,\infty), \; b \in (a+q,\infty). \tag{2.12}$$

2.3. Lemma. ([15, Lemma 2.3].) *Assume that $\sigma_1, \sigma_2 \in \mathbb{AC}^1_{\mathrm{D}}$, J_i, M_i, $\widetilde{J}_i$, $\widetilde{M}_i \in \mathbb{C}(\mathbb{R})$, $i = 1,2,\dots,m$, satisfy* (1.7), (1.8),

$$\begin{cases} x > \sigma_1(t_i) \implies \widetilde{J}_i(x) > \widetilde{J}_i(\sigma_1(t_i)) = J_i(\sigma_1(t_i)), \\ x < \sigma_2(t_i) \implies \widetilde{J}_i(x) < \widetilde{J}_i(\sigma_2(t_i)) = J_i(\sigma_2(t_i)), \quad i = 1,2,\dots,m \end{cases} \tag{2.13}$$

and

$$\begin{cases} y \le \sigma_1'(t_i) \implies \widetilde{M}_i(y) \le M_i(\sigma_1'(t_i)), \\ y \ge \sigma_2'(t_i) \implies \widetilde{M}_i(y) \ge M_i(\sigma_2'(t_i)), \quad i = 1,2,\dots,m. \end{cases} \tag{2.14}$$

Define

$$B = \{u \in \mathbb{C}_{\mathrm{D}} : u \textit{ satisfies } (1.3), (2.10), (2.3) \textit{ and one of the conditions } (2.16), (2.17), (2.18)\}, \tag{2.15}$$

where

$$u(s_u) < \sigma_1(s_u) \quad \textit{and} \quad u(t_u) > \sigma_2(t_u) \quad \textit{for some} \;\; s_u, t_u \in [0,T], \tag{2.16}$$

$$u \ge \sigma_1 \;\; \textit{on} \;\; [0,T] \quad \textit{and} \quad \inf_{t\in[0,T]} |u(t) - \sigma_1(t)| = 0, \tag{2.17}$$

$$u \le \sigma_2 \;\; \textit{on} \;\; [0,T] \quad \textit{and} \quad \inf_{t\in[0,T]} |u(t) - \sigma_2(t)| = 0. \tag{2.18}$$

Then each function $u \in B$ satisfies

$$\begin{cases} |u'(\xi_u)| < \rho_1 \quad \textit{for some} \;\; \xi_u \in [0,T], \;\; \textit{where} \\ \rho_1 = \dfrac{2}{t_1}\left(\|\sigma_1\|_\infty + \|\sigma_2\|_\infty\right) + \|\sigma_1'\|_\infty + \|\sigma_2'\|_\infty + 1. \end{cases} \tag{2.19}$$

2.4. Lemma. ([14, Lemma 2.4].) *Assume that* $\sigma_1,\ \sigma_2 \in \mathbb{AC}^1_{\mathrm{D}},\ J_i,\ \widetilde{J}_i \in \mathbb{C}(\mathbb{R})$, $i = 1, 2, \dots, m$, *satisfy* (1.7) *and* (2.13). *Then*

$$\min\{\sigma_1(\tau_u+), \sigma_2(\tau_u+)\} \le u(\tau_u+) \le \max\{\sigma_1(\tau_u+), \sigma_2(\tau_u+)\} \quad \textit{for some} \ \ \tau_u \in [0, T) \tag{2.20}$$

is true for each $u \in \mathbb{C}_{\mathrm{D}}$ *fulfilling* (1.3), (2.10) *and one of the conditions* (2.16) – (2.18).

3. Main result

3.1. Theorem. *Assume that* (1.4), (1.6), (1.7) *and* (1.8) *hold and let* $h \in \mathbb{L}[0, T]$ *be such that*

$$|f(t, x, y)| \le h(t) \ \ \textit{for a.e.} \ t \in [0, T] \ \ \textit{and all} \ (x, y) \in \mathbb{R}^2. \tag{3.1}$$

Then the problem (1.1) –(1.3) *has a solution* u *satisfying one of the conditions* (2.16) – (2.18).

Proof. • Step 1. *We construct a proper auxiliary problem.*

Let σ_1 and σ_2 be respectively lower and upper functions of (1.1)–(1.3) and let ρ_1 be associated with them as in (2.19). Put

$$\widetilde{h}(t) = 2\,h(t) + 1 \ \text{ for a.e. } t \in [0, T] \text{ and } \ \widetilde{\rho} = \rho_1 + \sum_{i=1}^{m} \big(|M_i(\sigma_1'(t_i))| + |M_i(\sigma_2'(t_i))|\big).$$

By Lemma 2.1, find $d \in (\widetilde{\rho}, \infty)$ satisfying (2.1). Furthermore, put $\rho_0 = \|\sigma_1\|_\infty + \|\sigma_2\|_\infty + 1$ and

$$q = \frac{T}{m} \max\left\{ \Big(\sum_{i=1}^{m} \max_{|y| \le d+1} |M_i(y)|\Big), d + 1 \right\} \tag{3.2}$$

and, by Lemma 2.2, find $c \in (\rho_0 + q, \infty)$ fulfilling (2.9). In particular, we have

$$c > \|\sigma_1\|_\infty + \|\sigma_2\|_\infty + q + 1, \quad d > \|\sigma_1'\|_\infty + \|\sigma_2'\|_\infty + 1. \tag{3.3}$$

Finally, for a.e. $t \in [0, T]$ and all $x, y \in \mathbb{R}$ and $i = 1, 2, \dots, m$, define functions

$$\widetilde{f}(t, x, y) = \begin{cases} f(t, x, y) - h(t) - 1 & \text{if } x \le -c - 1, \\ f(t, x, y) + (x + c)\,(h(t) + 1) & \text{if } -c - 1 < x < -c, \\ f(t, x, y) & \text{if } -c \le x \le c, \\ f(t, x, y) + (x - c)\,(h(t) + 1) & \text{if } c < x < c + 1, \\ f(t, x, y) + h(t) + 1 & \text{if } x \ge c + 1, \end{cases} \tag{3.4}$$

$$
(3.5)\qquad \widetilde{J}_i(x)=\begin{cases} x+q & \text{if } x\le -c-1,\\ J_i(-c)\,(c+1+x)-(x+q)\,(x+c) & \text{if } -c-1<x<-c,\\ J_i(x) & \text{if } -c\le x\le c,\\ J_i(c)\,(c+1-x)+(x-q)\,(x-c) & \text{if } c<x<c+1,\\ x-q & \text{if } x\ge c+1,\end{cases}
$$

$$
(3.6)\qquad \widetilde{M}_i(y)=\begin{cases} y & \text{if } y\le -d-1,\\ M_i(-d)\,(d+1+y)-y\,(y+d) & \text{if } -d-1<y<-d,\\ M_i(y) & \text{if } -d\le y\le d,\\ M_i(d)\,(d+1-y)+y\,(y-d) & \text{if } d<y<d+1,\\ y & \text{if } y\ge d+1\end{cases}
$$

and consider the auxiliary problem

$$
(3.7)\qquad (\phi(u'))'=\widetilde{f}(t,u,u'),\quad (2.10),\quad (2.3),\quad (1.3).
$$

Due to (1.6), $\widetilde{f}\in \mathrm{Car}([0,T]\times\mathbb{R})$, $\widetilde{J}_i$, $\widetilde{M}_i\in\mathbb{C}(\mathbb{R})$ for $i=1,2,\dots,m$, and, as in the proof of [15, Theorem 3.1], they satisfy the assumptions of Lemmas 2.1–2.4. According to (3.3)–(3.6) the functions σ_1 and σ_2 are respectively lower and upper functions of (3.7). By (3.1) we have

$$
(3.8)\qquad |\widetilde{f}(t,x,y)|\le \widetilde{h}(t)\ \text{ for a.e. } t\in[0,T] \text{ and all } (x,y)\in\mathbb{R}^2
$$

and

$$
(3.9)\qquad \begin{cases} \widetilde{f}(t,x,y)<0 & \text{for a.e. } t\in[0,T] \text{ and all } (x,y)\in(-\infty,-c-1]\times\mathbb{R},\\ \widetilde{f}(t,x,y)>0 & \text{for a.e. } t\in[0,T] \text{ and all } (x,y)\in[c+1,\infty)\times\mathbb{R}.\end{cases}
$$

• Step 2. *We construct a well-ordered pair of "big" lower/upper functions for* (3.7). Put

$$
(3.10)\qquad A^*=q+\sum_{i=1}^{m}\max_{|x|\le c+1}|\widetilde{J}_i(x)|
$$

and

$$
(3.11)\qquad \begin{cases} \sigma_4(0)=A^*+m\,q,\\ \sigma_4(t)=A^*+(m-i)\,q+\frac{m\,q}{T}\,t \ \text{ for } t\in(t_i,t_{i+1}],\ i=0,1,\dots,m,\\ \sigma_3(t)=-\sigma_4(t) \ \text{ for } t\in[0,T].\end{cases}
$$

Then $\sigma_3,\sigma_4\in\mathbb{AC}^1_{\mathrm{D}}$ and, by (3.5) and (3.10),

$$
(3.12)\qquad \sigma_3(t)<-A^*<-c-1,\quad \sigma_4(t)>A^*>c+1 \ \text{ for } t\in[0,T].
$$

In view of (3.2),

$$\sigma_3'(t) = -\frac{m\,q}{T} \le -(d+1) \quad \text{and} \quad \sigma_4'(t) = \frac{m\,q}{T} \ge d+1 \ \text{ for } t \in [0,T]. \tag{3.13}$$

Furthermore, by (3.5) and (3.9), (3.11), (3.12), we have

$$\sigma_4(t_i+) = A^* + (m-i)\,q + \frac{m\,q}{T}t_i = \sigma_4(t_i) - q = \widetilde{J}_i(\sigma_4(t_i))$$

and

$$0 = (\phi(\sigma_4'(t)))' < \widetilde{f}(t, \sigma_4(t), \sigma_4'(t)) \ \text{ for a.e. } t \in [0,T],$$

respectively. Moreover, $\sigma_4(0) = A^* + m\,q = \sigma_4(T)$ and $\sigma_4'(0) = \frac{m\,q}{T} = \sigma_4'(T)$ and, by virtue of (3.2) and (3.6),

$$\sigma_4'(t_i+) = \frac{m\,q}{T} = \sigma_4'(t_i) = \widetilde{M}_i(\sigma_4'(t_i)) \ \text{ for } i = 1, 2, \dots, m,$$

i.e. σ_4 is an upper function of (3.7). Finally, since $\sigma_3 = -\sigma_4$, we see that σ_3 is a lower function of (3.7).

Clearly,

$$\sigma_3 < \sigma_4 \ \text{ on } [0,T] \quad \text{and} \quad \sigma_3(\tau+) < \sigma_4(\tau+) \ \text{ for } \tau \in \mathrm{D}\,. \tag{3.14}$$

Having a from (1.10), let us define for $x \in \mathbb{C}^1_{\mathrm{D}}$ and $t \in [0,T]$

$$(\widetilde{\mathcal{N}}(x))(t) = \int_0^t \widetilde{f}(s, x(s), x'(s))\,\mathrm{d}s + \sum_{i=1}^m \big[\phi\big(\widetilde{M}_i(x'(t_i))\big) - \phi\big(x'(t_i)\big)\big]\,\chi_{(t_i,\,T]}(t),$$

$$(\widetilde{\mathcal{J}}(x))(t) = \sum_{i=1}^m \big[\widetilde{J}_i(x(t_i)) - x(t_i)\big]\,\chi_{(t_i,\,T]}(t)$$

and

$$\left\{ \begin{aligned} (\widetilde{\mathcal{F}}(x))(t) = {} & \int_0^t \phi^{-1}\Big(a\big(\widetilde{\mathcal{N}}(x), (\widetilde{\mathcal{J}}(x))(T)\big) + (\widetilde{\mathcal{N}}(x))(s)\Big)\,\mathrm{d}s \\ & + x(0) + x'(0) - x'(T) + (\widetilde{\mathcal{J}}(x))(t). \end{aligned} \right. \tag{3.15}$$

By [16, Theorem 3.5], $\widetilde{\mathcal{F}} : \mathbb{C}^1_{\mathrm{D}} \mapsto \mathbb{C}^1_{\mathrm{D}}$ is completely continuous and u is a solution of (3.7) whenever $\widetilde{\mathcal{F}}u = u$.

- Step 3. *We prove the first a priori estimate for solutions of* (3.7).
  Define

$$\Omega_0 = \{u \in \mathbb{C}_{\mathrm{D}} : \|u'\|_\infty < C^*,\ \sigma_3 < u < \sigma_4 \text{ on } [0,T], \\ \sigma_3(\tau+) < u(\tau+) < \sigma_4(\tau+) \text{ for } \tau \in \mathrm{D}\}, \tag{3.16}$$

where

$$C^* = 1 + \left\{ \|\widetilde{h}\|_1 + \left\{ \frac{\|\sigma_3\|_\infty + \|\sigma_4\|_\infty}{\Delta} \right\}_\phi \right\}_{\phi^{-1}} \tag{3.17}$$

and Δ is defined in (1.19). We are going to prove that for each solution u of (3.7) the estimate

$$u \in \mathrm{cl}(\Omega_0) \implies u \in \Omega_0 \tag{3.18}$$

is true. To this aim, suppose that u is a solution of (3.7) and $u \in \mathrm{cl}(\Omega_0)$, i.e. $\|u'\|_\infty \le C^*$ and

$$\sigma_3 \le u \le \sigma_4 \text{ on } [0,T]. \tag{3.19}$$

By the Mean Value Theorem, there are $\xi_i \in (t_i, t_{i+1})$, $i = 1, 2, \dots, m$, such that $|u'(\xi_i)| \le (\|\sigma_3\|_\infty + \|\sigma_4\|_\infty)/\Delta$. Hence, by (3.8), we get

$$\|u'\|_\infty < C^*, \tag{3.20}$$

where C^* is defined in (3.17). It remains to show that $\sigma_3 < u < \sigma_4$ on $[0,T]$ and $\sigma_3(\tau+) < u(\tau+) < \sigma_4(\tau+)$ for $\tau \in \mathrm{D}$. Assume the contrary. Then there exists $k \in \{3,4\}$ such that

$$u(\xi) = \sigma_k(\xi) \qquad \text{for some } \xi \in [0,T] \tag{3.21}$$

or

$$u(t_i+) = \sigma_k(t_i+) \quad \text{for some } t_i \in \mathrm{D}\,. \tag{3.22}$$

Case A. Let (3.21) hold for $k = 4$.

(i) If $\xi = 0$, then $u(0) = \sigma_4(0) = \sigma_4(T) = u(T) = A^* + q\,m$ which gives, in view of (1.3), (3.13) and (3.19),

$$u'(0) = u'(T) = \frac{m\,q}{T} = \sigma_4'(t) \text{ for } t \in [0,T].$$

Further, due to (3.9) and (3.12), we can find $\delta > 0$ such that $u > c+1$ on $[0,\delta]$ and

$$\phi(u'(t)) - \phi(u'(0)) = \int_0^t \widetilde{f}(s, u(s), u'(s))\,\mathrm{d}s > 0 \text{ for } t \in [0,\delta].$$

Hence $u'(t) > u'(0) = \sigma_4'(t)$ on $(0,\delta]$ which implies that $u > \sigma_4$ on $(0,\delta]$, contrary to (3.19).

(ii) If $\xi \in (t_i, t_{i+1})$ for some $t_i \in \mathrm{D}$, then $u'(\xi) = \sigma_4'(\xi) = \frac{mq}{T} = \sigma_4'(t)$ for $t \in [0,T]$ and we reach a contradiction as above.

(iii) If $\xi = t_i \in \mathrm{D}$, then $u(t_i) = \sigma_4(t_i)$ and, by (3.5), (3.12) and (3.3),

$$u(t_i+) = \sigma_4(t_i+) = \sigma_4(t_i) - q > c + 1 - q > \|\sigma_1\|_\infty + \|\sigma_2\|_\infty.$$

By virtue of (3.19) we have $u'(t_i+) \le \sigma_4'(t_i+)$ and $u'(t_i) \ge \sigma_4'(t_i)$. Now, since the last inequality together with (3.6) and (3.13) yield $u'(t_i+) \ge \sigma_4'(t_i+)$, we get $u'(t_i+) = \sigma_4'(t_i+) = \frac{mq}{T} = \sigma_4'(t)$ for $t \in [0,T]$. Similarly as above, this leads again to a contradiction.

Case B. Let (3.22) hold for $k = 4$, i.e. $u(t_i+) = \sigma_4(t_i+)$. By (3.5) and (3.12), $\widetilde{J}_i(u(t_i)) = \sigma_4(t_i+) = \sigma_4(t_i) - q > A^* - q$, wherefrom, with respect to (3.10), we get $u(t_i) > c+1$ and hence $\widetilde{J}_i(u(t_i)) = u(t_i) - q$. Therefore $u(t_i) = \sigma_4(t_i)$ and we can continue as in Case A (iii).

If (3.21) or (3.22) hold for $k = 3$, then we use analogical arguments as in Case A or Case B.

- Step 4. *We prove the second a priori estimate for solutions of* (3.7).
  Define sets

$$\Omega_1 = \{u \in \Omega_0 : \, u(t) > \sigma_1(t) \ \text{ for } \ t \in [0,T], \, u(\tau+) > \sigma_1(\tau+) \ \text{ for } \ \tau \in \mathrm{D}\},$$

$$\Omega_2 = \{u \in \Omega_0 : \, u(t) < \sigma_2(t) \ \text{ for } \ t \in [0,T], \, u(\tau+) < \sigma_2(\tau+) \ \text{ for } \ \tau \in \mathrm{D}\}$$

and $\widetilde{\Omega} = \Omega_0 \setminus \mathrm{cl}(\Omega_1 \cup \Omega_2)$. Then

$$\widetilde{\Omega} = \{u \in \Omega_0 : \, u \text{ satisfies (2.16)}\} \tag{3.23}$$

and, due to (1.18) and (3.16),

$$\Omega_0 = \Omega(\sigma_3, \sigma_4, C^*), \ \ \Omega_1 = \Omega(\sigma_1, \sigma_4, C^*) \ \text{ and } \ \Omega_2 = \Omega(\sigma_3, \sigma_2, C^*).$$

Moreover, by (1.6), we have $\Omega_1 \cap \Omega_2 = \emptyset$.

Consider c from Step 1. We will show that the estimates

$$u \in \mathrm{cl}(\widetilde{\Omega}) \implies \|u\|_\infty < c, \quad \|u'\|_\infty < d \tag{3.24}$$

are valid for each solution u of (3.7). Indeed, let u be a solution of (3.7) and let $u \in \mathrm{cl}(\widetilde{\Omega})$. Then $u \in B$, due to (3.18) and (2.15), and u satisfies (2.2)–(2.4). We have already noticed that $\widetilde{f}$, $\widetilde{J}_i$ and $\widetilde{M}_i$, $i = 1, 2, \ldots, m$, satisfy the corresponding assumptions of Lemmas 2.1–2.4. So, by Lemma 2.3, there is $\xi_u \in [0,T]$ such that (2.19) holds and by Lemma 2.1 the estimate (2.1) is true. Further, by Lemma 2.4,

u satisfies (2.11) with ρ_0 defined in Step 1. Finally, by Lemma 2.2, we have (2.9), i.e. each solution u of (3.7) satisfies (3.24).

- Step 5. *We prove the existence of a solution to the problem* (1.1)–(1.3).

Consider the operator $\widetilde{\mathcal{F}}$ defined by (3.15). We distinguish two cases: either $\widetilde{\mathcal{F}}$ has a fixed point in $\partial\widetilde{\Omega}$ or it has no fixed point in $\partial\widetilde{\Omega}$.

Assume that $\widetilde{\mathcal{F}}u = u$ for some $u \in \partial\widetilde{\Omega}$. Then u is a solution of (3.7) and, with respect to (3.24), we have $\|u\|_\infty < c$, $\|u'\|_\infty < d$, which means, by (3.4)–(3.6), that u is a solution of (1.1)–(1.3). Furthermore, due to (3.18), u satisfies (2.17) or (2.18).

Now, assume that $\widetilde{\mathcal{F}}u \neq u$ for all $u \in \partial\widetilde{\Omega}$. Then $\widetilde{\mathcal{F}}u \neq u$ for all $u \in \partial\Omega_0 \cup \partial\Omega_1 \cup \partial\Omega_2$. If we replace f, h, J_i, M_i, $i = 1, 2, \ldots, m$, α, β and γ respectively by $\widetilde{f}$, $\widetilde{h}$, $\widetilde{J}_i$, $\widetilde{M}_i$, $i = 1, 2, \ldots, m$, σ_3, σ_4 and C^* in Proposition 1.1, we see that the assumptions (1.14)–(1.17) and (1.19) are satisfied. Thus, by Proposition 1.1, we obtain that

$$\deg(\mathrm{I} - \widetilde{\mathcal{F}}, \Omega(\sigma_3, \sigma_4, C^*)) = \deg(\mathrm{I} - \widetilde{\mathcal{F}}, \Omega_0) = 1. \tag{3.25}$$

Similarly, we can apply Proposition 1.1 to show that

$$\deg(\mathrm{I} - \widetilde{\mathcal{F}}, \Omega(\sigma_1, \sigma_4, C^*)) = \deg(\mathrm{I} - \widetilde{\mathcal{F}}, \Omega_1) = 1 \tag{3.26}$$

and

$$\deg(\mathrm{I} - \widetilde{\mathcal{F}}, \Omega(\sigma_3, \sigma_2, C^*)) = \deg(\mathrm{I} - \widetilde{\mathcal{F}}, \Omega_2) = 1. \tag{3.27}$$

Using the additivity property of the Leray-Schauder topological degree we derive from (3.25)–(3.27) that

$$\deg(\mathrm{I} - \widetilde{\mathcal{F}}, \widetilde{\Omega}) = \deg(\mathrm{I} - \widetilde{\mathcal{F}}, \Omega_0) - \deg(\mathrm{I} - \widetilde{\mathcal{F}}, \Omega_1) - \deg(\mathrm{I} - \widetilde{\mathcal{F}}, \Omega_2) = -1.$$

Therefore, $\widetilde{\mathcal{F}}$ has a fixed point $u \in \widetilde{\Omega}$. By (3.24) we have $\|u\|_\infty < c$ and $\|u'\|_\infty < d$. This together with (3.4)–(3.6) and (3.23) yields that u is a solution to (1.1)–(1.3) fulfilling (2.16). □

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Irena Rachůnková, Department of Mathematics, Palacký University, 779 00 OLOMOUC, Tomkova 40, Czech Republic (e-mail: rachunko@inf.upol.cz)

Milan Tvrdý, Mathematical Institute, Academy of Sciences of the Czech Republic, 115 67 PRAHA 1, Žitná 25, Czech Republic (e-mail: tvrdy@math.cas.cz)