Second Order Periodic Problem with ϕ-Laplacian and Impulses

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Abstract. Existence principles for the BVP $(\phi(u'))' = f(t,u,u')$, $u(t_i+) = J_i(u(t_i))$, $u'(t_i+) = M_i(u'(t_i))$, $i = 1,2,\dots,m$, $u(0) = u(T)$, $u'(0) = u'(T)$ are presented. They are based on the method of lower/upper functions which need not be well-ordered.

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1. Formulation of the problem

Let $m \in \mathbb{N}$, $0 = t_0 < t_1 < \dots < t_m < t_{m+1} = T$ and $\mathrm{D} = \{t_1, t_2, \dots, t_m\}$. Define $\mathbb{C}_\mathrm{D}$ (or $\mathbb{C}^1_\mathrm{D}$) as the sets of functions $u : [0,T] \mapsto \mathbb{R}$,

$$u(t) = \begin{cases} u_{[0]}(t) & \text{if } t \in [0,t_1], \\ u_{[1]}(t) & \text{if } t \in (t_1,t_2], \\ \dots & \quad\dots \\ u_{[m]}(t) & \text{if } t \in (t_m,T], \end{cases}$$

where $u_{[i]}$ is continuous on $[t_i, t_{i+1}]$ (or continuously differentiable on $[t_i, t_{i+1}]$) for $i = 0,1,\dots,m$. We put $\|u\|_\mathrm{D} = \|u\|_\infty + \|u'\|_\infty$, where $\|u\|_\infty = \sup \mathrm{ess}_{t\in[0,T]}\, |u(t)|$. Then $\mathbb{C}_\mathrm{D}$ and $\mathbb{C}^1_\mathrm{D}$ respectively with the norms $\|.\|_\infty$ and $\|.\|_\mathrm{D}$ become Banach spaces. Further, $\mathbb{AC}_\mathrm{D}$ is the set of functions $u \in \mathbb{C}_\mathrm{D}$ which are absolutely continuous on each subinterval (t_i, t_{i+1}), $i = 0,1,\dots,m$.

We consider the problem

$$(\phi(u'(t)))' = f(t,u(t),u'(t)) \quad \text{a.e. on } [0,T], \tag{1.1}$$

$$u(t_i+) = J_i(u(t_i)), \quad u'(t_i+) = M_i(u'(t_i)), \quad i = 1,2,\dots,m, \tag{1.2}$$

$$u(0) = u(T), \quad u'(0) = u'(T), \tag{1.3}$$

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where $u'(t_i) = u'(t_i-) = \lim_{t \to t_i-} u'(t)$ for $i = 1, 2, \dots, m+1$, $u'(0) = u'(0+) = \lim_{t \to 0+} u'(t)$, f is an $\mathbb{L}_1$-Carathéodory function, functions J_i, M_i are continuous on $\mathbb{R}$ and ϕ is an increasing homeomorphism such that $\phi(0) = 0$ and $\phi(\mathbb{R}) = \mathbb{R}$. A typical example of a proper function ϕ is the p-Laplacian $\phi_p(y) = |y|^{p-2}\, y$, where $p > 1$.

A *solution* of the problem (1.1)–(1.3) is a function $u \in \mathbb{C}^1_{\mathrm{D}}$ such that $\phi(u') \in \mathbb{AC}_{\mathrm{D}}$ and (1.1)–(1.3) hold.

A function $\sigma \in \mathbb{C}^1_{\mathrm{D}}$ is called a *lower function* of (1.1)–(1.3) if $\phi(\sigma') \in \mathbb{AC}_{\mathrm{D}}$ and

$$\tag{1.4} \begin{cases} \phi(\sigma'(t))' \ge f(t, \sigma(t), \sigma'(t)) \quad \text{for a.e.} \quad t \in [0, T], \\ \sigma(t_i+) = J_i(\sigma(t_i)), \ \sigma'(t_i+) \ge M_i(\sigma'(t_i)), \ i = 1, 2, \dots, m, \\ \sigma(0) = \sigma(T), \ \sigma'(0) \ge \sigma'(T). \end{cases}$$

Similarly, a function $\sigma \in \mathbb{C}^1_{\mathrm{D}}$ with $\phi(\sigma') \in \mathbb{AC}_{\mathrm{D}}$ is an *upper function* of (1.1)–(1.3) if it satisfies the relations (1.4) but with reversed inequalities.

The aim of this paper is to offer existence principles for problem (1.1)–(1.3) in terms of lower/upper functions. Hence our basic assumption is the existence of lower/upper functions. We will suppose that either

(1.5) $\quad \sigma_1$ and σ_2 are respectively lower and upper functions of (1.1)–(1.3) such that $\sigma_1 \le \sigma_2$ on $[0, T]$

or

(1.6) $\quad \sigma_1$ and σ_2 are respectively lower and upper functions of (1.1)–(1.3) such that $\sigma_1 \not\le \sigma_2$ on $[0, T]$, i.e. $\sigma_1(\tau) > \sigma_2(\tau)$ for some $\tau \in [0, T]$.

Note that problems with ϕ-Laplacians and impulses have not been studied yet. As concerns problem (1.1), (1.3) (without impulses), there are various results about its solvability. For example the papers [4] and [19] present some results about the existence or multiplicity of periodic solutions of the equation

$$\tag{1.7} (\phi_p(u'))' = f(t, u)$$

under non resonance conditions imposed on f. The paper [10] presents general existence principles for the vector problem (1.1), (1.3). Using this the authors provide various existence theorems and illustrative examples. The vector case is also considered in [9], [11] and [12]. The existence of periodic solutions of the Liénard type equations with p-Laplacians has been proved in the resonance case under the Landesman-Lazer conditions in [5] and [6]. Multiplicity results of the Ambrosetti-Prodi type for this problem (with a real parameter) can be found in [8].

The papers which are devoted to the lower/upper functions method for the problem (1.1), (1.3) mostly deal with the condition (1.5), i.e. they assume well-ordered

σ_1/σ_2. We can refer to the papers [1] and [3] which study the problem (1.1), (1.3) under the Nagumo type two-sided growth conditions and to the paper [17] where the second order equation with a ϕ-Laplacian is considered provided a functional right-hand side of this equation fulfils one-sided growth conditions of the Nagumo type. The significance of the lower/upper functions method is shown in the papers [7] and [18] where this method is used in the investigation of singular periodic problems with a ϕ-Laplacian. The paper [2] is, to our knowledge, the only one presenting the lower/upper functions method for the problem (1.7), (1.3) (with a ϕ-Laplacian) under the assumption that $\sigma_1 \geq \sigma_2$, i.e. lower/upper functions are in the reverse order. If $\phi = \phi_p$ the authors get the solvability of (1.7), (1.3) for $1 < p \leq 2$, only. Therefore the existence principles (Theorems 2.3 and 2.4) which we state here for the impulsive problem (1.1)–(1.3) and the case (1.6) are new even for the non-impulsive problem (1.1), (1.3).

We will work with the following assumptions, where the sets A_i, $B(t) \subset \mathbb{R}$, $t \in [0,T]$, will be determined later, according to whether (1.5) or (1.6) is assumed:

$$\left\{\begin{array}{l} x > \sigma_1(t_i) \implies J_i(x) > J_i(\sigma_1(t_i)) \\ x < \sigma_2(t_i) \implies J_i(x) < J_i(\sigma_2(t_i)) \end{array}\right. \quad \text{for } x \in A_i,\ i = 1,2,\dots,m; \tag{1.8}$$

$$\left\{\begin{array}{l} y \leq \sigma_1'(t_i) \implies M_i(y) \leq M_i(\sigma_1'(t_i)), \\ y \geq \sigma_2'(t_i) \implies M_i(y) \geq M_i(\sigma_2'(t_i)), \end{array}\right. \quad i = 1,2,\dots,m; \tag{1.9}$$

$$\left\{\begin{array}{l} \text{there is } h \in \mathbb{L}_1 \text{ such that} \\ |f(t,x,y)| \leq h(t) \ \text{ for a.e. } t \in [0,T] \ \text{ and all } x, y \in \mathbb{R}; \end{array}\right. \tag{1.10}$$

$$\left\{\begin{array}{l} \text{there are } \omega : [0,\infty) \mapsto (0,\infty) \text{ continuous and } h \in \mathbb{L}_1 \text{ such that} \\ \displaystyle\int_0^\infty \frac{\mathrm{d}\, s}{\omega(s)} = \infty \text{ and } |f(t,x,y)| \leq \omega(\phi(|y|))\,(|y| + h(t)) \\ \text{for a.e. } t \in [0,T],\ \text{all } x \in B(t) \text{ and } |y| \geq 1, \end{array}\right. \tag{1.11}$$

$$\left\{\begin{array}{l} \text{there are } c_j,\ d_j \in \mathbb{R},\ c_j \leq \sigma_k'(t) \leq d_j \ \text{ on } (t_{j-1}, t_j],\ k = 1,2, \\ \text{such that } f(t,x,c_j) \leq 0,\ f(t,x,d_j) \geq 0 \ \text{ for a.e. } t \in (t_{j-1}, t_j] \\ \text{and all } x \in B(t),\ j = 1,2,\dots,m+1, \text{ and } c_1 \geq c_{m+1}, d_1 \leq d_{m+1}, \\ M_i(c_i) \leq c_{i+1},\ M_i(d_i) \geq d_{i+1},\ i = 1,2,\dots,m. \end{array}\right. \tag{1.12}$$

2. Main results

Below we formulate our main results:

I. Existence principles for well-ordered case

2.1. Theorem. *Assume that* (1.5), (1.8) *with* $A_i = [\sigma_1(t_i), \sigma_2(t_i)]$, $i = 1,2,\dots,m$, (1.9) *and* (1.11) *with* $B(t) = [\sigma_1(t), \sigma_2(t)]$ *hold. Then the problem* (1.1) – (1.3) *has* *a solution u satisfying*

$$\sigma_1 \le u \le \sigma_2 \quad on \ [0,T]. \tag{2.1}$$

2.2. Theorem. *Assume that* (1.5), (1.8) *with* $A_i = [\sigma_1(t_i), \sigma_2(t_i)]$, $i = 1,2,\dots,m$, (1.9) *and* (1.12) *with* $B(t) = [\sigma_1(t), \sigma_2(t)]$ *hold.*

Then the problem (1.1) − (1.3) *has a solution u satisfying* (2.1) *and*

$$c_j \le u'(t) \le d_j \quad for \ t \in (t_{j-1}, t_j], \ j = 1,2,\dots,m+1. \tag{2.2}$$

II. Existence principles for non-ordered case

2.3. Theorem. *Assume that* (1.6), (1.8) *with* $A_i = \mathbb{R}$, $i = 1,2,\dots,m$, (1.9) *and* (1.10) *hold. Then the problem* (1.1) − (1.3) *has a solution u satisfying*

$$|u(t_u)| \le \max\{|\sigma_1(t_u)|, |\sigma_2(t_u)|\} \quad for\ some \ \ t_u \in [0,T]. \tag{2.3}$$

2.4. Theorem. *Assume that* (1.6), (1.8) *with* $A_i = \mathbb{R}$, $i = 1,2,\dots,m$, (1.9) *and* (1.12) *with* $B(t) = \mathbb{R}$ *hold. Then the problem* (1.1) − (1.3) *has a solution u satisfying* (2.2) *and* (2.3).

Note that Theorems 2.2 and 2.4 impose no growth restrictions on f. For example, taking $f(t,x,y) = y\,(y^{2k}x^{2n}+1) - x^{2n-1} + e(t)$, where $e \in \mathbb{C}_{\mathrm{D}}$, $k, n \in \mathbb{N}$, we can check that there are $c_j \in (-\infty, 0)$ $d_j \in (0,\infty)$, $j = 1,2,\dots,m+1$, such that $c_1 \ge c_{m+1}$, $d_1 \le d_{m+1}$, $f(t,x,c_j) \le 0$ and $f(t,x,d_j) \ge 0$ for a.e. $t \in (t_{j-1}, t_j]$ and all $x \in \mathbb{R}$, $j = 1,2,\dots,m+1$.

3. A fixed point operator

We will transform the problem (1.1)–(1.3) into a fixed point problem in $\mathbb{C}^1_{\mathrm{D}}$. First, we borrow some ideas from [10] to get the following two lemmas.

3.1. Lemma. *For each* $\ell \in \mathbb{C}_{\mathrm{D}}$ *and* $d \in \mathbb{R}$, *the function*

$$\Psi_{\ell,d} : \mathbb{R} \mapsto \mathbb{R}, \quad \Psi_{\ell,d}(a) = d + \int_0^T \phi^{-1}\big(a + \ell(t)\big)\,\mathrm{d}\,t$$

has exactly one zero point $a(\ell,d)$ *in* $\mathbb{R}$.

Proof. Choose $\ell \in \mathbb{C}_{\mathrm{D}}$ and $d \in \mathbb{R}$. Since $\Psi_{\ell,d}$ is continuous, increasing on $\mathbb{R}$ and $\Psi_{\ell,d}(\mathbb{R}) = \mathbb{R}$, there is a unique real number $a(\ell,d)$ such that

$$\Psi_{\ell,d}(a(\ell,d)) = 0. \tag{3.1}$$

□

3.2. Lemma. *The mapping $a : \mathbb{C}_{\mathrm{D}} \times \mathbb{R} \mapsto \mathbb{R}$ defined by* (3.1) *is continuous and maps bounded sets into bounded sets.* [1]

Proof. (i) Assume that $\mathcal{A} \subset \mathbb{C}_{\mathrm{D}} \times \mathbb{R}$ and $\gamma \in (0,\infty)$ are such that $\|\ell\|_\infty + |d| \le \gamma$ for each $(\ell,d) \in \mathcal{A}$ and that there is a sequence $\{a(\ell_n,d_n)\}_{n=1}^\infty \subset a(\mathcal{A})$ such that $\lim_{n\to\infty} a(\ell_n,d_n) = \infty$ or $\lim_{n\to\infty} a(\ell_n,d_n) = -\infty$. Let the former possibility occur. Then, by (3.1), we have $0 = \lim_{n\to\infty} \Psi_{\ell_n,d_n}(a(\ell_n,d_n)) \ge \lim_{n\to\infty}(-\gamma + T\phi^{-1}\big(a(\ell_n,d_n)-\gamma)\big) = \infty$, a contradiction. The latter possibility can be argued similarly.
(ii) Let $\lim_{n\to\infty}(\ell_n,d_n) = (\ell_0,d_0)$ in $\mathbb{C}_{\mathrm{D}} \times \mathbb{R}$. By (i) the sequence $\{a(\ell_n,d_n)\}_{n=1}^\infty$ is bounded and hence we can choose a subsequence such that $\lim_{n\to\infty} a(\ell_{k_n},d_{k_n}) = a_0 \in \mathbb{R}$. By (3.1), we get

$$0 = \Psi_{\ell_{k_n},d_{k_n}}(a(\ell_{k_n},d_{k_n})) = d_{k_n} + \int_0^T \phi^{-1}\big(a(\ell_{k_n},d_{k_n}) + \ell_{k_n}(t)\big)\,\mathrm{d}\,t,$$

which, for $n \to \infty$, yields

$$0 = d_0 + \int_0^T \phi^{-1}\big(a_0 + \ell_0(t)\big)\,\mathrm{d}\,t.$$

Thus, with respect to Lemma 3.1, we have $a_0 = a(\ell_0,d_0) = \lim_{n\to\infty} a(\ell_n,d_n)$. □

3.3. Lemma. *The operator $\mathcal{N} : \mathbb{C}^1_{\mathrm{D}} \mapsto \mathbb{C}_{\mathrm{D}}$ given by*

$$(\mathcal{N}(x))(t) = \int_0^t f(s,x(s),x'(s))\,\mathrm{d}\,s + \sum_{i=1}^m \big[\phi\big(M_i(x'(t_i))\big) - \phi\big(x'(t_i)\big)\big]\,\chi_{(t_i,T]}(t) \tag{3.2}$$

is absolutely continuous. [2]

Proof. The continuity of $\mathcal{N}$ follows from the continuity of all the mappings involved in the right-hand side of (3.2). Furthermore, let $\mathcal{H} \subset \mathbb{C}^1_{\mathrm{D}}$ be bounded. We need to show that the closure $\overline{\mathcal{N}(\mathcal{H})}$ of $\mathcal{N}(\mathcal{H})$ in $\mathbb{C}_{\mathrm{D}}$ is compact. To this aim, let $\|x\|_{\mathrm{D}} \le \gamma < \infty$ for each $x \in \mathcal{H}$. Then there are $c \in (0,\infty)$ and $h \in \mathbb{L}_1$ such that

$$\sum_{i=1}^m \big[\phi\big(M_i(x'(t_i))\big) - \phi\big(x'(t_i)\big)\big] \le c \quad \text{and} \quad |f(t,x(t),x'(t))| \le h(t) \quad \text{a.e. on } [0,T]$$

for all $x \in \mathcal{H}$. Therefore

$$\|\mathcal{N}(x)\|_\infty \le \|h\|_1 + c \quad \text{for each} \;\; x \in \mathcal{H}. \tag{3.3}$$

[1] The norm of $(\ell,d) \in \mathbb{C}_{\mathrm{D}} \times \mathbb{R}$ is defined by $\|\ell\|_\infty + |d|$.
[2] As usual, χ_M stands for the characteristic function of the set $M \subset \mathbb{R}$.

Put $(\mathcal{N}_1(x))(t) = \int_0^t f(s, x(s), x'(s))\, \mathrm{d}\, s$. Then, for $t_1,\ t_2 \in [0, T]$, we have

$$|(\mathcal{N}_1(x))(t_2) - (\mathcal{N}_1(x))(t_1)| \le \Big| \int_{t_1}^{t_2} h(s)\, \mathrm{d}\, s \Big|,$$

wherefrom, by (3.3), we deduce that the functions in $\mathcal{N}_1(\mathcal{H})$ are uniformly bounded and equicontinuous on $[0, T]$. Hence, making use of the Arzelà-Ascoli Theorem in $\mathbb{C}$ (the space of functions continuous on $[0, T]$ with the norm $\|.\|_\infty$), we get that each sequence in $\mathcal{N}_1(\mathcal{H})$ contains a subsequence convergent with respect to the norm $\|.\|_\infty$. This shows that $\overline{\mathcal{N}_1(\mathcal{H})}$ is compact in $\mathbb{C}_{\mathrm{D}}$. We know that the operator $\mathcal{N}_2 = \mathcal{N} - \mathcal{N}_1$ is continuous. By (3.3), it maps bounded sets into bounded sets. Moreover, its values are contained in an m-dimensional subspace of $\mathbb{C}_{\mathrm{D}}$. Thus, $\overline{\mathcal{N}_2(\mathcal{H})}$ is compact in $\mathbb{C}_{\mathrm{D}}$. $\square$

3.4. Theorem. *Let* $a : \mathbb{C}_{\mathrm{D}} \times \mathbb{R} \mapsto \mathbb{R}$ *and* $\mathcal{N} : \mathbb{C}^1_{\mathrm{D}} \mapsto \mathbb{C}_{\mathrm{D}}$ *be respectively defined by* (3.1) *and* (3.2). *Furthermore define* $\mathcal{J} : \mathbb{C}^1_{\mathrm{D}} \mapsto \mathbb{C}^1_{\mathrm{D}}$ *by*

$$(\mathcal{J}(x))(t) = \sum_{i=1}^{m} \big[J_i(x(t_i)) - x(t_i)\big]\, \chi_{(t_i,\, T]}(t) \tag{3.4}$$

and

$$\begin{aligned}(\mathcal{F}(x))(t) = \int_0^t \phi^{-1}\Big(a\big(\mathcal{N}(x), (\mathcal{J}(x))(T)\big) + (\mathcal{N}(x))(s)\Big)\, \mathrm{d}\, s \\ + x(0) + x'(0) - x'(T) + (\mathcal{J}(x))(t).\end{aligned} \tag{3.5}$$

Then $\mathcal{F} : \mathbb{C}^1_{\mathrm{D}} \mapsto \mathbb{C}^1_{\mathrm{D}}$ *is an absolutely continuous operator. Moreover,* u *is a solution of the problem* (1.1) $-$ (1.3) *if and only if* $\mathcal{F}(u) = u$.

Proof. For $x \in \mathbb{C}^1_{\mathrm{D}}$ and $t \in [0, T]$, we have

$$(\mathcal{F}(x))'(t) = \phi^{-1}\big(a(\mathcal{N}(x), (\mathcal{J}(x))(T)) + (\mathcal{N}(x))(t)\big). \tag{3.6}$$

Since the mappings a, $\mathcal{N}$ and $\mathcal{J}$ included in (3.5) and (3.6) are continuous, it follows that $\mathcal{F}$ is continuous in $\mathbb{C}^1_{\mathrm{D}}$.

Choose an arbitrary bounded set $\mathcal{H} \subset \mathbb{C}^1_{\mathrm{D}}$. We will show that then the set $\overline{\mathcal{F}(\mathcal{H})}$ is compact in $\mathbb{C}^1_{\mathrm{D}}$. Let a sequence $\{v_n\} \subset \mathcal{F}(\mathcal{H})$ be given. It suffices to show that it contains a subsequence convergent in $\mathbb{C}^1_{\mathrm{D}}$. Let $\{x_n\} \subset \mathcal{H}$ be such that $v_n = \mathcal{F}(x_n)$ for $n \in \mathbb{N}$. By Lemma 3.3, there is a subsequence $\{x_{k_n}\}$ such that $\{\mathcal{N}(x_{k_n})\}$ is convergent in $\mathbb{C}_{\mathrm{D}}$. According to (3.3) and (3.4), there exists $\gamma \in (0, \infty)$ such that $\|\mathcal{N}(x)\|_\infty + |(\mathcal{J}(x))(T)| \le \gamma$ for all $x \in \mathcal{H}$. Hence, by Lemma 3.2, the sequence $\{a(\mathcal{N}(x_{k_n}), (\mathcal{J}(x_{k_n}))(T))\} \subset \mathbb{R}$ is bounded and we can choose a subsequence $\{x_{\ell_n}\} \subset \{x_{k_n}\}$ such that $\{a\big(\mathcal{N}(x_{\ell_n}), (\mathcal{J}(x_{\ell_n}))(T)\big) + \mathcal{N}(x_{\ell_n})\}$ is convergent in $\mathbb{C}_{\mathrm{D}}$. Consequently, $\{(\mathcal{F}(x_{\ell_n}))'\}$ and $\{\mathcal{F}(x_{\ell_n})\}$ are convergent in $\mathbb{C}_{\mathrm{D}}$, as well. Finally, by a direct computation we check that (1.1)–(1.3) is equivalent to the problem $u = \mathcal{F}(u)$. For more details, see our preprint [15]. $\square$

4. Proofs of the main results

Sketch of the proof of Theorem 2.1. We can modify the arguments and constructions of [13], where the case $\phi(y) \equiv y$ is considered. By virtue of Theorem 3.4, the problem (1.1)–(1.3) has a solution if and only if the operator $\mathcal{F}$ which is defined by (3.5) has a fixed point. To prove it we argue as follows: (i) we construct an auxiliary operator $\widetilde{\mathcal{F}}$ and prove that its Leray-Schauder topological degree is nonzero and consequently $\widetilde{\mathcal{F}}$ has a fixed point u; (ii) using the method of a priori estimates we show that u is a fixed point of $\mathcal{F}$ satisfying (2.1). Since the realization of these ideas is quite close to the arguments of [13], we skip it. Detailed computation can be found in our preprint [15]. $\square$

Proof of Theorem 2.2. Step 1. Define

$$\beta_j(y) = \begin{cases} c_j & \text{for } y < c_j, \\ y & \text{for } c_j \le y \le d_j, \\ d_j & \text{for } y > d_j \end{cases} \qquad j = 1, 2, \dots, m+1; \tag{4.1}$$

$$\widetilde{f}(t,x,y) = f(t,x,\beta_j(y)) + \frac{y - \beta_j(y)}{|y - \beta_j(y)| + 1} \tag{4.2}$$
$$\text{for a.e. } t \in (t_{j-1}, t_j],\ x, y \in \mathbb{R},\ j = 1, 2, \dots, m+1;$$

and

$$\widetilde{M_i}(y) = M_i(\beta_i(y)) + \frac{y - \beta_j(y)}{|y - \beta_j(y)| + 1} \quad \text{for } y \in \mathbb{R},\ i = 1, 2, \dots, m. \tag{4.3}$$

Now, consider the auxiliary problem

$$(\phi(u'(t)))' = \widetilde{f}(t, u(t), u'(t)) \quad \text{a.e. on } [0,T]; \tag{4.4}$$

$$u(t_i+) = J_i(u(t_i)),\ \ u'(t_i+) = \widetilde{M_i}(u'(t_i)), \quad i = 1, 2, \dots, m, \tag{4.5}$$

$$u(0) = u(T),\ \ \beta_1(u'(0)) = u'(T). \tag{4.6}$$

We see that $\widetilde{f}$ and $\widetilde{M_i}$ have the same properties as f and M_i. In particular, $\widetilde{f}$ satisfies (1.11) with $\omega(s) \equiv 1$, $\widetilde{M_i}$ fulfils (1.9) and σ_1/σ_2 are lower/upper functions for (4.4)–(4.6). Since we work with (4.6) instead of (1.3), we have to replace the expression $x(0) + x'(0) - x'(T)$ in (3.5) by $x(0) + \beta_1(x'(0)) - x'(T)$. Then we get the existence of a solution u of (4.4)–(4.6) satisfying (2.1) in the same way as in the proof of Theorem 2.1 for (1.1)–(1.3).

Step 2. Having the solution u of (4.4)–(4.6), it remains to show that (2.2) is true.

(i) Let $j \in \{1, 2, \dots, m+1\}$ and $\xi \in [t_{j-1}, t_j)$ be such that

$$\sup\{u'(t) : t \in [0,T]\} = u'(\xi+) > d_j. \tag{4.7}$$

Then there is $\delta > 0$ such that $(\xi, \xi + \delta) \subset (t_{j-1}, t_j)$ and $u' > d_j$ on $(\xi, \xi + \delta)$. By (1.12),

$$(\phi(u'(t)))' = f(t, u(t), d_j) + \frac{u'(t) - d_j}{u'(t) - d_j + 1} > 0 \quad \text{for a.e. } t \in (\xi, \xi + \delta),$$

i.e. $\phi(u'(t)) > \phi(u'(\xi+))$ and so $u'(t) > u'(\xi+)$ for each $t \in (\xi, \xi + \delta))$, which contradicts (4.7).

(ii) Assume that

$$\sup\{u'(t) : t \in [0, T]\} = u'(t_j) > d_j \quad \text{for some } \ t_j \in \mathrm{D}\,. \tag{4.8}$$

If $j = m+1$, i.e. $u'(T) > d_{m+1}$, then, by (1.12), we have also $u'(T) > d_1$. Since (4.1) and (4.6) imply $u'(T) \le d_1$, we get a contradiction.

If $j < m + 1$, then

$$\widetilde{M}_j(u'(t_j)) = M_j(d_j) + \frac{u'(t) - d_j}{u'(t) - d_j + 1} > M_j(d_j) \ge d_{j+1},$$

so $u'(t_j+) > d_{j+1}$. By part (i) we know that $u'(t) - d_{j+1}$ cannot achieve a positive maximum inside (t_j, t_{j+1}). Consequently, we have $u'(t_{j+1}) > d_{j+1}$. Repeating this procedure we get $u'(T) > d_{m+1}$ and a contradiction as before.

We have proved that $u'(t) \le d_j$ on $(t_{j-1}, t_j]$, $j = 1, 2, \ldots, m + 1$. The remaining inequalities in (2.2) can be derived analogously. Finally, since u fulfils (2.2), u is a solution of (1.1)–(1.3). □

Sketch of the proof of Theorem 2.3. We borrow ideas of [14], where non-ordered lower/upper functions to periodic impulsive problem without ϕ-Laplacian ($\phi(y) = y$) have been studied. Here, we define the operator $\mathcal{F}$ by (3.5). Then, according to $\mathcal{F}$, we construct auxiliary operators and compute their Leray-Schauder degrees by a similar procedure as in [14]. For this we need a priori estimates of solutions of corresponding auxiliary problems. Now we consider problems with ϕ-Laplacians but the basic evaluation of estimates of $\phi(u')$ are similar to those of u' in [14] and hence we omit their computation here. For details see our preprint [16]. □

Proof of Theorem 2.4. First, we will prove the following a priori estimate:

Claim. *There exist $a_j \in (0, \infty)$, $j = 1, 2, \ldots, m + 1$, such that for each function $u \in \mathbb{C}^1_{\mathrm{D}}$ satisfying* (1.2), (1.3), (2.2) *and* (2.3), *the estimates*

$$|u(t)| \le a_j \quad \textit{for} \ \ t \in (t_{j-1}, t_j], \ j = 1, 2, \ldots, m + 1 \tag{4.9}$$

are valid.

Indeed, let u satisfy the assumptions of Claim and let

$$\rho_0 = \max\{\|\sigma_1\|_\infty, \|\sigma_2\|_\infty\} \quad \text{and} \quad \gamma_i = \max\{|c_i|,\, |d_i|\},\ i = 1, 2, \ldots, m+1.$$

(i) If $t_u \in [0, t_1]$, then $|u(t)| \le \gamma_1 t_1 + \rho_0$ for $t \in [0, t_1]$. Put

$$a_1^0 = \gamma_1 t_1 + \rho_0 \quad \text{and} \quad b_1^0 = \max\{|J_1(x)| : x \in [-a_1^0, a_1^0]\}.$$

Then

$$|u(t)| \le \gamma_2 (t_2 - t_1) + b_1^0 \ \text{ for } \ t \in (t_1, t_2].$$

Further, put

$$a_2^0 = \gamma_2 (t_2 - t_1) + b_1^0 \quad \text{and} \quad b_2^0 = \max\{|J_2(x)| : x \in [-a_2^0, a_2^0]\}.$$

Then

$$|u(t)| \le \gamma_3 (t_3 - t_2) + b_2^0 \ \text{ for } \ t \in (t_2, t_3].$$

By induction we get that $|u(t)| \le a_i^0$ for $t \in (t_{i-1}, t_i]$, where

$$a_{i+1}^0 = \gamma_{i+1} (t_{i+1} - t_i) + \max\{|J_i(x)| : x \in [-a_i^0, a_i^0]\}, \quad i = 1, 2, \ldots, m.$$

(ii) If $t_u \in (t_j, t_{j+1}]$ for some $j \in \{1, 2, \ldots, m\}$, we get similarly as in (i) that

$$|u(t)| \le a_i^j \ \text{ for } \ t \in (t_{i-1}, t_i], \quad i = 1, 2, \ldots, m+1,$$

where

$$a_{j+1}^j = \gamma_{j+1} (t_{j+1} - t_j) + \rho_0,$$

$$a_{i+1}^j = \gamma_{i+1} (t_{i+1} - t_i) + \max\{|J_i(x)| : x \in [-a_i^j, a_i^j]\},\ i = 1, 2, \ldots, j-1, j+1, \ldots, m,$$

$$a_1^j = \gamma_1 t_1 + a_{m+1}^j.$$

Setting

$$a_j = \max\{\rho_0, a_j^0, a_j^1, \ldots, a_j^m\} \ \text{ for } \ j = 1, 2, \ldots, m+1,$$

we complete the proof of Claim.

Now, take β_j by (4.1) and for a_j of Claim put

$$\alpha_j(x) = \begin{cases} -a_j & \text{for } x < -a_j, \\ x & \text{for } -a_j \le x \le a_j, \\ a_j & \text{for } x > a_j \end{cases}$$

and

$$\widetilde{f}(t, x, y) = f(t, \alpha_j(x), \beta_j(y)) + \frac{y - \beta_j(y)}{|y - \beta_j(y)| + 1}$$
$$\text{for a.e. } t \in (t_{j-1}, t_j],\ \text{all } \ x, y \in \mathbb{R},\ j = 1, 2, \ldots, m+1.$$

Finally, define $\widetilde{M}_i$ by (4.3). We see that all assumptions of Theorem 2.3 are satisfied for the problem (4.4)–(4.6) and consequently it has a solution u satisfying (2.3). As in the proof of Theorem 2.2, Step 2, we get that u fulfils (2.2). Hence u satisfies (1.2), (1.3) and, by Claim, also (4.8). Therefore, u is a solution of (1.1)–(1.3). $\square$

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