LINEAR DISTRIBUTIONAL DIFFERENTIAL EQUATIONS IN THE SPACE OF REGULATED FUNCTIONS

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Abstract. In the paper existence and uniqueness results for the linear differential system on the interval $[0,1]$

$$A_1\Big(A_0x\Big)' - A_2'x = f' \tag{0.1}$$

with distributional coefficients and solutions from the space of regulated functions are obtained.

The paper deals with the linear differential system on the interval $[0,1]$

$$A_1\big(A_0x\big)' - A_2'x = f' \tag{0.1}$$

with distributional coefficients and solutions from the space of regulated functions. In particular, we assume that A_0 and A_1 are $n\times n$-matrix valued functions continuous on $[0,1]$ such that $\det\big(A_0(t)A_1(t)\big) \neq 0$ for all $t\in[0,1]$, while A_1 has a bounded variation on $[0,1]$. Furthermore, the $n\times n$-matrix valued function A_2 has a bounded variation on $[0,1]$ and f is regulated on $[0,1]$. Distributions are understood in the sense of L. Schwartz.

It will be shown that the system (0.1) is equivalent to the integral equation

$$y(t) - y(0) - \int_0^t \mathrm{d}[A(s)]y(s) = h(t) - h(0), \tag{0.2}$$

where $y(t) = A_0(t)x(t)$,

$$A(t) = \int_0^t A_1^{-1}(s)\mathrm{d}[A_2(s)]A_0^{-1}(s) \qquad \text{for} \quad t\in[0,1]$$

and

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$$h(t)=\int_0^t A_1^{-1}(s)\mathrm{d}[f(s)] \qquad \text{for} \quad t\in[0,1].$$

and the integrals are the Perron-Stieltjes ones. The equation (0.2) is a generalized linear differential equation (in the sense of J. Kurzweil). The basic results known for equations of the form (0.2) (cf. e.g. [STV], [Sch1] and [Tv2]) make it possible to prove theorems on existence and uniqueness of solutions and the variation-of-constants formula for distributional differential systems (0.1) and

$$P_1\big(P_0u^{(m-1)}\big)'+P_2'u^{(m-1)}+\dots+P_m'u'+P_{m+1}'u=q', \tag{0.3}$$

where P_0 and P_1 are $n\times n$-matrix valued functions defined and continuous on $[0,1]$, $\det P_0(t)\det P_1(t)\neq 0$ on $[0,1]$; $P_1,P_2,\dots,P_{m+1}$ are of bounded variation on $[0,1]$, q is regulated on $[0,1]$ and an n-vector valued function u is called a solution to the system (0.3) if $u,u',\dots,u^{(m-1)}$ are regulated on $[0,1]$ and $P_1\Big(P_0u^{(m-1)}\Big)'+P_2'u^{(m-1)}+\dots+P_m'u'+P_{m+1}'u-q'$ is the zero n-vector distribution.

The study of linear differential systems with distributional coefficients was initiated more than thirty years ago (cf. e.g. [Ku2]). The fundamental theory for systems of the form

$$x'-A'x=f'$$

where A and f are of bounded variation on $[0,1]$ and distributions are understood in the sequential sense has been established by J. Ligęza in [Li1]. Further generalizations were obtained by J. Ligęza (cf. [Li2] and [Li3]), R. Pfaff (cf. [Pf1] and [Pf2]) and J. Persson (cf. [Pe1] and [Pe2]). Linear and nonlinear distributional differential equations in the space of regulated functions were treated by J. Ligęza in [Li4]. Related results may be found also in [At],[Za-Se],[Pa-De] and [Mi]. In this paper we generalize the results of J. Ligęza and R. Pfaff concerning the systems of the first order. In the case of higher order systems our results are complementary to those from [Li2], [Li3], [Pf1] and [Pf2]. The equations treated by J. Persson in [Pe1] and [Pe2] slightly differ from those considered in this paper. (In their equivalent integral form analogous to (0.2) the Lebesgue-Stieltjes integrals over the closed interval $[0,t]$ appear instead of the Perron-Stieltjes integrals from 0 to t.)

1. Preliminaries

1.1. Basic notation and definitions. Throughout the paper $\mathbb{R}^n$ denotes the space of real column n-vectors, $\mathbb{R}^1=\mathbb{R}$, $\mathbb{N}$ stands for the set of positive integers. If $-\infty<a<b<\infty$, then $[a,b]$ and (a,b) denote the corresponding closed and open intervals, respectively. Furthermore, $[a,b)$ and $(a,b]$ are the corresponding half-open intervals. The sets $d=\{t_0,t_1,\dots,t_m\}$ of points in the closed interval $[a,b]$ such that $a=t_0<t_1<\dots<t_m=b$ are called *divisions* of $[a,b]$. If $M\subset\mathbb{R}$, then χ_M denotes the characteristic function of M.

Given a $k \times n$-matrix M, its elements are denoted by $m_{i,j}$, M^{-1} denotes its inverse and M^* stands for its transposition, i.e.

$$M = (m_{i,j})_{\substack{i=1,\dots,k\\ j=1,\dots,n}} \quad \text{and} \quad M^* = (m_{j,i})_{\substack{j=1,\dots,n\\ i=1,\dots,k}}.$$

(In particular, $y^* = (y_1, y_2, \dots, y_n)$.) Furthermore,

$$|M| = \max_{i=1,\dots,k} \sum_{j=1}^{n} |m_{i,j}|$$

is the norm of M. (In particular,

$$|y| = \max_{i=1,\dots,n} |y_i| \quad \text{and} \quad |y^*| = \sum_{j=1}^{n} |y_j|$$

for $y \in \mathbb{R}^n$.) The symbols I and 0 stand respectively for the identity and the zero matrix of the proper type.

Let $-\infty < a < b < \infty$. Any function $f : [a,b] \to \mathbb{R}$ which possesses finite limits $f(t+) = \lim_{\tau \to t+} f(\tau)$, $f(s-) = \lim_{\tau \to s-} f(\tau)$ for all $t \in [a,b)$ and $s \in (a,b]$ is said to be *regulated* on $[a,b]$. Any $k \times n$-matrix valued function F defined on $[a,b]$ and such that all its elements $f_{i,j}(t)$ $(i = 1,2,\dots,k\ j = 1,2,\dots,n)$ are regulated functions on $[a,b]$ or functions of bounded variation on $[a,b]$ is said to be a matrix valued function regulated on $[a,b]$ or of bounded variation on $[a,b]$, respectively. A $k \times n$-matrix valued function F defined on $[a,b]$ is of bounded variation on $[a,b]$ if and only if

$$\operatorname{var}_a^b F = \sup_d \sum_{j=1}^{m} |F(t_j) - F(t_{j-1})| < \infty,$$

where the supremum is taken over all divisions $d = \{t_0, t_1, \dots, t_m\}$ of the interval $[a,b]$. The number $\operatorname{var}_a^b F$ defined above is called the *variation* of the function F on the interval $[a,b]$. $\mathrm{B}V^{k\times n}(a,b)$ denotes the Banach space of $k \times n$-matrix valued functions of bounded variation on $[a,b]$ equipped with the norm

$$F \in \mathrm{B}V^{k\times n}(a,b) \to \|F\|_{\mathrm{B}V} = |F(a)| + \operatorname{var}_a^b F.$$

Instead of $\mathrm{B}V^{n\times 1}(a,b)$ we write $\mathrm{B}V^n(a,b)$. Let us notice that if $A \in \mathrm{B}V^{n\times n}(a,b)$ is such that $\det A(t) \neq 0$ on $[a,b]$ and the corresponding inverse function A^{-1} is bounded on $[a,b]$ ($|A^{-1}(t)| \le \alpha < \infty$ for all $t \in [a,b]$), then $A^{-1} \in \mathrm{B}V^{n\times n}(a,b)$ as well. Indeed, if $\Big\{t_0, t_1, \dots, t_m\Big\}$ is an arbitrary division of $[a,b]$, then

$$\sum_{j=1}^{m} |A^{-1}(t_j) - A^{-1}(t_{j-1})| = \sum_{j=1}^{m} |A^{-1}(t_j)\big[A(t_{j-1}) - A(t_j)\big]A^{-1}(t_{j-1})|$$

$$quad \le \alpha^2 \sum_{j=1}^{m} |A(t_j) - A(t_{j-1})| \le \alpha^2 (\operatorname{var}_0^1 A).$$

Hence $\operatorname{var}_0^1(A^{-1}) \le \alpha^2(\operatorname{var}_0^1 A)$.

The space of column n-vector valued functions regulated on $[a,b]$ is denoted by $\mathrm{G}^n(a,b)$ while $\mathrm{G}^n_{reg}(a,b)$ stands for the set of all functions $f \in \mathrm{G}^n(a,b)$ such that

$$f(t) = \tfrac{1}{2}[f(t-) + f(t+)] \quad \text{for all} \quad t \in (a,b), \tag{1.1.1}$$

$$f(a+) = f(a) \quad \text{and} \quad f(b-) = f(b). \tag{1.1.2}$$

The functions fulfilling (1.1.1) are usually called *regular* on (a,b) and the functions fulfilling both (1.1.1) and (1.1.2) are called regular on $[a,b]$. Given $f \in \mathrm{G}^n(a,b)$, $t \in [a,b)$, $s \in (a,b]$ and $r \in (a,b)$, we put

$$\Delta^+ f(t) = f(t+) - f(t), \quad \Delta^- f(s) = f(s) - f(s-) \quad \text{and} \quad \Delta f(r) = f(r+) - f(r-).$$

Obviously, $f \in \mathrm{G}^n(a,b)$ is regular on (a,b) if and only if $\Delta^- f(t) = \Delta^+ f(t)$ holds for all $t \in (a,b)$. The set of all $k \times n$-matrix valued functions regular on $[a,b]$ and of bounded variation on $[a,b]$ will be denoted by $\mathrm{B}V^{k\times n}_{\mathrm{reg}}$. For $x \in \mathrm{G}^n(a,b)$ we put

$$\|x\| = sup_{t\in[0,1]}|x(t)|.$$

It is well known that $\mathrm{G}^n(a,b)$ is a Banach space with respect to this norm (cf. [Hö1, Theorem 3.6]). Obviously, $\mathrm{G}^n_{reg}(a,b)$ is a closed subspace of $\mathrm{G}^n(a,b)$ and hence it is also a Banach space with respect to the same norm.

For more details concerning regulated functions or functions of bounded variation see [Au], [Hö1], [Fra] or [Hi], respectively.

As usual $\mathrm{L}^n_1(a,b)$ stands for the Banach space of measurable and Lebesgue integrable column n-vector valued functions on $[a,b]$ equipped with the norm

$$f \in \mathrm{L}^n_1(a,b) \to \|f\|_{\mathrm{L}} = \int_a^b |f(t)|\mathrm{d}t.$$

In the case $[a,b] = [0,1]$ we write simply G^n, G^n_{reg}, $\mathrm{B}V^{k\times n}$ and L^n_1 instead of $\mathrm{G}^n(0,1)$, $\mathrm{G}^n_{reg}(0,1)$, $\mathrm{B}V^{k\times n}(0,1)$ and $\mathrm{L}^n_1(0,1)$, respectively. Furthermore, $\mathrm{G}(a,b)$, $\mathrm{G}_{reg}(a,b)$, $\mathrm{B}V(a,b)$ and $\mathrm{L}_1(a,b)$ stand for $\mathrm{G}^1(a,b)$, $\mathrm{G}^1_{reg}(a,b)$, $\mathrm{B}V^1(a,b)$ and $\mathrm{L}^1_1(a,b)$, respectively.

1.2. Perron-Stieltjes integral with respect to regulated functions. The integrals which occur in this paper are the Perron-Stieltjes ones. For the original definition, see [Wa] or [Sa]. We use the equivalent summation definition due to J. Kurzweil (cf. [Ku1], [Ku3], [STV]).

Given a $p \times q$-matrix valued function F and a $q \times r$-matrix valued function G defined on $[a,b]$ and such that all the integrals

$$\int_a^b f_{i,k}(t)\mathrm{d}[g_{k,j}(t)] \quad (i = 1,2,\dots,p; j = 1,2,\dots,r)$$

exist (i.e. have finite values), the symbol

$$\int_a^b F(t)\mathrm{d}[G(t)]$$

stands for the $p \times r$-matrix M with the entries

$$m_{i,j} = \sum_{k=1}^{q} \int_a^b f_{i,k}(t)\mathrm{d}[g_{k,j}(t)] \quad (i = 1, 2, \dots, p;\ j = 1, 2, \dots, r).$$

The integrals

$$\int_a^b \mathrm{d}[F(t)]G(t) \quad \text{and} \quad \int_a^b F(t)\mathrm{d}[G(t)]H(t)$$

for matrix valued functions F, G, H of proper types are defined analogously. The basic properties of the Perron-Stieltjes integral with respect to regulated functions were described in [Tv1] and [Tv2]. Let us mention here some of its further properties needed later on. We shall formulate them for scalar functions. Their extension to the vector or matrix valued function is obvious.

Let functions f, g be regulated on $[a, b]$. If the integral

$$\int_a^b f(t)\mathrm{d}[g(t)]$$

has a finite value, then by Theorem 1.3.4 from [Ku1] the function

$$h : t \in [a, b] \to \int_a^t f(s)\mathrm{d}[g(s)] \in \mathbb{R}$$

is regulated on $[a, b]$. Let us note that the integral

$$\int_a^b f(t)\mathrm{d}[g(t)]$$

has a finite value if the functions f and g are both regulated on $[a, b]$ and at least one of them has a bounded variation on $[a, b]$ (cf. [Tv1, Theorem 2.8]). In this case the above mentioned Theorem 1.3.4 from [Ku1] implies that

$$h(t+) = h(t) + f(t)\Delta^+ g(t) \quad \text{and} \quad h(s-) = h(s) - f(s)\Delta^- g(s)$$

holds for all $t \in [a, b)$ and $s \in (a, b]$. Moreover, if $g \in \mathrm{B}V(a, b)$, then $h \in \mathrm{B}V(a, b)$ as well.

The following modifications of the integration-by-parts formula (cf. [Tv1, Theorem 2.15]) will be useful for our considerations.

1.2.1. Proposition. *Let* $a \le c < d \le b$, $f \in \mathrm{BV}(a,b)$,

$$\widetilde{f}(t) = \begin{cases} f(a+) & \text{for} \quad t = a \\ \frac{1}{2}\big[f(t+) + f(t-)\big] & \text{for} \quad t \in (a,b) \\ f(b-), & \text{for} \quad t = b \end{cases} \tag{1.2.1}$$

$$\alpha = \tfrac{1}{2}\Delta f(c) \quad \text{if} \;\; c > a, \qquad \alpha = 0 \quad \text{if} \;\; c = a \tag{1.2.2}$$

and

$$\beta = \tfrac{1}{2}\Delta f(d) \quad \text{if} \;\; d < b, \qquad \beta = 0 \quad \text{if} \;\; d = b. \tag{1.2.3}$$

Then $\widetilde{f} \in \mathrm{BV}_{\mathrm{reg}}(a,b)$ *and the relation*

$$\int_c^d \mathrm{d}[f]g = f(d)g(d) - f(c)g(c) - \int_c^d \widetilde{f}\mathrm{d}[g] + \beta\Delta^- g(d) - \alpha\Delta^+ g(c), \tag{1.2.4}$$

holds for each $g \in \mathrm{G}_R(a,b)$.

Proof. Let $\{t_0, t_1, \dots, t_m\}$ be an arbitrary division of $[a,b]$. Then

$$\begin{aligned}
&\sum_{j=1}^m |\widetilde{f}(t_j) - \widetilde{f}(t_{j-1})| \\
&\qquad \le \sum_{j=1}^m |f(t_j) - f(t_{j-1})| + |\Delta^+ f(a)| + \sum_{j=1}^{m-1} \Big(|\Delta^+ f(t_j)| + |\Delta^- f(t_j)|\Big) + |\Delta^- f(b)| \\
&\qquad \le 2\,\mathrm{var}_a^b\, f
\end{aligned}$$

and hence $\mathrm{var}_a^b\, \widetilde{f} \le 2\,\mathrm{var}_a^b\, f$.

Let $a < c < d < b$ and let an arbitrary $g \in \mathrm{G}_{\mathrm{reg}}(a,b)$ be given. Let us put $w(s) = f(s) - \widetilde{f}(s)$ on $[a,b]$, i.e. $w(a) = -\Delta^+ f(a)$, $w(s) = \frac{1}{2}[\Delta^- f(s) - \Delta^+ f(s)]$ for $s \in [a,b)$ and $w(b) = \Delta^- f(b)$. It follows easily that $w(s+) = 0$ on $[a,b)$ and $w(s-) = 0$ on $(a,b]$ and consequently (cf. [STV, Lemma I.4.23])

$$\int_c^d \mathrm{d}[w(s)]g(s) = \Delta^+ w(c)g(c) + \Delta^- w(d)g(d).$$

In particular,

$$\int_c^d \mathrm{d}[w(s)]g(s) = \tfrac{1}{2}\Big[\Delta^- f(d)g(d) - \Delta^+ f(d)g(d) - \Delta^- f(c)g(c) + \Delta^+ f(c)g(c)\Big].$$

By the integration-by-parts formula (cf. [Tv1, Theorem 2.15]) we have for any

$g \in \mathrm{G}_{reg}(a,b)$

$$\begin{aligned}
&\int_c^d \mathrm{d}[\widetilde{f}(s)]g(s) + \int_c^d \widetilde{f}(s)\mathrm{d}[g(s)] \\
&\quad = \widetilde{f}(d)g(d) - \widetilde{f}(c)g(c) + \Delta^-\widetilde{f}(d)\Delta^- g(d) - \Delta^+\widetilde{f}(c)\Delta^+ g(c) \\
&\quad = \tfrac{1}{2}\Big([f(d+) + f(d-)]g(d) - [f(c+) + f(c-)]g(c)\Big) \\
&\qquad + \tfrac{1}{2}\Big(\Delta f(d)\Delta^- g(d) - \Delta f(c)\Delta^+ g(c)\Big) \\
&\quad = \tfrac{1}{2}\Big(f(d+)g(d) + f(d-)g(d) - f(c+)g(c) - f(c-)g(c) \\
&\qquad + f(d+)\Delta^- g(d) - f(d-)\Delta^- g(d) - f(c+)\Delta^+ g(c) + f(c-)\Delta^+ g(c)\Big) \\
&\quad = \tfrac{1}{2}\Big(f(d+)[g(d) + \Delta^- g(d)] + f(d-)[g(d) - \Delta^- g(d)] \\
&\qquad - f(c+)\big[g(c) + \Delta^+ g(c)\big] - f(c-)\big[g(c) - \Delta^+ g(c)\big]\Big) \\
&\quad = \tfrac{1}{2}\Big(f(d+)g(d+) + f(d-)g(d-) - f(c+)g(c+) - f(c-)g(c-)\Big).
\end{aligned}$$

Hence

$$\begin{aligned}
&\int_c^d \mathrm{d}[f(s)]g(s) + \int_c^d \widetilde{f}(s)\mathrm{d}[g(s)] \\
&\quad = \int_c^d \mathrm{d}[\widetilde{f}(s)]g(s) + \int_c^d \widetilde{f}(s)\mathrm{d}[g(s)] + \int_c^d \mathrm{d}[w(s)]g(s) \\
&\quad = \tfrac{1}{2}\Big(f(d+)g(d+) + f(d-)g(d-) - f(c+)g(c+) - f(c-)g(c-)\Big) \\
&\qquad + \tfrac{1}{2}\Big(f(d)g(d) - f(d-)g(d) - f(d+)g(d) + f(d)g(d)\Big) \\
&\qquad - \tfrac{1}{2}\Big(f(c)g(c) - f(c-)g(c) - f(c+)g(c) + f(c)g(c)\Big) \\
&\quad = f(d)g(d) + \tfrac{1}{2}\big[f(d+)\Delta^+ g(d) - f(d-)\Delta^- g(d)\big] \\
&\qquad - f(c)g(c) - \tfrac{1}{2}\big[f(c+)\Delta^+ g(c) - f(c-)\Delta^- g(c)\big] \\
&\quad = f(d)g(d) - f(c)g(c) + \tfrac{1}{2}\Delta f(d)\Delta^- g(d) - \tfrac{1}{2}\Delta f(c)\Delta^+ g(c).
\end{aligned}$$

The proof of the remaining assertions of the proposition can be done in a quite similar way. ■

1.2.2. Proposition. *Let* $a \le c < d \le b$ *and* $f \in \mathrm{G}(a,b)$. *Let* $\widetilde{f}$, α *and* β *be defined by* (1.2.1), (1.2.2) *and* (1.2.3), *respectively. Then* $\widetilde{f} \in \mathrm{G}_{reg}(a,b)$ *and the relation* (1.2.4) *holds for each* $g \in \mathrm{BV}_{reg}(a,b)$.

Proof is analogous to that of the previous proposition. ■

1.3. Distributions. The distributions are considered in this paper in the sense of L. Schwartz. Let us recall some basic definitions and properties of distributions needed later on. For more details concerning distributions see e.g. [Ha] or [Ru].

In what follows $\mathscr{D}$ stands for the topological vector space of functions $\varphi : \mathbb{R} \to \mathbb{R}$ possessing for any $j \in \mathbb{N} \cup \{0\}$ a derivative $\varphi^{(j)}$ of the order j which is continuous on $\mathbb{R}$ and such that $\varphi^{(j)}(t) = 0$ for any $t \in \mathbb{R} \setminus [0,1]$. The space $\mathscr{D}$ is endowed with the topology in which the sequence $\varphi_k \in \mathscr{D}$ tends to $\varphi_0 \in \mathscr{D}$ in $\mathscr{D}$ if and only if

$$\lim_k \|\varphi_k^{(j)} - \varphi_0^{(j)}\| = 0$$

for all $j \in \mathbb{N} \cup \{0\}$. Linear continuous functionals on $\mathscr{D}$ are called *distributions* on $[0,1]$ and the elements of the space $\mathscr{D}$ are called *test functions*. The space of distributions on $[0,1]$ (i.e. the dual space to $\mathscr{D}$) is denoted by $\mathscr{D}^*$. Given a distribution $f \in \mathscr{D}^*$ and a test function $\varphi \in \mathscr{D}$, $\langle f, \varphi\rangle$ denotes the value of the functional f on φ. Any function $f \in \mathrm{L}_1$ will be identified with the distribution

$$f : \varphi \in \mathscr{D} \to \langle f, \varphi\rangle = \int_0^1 f(t)\varphi(t)\mathrm{d}t = \int_0^1 f\varphi\mathrm{d}t.$$

In particular, the zero element 0 of $\mathscr{D}^*$will be identified with the function vanishing a.e. on $[0,1]$. Obviously, if $f \in \mathrm{G}$, then f is the zero distribution ($f = 0$) if and only if $f(t+) = f(s-) = 0$ holds for any $t \in [0,1)$ and $s \in (0,1]$, and if $f \in \mathrm{G}_{reg}$, then $f = 0$ in $\mathscr{D}^*$if and only if $f(t) \equiv 0$ on $[0,1]$.

Given an arbitrary $f \in \mathscr{D}^*$, the distribution f' defined by

$$f' : \varphi \in \mathscr{D} \to \langle f', \varphi\rangle = -\langle f, \varphi'\rangle$$

is said to be the (*distributional*) *derivative* of f. Analogously, for any $j \in \mathbb{N}$,

$$f^{(j)} : \varphi \in \mathscr{D} \to \langle f^{(j)}, \varphi\rangle >= (-1)^j \langle f, \varphi^{(j)}\rangle$$

defines the j^{th} derivative of f. For absolutely continuous functions their classical derivatives coincide with their distributional derivatives, of course.

If $f \in \mathscr{D}^*$ then $\langle f, \varphi'\rangle = 0$ holds for all $\varphi \in \mathscr{D}$ if and only if there is a $c \in \mathbb{R}$ such that

$$\langle f, \varphi\rangle = c\int_0^1 \varphi(s)\mathrm{d}s$$

for all $\varphi \in \mathscr{D}$ (cf. Sec. 3 in [Ha]). It means that $f = c$ in the sense of distributions. In other words, $\langle f, \varphi'\rangle = 0$ for all $\varphi \in \mathscr{D}$ if and only if the distribution f may be identified with a function $f : [0,1] \to \mathbb{R}$ for which there is a $c \in \mathbb{R}$ such that $f(t) = c$ a.e. on $[0,1]$.

Given distributions $u, v \in \mathscr{D}^*$, the definition of the product uv is well-known in the following two cases [cf. [Ha]):

(i) if u, v and $uv \in \mathrm{L}_1$, then

$$uv : \varphi \in \mathscr{D} \to \langle uv, \varphi\rangle = \int_0^1 (u(t)v(t))\varphi(t)\mathrm{d}t;$$

(ii) if $u \in \mathscr{D}^*$ and v is infinitely times continuously differentiable, then

$$uv : \varphi \in \mathscr{D} \to \langle uv, \varphi\rangle = \langle u, v\varphi\rangle.$$

In addition, if $f \in \mathrm{G}_{reg}$ and $g \in \mathrm{BV}_{reg}$, then we put as in [Pa-De]

$$f'g : \varphi \in \mathscr{D} \to \langle f'g, \varphi \rangle = \int_0^1 (g(s)\varphi(s))\mathrm{d}[f(s)] \tag{1.3.1}$$

and

$$fg' : \varphi \in \mathscr{D} \to \langle fg', \varphi \rangle = \int_0^1 (f(s)\varphi(s))\mathrm{d}[g(s)]. \tag{1.3.2}$$

Let us note, that the definitions (1.3.1) and (1.3.2) are not contradictory to the corresponding definitions given by P. Antosik and J. Ligęza in [An-Li] on the basis of the sequential approach to distributions. In particular, it is easy to verify that for any $\tau \in (0,1)$ both (1.3.1) and (1.3.2) yield

$$H_\tau \delta_\tau = \tfrac{1}{2}\delta_\tau$$

for the product of the Dirac distribution δ_τ concentrated in τ with the corresponding regular Heaviside function H_τ ($H_\tau(t) = 0$ for $t < \tau$, $H_\tau(\tau) = \frac{1}{2}$, $H_\tau(t) = 1$ for $t > \tau$, $\delta_\tau = H'_\tau$).

The relations (1.3.1) and (1.3.2) obviously define linear continuous functionals on $\mathscr{D}$ which are compatible with the definitions (i) and (ii) also in the case that the regularity of the functions f and g is not supposed. However, the usual relation

$$(fg)' = f'g + fg' \tag{1.3.3}$$

then need not be true in general. Indeed, (1.3.3) holds if and only if

$$\int_0^1 (f(t)g(t))\varphi'(t)\mathrm{d}t + \int_0^1 \mathrm{d}[f(t)](g(t)\varphi(t)) + \int_0^1 (f(t)\varphi(t))\mathrm{d}[g(t)] = 0$$

is true for all $\varphi \in \mathscr{D}$, i.e. if and only if

(1.3.4)
$$\int_0^1 \Big[f(t)g(t) - \int_0^t \mathrm{d}[f(s)]g(s) - \int_0^t f(s)\mathrm{d}[g(s)]\Big]\varphi'(t)\mathrm{d}t = 0 \quad \text{for all} \quad \varphi \in \mathscr{D}.$$

If both f and g are regular on $[0,1]$, then by integration-by-parts formula (cf. Proposition 1.2.1)

$$f(t)g(t) - \int_0^t \mathrm{d}[f(s)]g(s) - \int_0^t f(s)\mathrm{d}[g(s)] = f(0)g(0) - \Delta^- f(t)\Delta^- g(t)$$

holds for each $t \in [0,1]$. It means that

$$f(t)g(t) - \int_0^t \mathrm{d}[f(s)]g - \int_0^t f\mathrm{d}[g(s)] = f(0)g(0) \quad \text{a.e. on} \quad [0,1]$$

and the relation (1.3.4) is true if both f and g are regular on $[0,1]$.

The space of column n-vector valued functions $\varphi(t) = (\varphi_j(t))_{j=1,\dots,n}$ such that $\varphi_j \in \mathscr{D}$ for any $j = 1, 2, \dots, n$ will be denoted by $\mathscr{D}^n$ and its dual space (which is the n-th cartesian power of $\mathscr{D}^*$) will be denoted by $\mathscr{D}^{n*}$. The elements of $\mathscr{D}^{n*}$ will be called *n-vector distributions*. Given $f = (f_1, f_2, \dots, f_n) \in \mathscr{D}^{n*}$ and $\varphi = (\varphi_1, \varphi_2, \dots, \varphi_n)^* \in \mathscr{D}^n$, the value of the functional f on φ is given by

$$\langle f, \varphi\rangle = \langle f_1, \varphi_1\rangle + \langle f_2, \varphi_2\rangle + \dots + \langle f_n, \varphi_n\rangle.$$

As in the scalar case, if $f \in \mathrm{L}_1^n$, then this function will be identified with the n-vector distribution

$$f : \varphi \in \mathscr{D}^n \to \langle f, \varphi\rangle = \int_0^1 \varphi^*(t) f(t)\,\mathrm{d}t.$$

Similarly, if $g \in \mathrm{G}^n$, then the distributional derivative g' of g is given by

$$g' : \varphi \in \mathscr{D}^n \to \langle g', \varphi\rangle = \int_0^1 \varphi^*(t)\,\mathrm{d}[g(t)].$$

An n-vector distribution f is said to be the *n-vector zero distribution* ($f = 0$) if all its entries are zero distributions. A $k\times n$-matrix A whose entries $a_{i,j}$ ($i = 1, 2, \dots, k$; $j = 1, 2, \dots, n$) are distributions is said to be a *$k \times n$-matrix distribution*. Given a $k \times n$-matrix distribution

$$A = (a_{i,j})_{\substack{i=1,\dots,k\\ j=1,\dots,n}},$$

the matrix

$$A' = (a'_{i,j})_{\substack{i=1,\dots,k\\ j=1,\dots,n}}$$

is called the *derivative* of A. Analogously we define the derivatives of vector distributions.

If a $k \times n$-matrix distribution

$$A = (a_{i,j})_{\substack{i=1,\dots,k\\ j=1,\dots,n}}$$

and an n-vector distribution $x = (x_j)_{j=1,\dots,n}$ are such that all products

$$a_{i,j}x_j, \quad i = 1, 2, \dots, k;\ j = 1, 2, \dots, n,$$

are defined, then the product Ax is defined as the k-vector distribution y with the elements

$$y_i = \sum_{j=1}^n a_{i,j}x_j, \quad i = 1, 2, \dots, k.$$

1.3.1. Proposition. *Let $A \in \mathrm{BV}^{k\times n}_{\mathrm{reg}}$ and $x \in \mathrm{G}^n_{reg}$. Then the distributional products Ax' and $A'x$ are defined, $Ax' = \xi'$ and $A'x = \eta'$, where $\xi \in \mathrm{G}^k_{\mathrm{reg}}$ and $\eta \in \mathrm{BV}^k_{\mathrm{reg}}$ are given by*

$$\xi(t) = \int_0^t A(s)\mathrm{d}[x(s)] \quad \text{and} \quad \eta(t) = \int_0^t \mathrm{d}[A(s)]x(s) \quad \text{for} \quad t \in [0,1], \tag{1.3.5}$$

i. e.

$$Ax' : \varphi \in \mathscr{D}^n \longrightarrow \int_0^1 \varphi^*(t)\mathrm{d}[\xi(t)],$$

$$A'x : \varphi \in \mathscr{D}^n \longrightarrow \int_0^1 \varphi^*(t)\mathrm{d}[\eta(t)].$$

Proof. For any $\varphi \in \mathscr{D}^n$ we have by (1.3.1)

$$\begin{aligned}\langle Ax', \varphi\rangle &= \sum_{i=1}^k \Big(\sum_{j=1}^n \langle a_{i,j}x'_j, \varphi_i\rangle\Big) = \\ &= \sum_{i=1}^k \Big(\int_0^1 \mathrm{d}\Big[\sum_{j=1}^n \Big(\int_0^t a_{i,j}(s)\mathrm{d}[x_j(s)]\Big)\Big]\varphi_i(t)\Big) = \\ &= \int_0^1 \varphi^*(t)\mathrm{d}\Big[\int_0^t A(s)\mathrm{d}[x(s)]\Big] = \int_0^1 \varphi^*(t)\mathrm{d}[\xi(t)],\end{aligned}$$

and by (1.3.2)

$$\begin{aligned}\langle A'x, \varphi\rangle &= \sum_{i=1}^k \Big(\sum_{j=1}^n \langle a'_{i,j}x_j, \varphi_i\rangle\Big) = \sum_{i=1}^k \Big(\sum_{j=1}^n \int_0^1 \mathrm{d}[a_{i,j}(t)x_j(t)\varphi_i(t)\Big) = \\ &= \sum_{i=1}^k \Big(\int_0^1 \mathrm{d}\Big[\sum_{j=1}^n \Big(\int_0^t \mathrm{d}[a_{i,j}(s)]x_j(s)\Big)\Big]\varphi_i(t)\Big) = \\ &= \int_0^1 \varphi^*(t)\mathrm{d}\Big[\int_0^t \mathrm{d}[A(s)]x(s)\Big] = \int_0^1 \varphi^*(t)\mathrm{d}[\eta(t)]\end{aligned}$$

where $\xi(t)$ and $\eta(t)$ are given by (1.3.5). By Theorem 1.3.4 of [Ku1] (cf. Sec. 1.2) both ξ and η are regulated on $[0,1]$. Furthermore, given an arbitrary division $\{0 = t_0 < t_1 < \cdots < t_m = 1\}$ of $[0,1]$, we have

$$\begin{aligned}\sum_{j=1}^m |\eta(t_j) - \eta(t_{j-1})| &= \sum_{j=1}^m \Big|\int_{t_{j-1}}^{t_j} \mathrm{d}[A(s)]x(s)\Big| \\ &\le \sum_{j=1}^m \big(\mathrm{var}_{t_{j-1}}^{t_j} A\big)\|x\| \le \|A\|_{\mathrm{BV}}\|x\| < \infty\end{aligned}$$

and hence $\mathrm{var}_0^1 \eta \le \|A\|_{\mathrm{BV}}\|x\| < \infty$. Moreover, since $\eta(t+) = \eta(t) + \Delta^+ A(t)x(t)$ and $\eta(t-) = \eta(t) - \Delta^- A(t)x(t)$, it follows immediately that $\eta \in \mathrm{BV}^n_R$. Analogously we can show that $\xi \in \mathrm{G}^n_{reg}$. ■

1.3.2. Proposition. *Let $A \in \mathrm{BV}_{\mathrm{reg}}^{n\times n}$ and let its inverse matrix valued function A^{-1} be defined and bounded on $[0,1]$. Then the distributional equalities*

$$A^{-1}Ax' = AA^{-1}x' = x'$$

hold for any $x \in \mathrm{G}_{\mathrm{reg}}^n$.

Proof. Let an arbitrary $x \in \mathrm{G}_{\mathrm{reg}}^n$ be given. Since under our assumptions $A^{-1} \in \mathrm{BV}_{\mathrm{reg}}^{n\times n}$ (cf. Sec. 1.1), by Proposition 1.3.1 the product $A^{-1}x'$ is defined for any $x \in \mathrm{G}_{\mathrm{reg}}^n$ and $A^{-1}x' = \zeta'$, where $\zeta \in \mathrm{G}_{\mathrm{reg}}^n$ is given by

$$\zeta(t) = \int_0^t A^{-1}(s)\mathrm{d}[x(s)], \quad t \in [0,1].$$

Furthermore, by Proposition 1.3.1 $Ax' = \xi' \in \mathrm{G}^n$, where

$$\xi(t) = \int_0^t A(s)\mathrm{d}[x(s)], \quad t \in [0,1].$$

By (1.3.1) and by the Substitution Theorem we have for any $\varphi \in \mathscr{D}^n$

$$\begin{aligned}\langle A^{-1}Ax', \varphi\rangle &= \langle A^{-1}\xi', \varphi\rangle \\ &= \int_0^1 \varphi^*(t)\mathrm{d}\Big[\int_0^t A^{-1}(s)\mathrm{d}\Big[\int_0^s A(\sigma)\mathrm{d}[x(\sigma)]\Big]\Big] \\ &= \int_0^1 \varphi^*(t)\mathrm{d}\Big[\int_0^t A^{-1}(s)A(s)\mathrm{d}[x(s)]\Big] = \int_0^1 \varphi^*(t)\mathrm{d}[x(t)] = \langle x', \varphi\rangle\end{aligned}$$

and

$$\begin{aligned}\langle AA^{-1}x', \varphi\rangle = \langle A\zeta', \varphi\rangle &= \int_0^1 \varphi^*(t)\mathrm{d}\Big[\int_0^t A(s)\mathrm{d}\Big[\int_0^s A^{-1}(\sigma)\mathrm{d}[x(\sigma)]\Big]\Big] \\ &= \int_0^1 \varphi^*(t)\mathrm{d}\Big[\int_0^t A(s)A^{-1}(s)\mathrm{d}[x(s)]\Big] = \int_0^1 \varphi^*(t)\mathrm{d}[x(t)] = \langle x', \varphi\rangle.\end{aligned}$$

This implies that $A^{-1}Ax' = AA^{-1}x' = x'$ holds. ■

2. Linear distributional differential equations of the first order

In this section we will consider the system

$$A_1\big(A_0x\big)' - A_2'x = f' \tag{2.1}$$

and the corresponding homogeneous system

$$A_1\big(A_0x\big)' - A_2'x = 0, \tag{2.2}$$

where the derivatives, products and equality are understood in the sense of distributions.

2.1. Assumptions. A_0 and A_1 are $n \times n$-matrix valued functions continuous on $[0,1]$ and such that $\det(A_0(t)A_1(t)) \neq 0$. Furthermore, $A_1 \in \mathrm{BV}^{n\times n}_{reg}$, $A_2 \in \mathrm{BV}^{n\times n}_{reg}$ and $f \in \mathrm{G}^n_{reg}$.

2.2. Definition. An n-vector valued function $x(t)$ is called a *solution* to the equation (2.1) on the interval $[0,1]$ if $x \in \mathrm{G}^n_{reg}$ and $A_1(A_0x)' - A_2'x - f'$ is the zero n-vector distribution.

Let us notice that under the assumptions 2.1 the products $A_1(A_0x)'$ and $A_2'x$ are well defined for any $x \in \mathrm{G}^n_{reg}$. Furthermore, $A_1^{-1} \in \mathrm{BV}^{n\times n}_{reg}$ (cf. Sec. 1.1) and hence according to Proposition 1.3.2 the equality

$$A_1^{-1}(A_1(A_0x)') = (A_0x)'$$

holds for any $x \in \mathrm{G}^n_{reg}$. Consequently, the equation (2.1) may be rewritten as

$$(A_0x)' - A_1^{-1}A_2'x - A_1^{-1}f' = 0. \tag{2.3}$$

By Proposition 1.3.1 we have $A_2'x = \eta_x'$, where $\eta_x \in \mathrm{BV}^n_{reg}$ is given by

$$\eta_x(t) = \int_0^t \mathrm{d}[A_2(s)]x(s), \quad t \in [0,1].$$

For any $x \in \mathrm{G}^n_{reg}$ and any $\varphi \in \mathscr{D}^n$ we have in virtue of the Substitution Theorem

$$\begin{aligned}\langle A_1^{-1}A_2'x, \varphi\rangle = \ \langle A_1^{-1}\eta_x', \varphi\rangle &= \int_0^1 \varphi^*(t)\mathrm{d}_t\Big[\int_0^t A_1^{-1}(s)\mathrm{d}[\eta_x(s)]\Big] \\ &= \int_0^1 \varphi^*(t)A_1^{-1}(t)\mathrm{d}[\eta_x(t)] = \int_0^1 \varphi^*(t)A_1^{-1}(t)\mathrm{d}_t\Big[\int_0^t \mathrm{d}[A_2(s)]x(s)\Big] \\ &= \int_0^1 \varphi^*(t)A_1^{-1}(t)\mathrm{d}[A_2(t)]x(t) = \int_0^1 \varphi^*(t)\mathrm{d}_t\Big[\int_0^t A_1^{-1}(s)\mathrm{d}[A_2(s)]x(s)\Big] \\ &= \int_0^1 \varphi^*(t)\mathrm{d}[\zeta_x(t)]\end{aligned}$$

or

$$A_1^{-1}(A_2'x) = \zeta_x', \tag{2.4}$$

where

$$\zeta_x(t) = \int_0^t A_1^{-1}(s)\mathrm{d}[A_2(s)]x(s) = \int_0^t \mathrm{d}[B(s)]x(s), \quad t \in [0,1]$$

and

$$B(t) = \int_0^t A_1^{-1}(s)\mathrm{d}[A_2(s)], \quad t \in [0,1]. \tag{2.5}$$

Obviously, $B \in \mathrm{BV}^{n\times n}$,

$$\Delta^+ B(0) = A_1(0)\Delta^+ A_2(0) = 0, \quad \Delta^- B(1) = A_1(1)\Delta^- A_2(1) = 0$$

and

$$\Delta^+ B(t) = A_1(t)\Delta^+ A_2(t) = A_1(t)\Delta^- A_2(t) = \Delta^- B(t) \quad \text{for all} \quad t \in (0,1),$$

i.e. $B \in \mathrm{BV}_{reg}^{n\times n}$. Hence $\zeta_x \in \mathrm{G}_{reg}^n$ for any $x \in \mathrm{G}_{reg}^n$.

Similarly, given an arbitrary $\varphi \in \mathscr{D}^n$, we have

$$\langle A_1^{-1} f', \varphi\rangle = \int_0^1 \varphi^*(t)\mathrm{d}_t\Big[\int_0^t A_1^{-1}(s)\mathrm{d}[f(s)]\Big] == \int_0^1 \varphi^*(t)\mathrm{d}[h(t)] = \langle h', \varphi\rangle,$$

where $h \in \mathrm{G}_{reg}^n$ is given by

$$h(t) = \int_0^t A_1^{-1}(s)\mathrm{d}[f(s)], \quad t \in [0,1]. \tag{2.6}$$

By (2.4) this implies that (2.3) holds if and only if

$$\big(A_0 x - \zeta_x - h\big)' = 0,$$

i.e. if and only if there exists a $c \in \mathbb{R}^n$ such that

$$A_0(t)x(t) - \int_0^t \mathrm{d}[B(s)]x(s) - h(t) - c = 0 \tag{2.7}$$

holds for a.e. $t \in [0,1]$. Since the left-hand side of (2.7) belongs to G_{reg}^n for any $x \in \mathrm{G}_{reg}^n$, this means that $x \in \mathrm{G}_{reg}^n$ is a solution to the equation (2.1) on $[0,1]$ if and only if there is a $c \in \mathbb{R}^n$ such that (2.7) is satisfied for all $t \in [0,1]$. Introducing a new unknown function $y(t) = A_0(t)x(t)$ we complete the proof of the following assertion.

2.3. Proposition. *Let the assumptions 2.1 be satisfied. An n-vector valued function $x \in \mathrm{G}_{reg}^n$ is a solution to the equation (2.1) on $[0,1]$ if and only if the function $y(t) = A_0(t)x(t)$ satisfies on $[0,1]$ the integral equation*

$$y(t) - y(0) - \int_0^t \mathrm{d}[A(s)]y(s) = h(t) - h(0), \tag{2.8}$$

where $A \in \mathrm{BV}_{reg}^{n\times n}$ is given by

$$A(t) = \int_0^t \mathrm{d}[B(s)]A_0^{-1}(s) = \int_0^t A_1^{-1}(s)\mathrm{d}[A_2(s)]A_0^{-1}(s), \quad , t \in [0,1] \tag{2.9}$$

and $h \in \mathrm{G}_{reg}^n$ is given by (2.6).

2.4. Corollary. *An n-vector valued function $x \in \mathrm{G}^n_{reg}$ is a solution to the equation (2.2) on $[0,1]$ if and only if the function $y(t) = A_0(t)x(t)$ satisfies on $[0,1]$ the integral equation*

$$y(t) - y(0) - \int_0^t \mathrm{d}[A(s)]y(s) = 0 \tag{2.10}$$

where $A \in \mathrm{BV}^{n\times n}_{reg}$ is given by (2.9).

2.5. Remark. Let us notice that for any solution $x \in \mathrm{G}^n_{reg}$ of the homogeneous equation (2.2) on $[0,1]$ the function $y(t) = A_0(t)x(t)$ has to be of bounded variation on $[0,1]$.

The integral equations (2.8) and (2.10) are *generalized linear differential equations* which are special cases of the generalized differential equations introduced by J. Kurzweil (cf. e.g. [Ku1]). In the case $A \in \mathrm{BV}^{n\times n}$ and $h \in \mathrm{BV}^n$ the fundamental results for such equations may be found in [STV], Chapter III. Corollary 2.4 enables us to transfer directly all the results known for the homogeneous generalized linear differential equation (2.10) to the equation (2.2). Let us summarize some of them in the next two propositions.

2.6. Proposition. *Let the assumptions 2.1 be satisfied and let $t_0 \in [0,1]$ be given. Then for any $c \in \mathbb{R}^n$ the equation (2.2) possesses a unique solution $x \in \mathrm{G}^n_{reg}$ with $x(t_0) = c$ if and only if the relations*

$$\begin{aligned} &\det\big(A_1(t)A_0(t) - \Delta^- A_2(t)\big) \neq 0 \quad \textit{for all} \quad t \in (t_0, 1], \\ &\det\big(A_1(t)A_0(t) + \Delta^+ A_2(t)\big) \neq 0 \quad \textit{for all} \quad t \in [0, t_0) \end{aligned} \tag{2.11}$$

hold.

If the conditions (2.11) are satisfied, then there exists a unique $n \times n$-matrix valued function $U(t,s)$ defined on

$$\Delta = \big\{(t,s); 0 \le t \le s \le t_0 \ \textit{or}\ \ t_0 \le s \le t \le 1\big\}$$

and such that the relation

$$U(t,s) = I + \int_s^t \mathrm{d}[A(\tau)]U(\tau,s), \tag{2.12}$$

with $A(t)$ given by (2.9), holds for all $(t,s) \in \Delta$.

Given an arbitrary $c \in \mathbb{R}^n$, the corresponding solution of the initial value problem (2.2), $x(t_0) = c$ is given by

$$x(t) = A_0^{-1}(t)U(t,t_0)A_0(t_0)c, \quad t \in [0,1].$$

Proof follows from Theorem III.1.4 and Theorem III.2.2 in [STV] and from the relations

$$\begin{aligned} I - \Delta^- A(t) &= I - \Delta^- B(t)A_0^{-1}(t) = I - A_1^{-1}(t)\Delta^- A_2(t)A_0^{-1}(t) = \\ &= A_1^{-1}(t)\Big[A_1(t)A_0(t) - \Delta^- A_2(t)\Big]A_0^{-1}(t), \quad t \in (0,1] \end{aligned}$$

and

$$I + \Delta^+ A(t) = A_1^{-1}(t)\Big[A_1(t)A_0(t) + \Delta^+ A_2(t)\Big]A_0^{-1}(t), \quad t \in [0,1).$$

■

2.7. Proposition. *Let the assumptions 2.1 be satisfied. Let*

$$\begin{aligned}(2.13)\qquad &\det\Big(A_1(t)A_0(t)+\Delta^+A_2(t)\Big)\neq 0 \quad \textit{for all} \quad t\in[0,1),\\ &\det\Big(A_1(t)A_0(t)-\Delta^-A_2(t)\Big)\neq 0 \quad \textit{for all} \quad t\in(0,1].\end{aligned}$$

Then the function $U(t,s)$ given by Proposition 2.6 is defined and fulfils (2.12) *for all* $(t,s)\in[0,1]\times[0,1]$. *Furthermore,*

$$(2.14)\qquad U(t,r)U(r,s)=U(t,s) \quad \textit{for all} \quad t,s,r\in[0,1]$$

and

$$(2.15)\qquad v(U)+\operatorname{var}_0^1 U(0,.)+\operatorname{var}_0^1 U(.,0)<\infty,$$

where $v(U)$ stands for the Vitali two-dimensional variation of U on $[0,1]\times[0,1]$.

bf. 2.8. Remark. It follows from (2.14) that under the assumptions of Proposition 2.7 the relation $U(t,s)U(s,t)=I$ holds for all $t,s\in[0,1]$. Hence

$$U(s,t)=U^{-1}(t,s) \quad \text{and} \quad \det U(t,s)\neq 0 \quad \text{for all} \quad t,s\in[0,1].$$

Furthermore, the relation (2.15) implies that U is bounded on $[0,1]\times[0,1]$ and there exists an $M<\infty$ such that

$$v(U)+\operatorname{var}_0^1 U(t,.)+\operatorname{var}_0^1 U(.,s)\leq M \quad \text{for all} \quad t,s\in[0,1].$$

The next assertion follows immediately from [Tv2, Proposition 2.5] whose assumption that the right-hand side of the nonhomogeneous generalized differential equation is left-continuous on $(0,1)$ was not utilized in the proof.

2.9. Theorem. *Let the assumptions 2.1 and* (2.11) *be satisfied. Then the equation* (2.1) *possesses for any $t_0\in[0,1]$ and any $c\in\mathbb{R}^n$ a unique solution $x\in\mathrm{G}^n_{reg}$ on* $[0,1]$ *such that $x(t_0)=c$. This solution is given on $[0,1]$ by*

$$\begin{aligned}(2.16)\qquad x(t)=&A_0^{-1}(t)U(t,t_0)A_0(t_0)c+A_0^{-1}(t)\int_{t_0}^t A_1^{-1}(s)\mathrm{d}[f(s)]-\\ &-A_0^{-1}(t)\int_{t_0}^t \mathrm{d}_s[U(t,s)]\Big(\int_{t_0}^s A_1^{-1}(r)\mathrm{d}[f(r)]\Big),\end{aligned}$$

where $U(t,s)$ is given by Proposition 2.6.

2.10. Remark. A theorem on existence and uniqueness of a solution to the equation (2.1) with $A_0(t)\equiv A_1(t)\equiv I$ has been established by J. Ligęza in [Li4] (cf. Theorem 3.4).

2.11. Corollary. *Let the assumptions 2.1 and* (2.11) *be satisfied and let* $c \in \mathbb{R}^n$ *be given. Then the corresponding solution of the initial value problem* (2.1), $x(0) = c$ *is given by*

(2.17)
$$x(t) = \begin{cases} c, & \text{if } t = 0 \\ A_0^{-1}(t)U(t,0)A_0(0)c + A_0^{-1}(t)\displaystyle\int_0^t \widetilde{U}(t,s)A_1^{-1}(s)\mathrm{d}[f(s)] & \\ \quad - A_0^{-1}(t)[U(t,t+) - U(t,t-)]A_1^{-1}(t)\Delta^- f(t), & \text{if } 0 < t < 1, \\ = A_0^{-1}(1)U(1,0)A_0(0)c + A_0^{-1}(1)\displaystyle\int_0^1 \widetilde{U}(1,s)A_1^{-1}(s)\mathrm{d}[f(s)], & \text{if } t = 1, \end{cases}$$

where $U(t,s)$ *is given by Proposition 2.6 and* $\widetilde{U}(t,s)$ *is given by*

$$\widetilde{U}(t,s) = \begin{cases} U(t,0+) & \text{if } s = 0, \\ \frac{1}{2}\big[U(t,s-) + U(t,s+)\big] & \text{if } 0 < s < 1, \\ U(t,1-) & \text{if } s = 1. \end{cases} \tag{2.18}$$

Proof follows from Theorem 2.9 by Proposition 1.2.1. ■

2.12. Remark. Let us notice that the conditions (2.13) are satisfied if

$$\Delta^+ A(0) = 0, \quad \Delta^- A(1) = 0, \tag{2.19}$$

and

$$\big[\Delta^+ A(t)\big]^2 = \big[\Delta^- A(t)\big]^2 = 0 \quad \text{for} \quad t \in (0,1).$$

In this case we moreover have for any $t \in (0,1)$

$$\big[I + \Delta^+ A(t)\big]\big[I - \Delta^- A(t)\big] = \big[I - \Delta^- A(t)\big]\big[I + \Delta^+ A(t)\big] = I,$$

i.e.

$$\big[I + \Delta^+ A(t)\big]^{-1} = \big[I - \Delta^- A(t)\big]$$

and the assertion (iv) of Theorem II.2.10 in [STV] implies that

$$\begin{aligned} U(t,t+) - U(t,t-) &= I + \Delta^+ A(t)]^{-1} - [I - \Delta^- A(t)]^{-1} = \\ &= -\Delta^- A(t) - \Delta^+ A(t) = -\Delta A(t) \end{aligned}$$

holds for each $t \in (0,1)$. Consequently, if (2.19) is true, then the formula (2.17) reduces in the case $t \in (0,1)$ to

$$\begin{aligned} x(t) = A_0^{-1}(t)U(t,0)A_0(0)c + A_0^{-1}(t)\int_0^t \widetilde{U}(t,s)A_1^{-1}(s)\mathrm{d}[f(s)] \\ + A_0^{-1}(t)\Delta A(t)A_1^{-1}(t)\Delta^- f(t), \quad t \in (0,1). \end{aligned}$$

3. Linear distributional differential equations of higher order

Let us consider the system

$$P_1\big(P_0u^{(m-1)}\big)' + P_2'u^{(m-1)} + \dots + P_m'u' + P_{m+1}'u = q', \tag{3.1}$$

where the $n\times n$-matrix valued functions $P_0, P_1, \dots, P_{m+1}$ and the n-vector valued function q fulfil the assumptions 3.1 and the solutions are defined by Definition 3.2.

3.1. Assumptions. P_0 and P_1 are $n\times n$-matrix valued functions defined and continuous on $[0,1]$, $\det\big(P_0(t)P_1(t)\big)\neq 0$ on $[0,1]$; $P_1, P_2, \dots, P_{m+1}\in \mathrm{BV}_{reg}^{n\times n}$ and $q\in G_R^n$.

3.2. Definition. An n-vector valued function u is called a *solution* to the system (3.1) if

$$u, u', \dots, u^{(m-1)} \in G_R^n$$

and

$$P_1\big(P_0u^{(m-1)}\big)' + P_2'u^{(m-1)} + \dots + P_m'u' + P_{m+1}'u - q'$$

is the zero n-vector distribution.

Let us denote $x_1 = u,\ x_2 = u', \dots, x_m = u^{(m-1)}$,

$$x = \begin{pmatrix} x_1\\ x_2\\ \dots\\ x_m\end{pmatrix},\quad A_0 = \begin{pmatrix} I & 0\\ 0 & P_0\end{pmatrix},\quad A_1 = \begin{pmatrix} I & 0\\ 0 & P_1\end{pmatrix}\quad \text{and}\quad f = \begin{pmatrix} 0\\ q\end{pmatrix},$$

where I stands for the unit $(m-1)n\times(m-1)n$-matrix and the symbol 0 denotes the zero matrix of the type corresponding to its position. Furthermore, let us put

$$A_2 = \begin{pmatrix} 0 & It & 0 & \dots & 0\\ 0 & 0 & It & \dots & 0\\ \dots & \dots & \dots & \dots & \dots\\ -P_{m+1} & -P_m & -P_{m-1} & \dots & -P_2\end{pmatrix},$$

where I and 0 stand for the unit $n\times n$-matrix and the zero $n\times n$-matrix, respectively.

Then A_0, A_1, A_2 and f fulfil the assumptions 2.1 and the system (3.1) is equivalent to the system

$$A_1\big(A_0x\big)' - A_2'x = f', \tag{3.2}$$

with solutions defined by Definition 2.2.

The next assertion is now an immediate consequence of Theorem 2.9.

3.3. Theorem. *Let the assumptions 3.1 be satisfied and let*

$$\det\big(P_1(t)P_0(t) - \Delta^- P_2(t)\big) \neq 0 \quad \text{on} \quad (0,1].$$

Then for all $c_0, c_1, \dots c_m \in \mathbb{R}^n$ *the system* (3.1) *possesses a unique solution* u *such that* $u(0) = c_0, u'(0) = c_1, \dots, u^{(m-1)}(0) = c_{m-1}$. *This solution is given by*

$$\begin{aligned} u(t) &= U_{1,1}(t,0)c_0 + U_{1,2}(t,0)c_1 + \cdots + U_{1,m}(t,0)c_{m-1} \\ &\quad - \int_0^t \mathrm{d}_s[U_{1,m}(t,s)]\Big(\int_0^s P_1^{-1}(r)\mathrm{d}[q(r)]\Big), \quad t \in [0,1], \end{aligned} \tag{3.4}$$

where $U_{i,j}(t,s)$, $i,j = 1,2,\dots,m$ *are the* $n \times n$*-matrix valued functions such that the* $mn \times mn$*-matrix valued function*

$$U(t,s) = \Big(U_{i,j}(t,s)\Big)_{i,j=1,2,\dots,m}$$

is the fundamental matrix solution corresponding to the system (3.2) *by Proposition 2.6.*

Furthermore, for $j = 2,3,\dots,m-1$ *and* $t \in [0,1]$ *we have*

$$\begin{aligned} u^{(j)}(t) &= U_{j,1}(t,0)c_0 + U_{j,2}(t,0)c_1 + \cdots + U_{j,m}(t,0)c_{m-1} \\ &\quad - \int_0^t \mathrm{d}_s[U_{j,m}(t,s)]\Big(\int_0^s P_1^{-1}(r)\mathrm{d}[q(r)]\Big), \end{aligned}$$

and

$$\begin{aligned} P_0(t)u^{(m-1)}(t) &= U_{m,1}(t,0)c_0 + U_{m,2}(t,0)c_1 + \cdots + U_{m,m}(t,0)c_{m-1} \\ &\quad + \int_0^t P_1^{-1}(s)\mathrm{d}[q(s)] - \int_0^t \mathrm{d}_s[U_{2,2}(t,s)]\Big(\int_0^s P_1^{-1}(r)\mathrm{d}[q(r)]\Big), \end{aligned}$$

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