Linear Distributional Differential Equations of the Second Order

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Abstract. The paper deals with the linear differential equation

$$(pu')' + q'u = f'' \tag{0.1}$$

with distributional coefficients and solutions from the space of regulated functions. Our aim is to get the basic existence and uniqueness results for the equation (0.1) and to generalize the known results due to F. V. Atkinson [At64]; J. Ligęza [Li73],[Li77], [Li86], [Li88], [Li89]; R. Pfaff [Pf80], [Pf81]; A. B. Mingarelli [Mi83], as well as the results from the paper [PeTv93] concerning the equation (0.1).

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1. Preliminaries

Throughout the paper $\mathbb{R}$ denotes the space of real numbers, $[0,T]$ is the closed interval $0 \le t \le T$, $(0,T)$ is the open interval $0 < t < T$, while $[0,T)$ and $(0,T]$ are the corresponding half-open intervals.

Any function $f : [0,T] \mapsto \mathbb{R}$ which possesses finite limits

$$f(t+) = \lim_{\tau \to t+} f(\tau) \quad \text{and} \quad f(s-) = \lim_{\tau \to s-} f(\tau)$$

for all $t \in [0,T)$ and $s \in (0,T]$ is said to be *regulated* on $[0,T]$. The space of functions regulated on $[0,T]$ is denoted by $\mathbb{G}$, while $\mathbb{G}_{\text{reg}}$ stands for the set of all functions $f \in \mathbb{G}$ such that

$$f(0+) = f(0),\ f(t) = \tfrac{1}{2}[f(t-) + f(t+)] \quad \text{for all} \quad t \in (0,T),\ f(T-) = f(T). \tag{1.1}$$

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Functions fulfilling (1.1) are usually called *regular* on $[0,T]$.

Given $f \in \mathbb{G}$, $t \in [0,T)$, $s \in (0,T]$ and $r \in (0,T)$, we put

$$\Delta^+ f(t) = f(t+) - f(t),\ \Delta^- f(s) = f(s) - f(s-) \quad \text{and} \quad \Delta f(r) = f(r+) - f(r-).$$

$\mathbb{BV}$ denotes the space of functions of bounded variation on $[0,T]$. The subspace of $\mathbb{BV}$ consisting of the functions of bounded variation on $[0,T]$ and regular on $[0,T]$ will be denoted by $\mathbb{BV}_{\text{reg}}$.

As usual, $\mathbb{L}_1$ stands for the space of measurable and Lebesgue integrable functions on $[0,T]$, $\mathbb{L}_\infty$ denotes the space of measurable and essentially bounded functions on $[0,T]$ and $\mathbb{AC}$ stands for the space of functions absolutely continuous on $[0,T]$.

The integrals which occur in this paper are the Perron-Stieltjes ones. Let us mention here some of their further properties often needed later on.

Let the functions f, g be regulated on $[0,T]$. If the integral

$$\int_0^T f(s)\,\mathrm{d}g(s)$$

has a finite value, then by [Ku57, Theorem 1.3.4] the function

$$h : t \in [0,T] \mapsto \int_0^T f\,\mathrm{d}g \in \mathbb{R}$$

is regulated on $[0,T]$. Let us note that the integral

$$\int_0^T f(s)\,\mathrm{d}g(s)$$

has a finite value if both the functions f, g are regulated on $[0,T]$ and at least one of them has a bounded variation on $[0,T]$ (cf. [Tv89, Theorem 2.8]). In this case the above mentioned [Ku57, Theorem 1.3.4] implies that

$$h(t+) = h(t) + f(t)\Delta^+ g(t) \quad \text{and} \quad h(s-) = h(s) - f(s)\Delta^- g(s)$$

holds for all $t \in [0,T)$ and $s \in (0,T]$. Moreover, if $g \in \mathbb{BV}$, then $h \in \mathbb{BV}$ as well.

1.1. Proposition. (Substitution Theorem) *Let $f, g, h : [0,T] \mapsto \mathbb{R}$ be such that h is bounded on $[0,T]$ and the integral*

$$\int_0^T f(t)\,\mathrm{d}[g(t)]$$

exists. Then the integral

$$\int_0^T h(t) f(t)\,\mathrm{d}[g(t)]$$

exists if and only if the integral

$$\int_0^T h(t)\,\mathrm{d}\Big[\int_0^T f(s)\,\mathrm{d}[g(s)]\Big]$$

exists, and in this case the relation

$$\int_0^T h(t)\,\mathrm{d}\Big[\int_0^T f(s)\,\mathrm{d}[g(s)]\Big] = \int_0^T h(t) f(t)\,\mathrm{d}[g(t)]$$

holds.

For the *proof* see [Tv89, Theorem 2.19]. □

1.2. Proposition. (Integration-by-parts formula) *If $f \in \mathbb{BV}$ and $g \in \mathbb{G}$, then both the integrals*

$$\int_0^T f(t)\,\mathrm{d}[g(t)] \quad \textit{and} \quad \int_0^T \mathrm{d}[f(t)]g(t)$$

exist and

$$\begin{aligned}\int_0^T f(t)\,\mathrm{d}[g(t)] &+ \int_0^T \mathrm{d}[f(t)]g(t)\\ &= f(T)g(T) - f(0)g(0) - \Delta^+ f(0)\Delta^+ g(0)\\ &\quad + \sum_{0<t<T} [\Delta^- f(t)\Delta^- g(t) - \Delta^+ f(t)\Delta^+ g(t)] + \Delta^- f(T)\Delta^- g(T).\end{aligned}$$

For the *proof* see [Tv89, Theorem 2.15]. □

Further properties of the Perron-Stieltjes integral with respect to regulated functions were described in [Tv89] and [Tv91]. (See also [SchTvVe79] and [PeTv93].)

Distributions considered in this paper are linear continuous functionals on the topological vector space $\mathscr{D}$ of functions $\varphi : \mathbb{R} \mapsto \mathbb{R}$ possessing for any $j \in N \cup \{0\}$ a derivative $\varphi^{(j)}$ of the order j which is continuous on $\mathbb{R}$ and such that $\varphi^{(j)}(t) = 0$ for any $t \in \mathbb{R} \setminus (0, T)$. The space $\mathscr{D}$ is endowed with the topology in which the sequence $\varphi_k \in \mathscr{D}$ tends to $\varphi_0 \in \mathscr{D}$ in $\mathscr{D}$ if and only if

$$\lim_k \|\varphi_k^{(j)} - \varphi_0^{(j)}\| = 0$$

for all non negative integers j. The space of distributions on $[0, T]$ (i.e. the dual space to $\mathscr{D}$) is denoted by $\mathscr{D}^*$. Given a distribution $f \in \mathscr{D}^*$ and a test function $\varphi \in \mathscr{D}$, the value of the functional f on φ is denoted by $\langle f, \varphi\rangle$. For any $f \in \mathbb{L}_{1\,1}$, the relation

$$\varphi \in \mathscr{D} \to \int_0^T f(t)\varphi(t)\,\mathrm{d}t$$

defines a distribution on $[0,T]$ which will be denoted by the same symbol f, i.e.

$$\langle f,\varphi\rangle = \int_0^T f(t)\varphi(t)\,\mathrm{d}t \quad \text{for all} \quad \varphi\in\mathscr{D}.$$

In this sense, the zero distribution $0\in\mathscr{D}^*$ on $[0,T]$ is identified with an arbitrary measurable function vanishing a.e. on $[0,T]$. Obviously, if $f\in\mathbb{G}$, then $f=0\in\mathscr{D}^*$ only if $f(t-)=f(s+)=0$ for all $t\in(0,T]$ and all $s\in[0,T)$. Consequently, if $f\in\mathbb{G}_{\text{reg}}$, then $f=0\in\mathscr{D}^*$ if and only if $f(t)=0$ for all $t\in[0,T]$. This means that for a given $g\in\mathbb{L}_1{}^T$ there may exist at most one function $f\in\mathbb{G}_{\text{reg}}$ such that $f(t)=g(t)$ a.e. on $[0,T]$.

Given two distributions $f,g\in\mathscr{D}^*$, $f=g$ means that $f-g=0\in\mathscr{D}^*$. In particular, for given functions $f,g:[0,T]\to\mathbb{R}$, $f=g$ holds if and only if $f(t)=g(t)$ a.e. on $[0,T]$. Whenever a relation of the form $f=g$ for distributions or functions f and g (written without arguments) occurs in the following text, it is understood as the equality in the above sense.

Given an arbitrary $f\in\mathscr{D}^*$, f' denotes its distributional derivative, i.e.

$$f' : \varphi\in\mathscr{D}\mapsto\langle f',\varphi\rangle = -\langle f,\varphi'\rangle.$$

Analogously, for $j\in N$,

$$f^{(j)} : \varphi\in\mathscr{D}\mapsto\langle f^{(j)},\varphi\rangle = (-1)^j\langle f,\varphi^{(j)}\rangle.$$

For absolutely continuous functions their distributional derivatives coincide with their classical derivatives, of course. It is well-known that if $f\in\mathscr{D}^*$, then $f'=0$ if and only if $f\in\mathbb{L}_{1\,1}$ and there exists $c_0\in\mathbb{R}$ such that $f(t)=c_0$ a.e. on $[0,T]$ (cf. [Ha52, Sec.3]). It follows easily that if k is a non negative integer, then $f^{(k)}=0\in\mathscr{D}^*$ if and only if there exist $c_0,c_1,\dots,c_{k-1}\in\mathbb{R}$ such that

$$f(t)=c_0+c_1t+\dots+c_{k-1}t^{k-1} \quad \text{a.e. on} \quad [0,T].$$

Let us notice that if $u\in\mathbb{G}_{\text{reg}}$ and $v\in\mathbb{G}_{\text{reg}}$ are such that $u'=v$, then $u\in\mathbb{AC}$. Indeed, for

$$w(t)=u(0)+\int_0^t v(s)\,\mathrm{d}s$$

we have $(w-u)'=0, w(0)=u(0)$ and consequently (as $w-u\in\mathbb{G}_{\text{reg}}$) $w(t)\equiv u(t)$ on $[0,T]$.

1.3. Definition. Let $f\in\mathbb{G}$ and $g\in\mathbb{BV}$ be such that

$$\Delta^+f(t)\Delta^+g(t)=\Delta^-f(t)\Delta^-g(t) \quad \text{for all} \quad t\in(0,T). \tag{1.2}$$

Then we define

$$f'g:\ \varphi\in\mathscr{D}\mapsto\langle f'g,\varphi\rangle=\int_0^T g(t)\varphi(t)\,\mathrm{d}[f(t)] \tag{1.3}$$

and

$$fg' : \varphi \in \mathscr{D} \mapsto \langle fg', \varphi \rangle = \int_0^T f(t)\varphi(t)\,\mathrm{d}[g(t)]. \tag{1.4}$$

1.4. Remark. Let us notice that the condition (1.2) is satisfied e.g. in the following cases :

(i) both f and g are regular on $[0,T]$,

(ii) at least one of the functions f or g is continuous on $(0,T)$,

(iii) one of the functions f, g is left-continuous on $(0,T)$, while the other is right-continuous on $(0,T)$.

If $f \in \mathbb{L}_{1\,1}$ and $g \in \mathbb{G}$, then (1.3) implies that the product fg is given by

$$fg : \varphi \in \mathscr{D} \mapsto \langle fg, \varphi \rangle = \int_0^T f(t)g(t)\varphi(t)\,\mathrm{d}t,$$

i.e. the product of the functions f and g is in such a case represented by the function $t \in [0,T] \mapsto f(t)g(t)$.

1.5. Lemma. *Let* $f \in \mathbb{G}$ *and* $g \in \mathbb{BV}$ *satisfy* (1.2). *Then*

$$f'g = \Big(\int_0^t \mathrm{d}[f(s)]g(s)\Big)' \tag{1.5}$$

and

$$fg' = \Big(\int_0^t \mathrm{d}[g(s)]f(s)\Big)'. \tag{1.6}$$

Proof. In virtue of Propositions 1.1 and 1.2 we have for any $\varphi \in \mathscr{D}$

$$\langle f'g, \varphi \rangle = \int_0^T \mathrm{d}\Big[\int_0^t \mathrm{d}[f(s)]g(s)\Big]\varphi(t) = -\int_0^T \Big(\int_0^T \mathrm{d}[f(s)]g(s)\Big)\varphi'(t),$$

i.e. (1.5) is true. The formula (1.6) could be verified analogously. □

1.6. Remark. It follows from Definition 1.3 and from the integration-by-parts formula (cf. Proposition 1.2) that for any couple of functions $f \in \mathbb{G}$, $g \in \mathbb{BV}$ fulfilling the condition (1.2) the relation

$$(fg)' = fg' + f'g$$

holds.

1.7. Remark. It is easy to see that for $\tau \in (0,T)$, $f(t) = 0$ for $t < \tau$, $f(\tau) = \frac{1}{2}$, $f(t) = 1$ for $t > \tau$ and $g = f'$, we obtain from Definition 1.3 $fg = \frac{1}{2}g$, i.e. Definition 1.3 seems not to be contradictory to the known definitions of the product of measures and regulated functions based on the sequential approach (cf. [Li88]).

2. Homogeneous equation

Let us consider the equation

$$(pu')' + q'u = 0, \tag{2.1}$$

where

$$p, q \in \mathbb{BV}_{\text{reg}}, \quad p(t) \neq 0 \text{ on } [0,T] \quad \text{and} \quad p^{-1} \in \mathbb{BV}. \tag{2.2}$$

2.1. Definition. A function $u : [0,T] \mapsto \mathbb{R}$ is called a *solution to the equation* (2.1) *on the interval* $[0,T]$ if $u \in \mathbb{G}_{\text{reg}}$ and $(pu')' + q'u$ is the zero distribution on $[0,T]$.

2.2. Proposition. *A function $u \in \mathbb{G}_{reg}$ is a solution to the equation* (2.1) *on* $[0,T]$ *if and only if $u \in \mathbb{AC}$, $u' \in \mathbb{L}_{1\,\infty}$ and there is $v \in \mathbb{BV}_{reg}$ such that the couple (u,v) is a solution on $[0,T]$ to the system of integral equations*

$$u(t) = u(0) + \int_0^t p^{-1}(s)v(s)\,\mathrm{d}s, \quad t \in [0,T] \tag{2.3}$$

$$v(t) = v(0) - \int_0^t \mathrm{d}[q(s)]u(s), \qquad t \in [0,T]. \tag{2.4}$$

Proof. a) Let $u \in \mathbb{G}_{\text{reg}}$ and $v \in \mathbb{BV}_{\text{reg}}$ fulfil (2.3) and (2.4) on $[0,T]$. Then obviously

$$u' = p^{-1}v \in \mathbb{L}_{1\,\infty}.$$

By Lemma 1.5 we have

$$v' = -q'u.$$

Moreover, making use of the Substitution Theorem (cf. Proposition 1.1) and of Lemma 1.5 we obtain

$$pu' = \Big(\int_0^t p(s)\Big[\mathrm{d}\int_0^s p^{-1}(\tau)v(\tau)\,\mathrm{d}\tau\Big]\Big)' = \Big(\int_0^t v(\tau)\,\mathrm{d}\tau\Big)' = v.$$

Hence

$$(pu')' + q'u = v' + q'u = 0.$$

b) Let $u \in \mathbb{G}_{\text{reg}}$ be a solution to (2.1) on $[0,T]$. Then

$$\Big(\int_0^t p(s)\,\mathrm{d}[u(s)] + \int_0^t \Big(\int_0^s \mathrm{d}[q(\tau)]u(\tau)\Big)\,\mathrm{d}s\Big)'' = 0.$$

Hence there are c_0 and $c_1 \in \mathbb{R}$ such that

$$\int_0^t p(s)\,\mathrm{d}[u(s)] + \int_0^t \Big(\int_0^s \mathrm{d}[q(\tau)]u(\tau) \Big)\,\mathrm{d}s - c_0 - c_1 t = 0 \tag{2.5}$$

holds for a.e. $t \in (0,T)$. The left-hand side of (2.5) being a regulated function which is regular on $[0,T]$, the relation (2.5) is true for a.e. $t \in [0,T]$ if and only if it is true for each $t \in [0,T]$. In particular, inserting $t = 0$ we get $c_0 = 0$. Moreover,

$$\begin{aligned}\int_0^t p(s)\,\mathrm{d}[u(s)] &= c_1 t - \int_0^t \Big(\int_0^s \mathrm{d}[q(\tau)]u(\tau) \Big)\,\mathrm{d}s \\ &= \int_0^t \Big(c_1 - \int_0^s \mathrm{d}[q(\tau)]u(\tau) \Big)\,\mathrm{d}s \quad \text{for all} \quad t \in [0,T].\end{aligned}$$

Let us denote

$$v(t) = c_1 - \int_0^t \mathrm{d}[q(s)]u(s), \quad \text{for} \quad t \in [0,T].$$

Then $v \in \mathbb{BV}_{\mathrm{reg}}$, $v(0) = c_1$ and the couple (u,v) fulfils the relations (2.4) and

$$\int_0^t p(s)\,\mathrm{d}[u(s)] = \int_0^t v(s)\,\mathrm{d}s \quad t \in [0,T]. \tag{2.6}$$

In particular,

$$\int_0^t p(s)\,\mathrm{d}[u(s)] \in \mathbb{AC}.$$

Furthermore, differentiating the relation (2.6) we get

$$pu' = v. \tag{2.7}$$

Making use of the Substitution Theorem (cf. Proposition 1.1) and of Lemma 1.5 we obtain from (2.7) that

$$\begin{aligned}p^{-1}v = p^{-1}(pu') &= p^{-1}\Big(\int_0^t p(s)\,\mathrm{d}[u(s)] \Big)' \\ &= \Big(\int_0^t p^{-1}(s)\,\mathrm{d}\Big[\int_0^s p(\tau)\,\mathrm{d}[u(\tau)] \Big] \Big)' = \Big(\int_0^T \mathrm{d}[u(s)] \Big)' = u',\end{aligned}$$

i.e.

$$u' = p^{-1}v. \tag{2.8}$$

Consequently $u' = p^{-1}v \in \mathbb{L}_{1\,\infty}$. It follows that $u \in \mathbb{AC}$ and the relation (2.8) is true if and only if the relation (2.3) is true. This completes the proof of the proposition. $\square$

2.3. Remark. It follows from the proof of Proposition 2.2 that for any solution u of the equation (2.1) on $[0,T]$ there exists a function $w \in \mathbb{BV}$ such that

$$u'(t) = w(t) \quad \text{a.e. on} \quad [0,T] \quad \text{and} \quad p(t)w(t) = v(t) \quad \text{on} \quad [0,T].$$

The system (2.3), (2.4) may be rewritten in the vector form

$$x(t) = x(0) + \int_0^t \mathrm{d}[A(s)]x(s), \tag{2.9}$$

where

$$x(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} \quad \text{and} \quad A(t) = \begin{pmatrix} 0 & \int_0^t p^{-1}(s)\,\mathrm{d}s \\ -q(t) & 0 \end{pmatrix}, \quad t \in [0,T]. \tag{2.10}$$

Obviously, $A(t)$ is a 2×2-matrix valued function of bounded variation on $[0,T]$. The system (2.9) is a generalized linear differential equation considered e.g. in [Sch92] (cf. also [SchTvVe79] or [Sch85]). Under our assumptions $A(0+) = A(0)$, $A(T-) = A(T)$,

$$\det\big(I - \Delta^- A(t)\big) = \det \begin{pmatrix} 1 & 0 \\ \Delta^- q(t) & 1 \end{pmatrix} = 1 \quad \text{for} \quad t \in (0,T] \tag{2.11}$$

and

$$\det\big(I + \Delta^+ A(t)\big) = \det \begin{pmatrix} 1 & 0 \\ -\Delta^+ q(t) & 1 \end{pmatrix} = 1 \quad \text{for} \quad t \in [0,T). \tag{2.12}$$

Hence the following assertion is an immediate consequence of [Sch92, Proposition 2.1 and Theorem 6.5] (cf. also [SchTvVe79, Theorem III.1.4]).

2.4. Theorem. *Let us assume* (2.2). *Then for any $u_0, v_0 \in \mathbb{R}$ and any $t_0 \in [0,T]$, there exists a unique solution $u \in \mathbb{AC}$ of the equation* (2.1) *on* $[0,T]$ *and a unique function $v \in \mathbb{BV}_{reg}$ such that*

$$p(t)u'(t) = v(t) \ \textit{a.e. on} \ [0,T], \quad u(t_0) = u_0 \quad \textit{and} \quad v(t_0) = v_0.$$

2.5. Remark. It follows easily from the basic properties of the Perron-Stieltjes integral that the relations

$$\Delta^+ v(0) = \Delta^- v(T) = 0 \quad \text{and} \quad \Delta v(t) = -\Delta q(t) u(t), \quad t \in (0,T)$$

hold for any couple of functions $u \in \mathbb{AC}$, $v \in \mathbb{BV}$ satisfying the system (2.3),(2.4) on $[0,T]$ (cf. e.g. [SchTvVe79, Proposition III.1.6]).

2.6. Remark. Theorem 2.4 could be obtained from Proposition 2.2 by making use of a somewhat modified version of [Mi83, Theorem I.3.1], as well.

2.7. Corollary. *There exists a unique system of functions* $\{u_1, v_1, u_2, v_2\}$ *possessing the following properties:*

$$u_1, u_2 \in \mathbb{AC}, \quad u_1', u_2' \in \mathbb{L}_{1\,\infty}, \quad v_1, v_2 \in \mathbb{BV}_{reg}, \tag{2.13}$$

$$v_i' + q' u_i = 0 \quad \textit{and} \quad p u_i' = v_i \quad (i = 1, 2), \tag{2.14}$$

and

$$u_1(0) = 1, v_1(0) = 0, u_2(0) = 0, v_2(0) = 1. \tag{2.15}$$

2.8. Definition. The system $\{u_1, v_1, u_2, v_2\}$ of functions possessing the properties (2.13)-(2.15) given by Corollary 2.7 will be called the *fundamental system of solutions to* (2.1) *on* $[0, T]$.

2.9. Corollary. *Let us assume* (2.2) *and let* $\{u_1, v_1, u_2, v_2\}$ *be the fundamental system of solutions to* (2.1) *on* $[0, T]$. *Then a function* $u \in \mathbb{G}_{reg}$ *is a solution to the equation* (2.1) *on* $[0, T]$ *if and only if there are* α *and* $\beta \in \mathbb{R}$ *such that*

$$u(t) = \alpha u_1(t) + \beta u_2(t) \quad \textit{on} \quad [0, T].$$

2.10. Proposition. *Let* $\{u_1, v_1, u_2, v_2\}$ *be the fundamental system of solutions to* (2.1) *on* $[0, T]$. *Then the relation*

$$u_1(t) v_2(t) - u_2(t) v_1(t) \equiv 1 \tag{2.16}$$

holds for all $t \in [0, T]$.

Proof. In virtue of (2.14) we have

$$(u_1 v_2 - u_2 v_1)' = u_1'(p u_2') - u_2'(p u_1') - u_1(q' u_2) - u_2(q' u_1).$$

Since $p \in \mathbb{BV}_{\text{reg}}$ and by (2.13) $u_1', u_2' \in \mathbb{L}_{1\,\infty}$, the products $u_1'(p u_2')$ and $u_2'(p u_1')$ are functions essentially bounded on $[0, T]$ and they are given for a.e. $t \in [0, T]$ by

$$\Big(u_1'(p u_2')\Big)(t) = u_1'(t) p(t) u_2'(t) \quad \text{and} \quad \Big(u_2'(p u_1')\Big)(t) = u_2'(t) p(t) u_1'(t),$$

respectively. Furthermore, by Definition 1.3 and Lemma 1.5 we have

$$u_1(q' u_2) = u_1 \Big(\int_0^t \mathrm{d}[q(s)] u_2(s) \Big)' = \Big(\int_0^t u_1(s) u_2(s) \mathrm{d}[q(s)] \Big)'$$

and

$$u_2(q' u_1) = u_2 \Big(\int_0^t \mathrm{d}[q(s)] u_1(s) \Big)' = \Big(\int_0^t u_1(s) u_2(s) \mathrm{d}[q(s)] \Big)'.$$

Thus

$$\Big(u_1 v_2 - u_2 v_1\Big)' = 0,$$

i.e. there exists a $c \in \mathbb{R}$ such that

$$u_1(t)v_2(t) - u_2(t)v_1(t) = c \tag{2.17}$$

holds for a.e. $t \in [0,T]$. As the left-hand side of the relation (2.17) is a regular function of bounded variation on $[0,T]$, it follows that (2.17) holds for each $t \in [0,T]$. Inserting $t = 0$ we obtain $c = 1$, i.e. the relation (2.16) is true. □

3. Nonhomogeneous equation

This section is devoted to the nonhomogeneous equation

$$(pu')' + q'u = f'', \tag{3.1}$$

where p and q fulfil the assumptions (2.2) and

$$f \in \mathbb{G}_{\text{reg}}. \tag{3.2}$$

3.1. Definition. A function $u : [0,T] \mapsto \mathbb{R}$ is said to be a *solution to the equation* (3.1) *on the interval* $[0,T]$ if $u \in \mathbb{G}_{\text{reg}}$ and $(pu')' + q'u - f''$ is the zero distribution on $[0,T]$.

3.2. Remark. Let us try similarly as in the classical case to find a particular solution y to the equation (3.1) in the form

$$y = \alpha u_1 + \beta u_2, \tag{3.3}$$

where u_1 and $u_2 \in \mathbb{AC}$ are functions from the fundamental system of solutions to the corresponding homogeneous equation

$$(pu')' + q'u = 0$$

given by Definition 2.8 and α and $\beta \in \mathbb{G}_{\text{reg}}$ are such that

$$\alpha' u_1 + \beta' u_2 = 0. \tag{3.4}$$

By the Substitution Theorem (cf. Proposition 1.1) and by Lemma 1.5 we have

$$\begin{aligned}
py' &= p(\alpha u_1' + \beta u_2') = p\Big(\int_0^t \alpha(s)\,\mathrm{d}[u_1(s)] + \int_0^t \beta(s)\,\mathrm{d}[u_2(s)]\Big)' \\
&= \Big(\int_0^t p(s)\,\mathrm{d}\Big[\int_0^s \alpha(s)\,\mathrm{d}[u_1(s)]\Big]\Big)' + \Big(\int_0^t p(s)\,\mathrm{d}\Big[\int_0^s \beta(s)\,\mathrm{d}[u_2(s)]\Big]\Big)' \\
&= \Big(\int_0^t p(s)\alpha(s)\,\mathrm{d}[u_1(s)] + \int_0^t p(s)\beta(s)\,\mathrm{d}[u_2(s)]\Big)' \\
&= \Big(\int_0^t \alpha(s)\,\mathrm{d}\Big[\int_0^s p(s)\,\mathrm{d}[u_1(s)]\Big]\Big)' + \Big(\int_0^t \beta(s)\,\mathrm{d}\Big[\int_0^s p(s)\,\mathrm{d}[u_2(s)]\Big]\Big)' \\
&= \alpha(pu_1') + \beta(pu_2') = \alpha v_1 + \beta v_2
\end{aligned}$$

and

$$\begin{aligned}
q'y &= q'(\alpha u_1 + \beta u_2) = \Big(\int_0^t \mathrm{d}[q(s)](\alpha(s)u_1(s) + \beta(s)u_2(s))\Big)' \\
&= \Big(\int_0^t \alpha(s)\,\mathrm{d}\Big[\int_0^s \mathrm{d}[q(\tau)]u_1(\tau)\Big]\Big)' + \Big(\int_0^t \beta(s)\,\mathrm{d}\Big[\int_0^s \mathrm{d}[q(\tau)]u_2(\tau)\Big]\Big)' \\
&= \alpha(q'u_1) + \beta(q'u_2).
\end{aligned}$$

Hence by (2.14)

$$\begin{aligned}
(py')' + q'y &= \alpha(v_1' + q'u_1) + \beta(v_2' + q'u_2) + \alpha' v_1 + \beta' v_2 \\
&= \alpha' v_1 + \beta' v_2,
\end{aligned}$$

i.e. the function y of the form (3.3) is a solution to (3.1) on $[0,T]$ if and only if the couple $\alpha, \beta \in \mathbb{G}_{\text{reg}}$ satisfies the relations (3.4) and

$$\alpha' v_1 + \beta' v_2 = f''. \tag{3.5}$$

Making use of Proposition 2.10 we could show that (3.4) and (3.5) are satisfied if

$$\alpha' = -f'' u_2 \quad \text{and} \quad \beta' = f'' u_1. \tag{3.6}$$

If we had $p^{-1}v_1$ and $p^{-1}v_2 \in \mathbb{BV}_{\text{reg}}$ or f were continuous on $[0,T]$, then the products $f'u_1' = f'(p^{-1}v_1)$ and $f'u_2' = f'(p^{-1}v_2)$ would be defined by

$$f'u_1' = \Big(\int_0^t \mathrm{d}[f(s)]p^{-1}(s)v_1(s)\Big)'$$

and

$$f'u_2' = \Big(\int_0^t \mathrm{d}[f(s)]p^{-1}(s)v_2(s)\Big)',$$

respectively. Then we would have

$$f''u_1 = (f'u_1)' - f'u_1' = \Big(f'u_1 - \int_0^t \mathrm{d}[f(s)]p^{-1}(s)v_1(s)\Big)'$$

and

$$f''u_2 = (f'u_2)' - f'u_2' = \Big(f'u_2 - \int_0^t \mathrm{d}[f(s)]p^{-1}(s)v_1(s)\Big)'.$$

This means that we could put

$$\alpha = -f'u_2 + \int_0^t \mathrm{d}[f(s)]p^{-1}(s)v_2(s) \tag{3.7}$$

and

$$\beta = \quad f'u_1 - \int_0^t \mathrm{d}[f(s)]p^{-1}(s)v_1(s) \tag{3.8}$$

and after inserting (3.7) and (3.8) into (3.3) we would get the following formula for a particular solution y of the given equation (3.1):

$$y(t) = \int_0^t \mathrm{d}[f(s)]p^{-1}(s)\big(v_2(s)u_1(t) - v_1(s)u_2(t)\big) \quad \text{on} \quad [0,T]. \tag{3.9}$$

The distributions α and β given respectively by (3.7) and (3.8) are in general not regular functions on $[0,T]$, of course. Nevertheless it may be proved that under our assumptions the formula (3.9) provides a solution to (3.1) on $[0,T]$ (without assuming that $p^{-1}v_1$ and $p^{-1}v_2 \in \mathbb{BV}_{\text{reg}}$ or f is continuous on $[0,T]$).

3.3. Theorem. *Let us assume* (2.2) *and* (3.2). *A function $u \in \mathbb{G}_{reg}$ is a solution to the equation* (3.1) *on* $[0,T]$ *if and only if there are α and $\beta \in \mathbb{R}$ such that*

$$\begin{aligned} u(t) &= \alpha u_1(t) + \beta u_2(t) \\ &\quad + \int_0^t \mathrm{d}[f(s)]p^{-1}(s)(v_2(s)u_1(t) - v_1(s)u_2(t)) \quad \textit{on} \quad [0,T], \end{aligned} \tag{3.10}$$

where $\{u_1, u_2, v_1, v_2\}$ is the fundamental system of solutions to the corresponding homogeneous equation $(pu')' + q'u = 0$ on $[0,T]$ given by Definition rm2.8.

Proof. a) Let the function u be given by (3.10). Then obviously $u \in \mathbb{G}_{\text{reg}}$. Furthermore,

$$\begin{aligned} u' &= \alpha u_1' + \beta u_2' \\ &\quad + \Big(\int_0^t \mathrm{d}[f(s)]p^{-1}(s)v_2(s)\Big)' u_1 - \Big(\int_0^t \mathrm{d}[f(s)]p^{-1}(s)v_1(s)\Big)' u_2 \\ &\quad + \Big(\int_0^t \mathrm{d}[f(s)]p^{-1}(s)v_2(s)\Big) u_1' - \Big(\int_0^t \mathrm{d}[f(s)]p^{-1}(s)v_1(s)\Big) u_2'. \end{aligned}$$

In virtue of the Substitution Theorem (cf. Proposition 1.1) and of Lemma 1.5 we have

$$p\Big(\Big(\int_0^t \mathrm{d}[f(s)]p^{-1}(s)v_2(s)\Big)' u_1\Big)$$

$$= p\Big(\int_0^t \mathrm{d}\Big[\int_0^s \mathrm{d}[f(\tau)]p^{-1}(\tau)v_2(\tau)\Big]u_1(s)\Big)' = p\Big(\int_0^t \mathrm{d}[f(s)]p^{-1}(s)v_2(s)u_1(s)\Big)'$$

$$= \Big(\int_0^t p(s)\mathrm{d}\Big[\int_0^s \mathrm{d}[f(\tau)]p^{-1}(\tau)v_2(\tau)u_1(\tau)\Big]\Big)' = \Big(\int_0^t \mathrm{d}[f(s)]v_2(s)u_1(s)\Big)'.$$

Analogously

$$p\Big(\Big(\int_0^t \mathrm{d}[f(s)]p^{-1}(s)v_1(s)\Big)' u_2\Big) = \Big(\int_0^t \mathrm{d}[f(s)]v_1(s)u_2(s)\Big)'.$$

Obviously,

$$p\Big(\Big(\int_0^t \mathrm{d}[f(s)]p^{-1}(s)v_2(s)\Big)u_1'\Big) \quad - p\Big(\Big(\int_0^t \mathrm{d}[f(s)]p^{-1}(s)v_1(s)\Big)u_2'\Big)$$

$$= \Big(\int_0^t \mathrm{d}[f(s)]p^{-1}(s)v_2(s)\Big)pu_1' - \Big(\int_0^t \mathrm{d}[f(s)]p^{-1}(s)v_1(s)\Big)pu_2'$$

$$= \Big(\int_0^t \mathrm{d}[f(s)]p^{-1}(s)v_2(s)\Big)v_1 \quad - \Big(\int_0^t \mathrm{d}[f(s)]p^{-1}(s)v_1(s)\Big)v_2.$$

Since by Proposition 2.10

$$\int_0^t \mathrm{d}[f(s)](v_2(s)u_1(s) - v_1(s)u_2(s)) = f(t) - f(0),$$

we have

$$pu' = \Big(\alpha + \int_0^t \mathrm{d}[f(s)]p^{-1}(s)v_2(s)\Big)v_1 + \Big(\beta - \int_0^t \mathrm{d}[f(s)]p^{-1}(s)v_1(s)\Big)v_2 + f' \tag{3.11}$$

and consequently, taking into account the definition of the functions u_1, u_2, v_1 and v_2, we obtain

$$\begin{aligned}(pu')' + q'u = \; & \Big(\alpha + \int_0^t \mathrm{d}[f(s)]p^{-1}(s)v_2(s)\Big)(v_1' + q'u_1) \\ & + \Big(\beta - \int_0^t \mathrm{d}[f(s)]p^{-1}(s)v_1(s)\Big)(v_2' + q'u_2) \\ & + \Big(\int_0^t \mathrm{d}[f(s)]p^{-1}(s)\big(v_2(s)v_1(s) - v_1(s)v_2(s)\big)\Big)' + f'' = f''.\end{aligned}$$

b) Let u and w be solutions to (3.1) on $[0,T]$ and let $y = u - w$. Then $(py')' + q'y = 0$ and the proof of this theorem is completed by making use of Corollary 2.9. □

Similarly as the corresponding homogeneous equation

$$(pu')' + q'u = 0$$

treated in the previous section the present equation can be rewritten as a system of two linear generalized differential equations, as well.

3.4. Proposition. *A function $u \in \mathbb{G}_{reg}$ is a solution to the equation* (3.1) *on* $[0,T]$ *if and only if there is a function $v \in \mathbb{BV}_{reg}$ such that*

$$u(t) - u(0) - \int_0^t p^{-1}(s)v(s)\,\mathrm{d}s = \int_0^t p^{-1}(s)\,\mathrm{d}[f(s)], t \in [0,T], \tag{3.12}$$

and

$$v(t) - v(0) + \int_0^t \mathrm{d}[q(t)]u(s) \quad = 0, \qquad t \in \ [0,T] \tag{3.13}$$

hold. For a given solution u of the equation (3.1) *on* $[0,T]$ *this function v is determined uniquely.*

Proof. a) Let $u \in \mathbb{G}_{\rm reg}$ be a solution to (3.1) on $[0,T]$. Then by Lemma 1.5 we have

$$\left(\int_0^t p(s)\,\mathrm{d}[u(s)] + \int_0^t \Big(\int_0^s \mathrm{d}[q(\tau)]u(\tau)\Big)\,\mathrm{d}s - f\right)'' = 0,$$

i.e. there are $c_0, c_1 \in \mathbb{R}$ such that

$$\int_0^t p(s)\,\mathrm{d}[u(s)] + \int_0^t \Big(\int_0^s \mathrm{d}[q(\tau)]u(\tau)\Big)\,\mathrm{d}s - f(t) - c_0 - c_1 t = 0 \ \text{ on } \ [0,T]. \tag{3.14}$$

(The left-hand side of (3.14) being regular on $[0,T]$, it equals 0 for a.e. $t \in [0,T]$ if and only if it equals 0 for all $t \in [0,T]$.)

In particular, we have $c_0 = -f(0)$ and

$$\int_0^t p(s)\,\mathrm{d}[u(s)] - f(t) + f(0) = \int_0^t v(s)\,\mathrm{d}s, \quad t \in [0,T], \tag{3.15}$$

where

$$v(t) = c_1 - \int_0^t \mathrm{d}[q(s)]u(s), \quad t \in [0,T]. \tag{3.16}$$

Obviously, $v(0) = c_1$ and hence the equations (3.16) and (3.13) coincide. Moreover, $v \in \mathbb{BV}_{\rm reg}$ and v given by (3.16) is the only function in $\mathbb{BV}_{\rm reg}$ such that (3.15) holds. Differentiating (3.15) we obtain

$$pu' - v = f'. \tag{3.17}$$

By (3.15) the function

$$r(t) = \int_0^t p(s)\,\mathrm{d}[u(s)] - f(t), t \in [0,T]$$

is absolutely continuous on $[0,T]$ and according to Lemma 1.5 we have

$$r' = pu' - f'.$$

Thus the product

$$p^{-1}r' = p^{-1}(pu' - f')$$

is by Definition 1.3 well defined. Making use of the Substitution Theorem (cf. Proposition 1.1) and of Lemma 1.5 we obtain

$$\begin{aligned} p^{-1}r' &= \Big(\int_0^t p^{-1}(s)\,\mathrm{d}[r(s)]\Big)' = \Big(\int_0^t \mathrm{d}[u(s)] - \int_0^t p^{-1}(s)\,\mathrm{d}[f(s)]\Big)' \\ &= u' - \Big(\int_0^t p^{-1}(s)\,\mathrm{d}[f(s)]\Big)', \end{aligned}$$

wherefrom by (3.17) the relation

$$u' - p^{-1}v = \Big(\int_0^t p^{-1}(s)\,\mathrm{d}[f(s)]\Big)' \tag{3.18}$$

follows. By Lemma 1.5 the relation (3.18) holds if and only if there is a $d \in \mathbb{R}$ such that

$$u(t) - \int_0^t p^{-1}(s)v(s)\,\mathrm{d}s - \int_0^t p^{-1}(s)\,\mathrm{d}[f(s)] = d \quad \text{on} \quad [0,T]. \tag{3.19}$$

(The left-hand side of (3.19) is regular on $[0,T]$.) Obviously $d = u(0)$ and hence the equations (3.19) and (3.12) coincide. We have shown that for any solution $u \in \mathbb{G}_{\mathrm{reg}}$ of (3.1) on $[0,T]$ there is a unique $v \in \mathbb{BV}_{\mathrm{reg}}$ such that (3.12) and (3.13) hold.

b) On the other hand, if $u \in \mathbb{G}_{\mathrm{reg}}$ and $v \in \mathbb{BV}_{\mathrm{reg}}$ fulfil (3.12) and (3.13), then by Lemma 1.5 the relations (3.18) and

$$v' + q'u = 0 \tag{3.20}$$

are true. It is easy to see that

$$p(p^{-1}v) = v.$$

Furthermore, by the Substitution Theorem (cf. Proposition 1.1) and Lemma 1.5

$$\begin{aligned} p(p^{-1}f') &= p\Big(\int_0^t p^{-1}(s)\,\mathrm{d}[f(s)]\Big)' = \Big(\int_0^t p(s)\,\mathrm{d}\Big[\int_0^s p^{-1}(\tau)\,\mathrm{d}[f(\tau)]\Big]\Big)' \\ &= \Big(\int_0^t \mathrm{d}[f(s)]\Big)' = f'. \end{aligned}$$

Consequently, the relations (3.18) and (3.20) yield

$$\begin{aligned}(pu')' + q'u &= \big(p(p^{-1}v) + p(p^{-1}f')\big)' - v' \\ &= v' + f'' - v' = f'',\end{aligned}$$

i.e. u is a solution to (3.1) on $[0,T]$. □

3.5. Remark. Results analogous to Theorem 3.3 and Proposition 3.4 for a system similar to (3.12),(3.13) and corresponding to the case that $f' \in \mathbb{BV}$ and f' is right-continuous on $(0,T]$ were given in [Mi83, Theorem I.3.4].

The system (3.12),(3.13) may be rewritten in the vector form

$$x(t) - x(0) - \int_0^t \mathrm{d}[A(s)]x(s) = g(t) - g(0), \quad t \in [0,T],$$

where

$$x(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}, \quad g(t) = \begin{pmatrix} \int_0^t p^{-1}(s)\,\mathrm{d}[f(s)] \\ 0 \end{pmatrix}, \quad t \in [0,T]$$

and the 2×2-matrix valued function $A(t)$ is given by (2.10), i.e.

$$A(t) = \begin{pmatrix} 0 & \int_0^t p^{-1}(s)\,\mathrm{d}s \\ -q(t) & 0 \end{pmatrix}, \quad t \in [0,T].$$

A has a bounded variation on $[0,T]$,

$$\det\Big(I - \Delta^- A(t)\Big) = 1$$

(cf. (2.11) and g is regulated and regular on $[0,T]$. Moreover, A and g are regular on $[0,T]$. Consequently, [Tv91, Proposition 2.5] (whose assumption on the left-continuity of $A(t)$ and $f(t)$ on $(0,T]$ was not exploited in the proof) ensures the existence and uniqueness of solutions of the corresponding initial value problems. This enables us to prove the following assertion.

3.6. Theorem. *Let us assume* (2.2) *and* (3.2) *and let* $\{u_1, u_2, v_1, v_2\}$ *be the fundamental system of solutions to the corresponding homogeneous equation* $(pu')' + q'u = 0$ *on* $[0,T]$ *given by Definition* 2.8. *Then for any* $u_0, v_0 \in \mathbb{R}$ *there exists a unique solution* $u \in \mathbb{G}_{reg}$ *of* (3.1) *on* $[0,T]$ *and a unique function* $v \in \mathbb{BV}_{reg}$ *such that* (3.15), $u(0) = u_0$ *and* $v(0) = v_0$ *hold. This solution* $u(t)$ *is given by* (3.10), *where* $\alpha = u_0$ *and* $\beta = v_0$. *Furthermore,*

$$\begin{aligned}v(t) = &\Big(u_0 + \int_0^t \mathrm{d}[f(s)]p^{-1}(s)v_2(s)\Big)v_1(t) \\ &+ (v_0 - \int_0^t \mathrm{d}[f(s)]p^{-1}(s)v_1(s)\Big)v_2(t) \\ &- \Delta^- f(t)p^{-1}(t)\Delta^- q(t) \quad \textit{for all} \quad t \in (0,T].\end{aligned} \tag{3.21}$$

Proof. It remains to show that in the formula (3.10) for $u(t)$ we may put $\alpha = u_0$ and $\beta = u_1$ and that the formula (3.21) is true. By (2.12) we have $u_1(0) = 1$ and $u_2(0) = 0$. Hence (3.10) implies that $u(0) = \alpha = u_0$. Furthermore, by (3.11) and (3.17)

$$v(t) = w_1(t)v_1(t) + w_2(t)v_2(t) \quad \text{a.e. on } [0,T], \tag{3.22}$$

where

$$w_1(t) = u_0 + \int_0^t \mathrm{d}[f(s)]p^{-1}(s)v_2(s), \quad t \in [0,T] \tag{3.23}$$

and

$$w_2(t) = \beta - \int_0^t \mathrm{d}[f(s)]p^{-1}(s)v_1(s), \quad t \in [0,T]. \tag{3.24}$$

It is easy to see that under our assumptions w_1 and $w_2 \in \mathbb{G}_{\mathrm{reg}}$ and hence (cf. also (2.12)

$$v(0) = v(0+) = w_1(0)v_1(0) + w_2(0)v_2(0) = \beta v_2(0) = \beta,$$

i.e. we may put $\beta = v_0$ in (3.24). Furthermore, by (3.22)-(3.24) we have

$$v(t-) = w_1(t)v_1(t-) + w_2(t)v_2(t-) - \Delta^- f(t)p^{-1}(t)(v_1(t-) - v_2(t-)) \quad \text{for all} \quad t \in (0,T).$$

Analogously

$$w(t+) = w_1(t)v_1(t+) + w_2(t)v_2(t+) + \Delta^+ f(t)p^{-1}(t)(v_1(t+) - v_2(t+)) \quad \text{for all} \quad t \in (0,T).$$

Since $\Delta^- f(t) = \Delta^+ f(t)$ on $(0,T)$ (f is regular on $[0,T]$), it follows that

$$\begin{aligned} v(t) = {} & \tfrac{1}{2}[v(t-) + v(t+)] = w_1(t)v_1(t) + w_2(t)v_2(t) \\ & + \tfrac{1}{2}\Delta^- f(t)p^{-1}(t)\big(v_2(t)\Delta v_1(t) - v_1(t)\Delta v_2(t)\big) \quad \text{for all} \quad t \in (0,T). \end{aligned}$$

Furthermore, we have

$$\Delta v_1(t) = -u_1(t)\Delta q(t) \quad \text{and} \quad \Delta v_2(t) = -u_2(t)\Delta q(t)$$

on $(0,T)$ (cf. Remark 2.5) and

$$u_1(t)v_2(t) - u_2(t)v_1(t) \equiv 1$$

on $[0,T]$ (cf. Proposition 2.10). Hence the relation

$$v(t) = w_1(t)v_1(t) + w_2(t)v_2(t) - \tfrac{1}{2}\Delta^- f(t)p^{-1}(t)\Delta q(t)$$

and consequently also the formula (3.21) hold for any $t \in (0,T)$. (Let us recall that the regularity of q implies that $\Delta q(t) = 2\Delta^- q(t)$ holds for all $t \in (0,T)$.)

Obviously, $v(T) = w_1(T)v_1(T) + w_2(T)v_2(T)$, which completes the proof. $\square$

4. Example

In this section we will consider the boundary value problem

$$u'' + q'u = \sum_{j=1}^{N} g_j \delta'_{\tau_j}, \tag{4.1}$$

$$u(0) = u_0, \quad u(T) = u_T, \tag{4.2}$$

where

$$0 < \tau_1 < \tau_2 < \cdots < \tau_N < T, \quad u_0 \in \mathbb{R},\ u_T \in \mathbb{R}$$

and functions g_j ($j = 1, 2, \ldots, N$) continuously differentiable on $[0, T]$ are given, $\delta_{\tau_j} = h'_{\tau_j}$, $j = 1, 2, \ldots, N$, and h_{τ_j}, $j = 1, 2, \ldots, N$, stand for the regular Heaviside functions with jumps at $t = \tau_j$, i.e.

$$h_{\tau_j}(t) = \begin{cases} 0 & \text{for } t < \tau_j, \\ \frac{1}{2} & \text{for } t = \tau_j, \\ 1 & \text{for } t > \tau_j. \end{cases}$$

A special case of the problem (4.1),(4.2) (with $q' = b$, where $b \in \mathbb{L}_{1\,\infty}$ is piecewise continuous and $N = 1$) was treated in [Ho85], where a procedure of its numerical solution was suggested.

Let us put

$$f(t) = \sum_{j=1}^{N} \left(\int_0^t g_j(s)\,\mathrm{d}[h_{\tau_j}(s)] - \int_0^t \left(\int_0^s g'_j(\sigma)\,\mathrm{d}[h_{\tau_j}(\sigma)] \right) \mathrm{d}s \right) \tag{4.3}$$

for $t \in [0, T]$. It is easy to verify that

$$f' = \sum_{j=1}^{N} \left(g_j \delta_{\tau_j} - \int_0^t g'_j(s)\,\mathrm{d}[h_{\tau_j}(s)] \right) \quad \text{and} \quad f'' = \sum_{j=1}^{N} g_j \delta'_{\tau_j}.$$

It means that the equation (4.2) is a special case of the equation (3.1) treated in the previous section.

Let $\{u_1, u_2, v_1, v_2\}$ be the fundamental system of solutions to the corresponding homogeneous equation $u'' + q'u = 0$ on $[0, T]$ defined by Definition 2.8. By Theorem 3.6, for any $u_0, v_0 \in \mathbb{R}$ there exists a unique solution $u \in \mathbb{G}_{\mathrm{reg}}$, $v \in \mathbb{BV}_{\mathrm{reg}}$ of the system

$$v' + q'u = 0, \quad u' - v = f' \tag{4.4}$$

on $[0,T]$ such that $u(0) = u_0$ and $v(0) = v_0$ hold. (The function $u(t)$ is then a solution to the equation (4.1), of course.) Inserting (4.3) into (3.10) and (3.21), where $p(t) \equiv 1$ on $[0,T]$, we obtain that this solution is given for $t \in [0,T]$ by

$$u(t) = \big(u_0 + a(t)\big)u_1(t) + \big(v_0 - b(t)\big)u_2(t) \quad \text{on} \quad [0,T] \tag{4.5}$$

and

$$v(t) = \big(u_0 + a(t)\big)v_1(t) + \big(v_0 - b(t)\big)v_2(t) + \gamma(t) \quad \text{on} \quad [0,T] \tag{4.6}$$

where

$$a(t) = \sum_{j=1}^{N} h_{\tau_j}(t)\big(g_j(\tau_j)v_2(\tau_j) + g_j'(\tau_j)u_2(\tau_j)\big),$$

$$b(t) = \sum_{j=1}^{N} h_{\tau_j}(t)\big(g_j(\tau_j)v_1(\tau_j) + g_j'(\tau_j)u_1(\tau_j)\big)$$

and

$$\gamma(t) = \sum_{j=1}^{N} (h_{\tau_j}(t)g_j'(\tau_j) - \Delta^- h_{\tau_j}(t)g_j(t)\Delta^- q(t)).$$

In particular, it is easy to see that $\Delta u(t) = 0$ for $t \notin \{\tau_1, \tau_2, \dots, \tau_N\}$. Furthermore, it follows from (4.4) and (4.5) that $\Delta u(\tau_k) = g_k(\tau_k)$ for $k = 1, 2, \dots, N$. Moreover, we have

$$\Delta^+ v(0) = \Delta^- v(T) = 0 \quad \text{and} \quad \Delta v(t) = -\Delta q(t)u(t) \quad \text{for all} \quad t \in (0,T).$$

Obviously, the function (4.5) fulfils (4.1) if and only if $v_0 \in \mathbb{R}$ is such that

$$u_2(T)v_0 = u_T - \big(u_0 + a(T)\big)u_1(T) + b(T)u_2(T).$$

Hence under the assumption that $u_2(T) \neq 0$ for any $u_0 \in \mathbb{R}$ and any $u_T \in \mathbb{R}$ the function $u(t)$ given on $[0,T]$ by

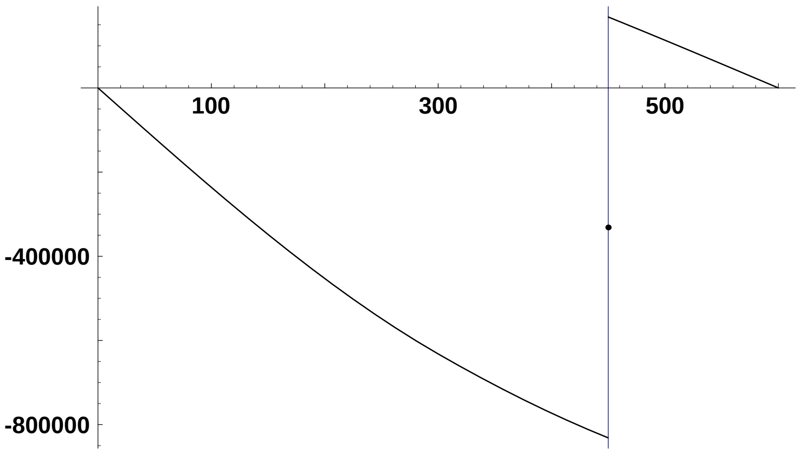
$$\begin{aligned} u(t) = &\Big(u_0 + a(t)\Big)u_1(t) \\ &+ \Big(\frac{u_1 - \big(u_0 + a(T)\big)u_1(T)}{u_2(T)} + \big[b(T) - b(t)\big]\Big)u_2(t) \end{aligned} \tag{4.7}$$

is the unique solution of the boundary value problem (4.1), (4.2).

The formula (4.7) enables us to get respectively precise or approximate values of the solution $u(t)$ of the boundary value problem (4.1), (4.2) once the precise or approximate values $u_1(t)$, $u_2(t)$, $t \in [0,T]$ and $v(\tau_j)$, $j = 1, 2, \dots, N$ are available. For example, the following graph and numerical values of the solution $u(t)$ were obtained from the formula (4.7) by means of the software system MATHEMATICA in the case $N = 1$, $u_0 = u_1 = 0$, $\tau_1 = 450$, $T = 600$, $g_1(t) \equiv M_0 = 10^6$, and

$$q'(t) = \begin{cases} \frac{50}{2.85 \times 2.1} 10^{-3} & \text{if } t < 300, \\ \frac{50}{5.7 \times 2.1} 10^{-3} & \text{if } t > 300. \end{cases}$$

Figure 1:



t	u(t)	t	u(t)
0	0.	350	-706107.543
50	-119379.035	449	-830379.019
100	-236269.111	450	-331468.874
150	-348233.159	451	167442.655
200	-452936.817	500	113340.584
250	-548197.095	525	85265.310
300	-632027.890	550	56967.483
400	-772819.890	600	-5.82 $\times 10^{-11}$

This example (with the same values of parameters as those chosen above) was numerically treated in [Ho85]. Acording to [Ho85] the solution $u(t)$ of the boundary value problem (4.1),(4.2) with

$$q'(t) = \frac{P}{E(t)I(t)}, \quad u_0 = u_1 = 0 \quad \text{and} \quad f'' = M_0\delta'_\tau$$

describes the binding moment in the beam of the length 1 subjected to the pressure (or pull) P at the ends $t = 0$ and $t = 1$ and to the revolution moment M_0 at the point $t = \tau$.

References

[At64] Atkinson F. V., *Discrete and Continuous Boundary Value Problems*, (Academic Press, New York, 1964).

[Ha52] Halperin I., *Introduction to the Theory of Distributions*, (University of Toronto Press, Toronto, 1952).

[Ho85] Holländer K., Die numerische Behandlung eines singulären Randwertproblems nach der Differenzenmethode, *ZAMM Z. Angew. Math. u. Mech.* **65** (1985), T269–T271.

[Hö75] Hönig Ch. S., *Volterra Stieltjes-Integral Equations*, (North Holland and American Elsevier, Mathematics Studies 16, Amsterdam and New York, 1975).

[Ku57] Kurzweil J., Generalized ordinary differential equations and continuous dependence on a parameter, *Czechoslovak Math. J.* **7 (82)** (1957), 418–449.

[Li73] Ligęza J., On generalized solutions of linear differential equations of order n, *Prace Mat. Uniw.Sląski v Katowicach* **3** (1973), 101–108.

[Li77] Ligęza J., On distributional solutions of some systems of linear differential equations *Časopis Pěst. Mat.* **102** (1977), 37–41.

[Li86] Ligęza J., *Weak Solutions of Ordinary Differential Equations*, (Uniwersytet Sląski, Katowice, 1986).

[Li88] Ligęza J., Product of measures and regulated functions, in *Generalized Functions and Convergence*, Memorial Volume for Professor Jan Mikusiński, eds. P. Antosik and A. Kamiński, Katowice, 13-18 June 1988, (World Scientific, Singapore, 1988), pp. 175–179.

[Li89] Ligęza J., Existence and uniqueness of distributional solutions of ordinary differential equations in the class of regulated functions, in *EQUADIFF 7 - Enlarged Abstracts* (7th Czechoslovak Conference on Differential Equations and Their Applications, Praha, 1989), pp. 48–50.

[Mi83] Mingarelli A. B., *Volterra-Stieltjes Integral Equations and Generalized Ordinary Differential Expressions*, (Lecture Notes in Mathematics 989, Springer-Verlag, Berlin, 1983)

[PaDe82] Pandit S. G., Deo S. G., *Differential Equations Involving Impulses*, (Lecture Notes in Mathematics 954, Springer-Verlag, Berlin, 1982).

[PeTv93] Pelant M, Tvrdý M., Linear distributional differential equations in the space of regulated functions, *Math. Bohem.* **118** (1993), 379–400.

[Pe84] Persson J., Linear distribution differential equations, *Comment. Math. Univ. St. Pauli* **33** (1984), 119–126.

[Pe88] Persson J., Fundamental theorems for linear measure differential equations, *Mathematica Scandinavica* **62** (1988), 19–43.

[Pf80] Pfaff R., Gewöhnliche lineare Differentialgleichungen n-ter Ordnung mit Distributionskoeffizienten, *Proc. Roy. Soc. Edinburgh* **85A** (1980), 291–298.

[Pf81] Pfaff R., Generalized systems of linear differential equations, *Proc. Roy. Soc. Edinburgh* **89A** (1981), 1–14.

[Sch85] Schwabik Š., *Generalized Differential Equations (Fundamental Results)*, (Rozpravy ČSAV, Řada MPV, 95 (6)) (Academia, Praha, 1985).

[Sch89] Schwabik Š., *Generalized Differential Equations (Special Results)*, (Rozpravy ČSAV, Řada MPV, 99 (3)) (Academia, Praha, 1989).

[Sch92] Schwabik Š., *Generalized Ordinary Differential Equations*, (World Scientific, Singapore, 1992).

[SchTvVe79] Schwabik Š., Tvrdý M., Vejvoda O., *Differential and Integral Equations: Boundary Value Problems and Adjoints*, (Academia and D. Reidel, Praha and Dordrecht, 1979).

[Tv89] Tvrdý M., Regulated functions and the Perron-Stieltjes integral, *Časopis Pěst. Mat.* **114** (1989), 187–209.

[Tv91] Tvrdý M., Generalized differential equations in the space of regulated functions (Boundary value problems and controllability), *Math. Bohem.* **116** (1991), 225–244.

[ZaSe91] Zavalishchin S. G., Sesekin A. N., *Impulse Processes, Models and Applications* (in Russian), (Nauka, Moscow, 1991)