DIFFERENTIAL AND INTEGRAL EQUATIONS WITH REGULATED SOLUTIONS

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Abstract. The paper deals with linear differential equations on the interval $[0,1]$ with distributional coefficients and solutions from the space of regulated functions. In particular, it is shown that if the $n \times n$-matrix valued function $\boldsymbol{A}(t)$ has a bounded variation on $[0,1]$ and the n-vector valued function $\boldsymbol{f}(t)$ is regulated on $[0,1]$ and both $A(t)$ and $f(t)$ are regular on $[0,1]$ (cf. (0.7)) then the system

$$\boldsymbol{x}' - \boldsymbol{A}'\boldsymbol{x} = \boldsymbol{f}',$$

where the derivatives and equality are understood in the distributional sense, is equivalent with the system of Volterra-Stieltjes integral equations

$$\boldsymbol{x}(t) - \boldsymbol{x}(0) - \int_0^t [\mathrm{d}\,\boldsymbol{A}(s)]\,\boldsymbol{x}(s) = \boldsymbol{f}(t) - \boldsymbol{f}(0), \quad t \in [0,1].$$

As a consequence the basic existence and uniqueness theorems for the given system are proved and the variation-of-constants formula for its solutions is obtained. Furthermore, analogous results for the second order distributional differential equations of the form

$$u'' + q'(t)u = f''(t)$$

are given, as well.

0. Introduction

Provided an $n \times n$-matrix valued function $\boldsymbol{P}(t)$ and an n-vector valued function $\boldsymbol{g}(t)$ are Lebesgue integrable on $[0,1]$ and the functions $\boldsymbol{A}(t)$ and $\boldsymbol{f}(t)$ are given by

$$\boldsymbol{A}(t) = \int_0^t \boldsymbol{P}(s)\mathrm{d}\,s \quad \text{and} \quad \boldsymbol{f}(t) = \int_0^t \boldsymbol{g}(s)\mathrm{d}\,s, \quad t \in [0,1],$$

an n-vector valued function $\boldsymbol{x}(t)$ is called a solution to the system

$$\boldsymbol{x}' - \boldsymbol{P}(t)\boldsymbol{x} = \boldsymbol{g}(t) \tag{0.1}$$

on $[0,1]$ in the Carathéodory sense if it is a solution to the integral equation

$$\boldsymbol{x}(t) - \boldsymbol{x}(0) - \int_0^t [\mathrm{d}\,\boldsymbol{A}(s)]\,\boldsymbol{x}(s) = \boldsymbol{f}(t) - \boldsymbol{f}(0), \quad t \in [0,1], \tag{0.2}$$

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Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TeX i.e. if it is absolutely continuous on $[0,1]$ and the equality

$$\boldsymbol{x}'(t) - \boldsymbol{P}(t)\boldsymbol{x}(t) = \boldsymbol{g}(t)$$

holds for a.e. $t \in [0,1]$.

Probably the most stimulating motivation for investigation of systems which maintain basic features of ordinary differential equations and at the same time admit discontinuous solutions was the desire to explain some phenomena occurring in the study of continuous dependence of solutions of ordinary differential equations on a parameter (cf. [Ku-Vo] and [Ku2]). Furthermore, it appeared e.g. that the proper adjoint problem to the boundary value problem (0.1),

$$\int_0^1 \big[\mathrm{d}\,\boldsymbol{K}(t)\big]\boldsymbol{x}(t) = 0$$

contains in general case when $\boldsymbol{K}(t)$ is of bounded variation on $[0,1]$ (and not continuous in general) an equation of the form (0.2) with the right hand side of bounded variation on $[0,1]$ and not continuous in general (cf.e.g. [Ha-Mo] or [Ve-Tv]).

There are essentially two ways of generalization of linear ordinary differential equations (0.1) to equations which admit discontinuous solutions. One of them consists in generalization of the concept of the derivative and leads to differential equations in the distributional sense. The first step in this direction is due to J. Kurzweil (cf. [Ku1]). Theorems on existence and uniqueness of solutions to differential systems of the form

$$\boldsymbol{x}' - \boldsymbol{A}'(t)\boldsymbol{x} = \boldsymbol{f}'(t), \tag{0.3}$$

where the $n \times n$-matrix valued function $\boldsymbol{A}(t)$ and the n-vector valued function $\boldsymbol{f}(t)$ are of bounded variation on $[0,1]$ and derivatives and equality are understood in the distributional sense, were obtained by J. Ligęza (cf. [Li1]). Related results may be found e.g. in [Li2], [Pa-De], [Per], [Pf] and [Za-Se]. The other way of generalizing of ordinary differential equations is based on the use of a more general integral in the integral equation (0.2) and leads to generalized differential (or differentio-Stieltjes integral) equations defined by their integral form (0.2). The study of generalized differential equations of the form (0.2), where the $n \times n$-matrix valued function $\boldsymbol{A}(t)$ and the n-vector valued function $\boldsymbol{f}(t)$ are of bounded variation on $[0,1]$ and the integral is the Perron-Stieltjes one, was initiated by J. Kurzweil (cf. e.g. [Ku2]), T. H. Hildebrandt (cf. [Hi2]) and others. J. Kurzweil introduced a generalized concept of the sum-type Stieltjes integral which is equivalent to the Perron-Stieltjes integral. (A survey of the basic properties of this integral may be found in [S-T-V], [Sch1] or [Sch3]. Some additional properties of the Perron-Stieltjes integral concerning the integration with respect to regulated functions were established in [Tv1].) Furthermore, in [S-T-V] a basic theory of generalized integral systems of the form

$$\boldsymbol{x}(t) - \int_0^t \big[\mathrm{d}\,\boldsymbol{K}(t,s)\big]\boldsymbol{x}(s) = \boldsymbol{f}(t), \tag{0.4}$$

$$\boldsymbol{x}(t) - \int_0^1 \big[\mathrm{d}\,\boldsymbol{K}(t,s)\big]\boldsymbol{x}(s) = \boldsymbol{f}(t) \tag{0.5}$$

and of boundary value problems for generalized differential equations of the form (0.2) (with solutions of bounded variation) was developed. Further results concerning related topics may be found in [Sch1],[Sch2] and [Sch3]. Similar problems were treated also by Ch. S. Hönig, L. Barbanti and L. Fichmann (cf. e.g. [Hö], [Ba], and [Fi]), who made use of the interior Dushnik integral and were looking for solutions in the space of regulated functions, i.e. of functions possessing at each point of the interval $[0,1)$ a limit from the right and at each point of the interval $(0,1]$ a limit from the left. The use of the interior Dushnik integral instead of the Perron-Stieltjes integral causes that the equations studied by Ch. S. Hönig, L. Barbanti and L. Fichmann, though written in the same form as the equations (0.2), (0.4) and (0.5), are not identical with them. In the space of regulated functions the generalized differential equations of the form (0.2) (with the Perron-Stieltjes integral) were investigated in the paper [Tv2].

From many points of view the "integral" approach appears to be more convenient then the "distributional" one. The integral equation (0.2) which should be satisfied at any point t of the interval $[0,1]$ seems to be more concrete than the rather symbolic equation (0.3). Due to the difficulties with the explicit definition of the distributional product of functions of bounded variation with derivatives of such functions the results for distributional differential systems (0.3) are not so explicit as those available for generalized differential systems (0.2). In particular, the variation-of-constants formula for distributional differential systems of the form (0.3) has not been established until now. However, there is one substantial advantage of the "distributional" approach: generalizations of higher order differential equations are formulated via distributional differential systems in a more natural way. As the known conditions ensuring the existence and uniqueness of a solution to the systems (0.2) and (0.3) are very similar, it is a general feeling that these two systems should be equivalent. Nevertheless, it seems that there are no results concerning the relationship between the systems (0.2) and (0.3) available. The first attempt in this direction was done by M. Pelant (cf.[Pel]).

The aim of this contribution is to show the equivalence (under certain assumptions) of the systems (0.2) and (0.3) and of the second order distributional differential equation

$$u'' + q'(t)u = f''(t) \tag{0.6}$$

and a certain Volterra-Stieltjes integral equation of the form (0.4). This enables us e.g. to get the variation-of-constants formulas for distributional differential equations (0.3) and (0.6), as well.

Throughout the paper $\mathbb{R}^n$ denotes the space of real column n-vectors, $\mathbb{R}^1 = \mathbb{R}$ and $\mathbb{N}$ stands for the set of positive integers. Given a $k \times n$-matrix $\boldsymbol{M}$, its elements are denoted by $m_{i,j}$, $\boldsymbol{M}^{-1}$ denotes its inverse, $\boldsymbol{M}^*$ is its transposition and $\|\boldsymbol{M}\| = max_{i=1,\dots k} \sum_{j=1}^{n} |m_{i,j}|$. The symbols $\mathbf{I}$ and $\mathbf{0}$ stand respectively for the identity and the zero matrix of the proper type.

Any function $f: [0,1] \mapsto \mathbb{R}$ which possesses finite limits

$$f(t+) = \lim_{\tau \to t+} f(\tau) \quad \text{and} \quad f(s-) = \lim_{\tau \to s-} f(\tau)$$

for all $t \in [0,1)$ and $s \in (0,1]$ is said to be *regulated* on $[0,1]$. Any $k \times n$-matrix valued function $\boldsymbol{F}$ defined on $[0,1]$ and such that all its elements $f_{i,j}(t)$, $i = 1,2,\dots,k; j = 1,2,\dots,n$ are regulated functions on $[0,1]$ or functions of bounded variation on $[0,1]$ is said to be a matrix valued function regulated on $[0,1]$ or of bounded variation on $[0,1]$, respectively. $\mathrm{BV}^{k,n}$ denotes the Banach space of $k \times n$-matrix valued functions of bounded variation on $[0,1]$ equipped with the norm

$$\boldsymbol{F} \in \mathrm{BV}^{k,n} \mapsto \|\boldsymbol{F}\|_{\mathrm{BV}} = |\boldsymbol{F}(0)| + \mathrm{var}_0^1\, \boldsymbol{F}.$$

The space of column n-vector valued functions regulated on $[0,1]$ is denoted by G^n and G^n_R stands for the set of all functions $f \in \mathrm{G}^n$ such that

$$\boldsymbol{f}(0+) = \boldsymbol{f}(0), \boldsymbol{f}(1-) = \boldsymbol{f}(1) \tag{0.7}$$

and

$$\boldsymbol{f}(t) = \frac{\boldsymbol{f}(t-) + \boldsymbol{f}(t+)}{2} \quad \text{for } t \in (0,1).$$

Functions fulfilling (0.7) are usually called *regular* on $[0,1]$. The set of all regular on $[0,1]$ $k \times n$-matrix valued functions of bounded variation on $[0,1]$ is denoted by $\mathrm{BV}^{k,n}_{reg}$. For $\boldsymbol{x} \in \mathrm{G}^n$ we put

$$\|\boldsymbol{x}\| = \sup_{t\in[0,1]} |\boldsymbol{x}(t)|.$$

It is well known that G^n is a Banach space with respect to this norm (cf. [Hö, Theorem 3.6]). Obviously, G^n_{reg} is a closed subspace of G^n and hence it is also a Banach space with respect to the same norm. For more details concerning regulated functions or functions of bounded variation see [Hö] or [Hi2], respectively.

As usual L^n_1 stands for the Banach space of measurable and Lebesgue integrable column n-vector valued functions on $[0,1]$. Instead of G^1, G^1_{reg}, $\mathrm{B}V^1$ and L^1_1, we write G, G_{reg}, $\mathrm{B}V$ and L_1, respectively.The integrals occurring in the sequel are the Perron-Stieltjes ones.

1. Distributions

In what follows $\mathscr{D}$ stands for the topological vector space of functions $\varphi : \mathbb{R} \mapsto \mathbb{R}$ possessing for any $j \in \mathbb{N} \cup \{0\}$ a derivative $\varphi^{(j)}$ which is continuous on $\mathbb{R}$ and such that $\varphi^{(j)}(t) = 0$ for any $t \in \mathbb{R} \setminus [0,1]$. The space $\mathscr{D}$ is endowed with the topology in which the sequence $\varphi_k \in \mathscr{D}$ tends to $\varphi_0 \in \mathscr{D}$ in $\mathscr{D}$ if and only if $\lim_k \|\varphi_k^{(j)} - \varphi_0^{(j)}\| = 0$ for all $j \in \mathbb{N} \cup \{0\}$. Linear continuous functionals on $\mathscr{D}$ are called *distributions* on $[0,1]$. The space of distributions on $[0,1]$ is denoted by $\mathscr{D}^*$. Given a distribution $f \in \mathscr{D}^*$ and $\varphi \in \mathscr{D}$, $< f, \varphi >$ denotes the value of the functional f on φ. Any function $f \in \mathrm{L}_1$ is identified with the distribution

$$f : \varphi \in \mathscr{D} \mapsto < f, \varphi > = \int_0^1 f(t)\varphi(t)\mathrm{d}\,t$$

In particular, the zero element 0 of $\mathscr{D}^*$ is identified with the function vanishing a.e. on $[0,1]$. Obviously, if $f \in \mathrm{G}$, then $f = 0$ if and only if $f(t+) = f(s-) = 0$ holds for any $t \in [0,1)$ and $s \in (0,1]$.

Given an arbitrary $f \in \mathscr{D}^*$, the distributional derivative of f is denoted by f', i.e.

$$f' : \varphi \in \mathscr{D} \mapsto < f', \varphi > = - < f, \varphi' > .$$

Analogously,

$$f'' : \varphi \in \mathscr{D} \mapsto < f'', \varphi > = < f, \varphi'' > .$$

If $f \in \mathscr{D}^*$ then $f' = 0$ if and only if there is a $c \in \mathbb{R}$ such that $f = c$, i.e. there is a function $g \in \mathrm{L}^1$ such that $f = g$ and $g(t) = c$ for a.e. $t \in [0,1]$. Analogously $f'' = 0$ if and only if there are $c_1, c_2 \in \mathbb{R}$ such that $f = c_1 + c_2 t$.

All multiplications of distributions which occur in this paper are justified by the following definitions based on the definition given in [Pa-De]:

If $f \in \mathrm{G}_{reg}$ and $g \in \mathrm{B}V_{reg}$, then

$$fg' : \varphi \in \mathscr{D} \mapsto < fg', \varphi > = \int_0^1 (f(t)\varphi(t))[\mathrm{d}\, g(t)] \tag{1.1}$$

and

$$f'g : \varphi \in \mathscr{D} \mapsto < f'g, \varphi > = \int_0^1 (g(t)\varphi(t))[\mathrm{d}\, f(t)]. \tag{1.2}$$

The relations (1.1) and (1.2) define linear continuous functionals on $\mathscr{D}$ which are compatible with the known definitions also in the case that the regularity of functions f and g is not supposed. However, the usual relation

$$(1.3) \qquad (f\,g)' = f'\,g + f\,g',$$

which is by (1.1) and (1.2) equivalent to the relation

$$\int_0^1 \Big[f(t)g(t) - \int_0^t \big[\mathrm{d}\, f(s)\big] g(s) - \int_0^t f(s)\big[\mathrm{d}\, g(s)\big] \Big] \varphi'(t)\mathrm{d}\, t = 0 \text{ for all } \varphi \in \mathscr{D},$$

then need not be true in general. The validity of (1.3) in the case that both f and g are regular on $[0,1]$ follows by the integration-by-parts theorem (cf. [Tv1, Theorem 2.15]).

The space of column n-vector valued functions $\boldsymbol{\varphi}(t) = (\varphi_j(t))_{j=1,\dots,n}$ such that $\varphi_j \in \mathscr{D}$ for any $j = 1, 2, \dots, n$ will be denoted by $\mathscr{D}^n$ while its dual space (which is the n-th cartesian power of $\mathscr{D}^*$) will be denoted by $\mathscr{D}^{n*}$. The elements of $\mathscr{D}^{n*}$ will be called *n-vector distributions.* Given $\boldsymbol{f} = (f_1, f_2, \dots, f_n) \in \mathscr{D}^{n*}$ and $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \dots, \varphi_n)^* \in \mathscr{D}^n$, the value of the functional $\boldsymbol{f}$ on $\boldsymbol{\varphi}$ is given by

$$< \boldsymbol{f}, \boldsymbol{\varphi} > = < f_1, \varphi_1 > + < f_2, \varphi_2 > + \dots + < f_n, \varphi_n > .$$

As in the scalar case, if $\boldsymbol{f} \in \mathrm{L}_1^n$, then f will be identified with the n-vector distribution

$$\boldsymbol{f} : \boldsymbol{\varphi} \in \mathscr{D}^n \mapsto < \boldsymbol{f}, \boldsymbol{\varphi} > = \int_0^1 \boldsymbol{\varphi}^*(t)\boldsymbol{f}(t)\mathrm{d}\, t.$$

Similarly, if $\boldsymbol{g} \in \mathrm{G}^n$, then the distributional derivative $\boldsymbol{g}'$ of $\boldsymbol{g}$ is given by

$$\boldsymbol{g}' : \boldsymbol{\varphi} \in \mathscr{D}^n \mapsto < \boldsymbol{g}', \boldsymbol{\varphi} > = \int_0^1 \boldsymbol{\varphi}^*(t)\big[\mathrm{d}\, \boldsymbol{g}(t)\big].$$

An n-vector distribution $\boldsymbol{f}$ is said to be the n-vector *zero* distribution ($\boldsymbol{f} = \boldsymbol{0}$) if all its entries are zero distributions. A $k \times n$-matrix $\boldsymbol{A}$ whose entries $a_{i,j}, i = 1, 2, \dots, k; j = 1, 2, \dots, n$ are distributions is said to be a *$k \times n$-matrix distribution.* Given a $k \times n$-matrix distribution $\boldsymbol{A} = (a_{i,j})_{i=1,\dots,k\supset=1,\dots,n}$, the matrix $\boldsymbol{A}' = (a'_{i,j})_{i=1,\dots,k\supset=1,\dots,n}$ is said to be the *derivative* of $\boldsymbol{A}$.

If a $k \times n$-matrix distribution $\boldsymbol{A} = (a_{i,j})_{i=1,\dots,k\supset=1,\dots,n}$ and an n-vector distribution $\boldsymbol{x} = (x_j)_{j=1,\dots,n}$ are such that all products $a_{i,j}x_j$, $i = 1, 2, \dots, k; j = 1, 2, \dots, n$ are defined, then the product $\boldsymbol{A}\boldsymbol{x}$ is defined as the k-vector distribution $\boldsymbol{y}$ with the elements $y_i = \sum_{j=1}^n a_{i,j}x_j$, $i = 1, 2, \dots, k$.

2. Linear distributional differential equations of the first order

In this section we shall consider the system

$$(2.1) \qquad \boldsymbol{x}' - \boldsymbol{A}'\boldsymbol{x} = \boldsymbol{f}'$$

and the corresponding homogeneous system

$$(2.2) \qquad \boldsymbol{x}' - \boldsymbol{A}'\boldsymbol{x} = \boldsymbol{0},$$

where the derivatives, products and equality are understood in the sense of distributions.

2.1. Assumptions. $\boldsymbol{A} \in \mathrm{BV}^{n,n}_{reg}$ and $\boldsymbol{f} \in \mathrm{G}^n_{reg}$.

2.2. Definition. An n-vector valued function $\boldsymbol{x}(t)$ is called a *solution* to the equation (2.1) on the interval $[0,1]$ if $\boldsymbol{x} \in \mathrm{G}^n_{reg}$ and

$$\boldsymbol{x}' - \boldsymbol{A}'\boldsymbol{x} - \boldsymbol{f}'$$

is the zero n-vector distribution.

It follows from the definition (1.1) that *under the assumptions 2.1 the product $\boldsymbol{A}'\boldsymbol{x}$ is defined and the relation*

$$\boldsymbol{A}'\boldsymbol{x} = \Big(\int_0^t [\mathrm{d}\,\boldsymbol{A}(s)]\,\boldsymbol{x}(s)\Big)' \tag{2.3}$$

holds for an arbitrary $\boldsymbol{x} \in \mathrm{G}^n_{reg}$. In fact, for any $\boldsymbol{\varphi} \in \mathscr{D}^n$ we have by (1.1) and by the substitution theorem (cf.[Tv1, Theorem 2.19])

$$\begin{aligned} < \boldsymbol{A}'\boldsymbol{x}, \boldsymbol{\varphi} > &= \sum_{i=1}^k \Big(\sum_{j=1}^n \int_0^1 [\mathrm{d}\, a_{i,j}(t)]\, x_j(t) \varphi_i(t) \Big) \\ &= \sum_{i=1}^n \int_0^1 \Big[\mathrm{d} \sum_{j=1}^n \int_0^t [\mathrm{d}\, a_{i,j}(s)]\, x_j(s)\Big] \varphi_i(t) = \int_0^1 \boldsymbol{\varphi}^*(t) [\mathrm{d}\,\boldsymbol{\xi}(t)], \end{aligned}$$

where the function $\boldsymbol{\xi}(t)$ given by

$$\boldsymbol{\xi}(t) = \int_0^t [\mathrm{d}\,\boldsymbol{A}(s)]\,\boldsymbol{x}(s) \quad \text{for} \quad t \in [0,1]$$

is regulated on $[0,1]$ by Theorem 1.3.4 of [Ku2]. Furthermore, given an arbitrary division $\{0 = t_0 < t_1 < \cdots < t_m = 1\}$ of $[0,1]$, we have

$$\begin{aligned} \sum_{j=1}^m |\boldsymbol{\xi}(t_j) - \boldsymbol{\xi}(t_{j-1})| &= \sum_{j=1}^m \Big| \int_{t_{j-1}}^{t_j} [\mathrm{d}\,\boldsymbol{A}(s)]\,\boldsymbol{x}(s) \Big| \\ &\le \sum_{j=1}^m \big(\mathrm{var}_{t_{j-1}}^{t_j} \boldsymbol{A} \big) \|\boldsymbol{x}\| \le \|\boldsymbol{A}\|_{\mathrm{BV}} \|\boldsymbol{x}\| < \infty \end{aligned}$$

and hence

$$\mathrm{var}_0^1\, \boldsymbol{\xi} \le \|\boldsymbol{A}\|_{\mathrm{BV}} \|\boldsymbol{x}\| < \infty.$$

Moreover, since $\boldsymbol{\xi}(t+) = \boldsymbol{\xi}(t) + \Delta^+ \boldsymbol{A}(t)\boldsymbol{x}(t)$ and $\boldsymbol{\xi}(t-) = \boldsymbol{\xi}(t) - \Delta^- \boldsymbol{A}(t)\boldsymbol{x}(t)$, it follows immediately that $\boldsymbol{\xi} \in \mathrm{BV}^n_{reg}$.

Consequently, the equation (2.1) may be rewritten as the relation

$$\Big(\boldsymbol{x} - \int_0^t [\mathrm{d}\,\boldsymbol{A}(s)]\,\boldsymbol{x}(s)\Big)' = \boldsymbol{0},$$

which holds if and only if there is $\boldsymbol{c} \in \mathbb{R}^n$ such that

$$\boldsymbol{x}(t) - \int_0^t [\mathrm{d}\,\boldsymbol{A}(s)]\,\boldsymbol{x}(s) - \boldsymbol{f}(t) - \boldsymbol{c} = \boldsymbol{0} \tag{2.4}$$

holds for a.e. $t \in [0,1]$. Since the left-hand side of (2.4) belongs to G^n_{reg} for any $\boldsymbol{x} \in \mathrm{G}^n_{reg}$, this means that $\boldsymbol{x} \in \mathrm{G}^n_{reg}$ is a solution to the equation (2.1) on $[0,1]$ if and only if there is a $\boldsymbol{c} \in \mathbb{R}^n$ such that (2.4) is satisfied for all $t \in [0,1]$. This completes the proof of the following assertion.

2.3. Proposition. *Let the assumptions 2.1 be satisfied. An n-vector valued function $\boldsymbol{x} \in \mathrm{G}^n_{\mathrm{reg}}$ is a solution to the equation* (2.1) *on* $[0,1]$ *if and only if it satisfies on* $[0,1]$ *the integral equation*

$$\boldsymbol{x}(t) - \boldsymbol{x}(0) - \int_0^t [\mathrm{d}\,\boldsymbol{A}(s)]\,\boldsymbol{x}(s) = \boldsymbol{f}(t) - \boldsymbol{f}(0). \tag{2.5}$$

By Proposition 2.3 any solution $\boldsymbol{x} \in \mathrm{G}^n_{\mathrm{reg}}$ of the homogeneous equation (2.2) on $[0,1]$ is of bounded variation on $[0,1]$. This enables us to transfer directly all results known for the homogeneous equation corresponding to (2.5) to the equation (2.2).

2.4. Proposition. *Let the assumptions 2.1 be satisfied. Let $t_0 \in [0,1]$ be given. Then the equation* (2.2) *possesses for any $\boldsymbol{c} \in \mathbb{R}^n$ a unique solution $\boldsymbol{x} \in \mathrm{G}^n_{\mathrm{reg}}$ such that $\boldsymbol{x}(t_0) = \boldsymbol{c}$ if and only if the relation*

$$\det\big(\mathbf{I} - \Delta^- \boldsymbol{A}(t)\big)\det\big(\mathbf{I} + \Delta^+ \boldsymbol{A}(s)\big) \neq 0 \quad \textit{for all} \;\; t \in (t_0, 1) \textit{ and } s \in (0, t_0) \tag{2.6}$$

holds.

If the conditions (2.6) *are satisfied, then there exists a unique $n \times n$-matrix valued function $\boldsymbol{U}(t,s)$ defined on*

$$\Delta = \big\{(t,s);\, 0 \le t \le s \le t_0 \quad \text{or} \quad t_0 \le s \le t \le 1\big\}$$

and such that

$$\boldsymbol{U}(t,s) = \mathbf{I} + \int_s^t [\mathrm{d}\,\boldsymbol{A}(\tau)]\,\boldsymbol{U}(\tau,s) \tag{2.7}$$

holds for all $(t,s) \in \Delta$.

Given an arbitrary $\boldsymbol{c} \in \mathbb{R}^n$, the corresponding solution of the initial value problem (2.2), *$\boldsymbol{x}(t_0) = \boldsymbol{c}$ is given by*

$$\boldsymbol{x}(t) = \boldsymbol{U}(t,t_0)\boldsymbol{c}, \quad t \in [0,1].$$

Proof follows from Theorem III.1.4 and Theorem III.2.2 in [S-T-V]. □

2.5.Theorem. *Let the assumptions 2.1 and* (2.6) *be satisfied. Then for any $t_0 \in [0,1]$ and any $\boldsymbol{c} \in \mathbb{R}^n$ the equation* (2.1) *possesses a unique solution $\boldsymbol{x} \in \mathrm{G}^n_{\mathrm{reg}}$ on* $[0,1]$ *such that $\boldsymbol{x}(t_0) = \boldsymbol{c}$. This solution is given by*

$$\boldsymbol{x}(t) = \boldsymbol{U}(t,t_0)\boldsymbol{c} + \boldsymbol{f}(s) - \boldsymbol{f}(0) - \int_{t_0}^t [\mathrm{d}_s\boldsymbol{U}(t,s)]\big(\boldsymbol{f}(s) - \boldsymbol{f}(0)\big), \quad t \in [0,1], \tag{2.8}$$

where $\boldsymbol{U}(t,s)$ is given by Proposition 2.5.

Proof. Let $\boldsymbol{x}(t)$ be defined by (2.8) and let $t_0 \in [0,1]$ and $t \in [t_0,1]$ be given. Let us put $\boldsymbol{V}(t,s) = \boldsymbol{U}(t,s)$ for $t_0 \le s \le t \le 1$, $\boldsymbol{V}(t,s) = \mathbf{I}$ for $t_0 \le t \le s \le 1$. By [S-T-V, Theorem III.2.10] we have

$$\mathrm{v}(\boldsymbol{V}) + \operatorname{var}_{t_0}^1 \boldsymbol{V}(t_0,.) + \operatorname{var}_{t_0}^1 \boldsymbol{V}(.,t_0) < \infty,$$

where $\mathrm{v}(\boldsymbol{V})$ stands for the Vitali two-dimensional variation of $\boldsymbol{V}$ over $[t_0,1] \times [t_0,1]$ (cf. [Hi2]). Furthermore,

$$\int_{t_0}^s [\mathrm{d}_r\boldsymbol{U}(s,r)]\,\boldsymbol{g}(r) = \int_{t_0}^t [\mathrm{d}_r\boldsymbol{V}(s,r)]\,\boldsymbol{g}(r)$$

holds for all $s \in [t_0, t]$ and any $\boldsymbol{g} \in \mathrm{G}^n$. Hence we have

$$\int_{t_0}^{t} [\mathrm{d}\,\boldsymbol{A}(s)] \int_{t_0}^{s} [\mathrm{d}_r \boldsymbol{U}(s,r)] \big(\boldsymbol{f}(r) - \boldsymbol{f}(0)\big) = \int_{t_0}^{t} [\mathrm{d}\,\boldsymbol{A}(s)] \int_{t_0}^{t} [\mathrm{d}_r \boldsymbol{V}(s,r)] \big(\boldsymbol{f}(r) - \boldsymbol{f}(0)\big)$$

whence by interchanging the integration order (cf. [Tv1, Theorem 2.20]) and making use of the substitution theorem (cf. [Tv2, Theorem 2.19]) and of (2.7) we obtain the following relations:

$$\begin{aligned}
\int_{t_0}^{t} [\mathrm{d}\,\boldsymbol{A}(s)] \int_{t_0}^{s} & [\mathrm{d}_r \boldsymbol{U}(s,r)] \big(\boldsymbol{f}(r) - \boldsymbol{f}(0)\big) \\
&= \int_{t_0}^{t} \Big[\mathrm{d}_s \int_{t_0}^{t} [\mathrm{d}\,\boldsymbol{A}(r)] \boldsymbol{V}(r,s)\Big] \big(\boldsymbol{f}(s) - \boldsymbol{f}(0)\big) \\
&= \int_{t_0}^{t} \Big[\mathrm{d}_s \int_{t_0}^{s} [\mathrm{d}\,\boldsymbol{A}(r)] \boldsymbol{U}(r,r)\Big] \big(\boldsymbol{f}(s) - \boldsymbol{f}(0)\big) \\
&\quad + \int_{t_0}^{t} \Big[\mathrm{d}_s \int_{s}^{t} [\mathrm{d}\,\boldsymbol{A}(r)] \boldsymbol{U}(r,s)\Big] \big(\boldsymbol{f}(s) - \boldsymbol{f}(0)\big) \\
&= \int_{t_0}^{t} [\mathrm{d}\,\boldsymbol{A}(s)] \big(\boldsymbol{f}(s) - \boldsymbol{f}(0)\big) + \int_{t_0}^{t} [\mathrm{d}_s \boldsymbol{U}(t,s)] \big(\boldsymbol{f}(s) - \boldsymbol{f}(0)\big).
\end{aligned}$$

Now it is easy to verify that (2.8) satisfies (2.5) for any $t \geq t_0$. Similarly we could show that (2.8) satisfies (2.5) also for $t \leq t_0$. □

3. Linear distributional differential equations of the second order

Let us consider the equation

$$u'' + q'u = f'' \tag{3.1}$$

where

$$q \in \mathrm{BV}_{reg} \quad \text{and} \quad f \in \mathrm{G}_{reg} \tag{3.2}$$

and solutions are defined by Definition 3.1.

3.1. Definition. A function $u : [0,1] \mapsto \mathbb{R}$ is called to be a *solution* to the equation (3.1) if $u \in \mathrm{G}_{reg}$ and $u'' + q'\,u - f''$ is the zero distribution.

Let q and f fulfill (3.2). Then by the definition of the distributional derivative we have for any $u \in \mathrm{G}_{reg}$

$$u'' : \varphi \in \mathscr{D} \mapsto \int_0^1 u(s)\varphi''(s)\mathrm{d}\,s \quad \text{and} \quad f'' : \varphi \in \mathscr{D} \mapsto \int_0^1 f(s)\varphi''(s)\mathrm{d}\,s.$$

Furthermore, by (1.1) we obtain

$$\begin{aligned}
q'\,u : \varphi \in \mathscr{D} \mapsto \int_0^1 [\mathrm{d}\,q(s)] u(s)\varphi(s) &= -\int_0^1 \Big[\mathrm{d} \int_0^t [\mathrm{d}\,q(s)] u(s)\Big] \varphi(s) \\
&= \int_0^1 \Big(\int_0^t \Big(\int_0^s [\mathrm{d}\,q(r)] u(r)\Big) \mathrm{d}\,s\Big) \varphi''(t)\mathrm{d}\,t.
\end{aligned}$$

Since according to the integration by parts theorem (cf. [Tv1, Theorem 2.15]) and the substitution theorem (cf. [Tv1, Theorem 2.19])

$$\int_0^t \Big(\int_0^s [\mathrm{d}\,q(r)] u(r)\Big) \mathrm{d}\,s = \int_0^t \Big[\mathrm{d} \int_0^s (t-r)[\mathrm{d}\,q(r)]\Big] u(s)$$

for any $t \in [0,1]$, it is easy to complete the proof of the following assertion.

3.2. Theorem. *Under the assumptions* (3.2) *the function* $u \in \mathrm{G}_{reg}$ *is a solution to the equation* (3.1) *on* $[0,1]$ *if and only if there are constants* $u_0, u_1 \in \mathbb{R}$ *such that it holds*

$$u(t) + \int_0^t \big[\mathrm{d}_s k(t,s)\big] u(s) = f(t) - f(0) + u_0 + u_1\, t, \quad t \in [0,1], \tag{3.3}$$

where

$$k(t,s) = \int_0^s (t-r)\big[\mathrm{d}\, q(r)\big] \quad \textit{for} \ \ 0 \le s \le t \le 1. \tag{3.4}$$

Let us put $k(t,s) = k(t,t)$ for $0 \le t \le s \le 1$. Then it is easy to verify that

$$\mathrm{v}(k) + \mathrm{var}_0^1\, k(0,.) + \mathrm{var}_0^1\, k(.,0) < 2\,\mathrm{var}_0^1\, q,$$

where $\mathrm{v}(k)$ stands for the Vitali variation of $k(t,s)$ over $[0,1] \times [0,1]$. Furthermore,

$$k(t,s) - k(t,s-) = (t-s)(q(s) - q(s-)) \quad \text{for} \ \ 0 < s \le t \le 1$$

and

$$k(t,s) - k(t,s-) = 0 \qquad \text{for} \ \ 0 \le t < s \le 1.$$

In particular, $k(t,t) - k(t,t-) = 0$ for all $t \in (0,1]$. Hence by [S-T-V, Theorem II.3.10] there exists a unique function $\Gamma(t,s)$ such that $\mathrm{v}(\Gamma) + \mathrm{var}_0^1\, \Gamma(0,.) + \mathrm{var}_0^1\, \Gamma(.,0) < \infty$,

$$\Gamma(t,s) = k(t,s) - k(t,0) + \int_0^t \big[\mathrm{d}_r k(t,r)\big]\Gamma(r,s) \quad \text{if} \ \ 0 \le s \le t \le 1$$

and

$$\Gamma(t,s) = \Gamma(t,t) \qquad \text{if} \ \ 0 \le t \le s \le 1.$$

Denoting

$$\Phi(t,s) = 1 + \Gamma(t,t) - \Gamma(t,s) \quad \text{for} \ \ t,s \in [0,1], \tag{3.5}$$

we obtain

$$\Phi(t,.) \in \mathrm{B}V_{reg} \quad \text{for any} \ \ t \in [0,1], \quad \Phi(t,t) - \Phi(t,t-) = 0 \quad \text{for any} \ \ t \in (0,1]$$

and

$$\begin{aligned} \Phi(t,s) &= 1 + \int_s^t \big[\mathrm{d}_r k(t,r)\big]\,\Phi(r,s) \quad && \text{for} \ \ 0 \le s \le t \le 1 \\ \Phi(t,s) &= 1 && \text{for} \ \ 0 \le t \le s \le 1. \end{aligned} \tag{3.6}$$

A similar argument as that used in the proof of Theorem 2.5 implies that for any $g \in \mathrm{G}$ the function

$$x(t) = g(t) - \int_0^t \big[\mathrm{d}_s \Phi(t,s)\big] g(s), \quad t \in [0,1] \tag{3.7}$$

is the unique solution of the Volterra-Stieltjes integral equation

$$x(t) + \int_0^t \big[\mathrm{d}_s k(t,s)\big] x(s) = g(t)$$

on $[0,1]$. Since the function $v(t)$ given by

$$v(t) = \int_0^t \big[\mathrm{d}_s k(t,s)\big] x(s) = \int_0^t \big[\mathrm{d}\, q(s)\big](t-s)x(s)$$

is continuous on $[0,1]$ for any $x \in \mathrm{G}$, any solution $u \in \mathrm{G}$ of (3.3) on $[0,1]$ has to be regular on $[0,1]$. Inserting $g(t) = f(t) - f(0) + u_0 + u_1 t$ into (3.7) and integrating by parts we complete the proof of the following assertion.

3.3. Theorem. *Let* (3.2) *hold and let the function* $k(t,s)$ *be given by* (3.4). *Then for any* $u_0 \in \mathbb{R}$ *and any* $u_1 \in \mathbb{R}$ *there exists a unique solution* $u \in \mathrm{G}_{reg}$ *of the equation* (3.3) *on* $[0,1]$ *such that* $u(0) = u_0$. *This solution is given by*

$$u(t) = \Phi(t,0)u_0 + \Big(\int_0^t \Phi(t,s)\mathrm{d}\,s\Big)u_1 + \int_0^t \Phi(t,s)\big[\mathrm{d}\,f(s)\big], \quad t \in [0,1], \tag{3.8}$$

$\Phi(t,s)$ *is defined by* (3.5).

3.4. Remark. Making use of similar arguments to those of the previous section we could obtain that the given equation (3.1) is equivalent to the equation

$$\big(u' + \int_0^t \big[\mathrm{d}\,q(s)\big]u(s) - f'\big)' = 0.$$

Hence, if $f' \in \mathrm{G}_{reg}$, then (3.1) may be rewritten as

$$u'(t) - u'(0) + \int_0^t \big[\mathrm{d}\,q(s)\big]u(s) = f'(t) - f'(0), \quad t \in [0,1]. \tag{3.9}$$

In particular, $u' \in \mathrm{G}_{reg}$ for any solution u of (3.1) on $[0,1]$. Making use of Theorem 3.3 it can be shown easily that if we assume $f' \in \mathrm{G}_{reg}$ in addition to the assumptions of Theorem 3.3, then for any u_0 and $u_1 \in \mathbb{R}$ there is a unique solution $u(t)$ of (3.1) on $[0,1]$ such that $u(0) = u_0$ and $u'(0) = u_1$ and this solution is given by (3.8).

3.5. Remark. Equations of the form (3.9) with $f' \in \mathrm{B}V$ were treated in [Mi].

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