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Existence of Nonnegative and Nonpositive Solutions for Second Order Periodic Boundary Value Problems

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Summary. In the paper we are interested in nonnegative and nonpositive solutions of the boundary value problem $u'' = f(t,u)$, $u(0) = u(1)$, $u'(0) = u'(1)$, where f fulfils the Carathéodory conditions on $[0,1] \times \mathbb{R}$. We generalize the results reached by M. N. Nkashama, J. Santanilla and L. Sanchez and present estimates for solutions. Besides, we apply our existence theorems to periodic boundary value problems for nonlinear Duffing equations whose right-hand sides have a repulsive or attractive singularity at the origin. We extend or generalize existence results by A. C. Lazer and S. Solimini and other authors. Moreover, we get some multiplicity results and in the case of a repulsive singularity we also admit a weak singularity, in constrast to the previous papers on this subject. Our proofs are based on the method of lower and upper functions and topological degree arguments and the results are tested on examples.

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0 . Introduction

In our previous papers [8] and [10] we have established a connection between fairly general lower and upper functions and the Leray-Schauder topological degree of an operator associated to the generalized periodic boundary value problem

$$u'' = f(t, u, u'), \quad u(a) = u(b), \quad u'(a) = w(u'(b)), \tag{0.1}$$

where $-\infty < a < b < \infty$, $f : [a, b] \times \mathbb{R}^2 \mapsto \mathbb{R}$ is a Carathéodory function and $w : \mathbb{R} \mapsto \mathbb{R}$ is continuous and nondecreasing. Using this connection, we have obtained a method providing an information about the solvability of (0.1) in terms of lower and upper functions. (See [8, Theorems 4.1-4.3].)

In this paper we study the special case of (0.1)

$$u'' = f(t, u), \quad u(0) = u(1), \quad u'(0) = u'(1). \tag{0.2}$$

We assume that f fulfils the Carathéodory conditions on $[0, 1] \times \mathbb{R}$, which means that (i) for each $x \in \mathbb{R}$ the function $f(., x)$ is measurable on $[0, 1]$; (ii) for almost every $t \in [0, 1]$ the function $f(t, .)$ is continuous on $\mathbb{R}$; (iii) for each compact set $\mathrm{K} \subset \mathbb{R}$ the function $m_{\mathrm{K}}(t) = \sup_{x \in \mathrm{K}} |f(t, x)|$ is Lebesgue integrable on $[0, 1]$.

The problem (0.2) was considered by M. N. Nkashama and J. Santanilla in [6], where a.o. the following three results concerning the existence of nonnegative and nonpositive solutions to the problem (0.2) were established:

0.1. Theorem ([6, Theorem 2.5])**.** *Suppose*

$$\liminf_{x \to \infty} f(t, x) \ge 0 \quad \textit{for a.e.} \quad t \in [0, 1] \tag{0.3}$$

with strict inequality on a subset of $[0, 1]$ *of positive Lebesgue measure. Furthermore, assume that there exist* $\alpha_+ \in (0, \infty)$ *and a function* $b \in \mathbb{L}[0, 1]$ *such that*

$$b(t) \le f(t, x) \le \alpha_+ x \quad \textit{for a.e.} \quad t \in [0, 1] \quad \textit{and all} \quad x \ge 0. \tag{0.4}$$

Then the problem (0.2) *has a nonnegative solution.*

0.2. Theorem ([6, p. 159])**.** *If inequalities* (0.3) *and* (0.4) *are replaced respectively by*

$$\liminf_{x \to -\infty} f(t, x) \ge 0 \quad \textit{for a.e.} \quad t \in [0, 1] \tag{0.5}$$

with strict inequality on a subset of $[0, 1]$ *of positive Lebesgue measure and*

$$b(t) \le f(t, x) \le -\alpha_-^2 x \quad \textit{for a.e.} \quad t \in [0, 1] \quad \textit{and all} \quad x \le 0 \tag{0.6}$$

with $\alpha_- \in (0, \pi]$, *then the problem* (0.2) *has a nonpositive solution.*

0.3. Theorem ([6, Theorem 2.7]) **.** *Suppose that the problem* (0.2) *has not the trivial solution and that all assumptions of both Theorem* 0.1 *and Theorem* 0.2 *are fulfilled. Then the problem* (0.2) *has at least two different solutions, one nonnegative and one nonpositive.*

In Section 2 of this paper, making use of the method of [8], we prove in Theorems 2.1, 2.5 and Corollaries 2.9 and 2.10 the existence of nonnegative and nonpositive solutions for (0.2) under assumptions weaker than (0.3)-(0.6). In particular, in Theorem 2.1 we use (2.1), (2.2) instead of (0.3), (0.4) and similarly in Corollary 2.9 we use (2.31), (2.32) instead of (0.5), (0.6). Moreover, Theorem 2.1 and Corollaries 2.7 and 2.8 generalize the assertions of [9, Theorem 2.1 and Theorem 2.3]. A comparison of the conditions used in our existence results with those from Theorems 0.1-0.3 is given in Theorems 2.13 and 2.14 (see also Examples 2.4 and 2.12).

The results presented in Section 2 can be applied also to periodic boundary value problems for nonlinear Duffing equations of the form (3.1) or (3.2) whose right-hand sides have a singularity at $x = 0$. Starting from the work [4] by Lazer and Solimini such problems have been studied by many authors (see e.g. [1], [2], [3], [5], [7] and [11]). Section 3 is devoted to this type of problems. First, we consider the case of an attractive singularity and in Corollaries 3.1 and 3.3 we extend results from [4] and [5]. Furthermore, we also get one related multiplicity result (Corollary 3.5). Our main result concerning a problem with a repulsive singularity is obtained in Corollary 3.7. Its goal consists, in contrast to the papers mentioned above, in that our results apply also to a weak singularity. The results of Section 3 are tested on periodic problems for the model equations

$$u'' + \frac{a}{u^\lambda} - bu = e(t) \quad \text{and} \quad u'' - \frac{a}{u^\lambda} + bu = e(t)$$

with $a > 0$, $\lambda > 0$, $b \in \mathbb{R}$ and $e \in \mathbb{L}[0,1]$ (see Examples 3.4, 3.6, 3.9 and 3.11). In particular, it turns out that in the case of repulsive restoring forces our Corollary 3.7 covers also the resonance case $b = \pi^2$ and so it gives the answer to an open question from [1, Example 3.9].

1 . Preliminaries

Throughout the paper we keep the following notation:

As usual, $\mathbb{C}[0,1]$ and $\mathbb{C}(0,\infty)$ are respectively the sets of functions continuous on $[0,1]$ and $(0,\infty)$, $\mathbb{L}[0,1]$ stands for the set of functions Lebesgue integrable on $[0,1]$, $\mathbb{L}_\infty[0,1]$ is the set of functions essentially bounded on $[0,1]$, $\mathbb{AC}[0,1]$ denotes the set of functions absolutely continuous on $[0,1]$ and $\mathbb{BV}[0,1]$ is the set of functions of bounded variation on $[0,1]$. Furthermore, for $x \in \mathbb{C}[0,1]$ and $y \in \mathbb{L}[0,1]$, we denote

$$\|x\|_{\mathbb{C}} = \sup_{t\in[0,1]} |x(t)|, \quad \overline{y} = \int_0^1 y(s)\,\mathrm{d}s \text{ and } \|y\|_{\mathbb{L}} = \int_0^1 |y(t)|\,\mathrm{d}t.$$

Finally, for a given $y \in \mathbb{L}[0,1]$, y^+ denotes its nonnegative part ($y^+(t) = \max\{y(t),0\}$ for a.e. $t \in [0,1]$) and y^- stands for its nonpositive part ($y^-(t) = \max\{y(t),0\}$ for a.e. $t \in [0,1]$).

By a *solution of the problem* (0.2) we understand a function $u : [0,1] \mapsto \mathbb{R}$ such that $u' \in \mathbb{AC}[0,1]$, $u''(t) = f(t,u(t))$ a.e. on $[0,1]$, $u(0) = u(1)$ and $u'(0) = u'(1)$.

We will use the definitions of lower and upper functions from [8] modified to the problem (0.2).

1.1. Definition. Functions $(\sigma_1,\rho_1) \in \mathbb{AC}[0,1]\times\mathbb{BV}[0,1]$ are called *lower functions of the problem* (0.2) if the singular part ρ_1^{sing} of ρ_1 is nondecreasing on $[0,1]$,

$$\sigma_1'(t) = \rho_1(t), \quad \rho_1'(t) \ge f(t,\sigma_1(t)) \quad \text{a.e. on } t \in [0,1]$$

and

$$\sigma_1(0) = \sigma_1(1), \quad \rho_1(0+) \ge \rho_1(1-).$$

Similarly, functions $(\sigma_2,\rho_2) \in \mathbb{AC}[0,1] \times \mathbb{BV}[0,1]$ are called *upper functions of the problem* (0.2) if the singular part ρ_2^{sing} of ρ_2 is nonincreasing on $[0,1]$,

$$\sigma_2'(t) = \rho_2(t), \quad \rho_2'(t) \le f(t,\sigma_2(t)) \quad \text{a.e. on } t \in [0,1]$$

and

$$\sigma_2(0) = \sigma_2(1), \quad \rho_2(0+) \le \rho_2(1-).$$

Let us formulate the existence theorem which is our main tool in this paper and which is contained in [8, Theorems 4.1 and 4.2].

1.2. Theorem. *Let (σ_1,ρ_1) and (σ_2,ρ_2) be respectively lower and upper functions of the problem* (0.2).

(I) *Suppose $\sigma_1(t) \le \sigma_2(t)$ on $[0,1]$. Then there is a solution u of the problem* (0.2) *such that $\sigma_1(t) \le u(t) \le \sigma_2(t)$ on $[0,1]$.*

(II) *Suppose $\sigma_1(t) \ge \sigma_2(t)$ on $[0,1]$ and*

$$f(t,x) \ge h(t) \quad \textit{for a.e. } t \in [0,1] \quad \textit{and all } x \in \mathbb{R}$$

or

$$f(t,x) \le h(t) \quad \textit{for a.e. } t \in [0,1] \quad \textit{and all } x \in \mathbb{R}$$

with $h \in \mathbb{L}[0,1]$. Then there is a solution u of the problem (0.2) *such that*

$$\sigma_2(t_u) \le u(t_u) \le \sigma_1(t_u) \quad \textit{for some } t_u \in [0,1].$$

We will need the following two lemmas giving apriori estimates for solutions of (0.2). The proof of the former would be quite analogous to that of [8, Lemma 1.1].

1.3. Lemma. *Let a function* $h \in \mathbb{L}[0,1]$ *and sets* $\mathscr{U}(t) \subset \mathbb{R}$, $t \in [0,1]$, *be such that*

$$f(t,x) \geq h(t) \quad \text{for a.e.} \quad t \in [0,1] \quad \text{and all} \quad x \in \mathscr{U}(t)$$

or

$$f(t,x) \leq h(t) \quad \text{for a.e.} \quad t \in [0,1] \quad \text{and all} \quad x \in \mathscr{U}(t).$$

Then $\|u'\|_{\mathbb{C}} \leq \|h\|_{\mathbb{L}}$ *holds for any solution* u *of the problem* (0.2) *such that* $u(t) \in \mathscr{U}(t)$ *for all* $t \in [0,1]$.

1.4. Lemma. *Let a function* $\alpha \in \mathbb{L}[0,1]$ *and a number* $A \in (0,\infty)$ *be such that* $\overline{\alpha} = 0$ *and*

$$f(t,x) \geq \alpha(t) \quad \text{for a.e.} \quad t \in [0,1] \quad \text{and all} \quad x \in [A,\infty). \tag{1.1}$$

Then the relation

$$u(t) - A \leq \frac{\|\alpha\|_{\mathbb{L}}}{4} \quad \text{on} \quad [0,1] \tag{1.2}$$

holds for any solution u *of the problem* (0.2) *satisfying*

$$u(t_u) < A \quad \text{for some} \quad t_u \in [0,1]. \tag{1.3}$$

1.5. Remark. Notice that for any $\alpha \in \mathbb{L}[0,1]$ such that $\overline{\alpha} = 0$ we have $\overline{\alpha^+} = \overline{\alpha^-}$ and thus $\|\alpha\|_{\mathbb{L}} = 2\,\overline{\alpha^+} = 2\,\overline{\alpha^-}$.

Proof of Lemma 1.4. We borrow some ideas from [9, Lemma 2.1]. Let u be a solution of the problem (0.2) and let (1.1) be valid. First, we shall show that its derivative satisfies the estimate

$$\|u'\|_{\mathbb{C}} \leq \frac{\|\alpha\|_{\mathbb{L}}}{2}. \tag{1.4}$$

Let $t \in [0,1]$ be such that $u(t) > A$ and $u'(t) > 0$. Then, in virtue of the periodicity of u and u', there is $t_1 \in [0,1]$ such that $u'(t_1) = 0$ and $u(s) \geq A$ for $s \in I_1$, where

$$I_1 = \begin{cases} [t,1] \cup [0,t_1] & \text{if} \quad t_1 < t, \\ [t,t_1] & \text{if} \quad t_1 > t. \end{cases}$$

In both cases, making use of (1.1) we get

$$u'(t) = -\int_{I_1} f(s,u(s))\mathrm{d}s \le \int_{I_1} \alpha^-(s)\mathrm{d}s \le \frac{\|\alpha\|_{\mathbb{L}}}{2} \tag{1.5}$$

(cf. Remark 1.5). Similarly, if $u'(t) < 0$, there is $t_2 \in [0,1]$ such that $u'(t_2) = 0$ and $u(s) \ge A$ for $s \in I_2$, where

$$I_2 = \begin{cases} [t_2, t] & \text{if } \ t_2 < t, \\ [0,t_0] \cup [t_2, 1] & \text{if } \ t_2 > t. \end{cases}$$

Consequently, using again (1.1) and Remark 1.5, for an arbitrary $t \in [0,1]$ we get

$$u'(t) = \int_{I_1} f(s,u(s))\mathrm{d}s \ge -\int_{I_1} \alpha^-(s)\mathrm{d}s \ge -\frac{\|\alpha\|_{\mathbb{L}}}{2},$$

wherefrom, with respect to (1.5), the validity of (1.4) follows.

Now, assume that u satisfies, in addition, (1.3) and that $u(s) > A$ holds for some $t \in [0,1]$. We can choose $s_1, s_2, s^* \in [0,1]$ in such a way that

$$s_1 < s_2, \quad u(s_1) = u(s_2) = A \quad \text{and} \quad u(s^*) = \max_{s \in [0,1]} u(s) > A.$$

Consequently, (1.4) yields

$$\begin{aligned} 2\,(u(s^*) - A) &= \big(u(s^*) - u(s_1)\big) + \big(u(s^*) - u(s_2)\big) + \big(u(1) - u(0)\big) \\ &\le \int_I |u'(s)|\mathrm{d}s \le \frac{\|\alpha\|_{\mathbb{L}}}{2}, \end{aligned}$$

where $I = [s_1, s_2]$ if $s^* \in (s_1, s_2)$ and $I = [0,1] \setminus [s_1, s_2]$ if $s^* > s_2$ or $s^* < s_1$. This completes the proof of (1.2). $\square$

2 . Nonnegative and nonpositive solutions

2.1. Theorem. *Suppose that there exist $r_1 \in \mathbb{R}$, $A_1 \in [r_1, \infty)$ and $\beta_1 \in \mathbb{L}[0,1]$ such that*

$$f(t, r_1) \le 0 \quad \textit{for a.e.} \quad t \in [0,1] \tag{2.1}$$

and

$$\overline{\beta_1} \ge 0 \quad \textit{and} \quad f(t,x) \ge \beta_1(t) \quad \textit{for a.e.} \quad t \in [0,1] \quad \textit{and all} \quad x \in [A_1, B_1], \tag{2.2}$$

where

$$B_1 - A_1 \geq \frac{\|\beta_1 - \overline{\beta_1}\|_{\mathbb{L}}}{4}.$$

Then the problem (0.2) *has a solution* u *satisfying*

$$r_1 \leq u(t) \leq B_1 \quad \textit{on} \quad [0,1]. \tag{2.3}$$

Proof. (i) First, assume

$$\overline{\beta_1} > 0. \tag{2.4}$$

For a.e. $t \in [0,1]$ let us put

$$\widetilde{f}(t,x) = \begin{cases} f(t,x) & \text{if } x \leq B_1, \\ f(t,B_1) & \text{if } x > B_1, \end{cases} \tag{2.5}$$

and consider the auxiliary problem

$$u'' = \widetilde{f}(t,u), \quad u(0) = u(1), \quad u'(0) = u'(1). \tag{2.6}$$

In view of (2.1) the constants $(r_1, 0)$ are lower functions of (2.6). If we put

$$\sigma_2(t) = A_1 + 2\,\|\beta_1\|_{\mathbb{L}} - t \int_0^1 \int_0^\tau \beta_1(s)\mathrm{d}s\,\mathrm{d}\tau + \int_0^t \int_0^\tau \beta_1(s)\mathrm{d}s\,\mathrm{d}\tau \quad \text{for } t \in [0,1],$$

then

$$\sigma_2''(t) = \beta_1(t) \quad \text{a.e. on } [0,1], \quad \sigma_2(0) = \sigma_2(1) \quad \text{and} \quad \sigma_2'(1) - \sigma_2'(0) = \overline{\beta_1}.$$

Since $A_1 \leq \sigma_2(t)$ on $[0,1]$, we get by (2.2) and (2.5) that $\sigma_2''(t) \leq \widetilde{f}(t, \sigma_2(t))$ a.e. on $[0,1]$, which means that (σ_2, σ_2') are upper functions to (2.6) and the assertion (I) of Theorem 1.2 yields the existence of a solution u of (2.6) for which the estimate

$$r_1 \leq u(t) \leq \sigma_2(t) \quad \text{on } [0,1]$$

is true. According to (2.4) there exists $t_0 \in [0,1]$ such that $u(t_0) < A_1$. Indeed, otherwise we would get a contradiction

$$0 = \int_0^1 u''(t)\mathrm{d}t \geq \overline{\beta_1} > 0.$$

Since

$$\widetilde{f}(t,x) > \widetilde{f}(t,x) - \overline{\beta_1} \geq \beta_1(t) - \overline{\beta_1} \quad \text{for a.e. } t \in [0,1] \text{ and all } x \geq A_1,$$

we see that $\widetilde{f}$ fulfils (1.1) with $\alpha = \beta_1 - \overline{\beta_1}$ and $A = A_1$ and so we can apply Lemma 1.4 to u and the problem (2.6) and get

$$u(t) - A_1 \le \frac{\|\beta_1 - \overline{\beta_1}\|_{\mathbb{L}}}{4} \le B_1 - A_1 \quad \text{on} \quad [0,1].$$

Therefore u satisfies (2.3) and it is a solution of (0.2), as well.
(ii) Now, let $\overline{\beta_1} = 0$. Consider the sequence of auxiliary problems

$$u'' = \widetilde{f}_n(t,u), \quad u(0) = u(1), \quad u'(0) = u'(1), \tag{2.7}$$

where

$$\widetilde{f}_n(t,x) = \begin{cases} f(t,x) & \text{if } x < A_1, \\ f(t,x) \ + \dfrac{1}{n}\Big(\dfrac{x - A_1}{x - A_1 + 1}\Big) & \text{if } x \in [A_1, B_1], \\ f(t,B_1) + \dfrac{1}{n}\Big(\dfrac{B_1 - A_1}{B_1 - A_1 + 1}\Big) & \text{if } x > B_1. \end{cases}$$

For $n \in \mathbb{N}$ we have

$$\widetilde{f}_n(t,x) \ge \beta_1(t) + \frac{1}{2\,n^2} \quad \text{for a.e. } \ t \in [0,1] \ \text{ and all } \ x \in [A_1 + \frac{1}{n}, \infty).$$

Now, the first part of the proof guarantees for each $n \in \mathbb{N}$ the existence of a solution u_n of (2.7) which satisfies

$$r_1 \le u_n(t) \le B_1 + \frac{1}{n} \quad \text{on} \quad [0,1]. \tag{2.8}$$

According to (2.8), the Arzelá-Ascoli Theorem and the Lebesgue Dominated Convergence Theorem, the sequence $\{u_n\}_{n=1}^{\infty}$ contains a subsequence $\mathbb{C}^1$-converging to a solution u of the problem (2.6). Since u fulfils (2.3), it is a solution of (0.2). $\square$

2.2. Remark. In Theorem 2.1 it is sufficient to suppose that f satisfies the Carathéodory conditions on $[0,1] \times [r_1, \infty)$ instead of on $[0,1] \times \mathbb{R}$, because we can replace f by its truncation

$$\widehat{f}(t,x) = \begin{cases} f(t,r_1) & \text{for } \ x < r_1, \\ f(t,x) & \text{for } \ x \ge r_1 \end{cases}$$

in the proof.

2.3. Remark. Notice that in the case that $\beta_1(t) = 0$ a.e. on $[0,1]$ we can put $B_1 = A_1$ and (2.2) reduces to the condition ensuring the existence of constant upper functions $(A_1, 0)$.

2.4. Example. With respect to Remark 2.2, Theorem 2.1 yields the existence of a nonnegative solution u to the problem

$$u'' = \frac{u}{u+1}\sin(\frac{3}{2}\pi t) + e(t), \quad u(0) = u(1), \quad u'(0) = u'(1) \tag{2.9}$$

for any $e \in \mathbb{L}[0,1]$ such that

$$e(t) \le 0 \text{ a.e. on } [0,1] \quad \text{and} \quad \overline{e} > -\frac{2}{3\pi}.$$

Notice that the right hand side of the differential equation in (2.9) does not satisfy the condition (0.3) of Theorem 0.1. On the other hand, the problem

$$u'' = a\,t\,u^k + e(t), \quad u(0) = u(1), \quad u'(0) = u'(1) \tag{2.10}$$

with $e(t) \le 0$ a.e. on $[0,1]$ and $a, k \in (0,\infty)$ provides an example when the assumptions of Theorem 2.1 are fulfilled, while for $k > 1$ the condition (0.4) of Theorem 0.1 fails to be satisfied.

In addition to the existence results, Theorem 2.1 enables us to get an estimate for the guaranteed solution. Indeed, in the case of (2.9) we have

$$\frac{-e^*}{1+e^*} \le u(t) \le \Big(\frac{2-3\pi\overline{e}}{2+3\pi\overline{e}} + \frac{2-3\pi\overline{e}}{6\pi}\Big) \quad \text{on} \quad [0,1],$$

where

$$e^* = \sup_{t\in[0,1]} \text{ess}\; e(t).$$

In particular, for $e(t) \equiv -\frac{1}{3\pi}$ we get

$$0.118 < \frac{1}{3\pi - 1} \le u(t) \le 3 + \frac{1}{2\,\pi} < 3.16 \quad \text{on} \quad [0,1].$$

Similarly, a solution u of (2.10) can be estimated as follows:

$$\sqrt[k]{\frac{-e^*}{a}} \le u(t) \le \sqrt[k]{\frac{-2\overline{e}}{a}} - \frac{\overline{e}}{2} \quad \text{on} \quad [0,1].$$

If we put $a = 1$, $k = 2$ and $e(t) = -\frac{1}{2\sqrt{t}}$, we get

$$0.71 < \frac{1}{\sqrt{2}} \le u(t) \le \sqrt{2} + \frac{1}{2} < 1.92 \quad \text{on} \quad [0,1]. \tag{2.11}$$

2.5. Theorem. *Suppose that there exist* $r_2 \in \mathbb{R}$, $A_2 \in [r_2, \infty)$ *and* $\beta_2 \in \mathbb{L}[0,1]$ *such that*

$$\overline{\beta_2} \le 0 \ \text{and} \ f(t,x) \le \beta_2(t) \ \text{for a.e.} \ t \in [0,1] \ \text{and all} \ x \in [A_2, B_2] \tag{2.12}$$

and

$$f(t,x) \ge -\pi^2(x - r_2) \ \text{for a.e.} \ t \in [0,1] \ \text{and all} \ x \in [r_2, B_2], \tag{2.13}$$

where

$$B_2 - A_2 \ge \frac{1}{2}\,\overline{m_2^+} \tag{2.14}$$

and

$$m_2(t) \ge \max\{\sup_{x \in [r_2, A_2]} f(t,x), \beta_2(t)\} \ \text{for a.e.} \ t \in [0,1]. \tag{2.15}$$

Then the problem (0.2) *has a solution* u *satisfying*

$$r_2 \le u(t) \le B_2 \quad \text{on} \ [0,1]. \tag{2.16}$$

Proof. First suppose

$$\overline{\beta_2} < 0. \tag{2.17}$$

For a.e. $t \in [0,1]$ put

$$\widetilde{f}(t,x) = \begin{cases} f(t, r_2) - \pi^2(x - r_2) & \text{if } x < r_2, \\ f(t,x) & \text{if } r_2 \le x \le B_2, \\ f(t, B_2) & \text{if } x > B_2, \end{cases} \tag{2.18}$$

and consider the auxiliary problem

$$u'' = \widetilde{f}(t,u), \quad u(0) = u(1), \quad u'(0) = u'(1). \tag{2.19}$$

We can see that

$$\widetilde{f}(t,x) \ge -\pi^2(B_2 - r_2) \ \text{for a.e.} \ t \in [0,1] \ \text{and all} \ x \in \mathbb{R}.$$

Furthermore, the assumption (2.13) implies that $\widetilde{f}(t, r_2) \ge 0$, and (2.12) yields

$$\widetilde{f}(t,x) \le \beta_2(t) \ \text{for a.e.} \ t \in [0,1] \ \text{and all} \ x \in [A_2, \infty). \tag{2.20}$$

Thus, if we put

$$\sigma_1(t) = A_2 + 2\|\beta_2\|_{\mathbb{L}} - t\int_0^1\int_0^\tau \beta_2(s)\mathrm{d}s\,\mathrm{d}\tau + \int_0^t\int_0^\tau \beta_2(s)\mathrm{d}s\,\mathrm{d}\tau$$

for $t \in [0,1]$, we obtain similarly as in the proof of Theorem 2.1 that the couples $(r_2, 0)$ and (σ_1, σ_1') are respectively upper and lower functions to (2.19) and $r_2 < \sigma_1(t)$ holds on $[0,1]$. By the assertion (II) of Theorem 1.2 with $h(t) \equiv -\pi^2 (B_2 - r_2)$, there exists a solution u of (2.19). We shall show that u satisfies (2.16).

In virtue of (2.13) and (2.18) we have

$$\widetilde{f}(t,x) + \pi^2(x - r_2) \geq 0 \quad \text{for a.e.} \quad t \in [0,1] \quad \text{and all} \quad x \in \mathbb{R}. \tag{2.21}$$

We can check that if we put

$$g(t,s) = \begin{cases} \dfrac{\sin(\pi(s-t))}{2\pi} & \text{for} \quad 0 \leq t \leq s \leq 1, \\ \dfrac{\sin(\pi(t-s))}{2\pi} & \text{for} \quad 0 \leq s \leq t \leq 1, \end{cases}$$

then g is the Green function of the problem

$$y'' + \pi^2 y = 0, \quad y(0) = y(1), \quad y'(0) = y'(1)$$

and $g(t,s) \geq 0$ on $[0,1] \times [0,1]$. Furthermore, the function $z(t) = u(t) - r_2$ fulfils the relations

$$z''(t) + \pi^2 z(t) = \widetilde{f}(t, u(t)) + \pi^2(u(t) - r_2)$$
$$\text{a.e. on} \quad [0,1], \; z(0) = z(1), \; z'(0) = z'(1)$$

and so, according to (2.21), we have

$$z(t) = \int_0^1 g(t,s)\big[\widetilde{f}(s,u(s)) + \pi^2(u(s) - r_2)\big]\,\mathrm{d}s \geq 0 \quad \text{on} \quad [0,1],$$

i.e.

$$u(t) \geq r_2 \quad \text{on} \quad [0,1]. \tag{2.22}$$

Now, assume $u(t) \geq A_2$ on $[0,1]$. Then, by (2.20), $u''(t) \leq \beta_2(t)$ for a.e. $t \in [0,1]$ and thus, according to (2.17), we get

$$0 = \int_0^1 u''(t)\,\mathrm{d}t \leq \overline{\beta_2} < 0,$$

a contradiction. It means that there is $t_0 \in [0,1]$ such that

$$u(t_0) < A_2. \tag{2.23}$$

According to (2.15) and (2.18) we have

$$\widetilde{f}(t,x) \le m_2(t) \le m_2^+(t) \quad \text{for a.e.} \quad t \in [0,1] \quad \text{and all} \quad x \in [r_2,\infty).$$

Since (2.22) holds, we can apply Lemma 1.3 with $h = m_2^+$ and $\mathscr{U}(t) \equiv [r_2,\infty)$ to the problem (2.19) and obtain

$$\|u'\|_{\mathbb{C}} \le \|m_2^+\|_{\mathbb{L}} = \overline{m_2^+}. \tag{2.24}$$

Owing to (2.23) we can argue similarly as in the proof of Lemma 1.4. Assume that $u(t) > A_2$ holds for some $t \in [0,1]$ and choose $s_1, s_2, s^* \in [0,1]$ in such a way that

$$s_1 < s_2, \quad u(s_1) = u(s_2) = A_2 \quad \text{and} \quad u(s^*) = \max_{s\in[0,1]} u(s) > A_2.$$

Using (2.24) and (2.14) we get

$$u(s^*) - A_2 \le \frac{1}{2}\,\overline{m_2^+} \le B_2 - A_2,$$

i.e. u fulfils (2.16), which also means that u solves (0.2).

If $\overline{\beta_2} = 0$, we can follow the second part of the proof of Theorem 2.1 with

$$\widetilde{f}_n(t,x) = \begin{cases} f(t,r_2) & \text{if } x < r_2, \\ f(t,x) & \text{if } x \in [r_2,A_2), \\ f(t,x) - \frac{x-A_2}{n(x-A_2+1)} & \text{if } x \in [A_2,B_2], \\ f(t,B_2) - \frac{B_2-A_2}{n(B_2-A_2+1)} & \text{if } x > B_2. \end{cases}$$

□

2.6. Remark. Theorem 2.5 applies also to the case $\beta_2(t) = 0$ a.e. on $[0,1]$. However, then the interval $[A_2, B_2]$ need not reduce to the degenerate one (cf. (2.14) and (2.15)). Nevertheless, by a slight modification of the proof of Theorem 2.5 we obtain the following two existence results which extend [9, Theorem 2.3].

2.7. Corollary. *Suppose that there exist $r_2 \in \mathbb{R}$, $A_2 \in [r_2,\infty)$ and $m_2 \in \mathbb{L}[0,1]$ such that*

$$f(t,A_2) \le 0 \qquad \text{for a.e.} \quad t \in [0,1], \tag{2.25}$$

$$f(t,x) \le m_2(t) \quad \text{for a.e.} \quad t \in [0,1] \quad \text{and all} \quad x \in [r_2,B_2] \tag{2.26}$$

and (2.13) *are satisfied, where B_2 is such that* (2.14) *is true. Then the problem* (0.2) *has a solution u fulfilling* (2.16).

Proof. We can use the arguments as in the first part of the proof of Theorem 2.5 with the only difference that now $\sigma_1(t) \equiv A_2$ on $[0,1]$. Moreover, since the assertion (II) of Theorem 1.2 guarantees the existence of a solution u of (2.19) with

$$r_2 \le u(t_0) \le A_2 \quad \text{for some } \; t_0 \in [0,1], \tag{2.27}$$

we need neither assume (2.17) nor derive (2.23). □

2.8. Corollary. *Suppose that there exist $r_2 \in \mathbb{R}$, $A_2 \in [r_2,\infty)$ and $k \in [0,2)$ such that* (2.25) *and*

$$f(t,x) \ge -k\,(x-r_2) \;\; \textit{for a.e.} \;\; t \in [0,1] \;\; \textit{and all} \;\; x \in [r_2, B_2] \tag{2.28}$$

are valid, where

$$B_2 \ge A_2\,\frac{2}{2-k} - r_2\,\frac{k}{2-k}. \tag{2.29}$$

Then the problem (0.2) *has a solution u fulfilling* (2.16).

Proof. In the same way as in the proofs of Theorem 2.5 and Corollary 2.7 we get a solution u of (2.19) satisfying (2.22) and (2.27). According to (2.18) and (2.28) we have

$$u''(t) = \widetilde{f}(t,u(t)) \ge -k\,(u(t)-r_2) \;\; \text{for a.e.} \;\; t \in [0,1].$$

Furthermore, Lemma 1.3 with $h(t) = -k\,(u(t)-r_2)$, $\mathscr{U}(t) \equiv [r_2,\infty)$ gives

$$\|u'\|_{\mathbb{C}} \le k\,(\overline{u}-r_2).$$

Since in virtue of (2.22) and (2.27) we have also

$$\overline{u} \le \int_0^1 \Big(u(t_0) + \Big|\int_{t_0}^s |u'(\tau)|\mathrm{d}\tau\Big|\Big)\mathrm{d}s \le A_2 + \frac{1}{2}\,\|u'\|_{\mathbb{C}},$$

the relation

$$\|u'\|_{\mathbb{C}} \le \frac{2\,k}{2-k}\,(A_2-r_2) \tag{2.30}$$

immediately follows. Thus, similarly as we deduced in the first part of the proof of Theorem 2.5 from (2.14) and (2.24) the validity of (2.16), we can now show that also (2.29) and (2.30) imply (2.16). □

Replacing x by $-x$ in Theorem 2.5 we get the dual assertion:

2.9. Corollary. *Suppose that there exist* $r_2 \in \mathbb{R}$, $A_2 \in [r_2, \infty)$ *and* $\beta_2 \in \mathbb{L}[0,1]$ *such that*

$$\overline{\beta_2} \geq 0 \text{ and } f(t,x) \geq \beta_2(t) \text{ for a.e. } t \in [0,1] \text{ and all } x \in [-B_2, -A_2] \tag{2.31}$$

and

$$f(t,x) \leq -\pi^2(x + r_2) \text{ for a.e. } t \in [0,1] \text{ and all } x \in [-B_2, -r_2], \tag{2.32}$$

where

$$B_2 - A_2 \geq \frac{1}{2}\overline{m_2^-}$$

and

$$m_2(t) = \min\{\inf_{x \in [-A_2, -r_2]} f(t,x), \beta_2(t)\} \text{ for a.e. } t \in [0,1].$$

Then the problem (0.2) *possesses a solution* u *such that*

$$-B_2 \leq u(t) \leq -r_2 \quad \text{on } [0,1].$$

□

Combining Theorem 2.1 and Corollary 2.9 we immediately obtain

2.10. Corollary. *Suppose that all assumptions of both Theorem* 2.1 *and Corollary* 2.9 *with* $r_1 \geq 0$ *and* $r_2 \geq 0$ *are fulfilled and that either* (0.2) *has not the trivial solution or* $r_1 + r_2 > 0$. *Then the problem* (0.2) *has at least two different solutions, one of them nonnegative and one nonpositive.* □

2.11. Remark. Dual assertions to Theorem 2.1 and Corollary 2.10 can be obtained by substituting $-x$ instead of x, as well.

In Theorem 2.5 it suffices to suppose that f fulfils the Carathéodory conditions on $[0,1] \times [r_2, \infty)$ instead of on $[0,1] \times \mathbb{R}$. A similar restriction of the Carathéodory conditions for f can be assumed in all the other existence theorems in this section and their dual versions.

2.12. Example. In Example 2.4 we have shown that the problem

$$u'' = tu^2 - \frac{1}{2\sqrt{t}}, \quad u(0) = u(1), \quad u'(0) = u'(1) \tag{2.33}$$

has a solution u which satisfies (2.11). Further, we can check that all assumptions of Corollary 2.9 are fulfilled. We can put $r_2 = \frac{1}{\sqrt{2}}$ and $A_2 = \sqrt{2}$. Then Corollary 2.9 implies the existence of a solution v of (2.33) with an estimate

$$-1.79 < -\sqrt{2} - \frac{3}{8} \leq v(t) \leq -\frac{1}{\sqrt{2}} < -0.71 \quad \text{on } [0,1].$$

On the other hand, we cannot get the existence of u and v from Theorem 0.3 because the right hand side of (2.33) fulfils neither (0.4) nor (0.6).

We will close this section by showing that Theorems 0.1-0.3 due to M. N. Nkashama and J. Santanilla are contained in our Theorem 2.1 and Corollaries 2.9 and 2.10, respectively.

Let μ denote the Lebesgue measure.

2.13. Theorem. *Suppose*

$$\liminf_{x\to\infty} f(t,x) \geq 0 \quad \textit{for a.e.} \quad t \in [0,1], \tag{2.34}$$

$$\mu(\{t \in [0,1] : \liminf_{x\to\infty} f(t,x) > 0\}) > 0 \tag{2.35}$$

and

$$b(t) \leq f(t,x) \quad \textit{for a.e.} \quad t \in [0,1] \quad \textit{and all} \quad x \in [0,\infty) \tag{2.36}$$

with some $b \in \mathbb{L}[0,1]$.

Then there exist $A_1 \in (0,\infty)$ *and* $\beta_1 \in \mathbb{L}[0,1]$ *such that*

$$\overline{\beta_1} > 0 \quad \textit{and} \quad f(t,x) \geq \beta_1(t) \quad \textit{for a.e.} \quad t \in [0,1] \quad \textit{and all} \quad x \in [A_1,\infty).$$

2.14. Theorem. *Suppose*

$$\liminf_{x\to-\infty} f(t,x) \geq 0 \quad \textit{for a.e.} \quad t \in [0,1],$$

$$\mu(\{t \in [0,1] : \liminf_{x\to-\infty} f(t,x) > 0\}) > 0$$

and

$$b(t) \leq f(t,x) \quad \textit{for a.e.} \quad t \in [0,1] \quad \textit{and all} \quad x \in (-\infty,0]$$

with some $b \in \mathbb{L}[0,1]$.

Then there exist $A_2 \in (0,\infty)$ *and* $\beta_2 \in \mathbb{L}[0,1]$ *such that*

$$\overline{\beta_2} > 0 \quad \textit{and} \quad f(t,x) \geq \beta_2(t) \quad \textit{for a.e.} \quad t \in [0,1] \quad \textit{and all} \quad x \in (-\infty,-A_2].$$

Because of the duality of these theorems we restrict ourselves to the proof of Theorem 2.13.

Proof of Theorem 2.13. Due to (2.35), there exists $\varepsilon > 0$ such that

$$\mu_\varepsilon = \mu(Q_\varepsilon) > 0,$$

where

$$Q_\varepsilon = \{t \in [0,1] : \liminf_{x\to\infty} f(t,x) > \varepsilon\}.$$

For $n \in \mathbb{N}$ and a.e. $t \in [0,1]$ we can define

$$\gamma(t,n) = \inf_{x \geq n} f(t,x) \tag{2.37}$$

and

$$D_n = \{t \in [0,1] : \gamma(t,n) > \varepsilon\}.$$

We have

$$D_n \subset D_{n+1} \quad \text{for all } \ n \in \mathbb{N} \quad \text{and} \quad Q_\varepsilon \subset \bigcup_{n=1}^{\infty} D_n.$$

Furthermore, there exists $n_1 \in \mathbb{N}$ such that

$$\mu(D_n) > \frac{\mu_\varepsilon}{2} \quad \text{for all } \ n \geq n_1. \tag{2.38}$$

Choose $m_2 \in \mathbb{N}$ and $\delta > 0$ in such a way that

$$\sup_{J \subset [0,1], \mu(J) < \delta} \left| \int_J b(s)\,\mathrm{d}s \right| + \frac{1}{m_2} < \frac{\mu_\varepsilon\,\varepsilon}{2} \tag{2.39}$$

and for $m, n \in \mathbb{N}$ denote

$$S_{n,m} = \{t \in [0,1] : \gamma(t,n) \geq -\frac{1}{m}\}.$$

Then

$$S_{n,m} \subset S_{n+1,m} \quad \text{for all } \ n, m \in \mathbb{N}. \tag{2.40}$$

Due to (2.34), for every $m \in \mathbb{N}$ we have

$$\mu\Big([0,1] \setminus \bigcup_{n=1}^{\infty} S_{n,m}\Big) = 0. \tag{2.41}$$

Further, according to (2.40) and (2.41), for a chosen m_2 there is $n_2 \in \mathbb{N}$ such that $n_2 \geq n_1$ and

$$\mu(S_{n_2,m_2}) > \mu([0,1]) - \delta = 1 - \delta. \tag{2.42}$$

Put $A_1 = n_2$ and

$$\beta_1(t) = \begin{cases} \varepsilon & \text{if } \ t \in D_{n_2}, \\ -\frac{1}{m_2} & \text{if } \ t \in S_{n_2,m_2} \setminus D_{n_2}, \\ b(t) & \text{if } \ t \in [0,1] \setminus S_{n_2,m_2}. \end{cases}$$

Now, from (2.38), (2.39) and (2.42) we conclude that

$$\overline{\beta_1} = \int_{D_{n_2}} \varepsilon\,\mathrm{d}t - \int_{S_{n_2,m_2} \setminus D_{n_2}} \frac{1}{m_2}\,\mathrm{d}t + \int_{[0,1] \setminus S_{n_2,m_2}} b(t)\,\mathrm{d}t$$

$$\geq \frac{\mu_\varepsilon \varepsilon}{2} - \frac{1}{m_2} - \left| \int_{[0,1] \setminus S_{n_2,m_2}} b(t)\,\mathrm{d}t \right| > 0.$$

Finally, according to (2.37) we have

$$f(t,x) \geq \beta_1(t) \quad \text{for a.e.} \quad t \in [0,1] \quad \text{and all} \quad x \in [A_1, \infty)$$

and this completes the proof of the theorem. □

2.15. Remark. The assertion of Theorem 2.13 remains valid also in the case that μ is not necessarily the Lebesgue measure, but it can be an arbitrary nonnegative measure on $[0,1]$.

If the function $f(t,x)$ is only supposed to be $\nu \times \lambda$-measurable on $[0,1] \times \mathbb{R}$, where ν is a nonnegative measure on $[0,1]$ and λ is the Lebesgue measure, the functions

$$\gamma(t,n) \quad \text{and} \quad \liminf_{x \to \infty} f(t,x)$$

need not be measurable. In this case we should replace the assumption (2.35) by

$$\nu_{out}(\{t \in [0,1] : \liminf_{x\to\infty} f(t,x) > 0\}) > 0, \tag{2.43}$$

where ν_{out} stands for the outer measure corresponding to ν. Theorem 2.13 can be then reformulated in the following assertion. Its proof would be analogous to that of Theorem 2.13. Only in the definition of $\gamma(t,n)$ the *essential infimum* should be used instead of infimum.

2.16. Proposition. *Suppose* (2.34), (2.36), (2.43) *and the* $\nu \times \lambda$*-measurability of* f *on* $[0,1] \times \mathbb{R}$, *where* ν *is a nonnegative measure and* λ *is the Lebesgue measure. Then the statement of Theorem* 2.13 *remains valid, with the exception that the inequality* $f(t,x) \geq \beta_1(t)$ *is valid for a.e.* $(t,x) \in [0,1] \times [0,\infty)$ *only.* □

3 . Applications to Lazer-Solimini singular problems

In this section we want to extend the results of Lazer and Solimini [4] concerning the existence of solutions to singular periodic boundary value problems

$$u'' + g(u) = e(t), \quad u(0) = u(1), \quad u'(0) = u'(1) \tag{3.1}$$

and

$$u'' - g(u) = e(t), \quad u(0) = u(1), \quad u'(0) = u'(1). \tag{3.2}$$

Under the hypotheses $g \in \mathbb{C}(0,\infty)$,

$$g(x) > 0 \quad \text{on} \quad (0,\infty), \tag{3.3}$$

$$g(0+) := \lim_{x \to 0+} g(x) = \infty \tag{3.4}$$

and

$$g(\infty) := \lim_{x \to \infty} g(x) = 0, \tag{3.5}$$

Lazer and Solimini proved in [4, Theorem 2.1] that the problem (3.1) has a positive solution for a given $e \in \mathbb{C}[0,1]$ if and only if it satisfies the condition $\overline{e} > 0$.

Having in mind Remarks 2.2 and 2.11, we can apply all existence theorems from Section 2 to the problems (3.1) and (3.2) provided r_1 and r_2 are strictly positive. First, as direct consequences of Theorem 2.1, we get the following two corollaries which contain the above result from [4].

3.1. Corollary. *Suppose that $g \in \mathbb{C}(0,\infty)$ and $e \in \mathbb{L}[0,1]$ are such that*

$$g(\infty) < \infty, \tag{3.6}$$

$$g(x) > g(\infty) \quad \textit{for all} \quad x > 0 \tag{3.7}$$

and

$$\textit{there exists} \quad r_1 \in (0,\infty) \ \textit{such that} \quad e(t) \le g(r_1) \quad \textit{for a.e.} \quad t \in [0,1]. \tag{3.8}$$

Then the condition $\overline{e} > g(\infty)$ is necessary and sufficient for the existence of a positive solution to (3.1).

Proof. First, suppose $\overline{e} > g(\infty)$ and for a.e. $t \in [0,1]$ and any $x \in \mathbb{R}$ put

$$f(t,x) = e(t) - \begin{cases} g(x) & \text{if} \ \ x \ge r_1, \\ g(r_1) & \text{if} \ \ x < r_1. \end{cases}$$

Then, in virtue of (3.8), f satisfies the assumption (2.1) of Theorem 2.1. Furthermore, according to (3.6), there is $A_1 \ge r_1$ such that (2.2) with $\beta_1(t) = e(t) - \overline{e}$ is also satisfied. By Theorem 2.1 this proves the existence of the desired solution.

On the other hand, if u is a positive solution to (3.1), then integrating the differential equation in (3.1) and making use of (3.7), we get

$$\overline{e} = \int_0^1 g(u(s))\,\mathrm{d}s > g(\infty).$$

□

3.2. Remark. In particular, if $g(\infty) = -\infty$, then the problem (3.1) has a solution for any $e \in \mathbb{L}[0,1]$ for which (3.8) is true.

3.3. Corollary. *Suppose that* $g \in \mathbb{C}(0, \infty)$ *and* $e \in \mathbb{L}[0,1]$ *satisfy* (3.8) *and*

$$\overline{e} - \limsup_{x \to \infty} g(x) > 0.$$

Then the problem (3.1) *has a positive solution.*

Proof follows the first part of the proof of Corollary 3.1. □

3.4. Example. Consider the problem

$$u'' + \frac{a}{u^\lambda} - bu = e(t), \quad u(0) = u(1), \quad u'(0) = u'(1) \tag{3.9}$$

with $a > 0$, $\lambda > 0$ and $b \geq 0$. By Corollary 3.3, if $b > 0$, then the problem (3.9) has a positive solution for any $e \in \mathbb{L}[0,1]$ such that

$$e^* = \sup_{t \in [0,1]} \text{ess } e(t) < \infty,$$

while in the case $b = 0$, the additional assumption $\overline{e} > 0$ is needed. Notice that if $b = 0$, then the condition $\overline{e} > 0$ is also necessary for the existence of a positive solution to (3.9).

Furthermore, as in Examples 2.4 and 2.12, using Theorem 2.1 we can derive estimates for the guaranteed positive solution u of (3.9). In particular, in the case $b = 0$ we get

$$\left(\frac{a}{e^*}\right)^{\frac{1}{\lambda}} \leq u(t) \leq \left(\frac{a}{\overline{e}}\right)^{\frac{1}{\lambda}} + \frac{\|e - \overline{e}\|_{\mathbb{L}}}{4} \quad \text{on} \quad [0,1].$$

The following immediate consequence of Theorem 1.2 enables us to consider the problem (3.9) also when $b < 0$.

3.5. Corollary. *Suppose that there exist positive numbers* $r_1 < r_2 < r_3 < r_4$ *and a function* $h \in \mathbb{L}[0,1]$ *such that* f *fulfils the Carathéodory conditions on* $[0,1] \times [r_1, \infty)$ *and*

$$f(t, r_1) < 0 \quad \text{and} \quad f(t, r_4) \leq 0 \quad \text{for a.e.} \quad t \in [0,1], \tag{3.10}$$

$$f(t, r_2) \geq 0 \quad \text{and} \quad f(t, r_3) \geq 0 \quad \text{for a.e.} \quad t \in [0,1] \tag{3.11}$$

and

$$f(t, x) \leq h(t) \quad \text{for a.e.} \quad t \in [0,1] \quad \text{and all} \quad x \in [r_1, \infty).$$

Then the problem (0.2) *has at least two positive solutions* u *and* v *satisfying*

$$r_1 \leq u(t) \leq r_2 \quad \text{on} \quad [0,1] \quad \text{and} \quad r_3 \leq v(t_v) \leq r_4 \quad \text{for some} \quad t_v \in [0,1]. \tag{3.12}$$

Proof. Let us denote

$$\widetilde{f}(t,x) = \begin{cases} f(t,r_1) & \text{for } \ x < r_1, \\ f(t,x) & \text{for } \ x \ge r_1. \end{cases}$$

Then Theorem 1.2 implies the existence of solutions u and v of the problem

$$u'' = \widetilde{f}(t,u), \quad u(0) = u(1), \quad u'(0) = u'(1) \tag{3.13}$$

satisfying (3.12). Let $\min_{t\in[0,1]} v(t) = v(t_0) < r_1$. In view of the periodic conditions in (3.13), we can suppose $t_0 \in [0,1)$ and $v'(t_0) = 0$. There exists $t_1 \in (t_0,1)$ such that $v'(t_1) \ge 0$ and $v(t) \le r_1$ for all $t \in [t_0,t_1]$. Then, by (3.10),

$$0 > \int_{t_0}^{t_1} \widetilde{f}(t,v(t))\,\mathrm{d}t = v'(t_1) - v'(t_0) \ge 0,$$

a contradiction. Thus $r_1 \le u(t)$ and $r_1 \le v(t)$ on $[0,1]$ and u,v are positive solutions to (0.2). □

3.6. Example. Assume that $\lambda > 0$, $a > 0$, $b < 0$ and $e \in \mathbb{L}_\infty[0,1]$ and denote

$$K = \min_{x>0} \left(\frac{a}{x^\lambda} - bx\right) \quad \text{and} \quad e_* = \inf_{t\in[0,1]} \text{ess}\ e(t). \tag{3.14}$$

Then

$$K = \left(\frac{|b|}{\lambda a}\right)^{\frac{\lambda}{\lambda+1}} (\lambda+1)\,a$$

and by Corollary 3.5, the problem (3.9) has at least two different positive solutions provided the condition $e_* > K$ holds.

If $e_* = K$, we get at least one positive solution for (3.9). Let us note that if $e^* = \sup \text{ess}_{t\in[0,1]}\, e(t) < K$, then the problem (3.9) has no positive solution because in such a case we have

$$e(t) - \frac{a}{x^\lambda} + bx < 0 \ \text{ for a.e. } \ t \in [0,1] \ \text{ and all } \ x > 0.$$

Theorem 6.1 and Corollary 6.1 of [5], which concern the case of continuous e and involve the stronger condition (3.4) instead of our condition (3.8), indicate that the above Corollaries 3.1 and 3.3 may be already known. However, the authors believe that the next assertion, which is a direct corollary of Theorem 2.5 and which concerns the problem (3.2) having a repulsive singularity at the origin, is new.

3.7. Corollary. *Suppose that* $g \in \mathbb{C}(0,\infty)$, $e \in \mathbb{L}[0,1]$,

$$\overline{e} + \limsup_{x\to\infty} g(x) < 0 \tag{3.15}$$

and there is $\eta > 0$ *such that*

$$e(t) + g(x) + \pi^2 x \geq \eta \quad \textit{for a.e.} \quad t \in [0,1] \quad \textit{and all} \quad x > \frac{\eta}{\pi^2}. \tag{3.16}$$

Then the problem (3.2) *has a positive solution* u *such that* $u(t) \geq \frac{\eta}{\pi^2}$ *on* $[0,1]$..

Proof. Denote $f(t,x) = g(x) + e(t)$. According to (3.16), f satisfies (2.13) with $r_2 = \frac{\eta}{\pi^2}$ and $B_2 > 0$ arbitrarily large. Furthermore, in view of (3.15), we can find $A_2 \geq r_2$ such that f satisfies (2.12) with $\beta_2(t) = e(t) - \overline{e}$. □

3.8. Remark. Provided $g \in \mathbb{C}(0,\infty)$ satisfies (3.3), (3.4), (3.5),

$$\int_0^1 g(x)\,\mathrm{d}x = \infty \tag{3.17}$$

(i.e. it has a *strong singularity* at $x = 0$) and $e \in \mathbb{L}[0,1]$, Lazer and Solimini proved in [4, Theorem 3.12] that the condition $\overline{e} < 0$ is necessary and sufficient for the existence of a positive solution to (3.2). This result has been extended by several authors, cf. e.g. [1], [2], [3], [5], [7] and [11], however all these papers concern the case of a strong singularity at the origin. Notice that Corollary 3.7 applies to (3.2) even if the assumption (3.17) is omitted.

3.9. Example. Consider the problem

$$u'' - \frac{a}{u^\lambda} + bu = e(t), \quad u(0) = u(1), \quad u'(0) = u'(1) \tag{3.18}$$

with $\lambda > 0$, $a > 0$ and $b \geq 0$. If $e \in \mathbb{L}[0,1]$, $b = 0$ and $\lambda \geq 1$ (i.e. the function

$$g(x) = \frac{a}{x^\lambda} - bx, \quad x > 0,$$

has a strong singularity at $x = 0$), then by [4, Theorem 3.12] the problem (3.18) has a positive solution if and only if the condition $\overline{e} < 0$ is satisfied, while in the case $\lambda \in (0,1)$ this condition need not ensure the existence of a positive solution to (3.18) (cf. [4, Theorem 4.1]). Further, if $e \in \mathbb{C}[0,1]$ and $\lambda \geq 1$, then by the result due to del Pino, Manásevich and Montero (cf. [1, Theorem 1.1]), the problem (3.18) has a positive solution whenever the condition

$$b \neq (k\pi)^2 \quad \text{for all} \quad k \in \mathbb{N}$$

is satisfied. It is worth mentioning that the resonance case of $b = \pi^2$ is covered neither by [1, Theorem 1.1] nor by [7, Theorem 1.2] even for the strong singularity $\lambda \geq 1$.

In comparison to these results, it should be pointed out that Corollary 3.7 applies also to the cases $\lambda \in (0,1)$ and $b = \pi^2$. In particular, for the problem (3.18) with $e \in \mathbb{L}[0,1]$, we get the existence of a positive solution in the following cases:

$$b = 0, \quad \overline{e} < 0 \quad \text{and} \quad e_* > -\Big(\frac{\pi^2}{\lambda a}\Big)^{\frac{\lambda}{\lambda+1}} (\lambda+1)\, a$$

or

$$b \in (0, \pi^2] \quad \text{and} \quad e_* > -\Big(\frac{\pi^2 - b}{\lambda a}\Big)^{\frac{\lambda}{\lambda+1}} (\lambda+1)\, a.$$

In particular, if $b = \pi^2$, then the problem (3.2) has a positive solution for any $e \in \mathbb{L}[0,1]$ such that $e_* > 0$. This result gives the answer to the open question from [1, Remark 1.2].

Finally, let us consider the problem (3.18) with $b < 0$. By a slight modification of the proof of Theorem 2.5 we get an assertion which can be applied to this case.

3.10. Corollary. *Suppose that there exist $r_2 \in [0, \infty)$, $r_1 \in (r_2, \infty)$ and $h \in \mathbb{L}[0,1]$ such that f fulfils the Carathéodory conditions on $[0,1] \times [r_2, \infty)$, $f(t,x) \geq h(t)$ for a.e. $t \in [0,1]$ and all $x \geq r_2$, $f(t, r_1) \leq 0$ a.e. on $[0,1]$, $f(t,x) \geq -\pi^2(x - r_2)$ for a.e. $t \in [0,1]$ and all $x \geq r_2$.*

Then the problem (0.2) has a solution u such that $r_2 \leq u(t)$ on $[0,1]$ and $u(t_u) \leq r_1$ for some $t_u \in [0,1]$.

Proof. Put

$$\widetilde{f}(t,x) = \begin{cases} f(t, r_2) - \pi^2(x - r_2) & \text{for } x < r_2, \\ f(t,x) & \text{for } x \geq r_2 \end{cases}$$

and consider the problem (3.13). As the couples $(r_2, 0)$ and $(r_1, 0)$ are respectively upper and lower functions to (3.13), by the assertion (II) of Theorem 1.2 there exists a solution u to (3.13) with $r_2 \leq u(t_u) \leq r_1$ for some $t_u \in [0,1]$. Following the proof of Theorem 2.5 we get that $u(t) \geq r_2$ on $[0,1]$, which completes the proof. □

3.11. Example. By the assertion (I) of Theorem 1.2, the problem (3.18) with $\lambda > 0$, $a > 0$, $b < 0$ and $e \in \mathbb{L}_\infty[0,1]$ such that

$$e^* = \sup_{t \in [0,1]} \text{ess}\ e(t) \leq -K = -\Big(\frac{|b|}{\lambda a}\Big)^{\frac{\lambda}{\lambda+1}} (\lambda+1)\, a$$

has a positive solution u. If, moreover, $e^* < -K$ and

$$e_* = \inf_{t\in[0,1]} \operatorname{ess} e(t) > -\Big(\frac{|b|+\pi^2}{\lambda a}\Big)^{\frac{\lambda}{\lambda+1}} (\lambda+1)\, a,$$

then by Corollary 3.10 the problem (3.18) has also another positive solution v which certainly does not coincide with u on $[0,1]$. (Notice that for

$$\inf_{t\in[0,1]} \operatorname{ess}\, e(t) > -K,$$

(3.18) cannot have any positive solution.)

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