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SOME RESULTS AND PROBLEMS IN THE MODAL SET THEORY MST

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Introduction

Modal set theory MST was introduced in [6], where were also developed some basic notions of the theory, in particular the notion of natural numbers. Further results were obtained in [7] and [8]. Several partial consistency statements are also proved there. In this paper we construct an arithmetical structure on natural numbers in a way which was sketched in [6]. This structure will be a model of PEANO's axioms. We introduce an axiom of absoluteness (A) and deduce some of its consequences, in particular that natural numbers form a set in the theory. This is supplemented by a list of some most important problems concerning MST. Almost all results of this paper are contained in the thesis [8] which are, however, written in Czech.

We do not recapitulate the motivation and the axiomatization of the system MST; for the notions, the definitions and the basic results the reader should consult [6]. There is only one difference in notation: almost all results in this paper are proved in MST together with the assumptions S4 + BF; so it is suitable to consider S4 and BF as axioms of MST itself as is done in this paper. In [6] S4 + BF were considered only as possible additional axioms and not as a part of MST.

In [6] and [8] connections with other related systems, namely those of FEFERMAN, FITCH and GILMORE, are discussed.

Chapter I. Basic set-theoretical notions

In this chapter we define the notions of ordered pair, relation and function.

We begin with a simple extension of the definition of decidable set.

1.1. Definition. A set x is *1-decidable* ($D^1(x)$ in symbols) iff $D(x)$ & $(\forall y \in x)\, D(y)$, and *2-decidable* ($D^2(x)$ in symbols) iff $D(x)$ & $(\forall y \in x)\, D^1(y)$. Sometimes we call a decidable set *0-decidable* and write $D^0(x)$ instead of $D(x)$.

The reason for this definition is rather pragmatic one: decidable sets are manageable hence if x will be 1-decidable also its members will be manageable, etc.

It can be easily observed that all notions introduced in this chapter are translations of the usual ones. They are only restricted to an appropriate domain to make them manageable and written in a carefull way.

2.1. Definition.

$$\begin{aligned}\mathrm{Pair}(a, b, c) \equiv (\exists t_1, t_2)\ (&t_1 \in c \;\&\; t_2 \in c \;\&\; a \in t_1 \;\&\; a \in t_2 \;\&\; b \in t_2\\ &\&\ (\forall s)\ (s \in c \to s = t_1 \lor s = t_2)\\ &\&\ (\forall s)\ (s \in t_1 \to s = a) \;\&\; (\forall s)\ (s \in t_2 \to (s = a \lor s = b)).\end{aligned}$$

[1]) I would like to thank PAVEL PUDLÁK for his assistance to my work.

2.2. **Lemma.** $\vdash D^1(c) \to \mathfrak{D}\,(\mathrm{Pair}(a, b, c))$.

Proof. It can be easily calculated:

$$\vdash D^1(c) \,\&\, \mathrm{Pair}(a, b, c) \to \Box \mathrm{Pair}(a, b, c),$$

since $D^1(c)$ implies $D(t_1)$ and $D(t_2)$. It remains to show:

$$\vdash D^1(c) \,\&\, \neg \mathrm{Pair}(a, b, c) \to \Box \neg \mathrm{Pair}(a, b, c).$$

We now formulate a general (meta)sublemma which stands behind the formula above and analogical formulas in the proofs of 3.2, 4.2 and 4.4.

Sublemma. *If* $\vdash D(t) \to \mathfrak{D}(\varphi(t))$, *then* $\vdash D^1(c) \to \mathfrak{D}((\exists t \in c)\, \varphi(t))$.

The proof of this statement is easy: It is enough to show that

$$\vdash D^1(c) \,\&\, (\exists t \in c)\, \varphi(t) \to \Box(\exists t \in c)\, \varphi(t)$$

and

$$\vdash D^1(c) \,\&\, (\forall t \in c) \neg \varphi(t) \to \Box(\forall t \in c) \neg \varphi(t).$$

For the proof of the first formula it is enough to observe that

$$\vdash D^1(c) \,\&\, (\exists t \in c)\, \varphi(t) \to (\exists t \in c)\, (D(t) \,\&\, \varphi(t)).$$

We deduce the second formula. Evidently

$$\vdash (\forall t \in c) \neg \varphi(t) \equiv (\forall t)\, (t \notin c \vee (t \in c \,\&\, \neg \varphi(t))).$$

Hence

$$\vdash D^1(c) \,\&\, (\forall t \in c) \neg \varphi(t) \to (\forall t)\, (t \notin c \vee (t \in c \,\&\, D(t) \,\&\, \neg \varphi(t)))$$

From the hypothesis of the sublemma then follows

$$\vdash D^1(c) \,\&\, (\forall t \in c) \neg \varphi(t) \to (\forall t)\, (\Box t \notin c \vee (\Box t \in c \,\&\, \Box \varphi(t))).$$

But

$$\vdash (\forall t)\, (\Box t \notin c \vee (\Box t \in c \,\&\, \Box \varphi(t))) \to \Box(\forall t \in c) \neg \varphi(t),$$

so we are done.

We write now the formula $\mathrm{Pair}(a, b, c)$ in the form $(\exists t_1, t_2 \in c)\, \psi(t_1, t_2, a, b, c)$, where ψ is an appropriate formula from 2.1. Using the sublemma we obtain $\vdash D^1(c) \to \mathfrak{D}(\mathrm{Pair}(a, b, c))$. $\Box$

3.1. **Definition.** A set r is a *relation* ($\mathrm{Rel}(r)$ in symbols) iff $(\forall c \in r)\, (\exists a, b)\, \mathrm{Pair}(a, b, c)$.

3.2. **Lemma.** $\vdash D^2(r) \to \mathfrak{D}(\mathrm{Rel}(r))$.

Proof. Observe that

$$\vdash \mathrm{Rel}(r) \equiv (\forall c)\, (\exists a, b)\, (c \notin r \vee (c \in r \,\&\, \mathrm{Pair}(a, b, c))).$$

So if $c \in r$, then the 2-decidability of r implies the 1-decidability of c. Using 2.2 we easily calculate $\vdash D^2(r) \,\&\, \mathrm{Rel}(r) \to \Box \mathrm{Rel}(r)$ and also $\vdash D^2(r) \,\&\, \neg \mathrm{Rel}(r) \to \Box \neg \mathrm{Rel}(r)$.

4.1. **Definition.** (i) A set f is a *function* ($\mathrm{Fuc}(f)$ in symbols) iff

$$\mathrm{Rel}(f) \,\&\, (\forall c_1, c_2, a, b_1, b_2)\, (c_1 \notin f \vee c_2 \notin f \vee \neg \mathrm{Pair}(a, b_1, c_1) \vee \neg \mathrm{Pair}(a, b_2, c_2) \vee c_1 = c_2).$$

(ii) A set d is a *domain of* f ($\mathrm{Dom}(d, f)$ in symbols) iff

$$(\forall a)\, (a \notin d \vee (\exists b, c)\, (c \in f \,\&\, \mathrm{Pair}(a, b, c))) \,\&\, (\forall c, a, b)\, (c \notin f \vee \neg \mathrm{Pair}(a, b, c) \vee a \in d).$$

4.2. Lemma.

(i) $\vdash D^2(f) \rightarrow \mathfrak{D}(\mathrm{Fuc}(f))$.

(ii) $\vdash D^2(f) \mathbin{\&} D(d) \rightarrow \mathfrak{D}(\mathrm{Dom}(d, f))$.

We skip the proof of this lemma since it is analogical to those of 2.2 and 3.2.

4.3. Definition.

(i) Relations r, s are *inverse* ($\mathrm{Inv}(r, s)$ in symbols) iff

$$(\forall c)\ (c \in r \equiv (\exists c', a, b)\ (c' \in s \mathbin{\&} \mathrm{Pair}(a, b, c') \mathbin{\&} \mathrm{Pair}(b, a, c))).$$

(ii) A set f is a $1-1$ *function* ($\mathrm{MFuc}(f)$ in symbols) iff

$$\mathrm{Fuc}(f) \mathbin{\&} (\forall c_1, c_2 \in f)\ (\forall a_1, b, a_2)\ (\mathrm{Pair}(a_1, b, c_1) \mathbin{\&} \mathrm{Pair}(a_2, b, c_2) \rightarrow a_1 = a_2).$$

(iii) A set d *contains a range of* f ($\mathrm{Rng}(d, f)$ in symbols) iff

$$(\forall c, a, b)\ (c \in f \mathbin{\&} \mathrm{Pair}(a, b, c) \rightarrow b \in d).$$

We mention (without the evident proof):

4.4. Lemma.

(i) $\vdash D^2(r) \mathbin{\&} D^2(s) \rightarrow \mathfrak{D}(\mathrm{Inv}(r, s))$.

(ii) $\vdash D^2(f) \rightarrow \mathfrak{D}(\mathrm{MFuc}(f))$.

(iii) $\vdash D(d) \mathbin{\&} D^2(f) \rightarrow \mathfrak{D}(\mathrm{Rng}(d, f))$.

The following three results are proved for the purposes of chapter II.

4.5. Lemma.

$$\vdash (\forall f, d)\ (D^2(f) \mathbin{\&} \mathrm{Fuc}(f) \mathbin{\&} D(d) \mathbin{\&} \mathrm{Dom}(d, f)$$
$$\rightarrow (\forall a \notin d)\ (\forall b)\ (\exists c, g)\ (\mathrm{Pair}(a, b, c) \mathbin{\&} D^1(c) \mathbin{\&} c \in g \mathbin{\&} D^2(g)$$
$$\mathbin{\&} \mathrm{Fuc}(g) \mathbin{\&} (\exists d')\ (D(d') \mathbin{\&} \mathrm{Dom}(d', g) \mathbin{\&} (\forall t)\ (t \in d' \equiv t \in d \vee t = a)))).$$

Informally: *Any 2-decidable function with decidable domain can be prolonged by one new pair to a 2-decidable function with decidable domain.*

Proof. Let us argue informally: Let f be a 2-decidable function, d its decidable domain, $a \notin d$ and b any sets. By [6], 4.6. (iii), there exists a 1-decidable c such that $\mathrm{Pair}(a, b, c)$. Again using [6], 4.7. (iv), we add c into f and a into d to obtain a 2-decidable g and a decidable d'. □

4.6. Lemma.

$$\vdash \forall \alpha (N(\alpha) \rightarrow (\forall f)\ (D^2(f) \mathbin{\&} \mathrm{MFuc}(f) \mathbin{\&} \mathrm{Dom}(\alpha, f) \rightarrow (\exists g)\ (D^2(g) \mathbin{\&} \mathrm{MFuc}(g) \mathbin{\&} \mathrm{Inv}(f, g)))).$$

Informally: *Any 2-decidable $1-1$ function whose domain is a natural number has a 2-decidable $1-1$ invers.*

Proof. By induction on α. If α is decidable empty, then the statement is clear. Let the statement hold for α, α' be a successor of α (i.e. $\mathrm{Suc}(\alpha, \alpha')$, see [6], 4.9) and f be a 2-decidable $1-1$ function with domain α'. Let $c \in f$ such that $\mathrm{Pair}(\alpha, x, c)$. By [6], 4.7, there exists a 2-decidable $1-1$ function g with domain α which is a "shortening" of f by the pair c. By induction hypothesis there exists a 2-decidable $1-1$ invers of g, say h. Using 4.5 we can prolong this function h to a function k by a 1-decidable set c' such that $\mathrm{Pair}(x, \alpha, c')$. The function k is the required one. □

4.7. Lemma.

$$\vdash (\forall\alpha)\,(\mathrm{N}(\alpha) \to (\forall f, g)\,(\mathrm{D}^2(f)\ \&\ \mathrm{D}^2(g)\ \&\ \mathrm{MFuc}(f)\ \&\ \mathrm{MFuc}(g)\ \&\ \mathrm{Rng}(\alpha, f)\ \&\ \mathrm{Dom}(\alpha, g)$$
$$\to (\exists h)\,(\mathrm{D}^2(h)\ \&\ \mathrm{MFuc}(h)$$
$$\&\ (\forall c)\,(c \in h \equiv (\exists a_1, a_2, a_3)\,(\exists c_1 \in f)\,(\exists c_2 \in g)\,(\mathrm{Pair}(a_1, a_2, c_1)$$
$$\&\ \mathrm{Pair}(a_2, a_3, c_2)\ \&\ \mathrm{Pair}(a_1, a_3, c))))).$$

Informally: *Any 2-decidable* $1-1$ *functions* f, g *with range resp. domain* α *have a 2-decidable* $1-1$ *composition.*

(The lemma surely holds also for non $1-1$ functions but the proof is more complicated and this version suffices for all our purposes.)

Proof. By induction on α. If α is decidable empty, it is trivial. Let the statement hold for α, let α' be a successor of α and f, g 2-decidable $1-1$ functions such that $\mathrm{Rng}(\alpha', f)$ and $\mathrm{Dom}(\alpha', g)$. Let $c_1 \in f$, $c_2 \in g$ be pairs with second resp. first coordinate α. If they do not exists we are, by induction hypothesis, done. If there exist at least one such pair, we shorten f by c_1 resp. g by c_2. This can be done by [6], 4.7. Let $\bar f, \bar g$ be the resulting functions having range resp. domain in α. By induction hypothesis there exists their composition $\bar h$. Using 4.4 we prolong $\bar h$ by an appropriate pair. The resulting function h is the required composition of f and g. □

Chapter II. Arithmetical structure

In this chapter we carefully construct an arithmetical structure on the domain N as was sketched in [6]. Recall that letters $\alpha, \beta, \ldots$ range over natural numbers.

1.1. Definition.

(i) $\alpha \leqq_{\mathrm{N}} \beta$ iff $(\exists f)\,(\mathrm{D}^2(f)\ \&\ \mathrm{MFuc}(f)\ \&\ \mathrm{Dom}(\alpha, f)\ \&\ \mathrm{Rng}(\beta, f))$.

(ii) $\alpha =_{\mathrm{N}} \beta$ iff $\alpha \leqq_{\mathrm{N}} \beta\ \&\ \beta \leqq_{\mathrm{N}} \alpha$.

1.2. Lemma. *For* α, β, γ *natural numbers*

(i) a) $\alpha \leqq_{\mathrm{N}} \alpha$, b) $\alpha \leqq_{\mathrm{N}} \beta\ \&\ \beta \leqq_{\mathrm{N}} \gamma \to \alpha \leqq_{\mathrm{N}} \gamma$, c) $\alpha \leqq_{\mathrm{N}} \beta \vee \beta \leqq_{\mathrm{N}} \alpha$;

(ii) a) $\alpha =_{\mathrm{N}} \alpha$, b) $\alpha =_{\mathrm{N}} \beta\ \&\ \beta =_{\mathrm{N}} \gamma \to \alpha =_{\mathrm{N}} \gamma$, c) $\alpha =_{\mathrm{N}} \beta \to \beta =_{\mathrm{N}} \alpha$;

(iii) a) $\alpha =_{\mathrm{N}} \beta\ \&\ \alpha \leqq_{\mathrm{N}} \gamma \to \beta \leqq_{\mathrm{N}} \gamma$, b) $\alpha =_{\mathrm{N}} \beta\ \&\ \gamma \leqq_{\mathrm{N}} \alpha \to \gamma \leqq_{\mathrm{N}} \beta$.

Proof. We argue informally. (i) a) By induction on α one proves that there is a 2-decidable identity function on α. b) follows from I.4.7. c) By induction on α. If α is decidable empty, it is evident. Let the statement hold for α, let α' be a successor of α and β any natural number. By induction hypothesis we know: $\beta \leqq_{\mathrm{N}} \alpha \vee \alpha \leqq_{\mathrm{N}} \beta$. In the first case also $\beta \leqq_{\mathrm{N}} \alpha'$. Let $\alpha \leqq_{\mathrm{N}} \beta$ and f be the witnessing function. Let us consider two possibilities: The range of f is not resp. is the whole β. In the first situation we can prolong f, by I.4.5, to a 2-decidable $1-1$ function from α' into β. In the second situation, by I.4.6, there exists a 2-decidable $1-1$ invers g of f which witness $\beta \leqq_{\mathrm{N}} \alpha'$. The rest of lemma easily follows from the facts just proved. □

We now define addition and multiplication. Let us write $\mathrm{F}(f, a, b)$ instead of $(\exists c \in f)\,\mathrm{Pair}(a, b, c)$.

2.1. Definition.

(i) α is *zero* ($\mathrm{Nul}(\alpha)$ in symbols) iff $\mathrm{D}(\alpha)\ \&\ (\forall t)\, t \notin \alpha$.

(ii) γ is a *sum of* α *and* β ($\Sigma(\alpha, \beta, \gamma)$ in symbols) iff

$$(\exists f, \alpha')\,(\mathrm{D}^2(f)\ \&\ \mathrm{MFuc}(f)\ \&\ \mathrm{N}(\alpha')\ \&\ \mathrm{Suc}(\alpha, \alpha')\ \&\ \mathrm{Dom}(\alpha', f)$$
$$\&\ (\forall \delta_1, \delta_2 \in \alpha')\,((\mathrm{Nul}(\delta_1) \rightarrow \mathrm{F}(f, \delta_1, \beta))$$
$$\&\ (\forall \eta_1, \eta_2)\,(\mathrm{Suc}(\delta_1, \delta_2)\ \&\ \mathrm{F}(f, \delta_1, \eta_1)\ \&\ \mathrm{F}(f, \delta_2, \eta_2) \rightarrow \mathrm{Suc}(\eta_1, \eta_2)))$$
$$\&\ \mathrm{F}(f, \alpha, \gamma)).$$

(iii) γ is a *product of* α *and* β ($\Pi(\alpha, \beta, \gamma)$ in symbols) iff

$$(\exists f, \alpha')\,(\mathrm{D}^2(f)\ \&\ \mathrm{MFuc}(f)\ \&\ \mathrm{N}(\alpha')\ \&\ \mathrm{Suc}(\alpha, \alpha')\ \&\ \mathrm{Dom}(\alpha', f)$$
$$\&\ (\forall \delta_1, \delta_2 \in \alpha')\,((\mathrm{Nul}(\delta_1)\ \&\ \mathrm{F}(f, \delta_1, \delta_2) \rightarrow \mathrm{Nul}(\delta_2))$$
$$\&\ (\forall \eta_1, \eta_2)\,(\mathrm{Suc}(\delta_1, \delta_2)\ \&\ \mathrm{F}(f, \delta_1, \eta_1)\ \&\ \mathrm{F}(f, \delta_2, \eta_2) \rightarrow \Sigma(\eta_1, \beta, \eta_2)))$$
$$\&\ \mathrm{F}(f, \alpha, \gamma)).$$

To finish the construction of arithmetical structure it would be necessary to prove:

(i) $=_{\mathrm{N}}$, $\leqq_{\mathrm{N}}$, Nul, Σ and Π have the usual recursive properties,

(ii) $=_{\mathrm{N}}$ is a congruence w.r.t. $\leqq_{\mathrm{N}}$, Nul, Σ and Π,

(iii) $(\forall x, y)\,(\mathrm{N}(x)\ \&\ \mathrm{N}(y) \rightarrow (\exists u, v)\,(\mathrm{N}(u)\ \&\ \mathrm{N}(v)\ \&\ \Sigma(x, y, u)\ \&\ \Pi(x, y, v)))$.

(Observe that since by [6], 7.7 (i.e. induction), induction holds for *all* formulas, we have no special problems with "arithmetical" one.)

The way of such a proof is evident (and already sketched): The results of chapter I enable us to manipulate sufficiently with functions and so to prove induction steps for any required property. Induction then will imply the general satisfaction of the properties.

Let us content with an informal summary:

3.1. Theorem. *The formulas* $=_{\mathrm{N}}$, $\leqq_{\mathrm{N}}$, Nul, Σ *and* Π *define an arithmetical structure on the domain of natural numbers* N *which satisfy Peano's arithmetic.*

Chapter III. Absoluteness

We proved in Chapter I that a number of important formulas are $\Box$-decidable (if they speak about 2-decidable sets). On the other side a lot of useful notions, e.g. decidability, fulfil only the weaker condition $\mathrm{D}(x) \rightarrow \Box\mathrm{D}(x)$.

1.1. Metadefinition. A formula φ is *absolute* iff $\varphi \rightarrow \Box\varphi$ holds.

We shall use the abbreviation $\mathfrak{Abs}(\varphi)$ for $\varphi \rightarrow \Box\varphi$. The notion absolute corresponds with W. Reinhardt's "weakly decidable" and S. Feferman's "persistent", see [9] and [3]. Trivially: φ is $\Box$-decidable iff both φ and $\neg\varphi$ are absolute.

The main result of this chapter is that all Σ_1^0-arithmetical sentences are absolute.

1.2. Lemma. *For any formulas* φ, ψ.

(i) $\vdash \mathfrak{Abs}(\varphi)\ \&\ \mathfrak{Abs}(\psi) \rightarrow \mathfrak{Abs}(\varphi \vee \psi)\ \&\ \mathfrak{Abs}(\varphi\ \&\ \psi)\ \&\ \mathfrak{Abs}(\exists x\varphi)\ \&\ \mathfrak{Abs}(\forall x\varphi)$.

(ii) $\vdash \mathfrak{Abs}(\Box\varphi)$.

1.3. Lemma. *For* $i = 0, 1, 2$: $\vdash \mathfrak{Abs}(\mathrm{D}^i(x))$.

1.4. Proposition. $\vdash \mathfrak{Abs}(\mathrm{N}(x))$.

This proposition was proved in [6].

Already before Chapter I was reader willing to believe that it is possible to define correctly such notions as pair, relation or function and prove their most basic properties. Boring work done in preceeding two chapters will be now used for the result that all these notions are absolute; the result to which it would be difficult to believe without a proof.

2.1. Metadefinition.

(i) *The class of Σ_0^0-arithmetical formulas* is the least class X of formulas which satisfies the following conditions:

a) $\Sigma(x, y, z)$, $\Pi(x, y, z)$, $x =_{\mathrm{N}} y$, $x \leqq_{\mathrm{N}} y$ and $\mathrm{Nul}(x)$ are in X,

b) if φ, ψ are in X, then $\neg\varphi$, $\varphi \mathbin{\&} \psi$, $\varphi \vee \psi$, $\varphi \rightarrow \psi$ and $\varphi \equiv \psi$ are in X,

c) *if φ is in X, then* $(\exists x \in y)\, \varphi$ and $(\forall x \in y)\, \varphi$ are in X.

(ii) *The class of Σ_1^0-arithmetical formulas* is the class of all formulas of the form

$(\exists x_1, \ldots, x_k)\, (\mathrm{N}(x_1) \mathbin{\&} \ldots \mathbin{\&} \mathrm{N}(x_k) \mathbin{\&} \varphi)$, where φ is a Σ_0^0-arithmetical formula.

2.2. Theorem. *Let φ be a formula and $x_1, \ldots, x_k$ all its free variables. Then the following holds:*

(i) *If φ is Σ_1^0-arithmetical, then* $\vdash \mathrm{N}(x_1) \mathbin{\&} \ldots \mathbin{\&} \mathrm{N}(x_k) \rightarrow \mathfrak{Abs}(\varphi)$.

(ii) *If φ is Σ_0^0-arithmetical, then* $\vdash \mathrm{N}(x_1) \mathbin{\&} \ldots \mathbin{\&} \mathrm{N}(x_k) \rightarrow \mathfrak{D}(\varphi)$.

Proof. The proof will be divided into several steps and sublemmas.

Sublemma A.

$$\vdash \mathrm{N}(\alpha) \mathbin{\&} \mathrm{N}(\beta) \mathbin{\&} \mathrm{N}(\gamma) \rightarrow \mathfrak{Abs}(\alpha \leqq_{\mathrm{N}} \beta) \mathbin{\&} \mathfrak{Abs}(\alpha =_{\mathrm{N}} \beta) \mathbin{\&} \mathfrak{Abs}(\mathrm{Nul}(\alpha)) \mathbin{\&} \mathfrak{Abs}(\Sigma(\alpha, \beta, \gamma)) \mathbin{\&} \mathfrak{Abs}(\Pi(\alpha, \beta, \gamma)).$$

This sublemma follows by a detailed inspection of the formal definitions. From Chapter I we know that all notions used in the definitions of formulas above are in natural number-instances absolute. The formula $\mathrm{D}^2(x)$ is absolute by 1.3 and finally $\mathrm{N}(x) \rightarrow \mathrm{D}^2(x)$.

Let us define $x <_{\mathrm{N}} y$ by $(x \leqq_{\mathrm{N}} y \mathbin{\&} \neg x =_{\mathrm{N}} y)$.

Sublemma B. *Let φ be a Σ_0^0-arithmetical formula without $\neg$ and with free variables among $x_1, \ldots, x_k$. Then there exists a $\Sigma_0^0(x <_{\mathrm{N}} y)$-arithmetical formula ψ without $\neg$ and with the same free variables as φ such that* $\mathrm{N}(x_1) \mathbin{\&} \ldots \mathbin{\&} \mathrm{N}(x_k) \rightarrow ((\neg\varphi) \equiv \psi)$.

The $\Sigma_0^0(x <_{\mathrm{N}} y)$-arithmetical formulas are defined in the same way as the Σ_0^0-arithmetical formulas by adopting $x <_{\mathrm{N}} y$ among the basic atomic formulas, i.e. belonging to 2.1. (i).

The proof of this sublemma goes by induction on the complexity of φ. It is clear that it suffices to consider the following facts:

$$\neg\alpha =_{\mathrm{N}} \beta \equiv \alpha <_{\mathrm{N}} \beta \vee \beta <_{\mathrm{N}} \alpha, \quad \neg\alpha \leqq_{\mathrm{N}} \beta \equiv \beta <_{\mathrm{N}} \alpha,$$

$$\neg\Sigma(\alpha, \beta, \gamma) \equiv (\exists\delta \in \gamma)\, \Sigma(\alpha, \beta, \delta) \vee (\exists\delta \in \beta)\, \Sigma(\alpha, \delta, \gamma) \vee (\exists\delta \in \alpha)\, \Sigma(\delta, \beta, \gamma),$$

$$\neg\Pi(\alpha, \beta, \gamma) \equiv (\exists\delta \in \gamma)\, \Pi(\alpha, \beta, \delta) \vee (\exists\delta \in \beta)\, \Pi(\alpha, \delta, \gamma) \vee (\exists\delta \in \beta)\, (\exists\varrho \in \alpha)\, (\exists\eta \in \gamma)\, (\Pi(\alpha, \delta, \eta) \mathbin{\&} \Sigma(\eta, \varrho, \gamma)).$$

These equivalences are proved by induction on α, β, γ.

Sublemma C. $\vdash \mathrm{N}(\alpha) \,\&\, \mathrm{N}(\beta) \to \mathfrak{Abs}(\alpha <_{\mathrm{N}} \beta)$.

For the proof we observe that

$$\alpha <_{\mathrm{N}} \beta \equiv (\exists f, \gamma)\, (\mathrm{D}^2(f) \,\&\, \mathrm{MFuc}(f) \,\&\, \mathrm{Dom}(\alpha, f) \,\&\, \mathrm{Rng}(\gamma, f) \,\&\, \mathrm{N}(\gamma) \,\&\, \mathrm{Suc}(\gamma, \beta)).$$

This can be proved by induction on α and β. But the right-hand side is absolute; this follows from the results of Chapter I.

Now we are ready to prove the theorem. At first we prove part (ii). By Sublemmas A and C and by 1.2 (i) we know that natural number-instances of all $\Sigma^0_0(x <_{\mathrm{N}} y)$-arithmetical formulas without $\neg$ are absolute; Sublemma B implies that each Σ^0_0-arithmetical formula is equivalent to such a formula and hence is also absolute. Using the trivial observation after the definition 1.1 we conclude the part (ii) of the theorem.

The part (i) is an easy consequence of part (ii), 1.2 (i) and 1.4. □

2.3. Corollary. *Let φ, ψ be Σ^0_1-arithmetical formulas with the same free variables among $x_1, \ldots, x_k$ and let be $\vdash \mathrm{N}(x_1) \,\&\, \ldots \,\&\, \mathrm{N}(x_k) \to \varphi \equiv \neg\psi$. Then $\vdash \mathrm{N}(x_1) \,\&\, \ldots \,\&\, \mathrm{N}(x_k) \to \mathfrak{D}(\varphi) \,\&\, \mathfrak{D}(\psi)$.* (Shortly: *$\Delta^0_1$-arithmetical sentences are decidable.*)

As an easy observation we obtain:

3.1. Lemma. $\vdash \mathfrak{Abs}(\varphi(t)) \to (\exists y)\, (\forall t)\, (\Box t \in y \equiv \varphi(t))$.

In the light of the preceeding results the following axiom is suggested:

Axiom of absoluteness.

(A). $\quad x \in y \to \Box x \in y.$

The interest in this axiom is given by some of its corrolaries, a few of which we will discuss in this section.

Let us begin trivially:

3.2. Lemma. (A) $\vdash \mathfrak{Abs}(\varphi(t)) \to (\exists y)\, (\forall t)\, (t \in y \equiv \varphi(t))$.

3.3. Corrolaries.

(i) (A) $\vdash (\exists y)\, (\forall t)\, (t \in y \equiv \mathrm{N}(t))$.

(ii) (A) $\vdash (\forall x)\, (\exists y)\, (\forall t)\, (t \in y \equiv (\exists z \in x)\, t \in z)$.

(iii) (A) $\vdash (\forall x, y)\, (\exists z)\, (\forall t)\, (t \in z \equiv t \in x \,\&\, t \in y)$.

(iv) (A) $\vdash (\forall x, y)\, (\exists z)\, (\forall t)\, (t \in z \equiv t \in x \vee t \in y)$.

(v) (A) $\vdash (\forall x, y)\, (\exists z)\, (\forall t)\, (t \in z \equiv t \in x \vee t = y)$.

(vi) (A) $\vdash$ "*f is a 2-decidable $1-1$ function $\to f$ has a 2-decidable $1-1$ invers*".

(vii) (A) $\vdash$ "*f, g are 2-decidable functions*

$\to$ there exists a 2-decidable composition of f and g".

(viii) (A) $\vdash (\exists y)\, (\forall t)\, (t \in y \equiv \mathrm{D}^i(t))$, *for* $i = 0, 1, 2$.

(ix) (A) $\vdash (\exists y)\, (\forall t)\, (t \in y \equiv \mathrm{N}(t)\, \varphi(t))$, *where $\varphi(t)$ is any Σ^0_1-arithmetical formula with the only free variable t.*

Observe that (in the presence of axiom (A)) (i), (ii) and (v) opens the possibility of construction of ordinals, i.e. of successors and supremes. Axiom (A) also considerably simplifies the construction of the arithmetical structure, compare (vi) and (vii) with the much more complicated I.4.6 and I.4.7.

We finish this chapter with

3.4. Lemma. *Axiom* (A) *is not derivable in* MST (*provided* MST *is consistent*).

Proof. For any formula φ we define its *dual* φ^{d} as follows: (i) $(x \in y)^{\mathrm{d}} = (x \notin y)$, (ii) $(x = y)^{\mathrm{d}} = (x = y)$, (iii) d commutes with all connectives and quantifiers.

Claim. *If* $\vdash \varphi$ *then also* $\vdash \varphi^{\mathrm{d}}$.

The claim is proved by induction on the number of steps in the proof of φ. The only nontrivial case are the instances of MCA. But easily $\mathrm{MCA}(\varphi)^{\mathrm{d}} \equiv \mathrm{MCA}(\neg \varphi^{\mathrm{d}})$, where $\mathrm{MCA}(\varphi)$ stands for an instance of MCA for a formula φ.

By claim $\vdash$(A) would imply $\vdash x \notin y \to \Box x \notin y$ and hence $\vdash \mathfrak{D}(x \in y)$. But this is in contradiction with [6], 2.2. □

Chapter IV. Some metamathematical observations

In this chapter we present a few metamathematical observations and define a semantics for MST. The results of this chapter do not depend on the non-logical assumptions of the theory MST. They are easy and well known. We present them for the sake of completeness and for some technical reasons.

It is easy to observe that equivalence of two formulas φ and ψ, $\varphi \equiv \psi$, does not imply their mutual substituability into other formulas. Take, for example, $\varphi = (x = x)$, $\psi = (x \in x)$ and $\theta = \Box \psi$. If $\vdash \varphi \equiv \psi \to \theta \equiv \theta(\varphi/\psi)$ were true, then we would have

$$\vdash x \in x \equiv x = x \to \Box x \in x \equiv \Box x = x,$$

and analogically

$$\vdash x \notin x \equiv x = x \to \Box x \notin x \equiv \Box x = x.$$

Since $(x \in x \equiv x = x) \vee (x \notin x \equiv x = x)$ is a tautology, the following would then hold:

$$\vdash (\Box x \in x \equiv \Box x = x) \vee (\Box x \notin x \equiv \Box x = x).$$

Using (LP) we obtain $\vdash \Box x \in x \vee \Box x \notin x$ which is a contradiction with [6], 2.2. So the sufficient condition for the mutual substituability should be stronger.

1.1. Lemma. *Let* φ, ψ, θ *be any formulas and* θ' *be a formula which arise from* θ *by substituting* ψ *for some* (*one or more*) *occurrences of* φ *in* θ. *Then* $\vdash \Box(\overline{\varphi \equiv \psi}) \to \theta \equiv \theta'$, *where* $\overline{\varphi \equiv \psi}$ *is the universal closure of* $\varphi \equiv \psi$.

The proof is easy by induction on the depth of θ.

Corollary. *If* $\vdash \varphi \equiv \psi$, *then* φ *is substituable for* ψ.

The deduction lemma usually enable us to convert provability of φ in some finite extension (say by axiom ψ) of a theory to the provability of the implication $\psi \to \varphi$ in the original theory. Since always $\varphi \vdash \Box \varphi$ but not necessarily $\vdash \varphi \to \Box \varphi$, it is expectable that the modal case is again a little bit complicated.

1.2. Lemma. *For any sentence* φ *and any formula* ψ, $\varphi \vdash \psi$ *iff* $\vdash \Box\varphi \to \psi$.

The proof goes by simple induction on the number of steps in the proof of ψ.

In the following we assume that MST is consistent. Since MST contains PEANO's arithmetic it can formalize its own syntax. Although the language of the theory does not contain terms (in particular, numerals) it can be constructed a predicate of provability based on "pseudoterms" (cf. [1]). One can think of such a predicate Pr(x) in the same way as of the usual one. Since Pr(x) fulfils conditions sufficient for GÖDEL's second incompleteness theorem one can prove:

2.1. $\nvdash \mathrm{Pr}(\ulcorner x \neq x \urcorner)$.

On the other side there is yet another, informal, "predicate of provability", the modality $\Box$. It satisfies

(i) $\vdash\varphi$ implies $\vdash\Box\varphi$, (ii) $\vdash\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$, (iii) $\vdash\Box\varphi \to \Box\Box\varphi$,

but also $\vdash \neg\Box x \neq x$. Thus there is not an appropriate "diagonal" formula needed for the GÖDEL's proof.

Hence the interesting question arises: what is the relation between Pr(x) and $\Box$? The following two results contribute to this question. A relevant paper here is [9].

2.2. Lemma. *There is no (arithmetical) formula* $\varphi(x)$ *such that* $\vdash\varphi(\ulcorner\theta\urcorner) \equiv \Box\theta$ *holds for any* θ, *where* $\ulcorner\theta\urcorner$ *stands for a "pseudonumeral" of a Gödel number of* θ).

Proof. Suppose that $\varphi(x)$ defines $\Box$, i.e. $\vdash\varphi(\ulcorner\theta\urcorner) \equiv \Box\theta$ holds generally. Then $\varphi(x)$ satisfy LÖB's condition (i), (ii) and (iii) above. Also it is possible to find a diagonal formula for $\neg\varphi(x)$ (since φ is arithmetical) and so $\nvdash\neg\varphi(\ulcorner x \neq x\urcorner)$. But $\vdash\neg\Box x \neq x$. A contradiction. $\Box$

2.3. Lemma. *It is not the case that for all* φ

$$\vdash \mathrm{Pr}(\ulcorner\varphi\urcorner) \to \Box\varphi.$$

Proof. $\vdash\neg\Box x \neq x \to \neg\mathrm{Pr}(\ulcorner x \neq x\urcorner)$. $\Box$

It seems that the modality $\Box$ is nearer to the "real" provability than Pr(x).

Let us repeat that the decidability of φ, i.e. $\vdash\varphi$ or $\vdash\neg\varphi$, implies a provable $\Box$-decidability of φ, i.e. $\vdash\mathfrak{D}(\varphi)$. Let us call φ *essentially* $\Box$*-undecidable* iff $\vdash\neg\mathfrak{D}(\varphi)$.

2.4. Lemma. *Let* φ *be essentially* $\Box$*-undecidable. Then both theories* MST + φ *and* MST + $\neg\varphi$ *are inconsistent.*

Proof. By the N-rule φ is provably $\Box$-decidable in both theories above. This is a contradiction that φ is essentially $\Box$-undecidable in MST. $\Box$

If we could find such a formula φ we could call MST *essentially incomplete*. The problem surely remains whether there is such a formula. (The obvious candidate is $r \in r$, where r is a set from RUSSELL's paradox, but r is not a constant of the language and even cannot be consistently added into it, cf. [6]).

In the rest of this section we describe a semantics which is complete w.r.t. MST. A semantics for modal systems is usually formulated in terms of KRIPKE's models with worlds and a relation of alternativeness. We will use a concept of [5] which differs from KRIPKE's original one in a different handling with free variables. For example, while a formula dual to the BARCAN's is derivable (in our sense) already in T in

KRIPKE's formulation it is not a theorem even of S5. Definition below of a KRIPKE's universum differs also slightly from that of [5], we will have a distinguished world.

We will not go into details since this material is familiar.

3.1. Definition. $(W, \models)$ *is a model of* MST iff $(W, \models)$ is a KRIPKE's universum,

(i) whose worlds are first-order structures for the language $\langle =, \in \rangle$ (= is absolute),

(ii) whose all worlds have the same domain,

(iii) whose relation of alternativeness is reflexive and transitive,

(iv) which has a distinguished world denoted by W_0,

(v) for any axiom φ of MST $W_0 \models \Box\bar{\varphi}$, where $\bar{\varphi}$ is the universal closure of φ

(for details see [5]).

3.2. Theorem. *Any formula φ is a consequence of* MST *iff φ holds in all models of* MST.

Proof. The "only if" part follows easily from the deduction lemma 1.2. For the "if" part it is sufficient to prove the compactness. Let $\psi_0, \psi_1, \ldots$ enumerate the universal closures of all axioms of MST and let for any $k \geqq 0$, $\psi_0 \,\&\, \ldots \,\&\, \psi_k \,\&\, \neg\varphi$ have a model W^k. On W^k's we can look as on many-sorted first-order structures and thus by standard argument we can produce their ultraproduct W such that $W \models \psi_0 \,\&\, \psi_1 \,\&\, \psi_2 \,\&\, \ldots \,\&\, \neg\varphi$. $\Box$

Chapter V. Problems and suggestions

In this chapter we consider several problems and suggestions connected with the development of the theory. Some of them are formulated strictly others are more or less vague.

1. Consistency. Surely, the main problem is to prove the consistency of the theory MST. Without this result the whole program is rather risky undertaking.

2. Models. The second basic problem is to learn how to construct models of MST. In the semantics of IV.3 is clear the meaning of the modality $\Box$ but it gives no idea what should be $\in$. Some proof of consistency could lead to some more effective description of $\in$. Such a description is necessary if semantics should help in solving some problems about MST.

3. Power of the theory. How strong is the theory (w.r.t. some well-known systems)? In particular, does MST interpret at least ACA_0?

4. Extensions of the theory. With the above problem the question of possible extensions of MST is connected. In the flavour of the theory new axioms of logical nature are rather than some having explicit mathematical meaning. Kripke-style semantics suggests to add axioms which would force some, intuitively satisfactory, form of the universum of worlds, i.e. of the relation of alternativeness. E.g. the axioms which force the relation of alternativeness to be a linear order. Another interesting possibility is to extend the language of the theory by adding class-variables and the assumptions by an appropriate class-existence axiom. Such a theory would easily interpret second order arithmetic.

5. Modal complexity. Let us define the *modal complexity* $\mathrm{mc}(\varphi)$ of the formula φ as follows:

(i) $\mathrm{mc}(\varphi) = 0$ iff φ is nonmodal,

(ii) $\mathrm{mc}(\varphi \vee \psi) = \mathrm{mc}(\varphi \,\&\, \psi) = \mathrm{mc}(\varphi \to \psi) = \mathrm{mc}(\varphi \equiv \psi) = \max(\mathrm{mc}(\varphi), \mathrm{mc}(\psi))$,

(iii) $\mathrm{mc}(\neg\varphi) = \mathrm{mc}(\exists x\varphi) = \mathrm{mc}(\forall x\varphi) = \mathrm{mc}(\varphi)$,

(iv) $\mathrm{mc}(\Box\varphi) = 1 + \mathrm{mc}(\varphi)$.

Now we can define natural fragments MST_i, $i < \omega$, of the theory MST: MST_i have instances of MCA only for formulas of modal complexity at most i, e.g. MST_0 contains only MCA-instances for nonmodal formulas. In [6] we proved that MST_0 is consistent. Problem: Does there exists an i such that $\mathrm{MST}_i = \mathrm{MST}$, i.e. is the hierarchy MST_0, $\mathrm{MST}_1, \ldots$ proper? Let us remark that everything important so far was proved in MST_1. Thus it is possible that $\mathrm{MST}_1 = \mathrm{MST}$. An argument for this claim could be the following: Let $\mathrm{MCA}(\varphi)$ be the instance of the scheme MCA for a formula φ such that $\mathrm{mc}(\varphi) > 1$. $\mathrm{MCA}(\psi)$ implies that $\Box\psi$ is equivalent to some formula of the form $\Box t \in y$. In particular, try to replace subformulas of φ of the form $\Box\psi$, $\mathrm{mc}(\Box\psi) \geqq 2$, by equivalent formulas of the form $\Box t \in y$. Thus the modal complexity of φ would be reduced until formula of the modal complexity 1 is reached. The fail in this argument is the point that for mutual substituability of two formulas ($\Box\psi$ and $\Box t \in y$ above) we need more than only their equivalence (see IV.1.1), they should be knowable equivalent. Thus instead of MCA we would need a scheme $(\exists y)\,(\forall t)\,\Box((\Box\varphi(t) \equiv$ $\equiv \Box t \in y) \,\&\, (\Box\neg\varphi(t) \equiv \Box t \notin y))$ which is, however, inconsistent (cf. [6]).

6. Terms. An important question is the problem about conditions under which terms can be introduced.

7. Disjunction property. Many intuitionistic systems fulfil the condition that if the disjunction $\varphi \vee \psi$ is provable, then one of the disjuncts φ or ψ is provable (cf. [4]). The modal transform of this situation is: if $\Box\varphi \vee \Box\psi$ is provable, then so φ or ψ is provable. Does MST obey this (disjunction) property? Let us sketch an interesting consequence of the affirmative answer: By [8], $\vdash\mathfrak{D}(\mathrm{N}(x))$ implies that $\vdash\mathfrak{D}(\psi)$ for any arithmetical sentence ψ. By the disjunction property would be $\vdash\psi$ or $\vdash\neg\psi$ for any such sentence. Thus if MST were consistent and obeyed disjunction property, then a recursively-enumerable completion of PA could be found in its theorems. That is evidently a contradiction. Hence the disjunction property for MST implies that $\nvdash\mathfrak{D}(\mathrm{N}(x))$.

8. Infinity. The interpretation of Peano's arithmetic in MST brings into the theory sufficient aparatus for an interpretation of finite mathematics. On the other side MST proves the existency of infinite sets in the usual classical sense, e.g. of the universal set or (by presence of axiom (A)) of the set of natural numbers. However, it seems that in the context of the modal calculus the cardinality of a set is not a sufficient expression of its "infiniteness". The position of the decidable empty set and of a decidable universal set in the set-universe is probably equivalently complex. A more natural expression of "infiniteness" of sets seems to be their "complexity" in the sense of the modality $\Box$. Thus an undecidable set is more infinite than a decidable set. So the aim is to define some relation of order for complexities of sets. This question was already touched in [7] and [8] where two definitions were suggested.

Roughly speaking these definitions correspond to recursive invariance and m-reducibility of sets from the recursion theory. Let us sketch the later one.

We define: $a \precsim b$ iff

$$(\exists f, v)\,(D^2(f)\ \&\ \mathrm{Rel}(f)\ \&\ (\forall t)\,(t \in v)\ \&\ D(v)\ \&\ \mathrm{Dom}(v, f)$$
$$\&\ (\forall x, x', y, y', z)\,(x \in f\ \&\ x' \in f\ \&\ \mathrm{Pair}(z, y, x)\ \&\ \mathrm{Pair}(z, y', x') \rightarrow y = y')$$
$$\&\ (\forall x, y)\,(F(f, x, y) \rightarrow (\Box x \in a \equiv \Box y \in b)\ \&\ (\Box x \notin a \equiv \Box y \notin b))).$$

Informally: $a \precsim b$ iff there exists a 2-decidable function on the set-universe which translates the "elementhood" of a into that of b in the sense of the formula above.

It is easy to prove, under axiom (A), that $\precsim$ is a quasi-order and any two nonempty decidable sets with nonempty complement are in relation $\precsim$ (the slight modification above of the notion of function is needed for the proof that there exists a 2-decidable identity function on the set-universe).

Final remark. Our interest in the theory MST lies mainly in the problem of consistency of some relatively unrestricted comprehension scheme which would have some mathematical power. The aparatus of the theory together with the modal language can be found to be suitable for formalization of some intuitive notions (without classical or intuitionistic counterparts, cf. [8]) and to contribute in this way also to understanding of their classical interpretations. As examples we can mention the results of W. Reinhardt or S. Shapiro. Such applications of the theory lie in the means of expression of the language of the theory and also in a point that some object exists in a classical form and also in its "decidable" version which often like to be a "kernel" of the classical concept.

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