Linear Algebra and Geometry of Simplicial Finite Elements in Four Space Dimensions

Jan Brandts

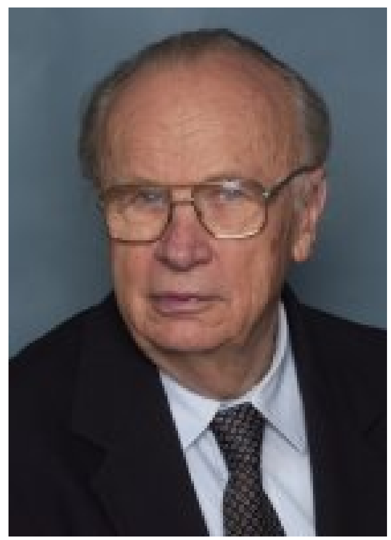
University of Amsterdam, Netherlands

Prague, Sunday May 28, 2006

Joint work with M. Křížek (Prague) and S. Korotov (Helsinki)

Professor Ivo Babuška 80 years young and enthousiastic

That is, young enough to help us youngsters through the next twenty years of numerical mathematics and engineering.



congratulations !

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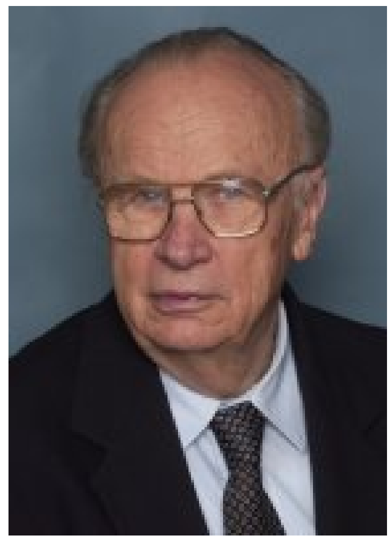

C O N G R A T U L A T I O N S !

A Finite Quantum Method for the 6D Hawking-Penrose Equations satisfying the 2021 Babuška Reliability Condition

Jan Brandts

Netherlands State University, European Union

Prague, Plutoday May 25, 2026

Joint work with M. Křížek and S. Korotov, both European Union

Summary

Already since 2018, the Finite Quantum Method completely outperforms the FDM and FEM. Applied to the the 6D Hawking-Penrose Equations* we arrive at a fully discrete Theory of Everything. We show that the FQM satisfies the 2021 Babuška† Reliability Condition and comment on how to solve the $n \times n = 1/h$ system of equations using Matlab, where h is Planck's constant.

References

- I. Babuška et al. (2013). The Finite Quantum Method.
- S. Hawking and R. Penrose (2017), The Theory of Everything.
- I. Babuška (2021). Reliability of Finite Quantum Methods.

*Hawking and Penrose got the Nobel Prizes for Peace and Physics in 2025

†without Brezzi, who retired already in 2010 at the early age of 65

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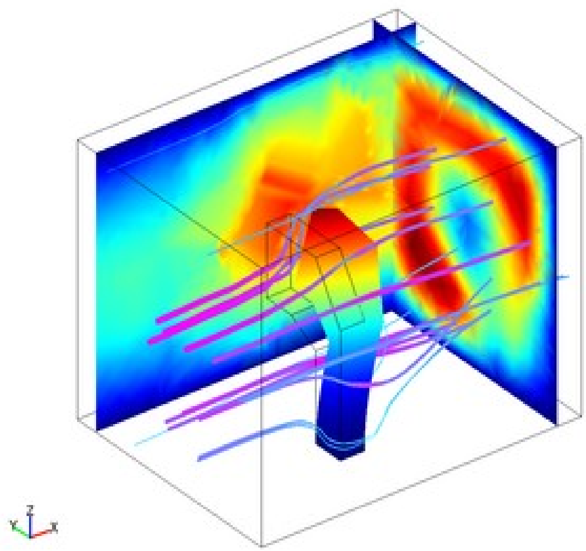
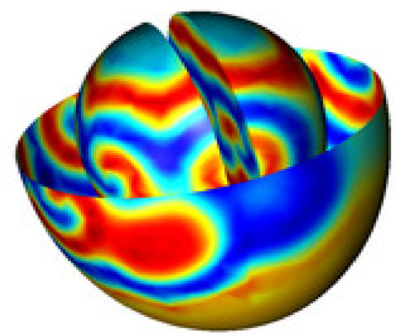
Motivation

Simplicial finite elements in two and three space dimensions are by now well *understood* and *coded* in many different applications

FEM models of fluid structure interaction (left) and electric charges in the heart, solved in FEMLAB

Understood:

Finite element theory is embedded more and more in pure mathematics, like in

- Differential Geometry
- Homology

Arnold, Falk, Winther: Finite element exterior calculus, homological techniques, and applications, *Acta Numerica* 2006

Tuesday 09.00 - 09.45:
Mark Ainsworth: Diagonal scaling of discrete differential forms

Coded:

Day by day, commercial software is becoming more powerful and easier to use. FEMLAB from COMSOL is being used by many people who have no mathematical knowledge at all of the theory.

Prague, October 27, 2006

For the latest developments, hands-on training, networking.

Time has come to look ahead into four dimensions

Computational resources are rapidly becoming powerful enough to realize four dimensional simplicial finite elements

Potential applications range from fundamental physics to financial mathematics. One day there may really be 6D Hawking-Penrose equations to solve!

Moreover:

It gives further insights in the finite element method in two and three space dimensions if proofs and constructions can be given that are independent of the spatial dimension.

Example 1: Supercloseness and superconvergence

Supercloseness and superconvergence are a tool for a posteriori error estimation in finite element methods

$$|u_h - L_h u|_{H^1} \leq Ch^2 |u|_{H^3}.$$

For simplicial elements, proofs can be found in:

$n = 1$: **Tong** (1956)

$n = 2$: **Oganesian & Ruhovets** (1969)

$n = 3$: **Chen (1980), Kantchev & Lazarov** (1986)

Even though the mesh conditions for $n = 2$ and $n = 3$ are similar, the proofs in these papers are very different from each other.

Dimensions $n \geq 4$

The directional derivative of a continuous piecewise linear function along an edge of a simplex is constant on the patch of simplices sharing that edge

If these patches are point-symmetric with respect to their center of gravity, this results in certain cancellations of error terms

Brandts & Křížek (2003)

Example 2: Strengthened Cauchy-Schwarz inequalities

Let V_{2h} and V_h the continuous piecewise linear functions with respect to a triangulation/tetrahedralization $\mathcal{T}_{2h}$ and its uniform refinement $\mathcal{T}_h$. Write

$$V_h = V_{2h} \oplus W_h.$$

For all $v \in V_{2h}, w_h \in W_h$ the strengthened CS inequality holds

$$|a(v_{2h}, w_h)|^2 \leq \gamma^2 a(v_{2h}, v_{2h}) a(w_h, w_h) \text{ with } \gamma^2 < 1.$$

$n = 1 :\ \gamma^2 = 0/1$

$n = 2 :\ \gamma^2 = 1/2$: **Axelsson (1982)**

$n = 3 :\ \gamma^2 = 3/4$: **Blaheta (2003)**

Dimensions $n \geq 4$

The value of γ_n in n space dimensions is

$$\gamma_n = \sqrt{1 - \left(\frac{1}{2}\right)^{n-1}}$$

which extends the tabular in the following way

dim	γ^2	dim	γ^2
1	0/1	4	07/08
2	1/2	5	15/16
3	3/4	6	31/32

Notice that $\gamma_n \to 1$ for $n \to \infty$.

Brandts, Korotov, Křížek (2004)

Questions that may rise

What does a simplicial partition in more than four dimensions look like? When are the patches point-symmetric? How can you refine a partition, globally or locally? What do the relevant angle properties depend on?

The n-simplex S in linear algebraic terms

S is the convex hull of the origin and n linearly independent vectors p_j

$$P = (p_1| \dots |p_n)$$

Let

$$Q^* = P^{-1}, \text{ with } \; Q = (q_1| \dots |q_n)$$

Since $Q^*P = I$ each q_j is normal to the convex hull F_j of the p_i with $i \neq j$ which is the facet F_j of S opposite p_j.

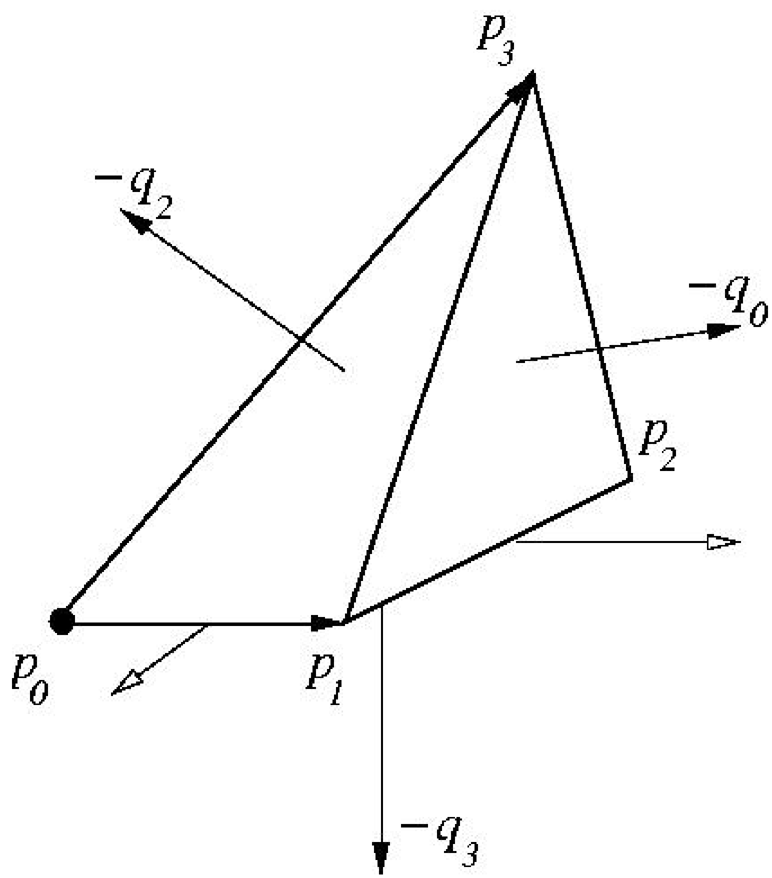
Notice

$\ell_j : x \mapsto q_j^* x$ is the linear nodal basis function for p_j and

$$(\nabla \ell_j, \nabla \ell_i) = q_j^* q_i |S|.$$

From this we also see that $q_0 = -Qe$, where $e = (1, \dots, 1)^*$.

Illustration for $n = 3$



Example 3: Assuring the discrete maximum principle

Sufficient for the DMP to hold is

$$0 \geq (\nabla \ell_j, \nabla \ell_i) = q_j^* q_i |S| = -\|q_i\| \|q_j\| |S| \cos(F_i, F_j)$$

Since the height h of S measured from F_j equals

$$h = \frac{p_j^* q_j}{\|q_j\|} = \frac{1}{\|q_j\|} \text{ and thus } |S| = \frac{|F_j|}{n\|q_j\|}$$

the DMP holds if

$$0 \leq \frac{|F_i||F_j| \cos(F_i, F_j)}{n^2 |S|}$$

$n = 2$: **Santos** (1986)

$n = 3$: **Křížek and Lin** (1995)

$n \geq 4$: **Brandts, Korotov Křížek and** (2006)

Non-obtuse and acute simplices

The DMP holds if $q_j^* q_i < 0$ for each pair of distinct normal vectors. Those numbers are the off-diagonal entries of

$$(q_0|Q)^*(q_0|Q) = \left[\begin{array}{c|c} q_0^* q_0 & q_0^* Q \\ \hline Q^* q_0 & Q^* Q \end{array}\right]$$

Recalling that $q_0 = -Qe$ with $e = (1, \ldots, 1)^*$ and thus that

$$Q^* q_0 = -Q^* Q e$$

it suffices to consider off-diagonal entries and row sums of

$$Q^* Q$$

only. Recall that $Q = P^{-*}$ can be any non-singular matrix.

Ortho-simplices and path-simplices

An ortho-simplex is a simplex having n mutually orthogonal edges. If those edges form a path, the simplex is called a path-simplex.

L. Schläfli, *Theorie der vielfachen Kontinuität aus dem Jahre 1852*; In: Gesammelte mathematische Abhandlungen, Birkhäuser, Basel, 1950.

Schläfli used the alternative name *orthoscheme*.

Lobachevsky: pyramid ($\mathbb{R}^3$)

Wythoff: double-rectangular ($\mathbb{R}^3$)

Schoute: polygonometry ($\mathbb{R}^n$)

Eppstein: path simplex ($\mathbb{R}^n$) (tetrahedron)

More about the path-simplex

The canonical path-simplex is represented by

$$P = \begin{bmatrix} 1 & \cdots & \cdots & 1 \\ & \ddots & & \vdots \\ & & \ddots & \vdots \\ & & & 1 \end{bmatrix} \quad \text{where} \quad Q = P^{-*} = \begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & \ddots & \ddots & \\ & & -1 & 1 \end{bmatrix}$$

which can be used to prove that path-simplices are non-obtuse

Rajan (1991) ortho-simplex + self-centered $\Leftrightarrow$ path-simplex

Corollary: A dissection into path-simplices is Delaunay

Dissection into path-simplices

Freudenthal (1942) The n-cube dissects into $n!$ path-simplices. However, the simplexity of the n-cube is

$$1, 2, 5, 16, 67, 308, 1493, ...$$

Hadwiger (1957) Conjecture: Every n-simplex can be dissected into finitely many path-simplices

- $n = 2$: trivially into 2
- $n = 3$: Lenhardt (1960), 12
- $n = 4$: A.B. Charsischwili (1982), into 730
- $n = 4$: H. Kaiser (1986), into 610
- $n = 4$: K. Tschirpke (1993), into 500
- $n = 5$: K. Tschirpke (1993), into $\leq$ 12.598.800

Uniform refinement in higher dimensions

$K = [0, 1]^n$ can be subdivided into $n!$ simplices of dimension n,

$$S_\sigma = \{x \in \mathbb{R}^n \mid 0 \leq x_{\sigma(1)} \leq \cdots \leq x_{\sigma(n)} \leq 1\},$$

where σ ranges over all permutations of $1, \ldots, n$.

We may define uniform refinement of those simplices by:

- uniformly refine the cube into 2^n subcubes
- Subdivide each subcube into $n!$ simplices
- Each S_σ consists of 2^n smaller simplices

Freudenthal (1942)

Kuhn (1960)

Blaheta (2003)

Example 4: bi-(tri-)section of the two-(three-)pathsimplex

A right triangle can be subdivided into two right triangles, and this process can be locally towards a vertex at the long diagonal

A path-tetrahedron (3-orthoscheme) can be dissected into three sub- orthoschemes

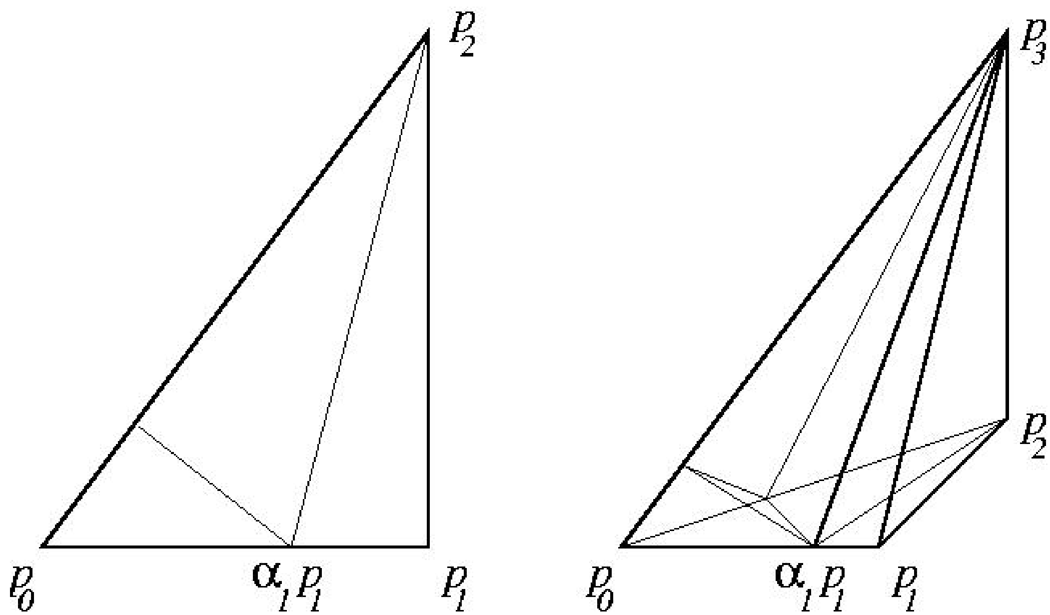
Coxeter (1989) Trisecting an orthoscheme, *Computers Math. Applic.*, 17(1-3):59–71

Repeating this procedure results in local refinement towards a vertex at the long diagonal

Korotov, Křížek (2003), *Local nonobtuse tetrahedral refinements of a cube*, Appl. Math. Lett. 16:1101 - 1104.

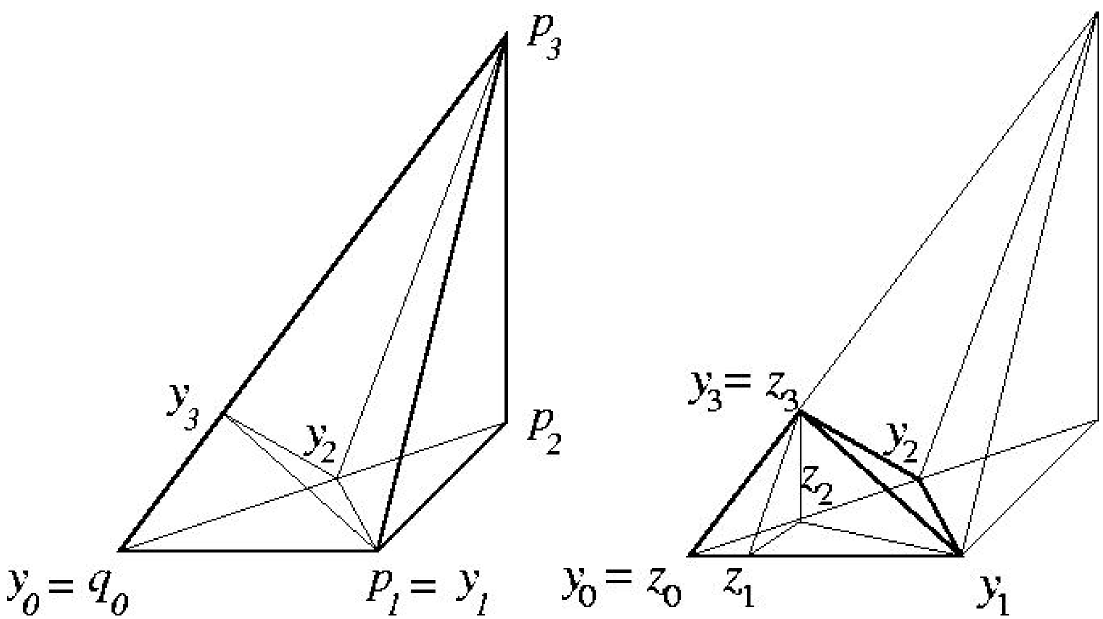
Dimensions $n \geq 4$

Inductively dissect into $n+1$ path-subsimplices such that each orthogonal path ends at $\alpha_1 p_1$ with $0 < \alpha_1 < 1$.



Then consider the degenerate case $\alpha_1 = 1$, resulting in n path-subsimplices.

Repeating this procedure results in local refinement towards a vertex at the long diagonal



The bottom left simplex after two refinements is similar to the original one.

Conclusions

We have a number of ingredients available to write an efficient linear finite element code for elliptic equations on four-dimensional domains.

Such a code could include:

- Uniform refinement or refinement towards a particular node
- Optimal complexity algebraic multigrid solvers
- Error estimation based on superconvergence
- A condition to assure the discrete maximum principle

Work in progress and future work

To prove Hadwiger's Conjecture that each simplex can be subdivided into a finite number of path-simplices.

To prove equivalence of several regularity conditions for families of simplicial partitions in two and three dimensions, to generalize them to $n \geq 4$, and to design refinement procedures satisfying those conditions for $n \geq 4$.

Coding in Matlab of four-dimensional examples.

Discretizing the Hodge Laplacian in four dimensions using mixed finite elements using Nedelec's edge–face–facet elements.