LOCAL INTERPOLATION BY ISOPARAMETRIC QUADRATIC LAGRANGE FINITE ELEMENTS

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1 Dimension one

A *reference quadratic Lagrange finite element* $\hat{\mathcal{K}}$ consists of

a) the *interval* $\hat{K} = [-1, 1]$,

b) the *local space* $\hat{\mathcal{L}}$ of polynomials of degree two or less on $\hat{K}$,

c) the set of *parameters* relating the values $\hat{p}(-1), \hat{p}(0), \hat{p}(1)$ to each $\hat{p} \in \hat{\mathcal{L}}$.

For arbitrary $a < b < c$, we define a *discretisation step* $h = \max(b - a, c - b)$, a *centre* $\tilde{b} = (a + c)/2$ and a (unique) function F_h from $\hat{\mathcal{L}}$ with parameters a, b, c. If F_h is an injection, i.e.

$$\frac{3a + c}{4} \le b \le \frac{a + 3c}{4}, \tag{1}$$

then we call F_h a *transform* and we denote by G_h the transform inverse to F_h.

An *isoparametric quadratic Lagrange finite element* $\mathcal{K}_h$ *in* 1D is determined by a transform F_h with parameters $a < b < c$. It consists of

a) the *interval* $K_h = [a, c]$,

b) the *local space* $\mathcal{L}_h$ of functions

$$p_h(x) = \hat{p}\left(G_h(x)\right) \quad \forall\, \hat{p} \in \hat{\mathcal{L}}. \tag{2}$$

c) the set of *parameters* relating the values $p_h(a), p_h(b), p_h(c)$ to each $p_h \in \mathcal{L}_h$.

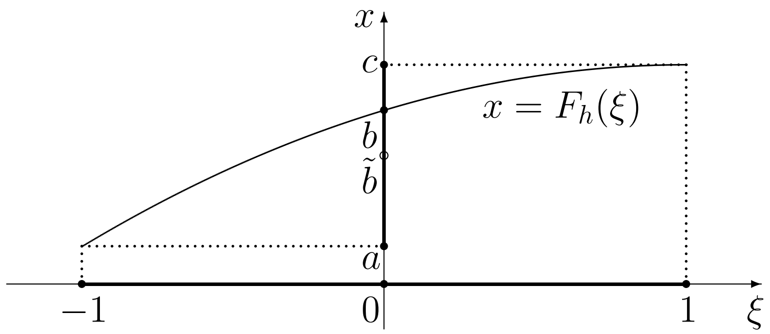

Figure 1

For $\nu \in (0,1)$ fixed, we call a transform F_h *regular* whenever

$$\frac{d}{d\xi} F_h(\xi) \geq \nu h \quad \text{in} \quad [-1,1].$$

Theorem 1. Let $p \geq 1$ be arbitrary. For every constant $C_1 > 0$ there exists a constant $C_2 > 0$ such that for every regular transform F_h satisfying

$$|b - \tilde{b}| \leq C_1 h^p$$

we have

$$|v - \Pi_h v|_{m,K_h} \leq C_2\, h^{1-m} \left(h^2 |v|_{3,K_h} + h^p |v|_{2,K_h}\right)$$

for $m = 0, 1, 2$ and for all $v \in H^3(K_h)$.

2 Dimension two

A *reference quadratic Lagrange finite element* $\hat{\mathcal{K}}$ consists of

(a) the *unit triangle* $\hat{K}$,

(b) the *local space* $\hat{\mathcal{L}}$ of quadratic polynomials on $\hat{K}$,

(c) the set of *parameters* $\hat{p}(\hat{a}^1), \ldots, \hat{p}(\hat{a}^6)$ for every $\hat{p} \in \hat{\mathcal{L}}$.

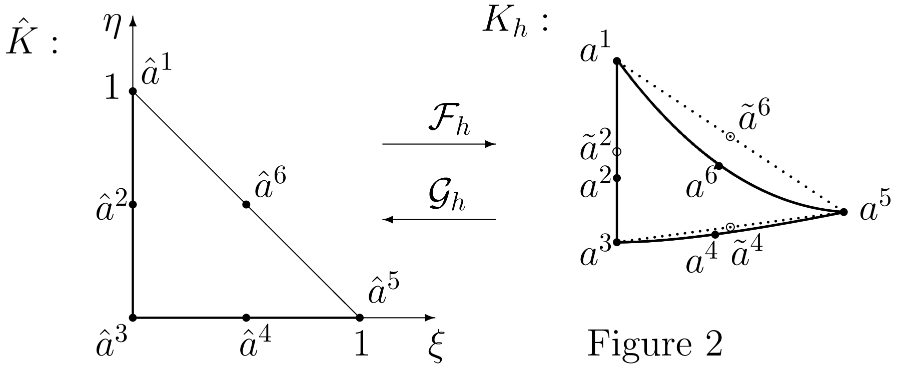

Figure 2

We put $h = \max(|a^1a^3|, |a^3a^5|, |a^5a^1|)$ and define a map $\mathcal{F}_h = (F_h, G_h) \in \hat{\mathcal{L}} \times \hat{\mathcal{L}}$ by $\mathcal{F}_h(\hat{a}^i) = a^i$ for $i = 1, \ldots, 6$.

Agreement. We consider such points $a^1, \ldots, a^6$ only for which the curve $\mathcal{F}_h(\partial\hat{K})$ is simple and its orientation determined by increasing indices of $a^1, \ldots, a^6$ is positive.

If $\mathcal{F}_h$ is an injection then we call $\mathcal{F}_h$ a *transform* and we denote by $\mathcal{G}_h$ the *transform* inverse to $\mathcal{F}_h$.

An *isoparametric quadratic Lagrange finite element* $\mathcal{K}_h$ in 2D is determined by a transform $\mathcal{F}_h$. It consists of

(a) the "triangle" $K_h = \mathcal{F}_h(\hat{K})$,

(b) the *local space* $\mathcal{L}_h$ of functions

$$p_h(x, y) = \hat{p}(\mathcal{G}_h(x, y)) \quad \forall \hat{p} \in \hat{\mathcal{L}},$$

(c) the set of *parameters* relating the values $p_h(a^1), \ldots, p_h(a^6)$ to any $p_h \in \mathcal{L}_h$.

Let $\nu \in (0,1)$ be fixed. A transform $\mathcal{F}_h$ is said to be *regular* whenever $J(\xi,\eta) \geq \nu h^2$ for all $(\xi,\eta) \in \hat{K}$.

Theorem 2. Let $p \geq 1$ be arbitrary. For every constant $C_1 > 0$ there exists a constant $C_2 > 0$ such that for every regular transform $\mathcal{F}_h$ such that $\left|\tilde{a}^2 a^2\right| + \left|\tilde{a}^4 a^4\right| + \left|\tilde{a}^6 a^6\right| \leq C h^p$, we have

$$|v - \Pi_h v|_{m,K_h} \leq C_2\, h^{1-m} \left(h^2 |v|_{3,K_h} + h^p |v|_{2,K_h}\right)$$

for $m = 0, 1, 2$ for all $v \in H^3(K_h)$.

$$\text{We put } D(abc) = \frac{1}{2} \begin{vmatrix} a_1 - c_1 & a_2 - c_2 \\ b_1 - c_1 & b_2 - c_2 \end{vmatrix} \quad \forall a,b,c \in \Re^2.$$

Theorem 3. The Jacobian J of a map $\mathcal{F}_h$ related to the points $a^1, \ldots, a^6$ is of the form

$$J(\xi,\eta) = A + B\xi + C\eta + D\xi^2 + E\xi\eta + F\eta^2 \quad \text{where}$$

$$\begin{aligned}
A &= 2[16D_{234} - 4(D_{134} + D_{235}) + D_{135}],\\
B &= 8[-12D_{234} + 5D_{235} + 4D_{346} + 2D_{134} - D_{356} - D_{345} - D_{135}],\\
C &= 8[-12D_{234} + 5D_{134} + 4D_{236} + 2D_{235} - D_{136} - D_{123} - D_{135}],\\
D &= 32[2(D_{234} - D_{346}) + D_{356} + D_{345} - D_{235}],\\
E &= 32[4D_{234} - 2(D_{235} + D_{134}) + D_{135}],\\
F &= 32[2(D_{234} - D_{236}) + D_{136} + D_{123} - D_{134}].
\end{aligned}$$

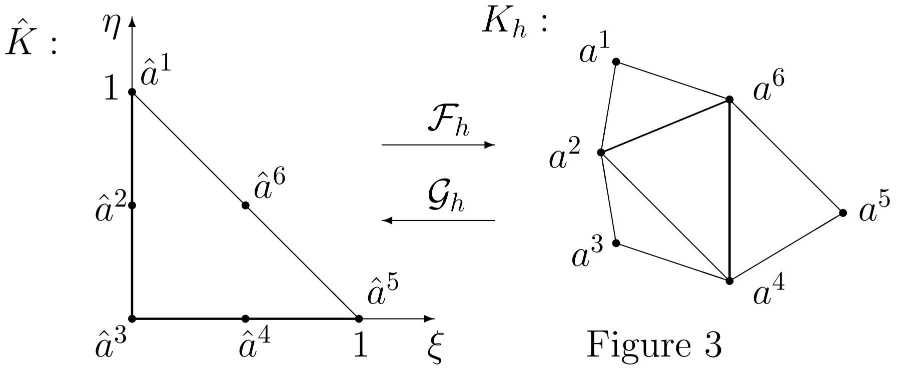

Figure 3

If we approximate the solution of the problem

$$-\Delta u = f \ \text{ in } \ \Omega \subseteq \Re^2, \quad u = 0 \ \text{ on } \ \partial\Omega$$

with piecewise smooth boundary $\partial\Omega$ by linear triangular finite elements on a polyhedral approximation Ω_h of the domain Ω, optimal rate of convergence in H^1 norm is $O(h)$. The use of quadratic finite elements on the same triangles leads to $O(h^{\frac{3}{2}})$ in H^1 norm. To

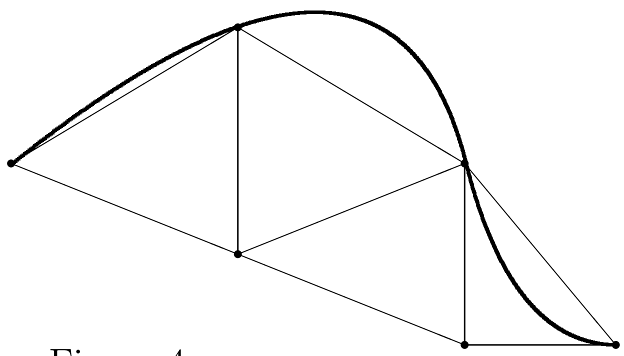

Figure 4

obtain $O(h^2)$, $\partial\Omega$ has to be approximated more exactly. The isoparametric quadratic Lagrange finite element is a standard tool for this purpose; it gives us an approximate solution with an error $O(h^2)$ in the H^1 norm.

In this case, we have

$$a^2 = \frac{1}{2}(a^1 + a^3), \quad a^4 = \frac{1}{2}(a^3 + a^5)$$

and the Jacobian attains the form

$$J(\xi, \eta) = 8D_{234} + 16\xi(D_{346} - D_{234}) + 16\eta(D_{236} - D_{234}).$$

J is positive on $\hat{K}$ if and only if

$$2D_{346} > D_{234}, \quad 2D_{236} > D_{234}. \tag{3}$$

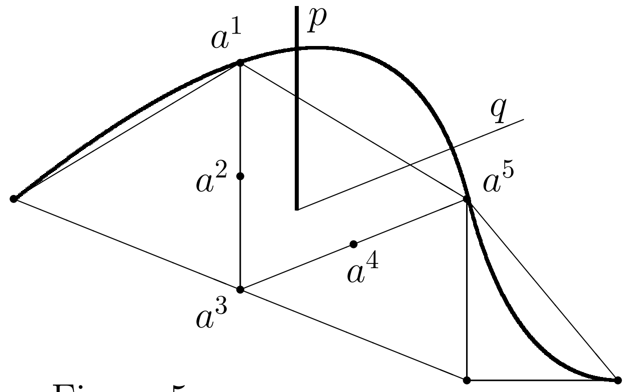

Figure 5

Jordan W.B., 1970.

For any given points $a^1, \ldots, a^5$, we define the *admissible set*

$$\begin{aligned}\mathcal{A}d &= \mathcal{A}d(a^1, \ldots, a^5) \\ &= \{a^6\,;\, \mathcal{F}_h \text{ related to } a^1, \ldots, a^6 \text{ is a transform}\}.\end{aligned}$$

For any $a^1, \ldots, a^5$ such that $a^2 \in \overline{a^1a^3}$ and $a^4 \in \overline{a^3a^5}$, we put

$$u = \frac{|a^2a^3|}{|a^1a^3|}, \quad v = \frac{|a^3a^4|}{|a^3a^5|}, \; U = \frac{D_{634}}{D_{134}} \text{ and } V = \frac{D_{236}}{D_{235}}.$$

Theorem 4. Assume that the points $a^1, \ldots, a^5$ satisfy $a^2 \in \overline{a^1a^3}$, $a^4 \in \overline{a^3a^5}$ and $\frac{1}{4} < u < \frac{3}{4}$, $\frac{1}{4} < v < \frac{3}{4}$.

If $u + v \leq 1$ then

$$\mathcal{A}d = \left\{a^6\,;\, \frac{1}{4} < U, \; \frac{1}{4} < V\right\}$$

If $u + v > 1$ then the admissible set is a convex set bounded by the lines $p \equiv U = 1/4$, $q \equiv V = 1/4$ and by the ellipse with centre $S = (2u - 1/2, 2v - 1/2)$ touching the line p, q in the point

$$T_p = \left(\frac{1}{4}, \frac{12u + 8v - 9)}{4(4u - 1)}\right), \; T_q = \left(\frac{8u + 12v - 9)}{4(4v - 1)}, \frac{1}{4}\right),$$

respectively. See Fig. 6.

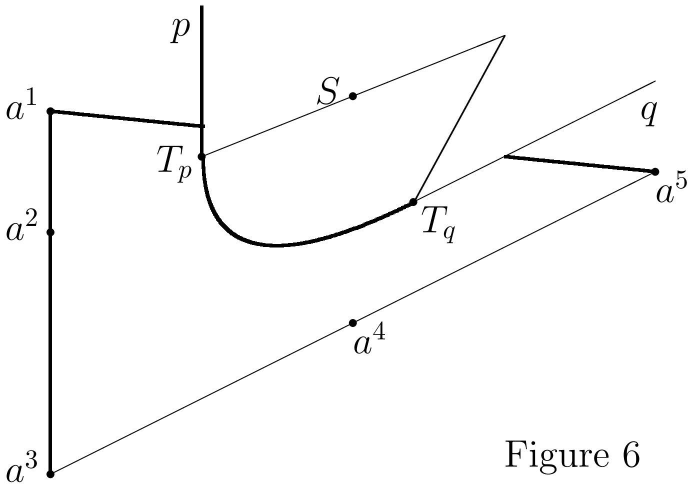

Figure 6