Finite volume WLSQR scheme for transonic flows

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Outline

- High order FVM scheme for compressible flows
  - Basic finite volume scheme
  - Finite volume scheme with a reconstruction
- Weighted least square reconstruction
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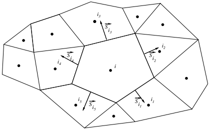
  - Analysis of WLSQR reconstruction
  - Numerical experiments
- Applications
  - 2D flows through turbine cascades
  - 3D flows through turbine cascades

Basic finite volume scheme



- unstructured meshes, general elements,
- more complicated coding, but
- adaptivity, complex geometry.

Explicit 1. order FVM for $u_t + f(u)_x + g(u)_y = 0$

$$\mathrm{u}_i^{n+1} = \mathrm{u}_i^n - \Delta t R^1(\mathrm{u}^n)_i,$$

$$R^1(\mathrm{u}^n)_i = \frac{1}{\mu(C_i)} \left[\sum_{j \in N_i} H(\mathrm{u}_i^n, \mathrm{u}_j^n, \vec{S}_{i,j}) + \sum_b \sum_{e \in B_i^b} H^b(\mathrm{u}_i^n, \vec{S}_e) \right],$$

$$H(\mathrm{u}_i^n, \mathrm{u}_j^n, \vec{S}_{ij}) \approx \int_{e_{ij}} (f(u)n_x + g(u)n_y)\, dS.$$

Finite volume scheme with a reconstruction

- cell-wise interpolation polynomial $P_i(\vec{x}; u^n)$,
- interpolation to the cell interfaces,
- evaluation of fluxes using interpolated values.

Explicit high order FVM

$$\mathrm{u}_i^{n+1} = \mathrm{u}_i^n - \Delta t R^2(\mathrm{u}^n)_i,$$

$$R^2(\mathrm{u}^n)_i = \frac{1}{\mu(C_i)} \left[\sum_{j \in N_i} H(P_i(\vec{x}_{ij}; \mathrm{u}^n), P_j(\vec{x}_{ij}; \mathrm{u}^n), \vec{S}_{ij}) + \ldots \right].$$

Piecewise polynomial reconstruction

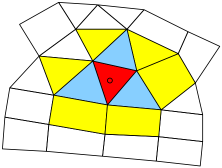
Cell-wise interpolation polynomial $P_i(\vec{x}; u^n)$

- conservativity: $\iint_{C_i} P_i \, d\vec{x} = \mu(C_i) u_i^n$,
- accuracy: $\iint_{C_j} P_i \, d\vec{x} \approx \mu(C_j) u_j^n$ for cells in the vicinity of C_i.

Construction of P

- Least squares $\Rightarrow$ simple but unstable
- LSQR with limiters $\Rightarrow$ stable but the accuracy is limited
- ENO/WENO $\Rightarrow$ uniformly high order but very complicated

Weighted least square reconstruction



Goals:

- uniformly high order of accuracy,
- simple implementation for 2D/3D,
- good convergence to steady state.

Constrained least square method

$$P_i = \arg\min \sum_{j \in \mathcal{N}_i^2} \left(\mu(C_j) u_j^n - \iint_{C_j} P_i \, d\vec{x} \right)^2 \cdot w_{ij}(u^n)^2,$$

$$\iint_{C_i} P_i \, d\vec{x} = \mu(C_i) u_i^n \text{ and } w_{ij}^2(u^n) = h^{-r} / \left[\left| \frac{u_i^n - u_j^n}{h} \right|^p + h^q \right].$$

Piecewise linear WLSQR reconstruction in 1D

Interpolation polynomial:

$$P_i(x) = u_i + \sigma_i x, \text{ for } x \in (x_{i-1/2}, x_{i+1/2}).$$

System for σ_i:

$$\begin{aligned} w_{i+1/2} u_{i+1} &= w_{i+1/2} \left(u_i + h\sigma_i \right), \\ w_{i-1/2} u_{i-1} &= w_{i-1/2} \left(u_i - h\sigma_i \right). \end{aligned}$$

Least-square solution for σ_i:

$$\sigma_i = \frac{w_{i+1/2}^2 (u_{i+1} - u_i) + w_{i-1/2}^2 (u_i - u_{i-1})}{h(w_{i+1/2}^2 + w_{i-1/2}^2)}.$$

Analysis for smooth data

Rewrite

$$\sigma_i = \alpha_i \frac{u_{i+1} - u_i}{h} + \beta_i \frac{u_i - u_{i-1}}{h},$$

with $\alpha_i, \beta_i \in [0,1]$, $\alpha_i + \beta_i = 1$, then:

$$\sigma_i = u_x(x_i) + \mathcal{O}(h),$$

and finally: $(TV(u) = \sum |u_i - u_{i-1}|)$

Lemma (Accuracy and stability for smooth data)

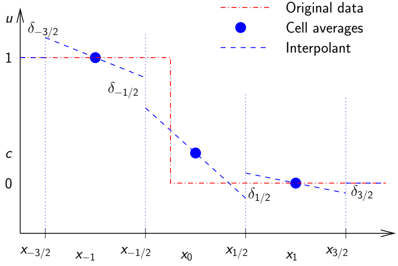
Let $u(x) \in C^2$, then

$$P(x;u) = u(x) + \mathcal{O}(h^2),$$
$$TV(P(x;u)) \leq TV(u) + \mathcal{O}(h)$$

Analysis for discontinuous data

Simple discontinuity:

- $u(x) = 1$ for $x < x_s$, and
- $u(x) = 0$ for $x \geq x_s$.
- arbitrary x_s.



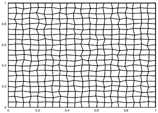
Lemma (TV stability for simple discontinuity)

Let u is simple discontinuity and $p+q \geq 0$ and $p > 1$. Then

$$TV(P(x;u)) \leq TV(u) + 6h^{1+q/p}.$$

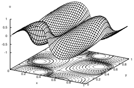
Numerical analysis for scalar problem $(p, q, r = 4, -2, 3)$

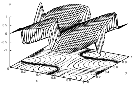
Linear problem: $u_t + u_x + u_y = 0, \quad u_0 = \sin(2\pi x)\cos(2\pi y).$



	P0 (first order)		P1 (second order)		P2 (third order)	
$1/N$	$\|\|e\|\|_1$	order	$\|\|e\|\|_1$	order	$\|\|e\|\|_1$	order
0.1	0.339084	-	0.141348	-	0.134682	-
0.05	0.253544	0.42	0.035086	2.01	0.021605	2.64
0.025	0.157564	0.68	0.007567	2.21	0.002843	2.93
0.0125	0.088477	0.83	0.001584	2.25	0.000377	2.92

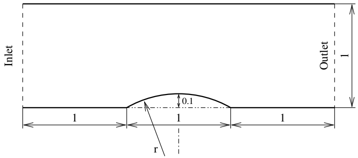
Non-linear problem: $u_t + uu_x + uu_y = 0, \quad u_0 = \sin(2\pi x)\cos(2\pi y).$



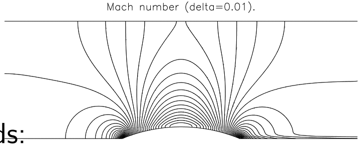


	P0 (first order)		P1 (second order)		P2 (third order)	
$1/N$	$\|\|e\|\|_1$	order	$\|\|e\|\|_1$	order	$\|\|e\|\|_1$	order
	Smooth data $(t = 0.1)$					
0.1	0.054867	-	0.017641	-	0.012703	-
0.05	0.040623	0.43	0.008839	1.00	0.002686	2.24
0.025	0.024009	0.76	0.001963	2.41	0.000648	2.05
0.0125	0.013414	0.84	0.000379	2.37	0.000116	2.48
0.00625	0.007095	0.92	0.000081	2.23	0.000017	2.77
	Non-smooth data $(t = 0.25)$					
0.1	0.112414	-	0.049627	-	0.047704	-
0.05	0.069466	0.69	0.018373	1.43	0.018493	1.36
0.025	0.039077	0.83	0.011098	0.73	0.009987	0.89
0.0125	0.021665	0.85	0.005554	1.00	0.004837	1.05

Numerical analysis for GAMM channel ($p, q, r = 4, -2, 3$)



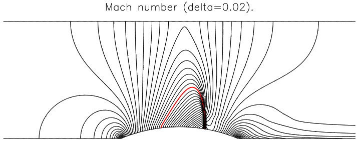
Subsonic case: $M_1 = 0.5$



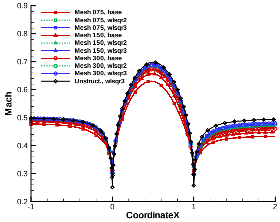
Grids:

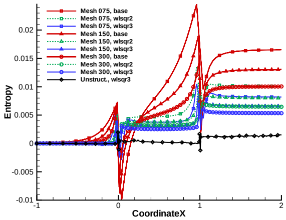
- structured, 75×25, 150×50, 300×100
- unstructured with adaptivity.

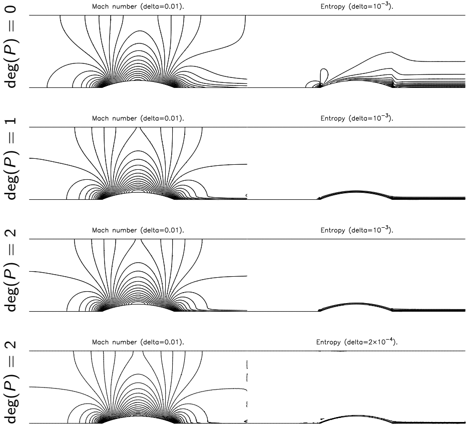
Transonic case: $M_1 = 0.675$



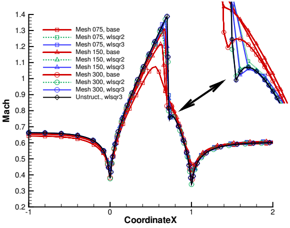
Order of accuracy for GAMM channel problem (subsonic)

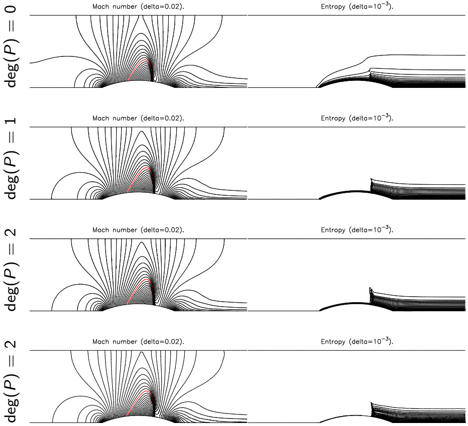


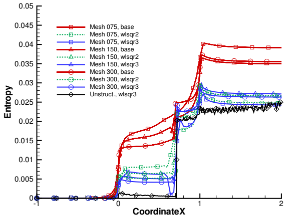




Order of accuracy for GAMM channel problem (transonic)







Order of accuracy for GAMM channel problem

Estimated order of accuracy

- 3 consecutive grids 75×25, 150×50, 300×100 cells,
- $EOA = \log_2(||\rho_h - P^h_{h/2}\rho_{h/2}||_1) - \log_2(||\rho_{h/2} - P^{h/2}_{h/4}\rho_{h/4}||_1)$.

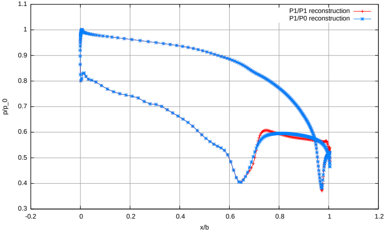
	P0 (1st order)	P1 (2nd order)	P2 (3rd order)
Subsonic	0.82	1.41	1.42
Transonic	0.90	1.46	1.30

Problems:

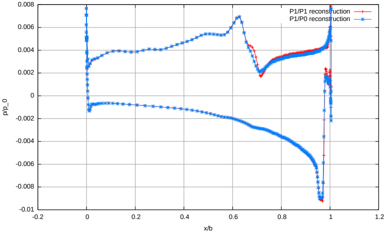
- Piecewise linear approximation of boundary!
- Implementation of boundary conditions?

2D flows through a turbine cascade

- 2D turbine cascade SE 1050 of Škoda Plzeň,
- $M_{2i} = 0.906$, $Re = 1.38 \cdot 10^6$,
- 2D RANS - P1 interpolation,
- TNT $k - \omega$ model - P0 or P1 interpolation.

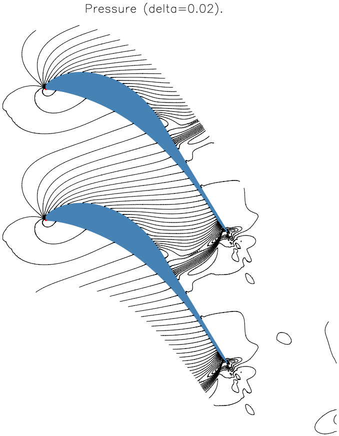


Distribution of the pressure

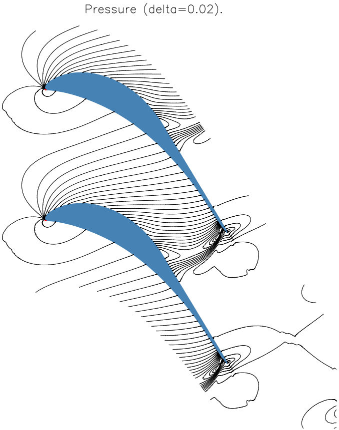


Distribution of the friction coeff.

2D flows through a turbine cascade (cont.)

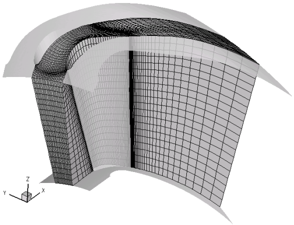


- P1 for ρ, ρu, ρv, e,
- P1 for ρk, $\rho \omega$.



- P1 for ρ, ρu, ρv, e,
- P0 for ρk, $\rho \omega$.

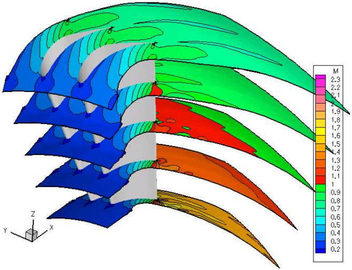
3D flows through turbine cascades



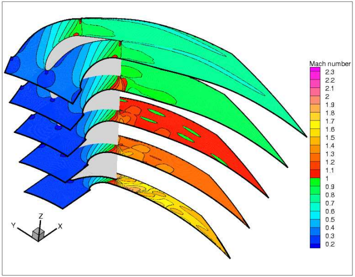
View of one inter-blade channel with structured mesh.

- Inviscid flow,
- Subsonic inlet,
- Outlet with given $p_2(r)$,
- Structured single-block mesh with
  - $200 \times 40 \times 40$ cells (fine),
  - $100 \times 20 \times 20$ cells (coarse),

3D flows through turbine cascades



WLSQR method, AUSM flux, coarse mesh.



TVD MC method, fine mesh.

Conclusion

Properties of WLSQR method:

- High order method for transonic flows,
- simple for unstructured meshes even in 3D,
- good convergence to steady state,
- preliminary analytical results.

To do:

- analysis of order of accuracy,
- analysis of stability for high AR cells,
- higher order approximation of boundaries,
- proof of convergence **:)**.