Modelling of a Steady Flow in a Cascade with Separate Boundary Conditions for Vorticity and Bernoulli's Pressure on the Outflow

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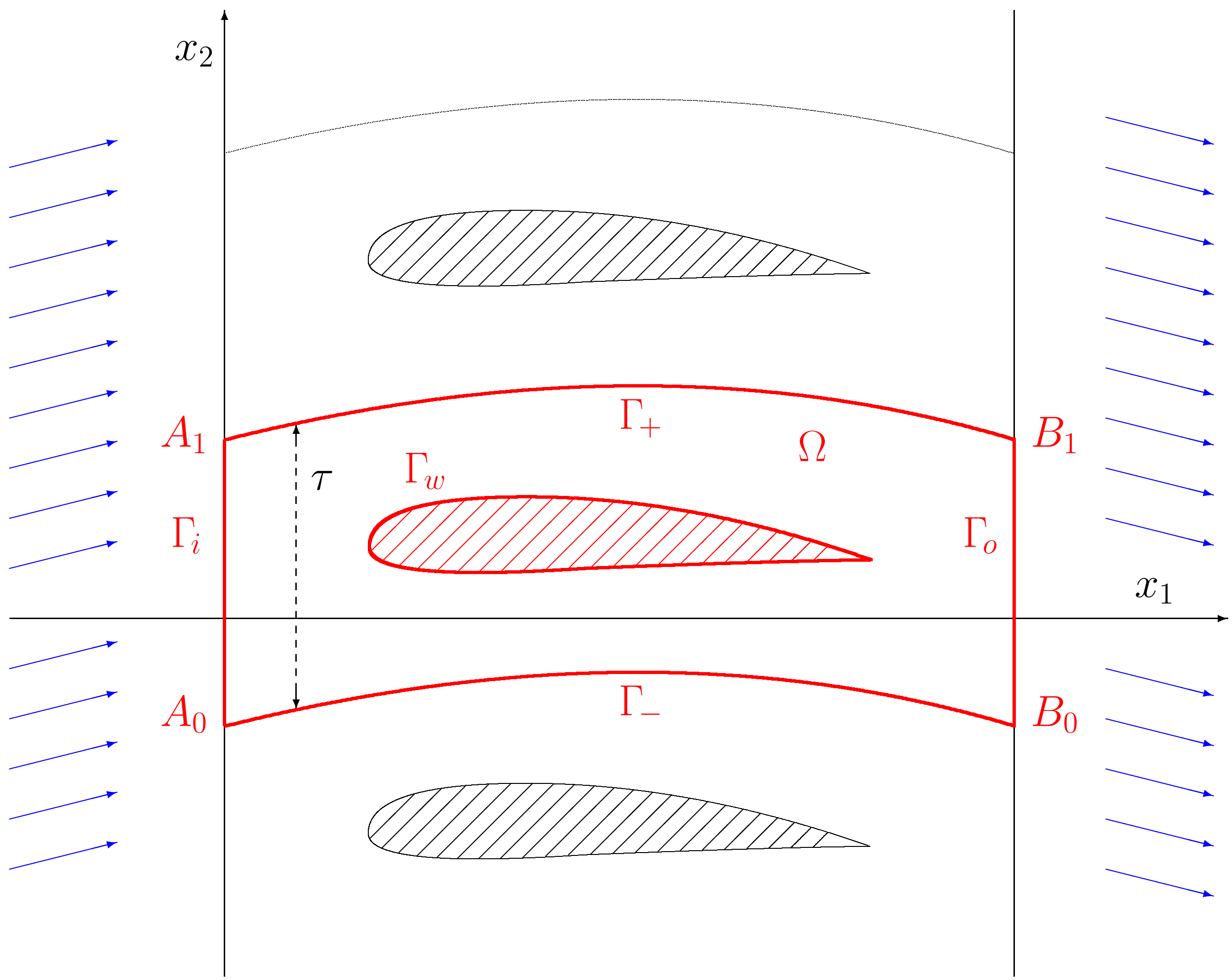

x2
x1
A1
A0
B1
B0
Γi
Γo
Γ+
Γ−
Γw
Ω
τ

We consider a flow through a cascade of profiles (2D model of a blade machine). We assume that the fluid is viscous, incompressible and Newtonian, and the flow is 2–dimensional.

1. Used equations

We use the 2–dimensional **Navier–Stokes equations**

$$\begin{aligned} u_1\,(\partial_1 u_1) + u_2\,(\partial_2 u_1) &= -\partial_1 p + \nu\,\Delta u_1 + f_1 \\ u_1\,(\partial_1 u_2) + u_2\,(\partial_2 u_2) &= -\partial_2 p + \nu\,\Delta u_2 + f_2 \end{aligned}$$

and the **equation of continuity**

$$\boxed{\operatorname{div} \boldsymbol{u} = 0} \tag{1}$$

where $\boldsymbol{u} = (u_1, u_2)$ is the **velocity**.

Using equation (1), the Navier–Stokes equations can also be written in the form

$$
\begin{aligned}
-u_2\,(\partial_1 u_2) + u_2\,(\partial_2 u_1) &= -\partial_1 q - \nu\,\partial_2(\partial_1 u_2 - \partial_2 u_1) + f_1 \\
u_1\,(\partial_1 u_2) - u_1\,(\partial_2 u_1) &= -\partial_2 q + \nu\,\partial_1(\partial_1 u_2 - \partial_2 u_1) + f_2
\end{aligned}
$$

where $q = p + \frac{1}{2}\,(u_1^2 + u_2^2)$ is the so called **Bernoulli pressure**, or in the equivalent form of one vector equation

$$
\boxed{\omega(\boldsymbol{u})\,\boldsymbol{u}^{\perp} \;=\; -\nabla q + \nu\,(-\partial_2, \partial_1)\,\omega(\boldsymbol{u}) + \boldsymbol{f}} \tag{2}
$$

where $\omega(\boldsymbol{u}) = \partial_1 u_2 - \partial_2 u_1$ is the **vorticity**, $\boldsymbol{f} = (f_1, f_2)$ is the **external body force** and

$$
\boldsymbol{u}^{\perp} \;=\; (-u_2, u_1).
$$

2. Boundary conditions

The **inhomogeneous Dirichlet condition on the inflow**:

$$\boldsymbol{u} = \boldsymbol{g} \qquad \text{on } \Gamma_i \tag{3}$$

where $\boldsymbol{g}$ is a given velocity on Γ_i.

The **conditions of periodicity on Γ_- and Γ_+**:

$$\boldsymbol{u}(x_1, x_2 + \tau) = \boldsymbol{u}(x_1, x_2) \tag{4}$$

$$\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}}(x_1, x_2 + \tau) = -\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}}(x_1, x_2) \qquad \text{for } (x_1, x_2) \in \Gamma_- \tag{5}$$

$$q(x_1, x_2 + \tau) = q(x_1, x_2) \tag{6}$$

The **homogeneous Dirichlet condition on the profile**:

$$\boldsymbol{u} = \boldsymbol{0} \qquad \text{on } \Gamma_w \tag{7}$$

3. The weak formulation

We need the following function spaces:

- $H^1(\Omega)$ (respective $H^1(\Omega)^2$) is the Sobolev space of scalar (respective vector) functions, defined a.e. in Ω, with the norm $\| \,.\, \|_1$.
- $H^s(\Gamma_i)^2$ (for $0 < s < 1$) is the Sobolev–Slobodetski space of vector functions, defined a.e. in Γ_i, with the norm $\| \,.\, \|_{s;\Gamma_i}$.
- $\mathcal{X} = \{\boldsymbol{v} \in C^\infty(\overline{\Omega})^2;\ \ \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_i \cup \Gamma_w,\ \ \boldsymbol{v}(x_1, x_2 + \tau) = \boldsymbol{v}(x_1, x_2)\ \ \forall\, (x_1, x_2) \in \Gamma_-\}$
- $\mathcal{V} = \{\boldsymbol{v} \in \mathcal{X};\ \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega\}$
- H ... the closure of $\mathcal{V}$ in $L^2(\Omega)^2$
- V ... the closure of $\mathcal{V}$ in $H^1(\Omega)^2$

The spaces $\boldsymbol{H}$ and $\boldsymbol{V}$ can be characterized:

$\boldsymbol{H}$ is the space of functions $\boldsymbol{v} \in L^2(\Omega)^2$ such that

- $\operatorname{div} \boldsymbol{v} = 0$ in the sense of distributions in Ω,
- $\boldsymbol{v} \cdot \boldsymbol{n} = 0$ in the sense of traces on $\Gamma_i \cup \Gamma_w$,
- $\boldsymbol{v}(x_1, x_2 + \tau) \cdot \boldsymbol{n} = -\boldsymbol{v}(x_1, x_2) \cdot \boldsymbol{n}$ for $(x_1, x_2) \in \Gamma_-$ in the sense of traces

$\boldsymbol{V}$ is the space of functions $\boldsymbol{v} \in H^1(\Omega)^2$ such that

- $\operatorname{div} \boldsymbol{v} = 0$ a.e. in Ω,
- $\boldsymbol{v} = \mathbf{0}$ on $\Gamma_i \cup \Gamma_w$,
- $\boldsymbol{v}(x_1, x_2 + \tau) = \boldsymbol{v}(x_1, x_2)$ for $(x_1, x_2) \in \Gamma_-$

$$-\nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\,\boldsymbol{u} + \nabla p = \boldsymbol{f}$$

Now we multiply this equation by an arbitrary test function $\boldsymbol{v} = (v_1, v_2) \in V$ and integrate in Ω. We get

$$-\int_\Omega \nu\Delta\boldsymbol{u}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} + \int_\Omega (\boldsymbol{u}\cdot\nabla)\,\boldsymbol{u}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} + \int_\Omega \nabla p\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}$$

we apply Green's theorem

$$\nu\int_\Omega \sum_{i,j=1}^{2} \frac{\partial u_i}{\partial x_j}\frac{\partial v_i}{\partial x_j}\,\mathrm{d}\boldsymbol{x} - \nu\int_{\partial\Omega}\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}}\cdot\boldsymbol{v}\,\mathrm{d}S + \int_\Omega \sum_{i,j=1}^{2}\frac{\partial u_i}{\partial x_j}\,v_i\,\mathrm{d}\boldsymbol{x}$$

$$-\int_\Omega p\,\mathrm{div}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} + \int_{\partial\Omega} p\cdot\boldsymbol{v}\,\boldsymbol{n}\,\mathrm{d}S \;=\; \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\mathrm{d}\boldsymbol{x}$$

because $\mathrm{div}\,\boldsymbol{v} = 0$ for all $\boldsymbol{v}\in V$, we have

$$-\int_\Omega p\,\mathrm{div}\,\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} \;=\; 0$$

using fact that $\boldsymbol{v}|_{\Gamma_i\cup\Gamma_w}$, conditions of periodicity and relation $\boldsymbol{n}(x_1,x_2) = -\boldsymbol{n}(x_1, x_2+\tau)$ for $(x_1,x_2)\in\Gamma_-$, only this term remain on boundary

$$\int_{\Gamma_o}\left(-\nu\,\frac{\partial\boldsymbol{u}}{\partial\boldsymbol{n}} + p\,\boldsymbol{n}\right)\,\boldsymbol{v}\,\mathrm{d}S$$

For our form of Navier–Stokes equations we obtain this term in boundary integral:

$$\int_{\Gamma_o} \omega(\boldsymbol{u})\, v_2 - q\, v_1 \,\mathrm{d}S$$

for all $\boldsymbol{v} \in V$. Let us take $\boldsymbol{v} = (v_1, 0)$, we have

$$-q\, v_1 + (h_1) = 0$$

and, using this and an arbitrary function $\boldsymbol{v}$,

$$\omega(\boldsymbol{u})\, v_2 + (h_2) = 0$$

From these equations our conditions on Γ_o follows in this way

$$q = h_1 \qquad -\omega(\boldsymbol{u}) = h_2$$

The **condition** used **on the outflow** Γ_o arises from the weak formulation of the problem, similarly as the so called "do nothing" condition. If the weak solution $\boldsymbol{u}$ is "smooth enough" then there exists a scalar function q such that $\boldsymbol{u}$, q is a classical solution of the equations (1), (2) and $\boldsymbol{u}$, q satisfy

$$\boxed{q = h_1} \qquad \boxed{-\,\omega(\boldsymbol{u}) = h_2} \qquad \text{on } \Gamma_o \tag{8}$$

where $\boldsymbol{h} = (h_1, h_2)$ is a given function on Γ_o.

3.1 Formal derivation of the weak formulation

In order to derive formally the weak formulation of the problem, we multiply equation (2) by an arbitrary test function $\boldsymbol{v} = (v_1, v_2) \in V$ and integrate in Ω. We obtain

$$\begin{aligned}&\int_\Omega \omega(\boldsymbol{u})\,\boldsymbol{u}^\perp \cdot \boldsymbol{v}\,\mathrm{d}\boldsymbol{x}\\&= -\int_\Omega \nabla q \cdot \boldsymbol{v}\,\mathrm{d}\boldsymbol{x} + \nu \int_\Omega \nabla^\perp \omega(\boldsymbol{u}) \cdot \boldsymbol{v}\,\mathrm{d}\boldsymbol{x} + \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\,\mathrm{d}\boldsymbol{x}.\end{aligned}$$

Where $\nabla^\perp = (-\partial_2, \partial_1)$. If we apply Green's theorem and use all the boundary conditions (3)–(8), we finally arrive at the equation

$$\begin{aligned}&\int_\Omega \omega(\boldsymbol{u})\,\boldsymbol{u}^\perp \cdot \boldsymbol{v}\,\mathrm{d}\boldsymbol{x} + \nu \int_\Omega \omega(\boldsymbol{u}) \cdot \omega(\boldsymbol{v})\,\mathrm{d}\boldsymbol{x}\\&= \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\,\mathrm{d}\boldsymbol{x} - \int_{\Gamma_o} \boldsymbol{h} \cdot \boldsymbol{v}\,\mathrm{d}S.\end{aligned}$$

This integral equation can be written in a simple form

$$\boxed{a(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v})_0 + b(\boldsymbol{h}, \boldsymbol{v})}$$

where $(.\,,\,.)_0$ denotes the scalar product in $L^2(\Omega)$ or in $L^2(\Omega)^2$ and

$$\begin{aligned}
a_1(\boldsymbol{u},\boldsymbol{v}) &= (\,\omega(\boldsymbol{u}),\,\omega(\boldsymbol{v})\,)_0,\\
a_2(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) &= \int_\Omega \omega(\boldsymbol{u})\,\boldsymbol{v}^\perp\cdot\boldsymbol{w}\,\mathrm{d}\boldsymbol{x},\\
a(\boldsymbol{u},\boldsymbol{v}) &= a_1(\boldsymbol{u},\boldsymbol{v})+a_2(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}),\\
b(\boldsymbol{h},\boldsymbol{v}) &= -\int_{\Gamma_o}\boldsymbol{h}\cdot\boldsymbol{v}\,\mathrm{d}S.
\end{aligned}$$

All these forms are defined for $\boldsymbol{u},\ \boldsymbol{v},\ \boldsymbol{w}\in H^1(\Omega)^2,\ \boldsymbol{f}\in L^2(\Omega)^2$ and $\boldsymbol{h}\in L^2(\Gamma_o)^2$.

3.2 The weak formulation of the problem (1)–(8)

Definition 1. *Let* $\boldsymbol{g} \in H^s(\Gamma_i)^2$ *(for some* $s \in (\frac{1}{2}, 1]$*) satisfy the condition* $\boldsymbol{g}(A_1) = \boldsymbol{g}(A_0)$. *Let* $\boldsymbol{f} \in L^2(\Omega)^2$ *and* $\boldsymbol{h} \in L^2(\Gamma_o)^2$. *The* **weak solution** *of the problem (1)–(8) is a vector function* $\boldsymbol{u} \in H^1(\Omega)^2$ *which satisfies the identity*

$$a(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v})_0 + b(\boldsymbol{h}, \boldsymbol{v}) \tag{9}$$

for all test functions $\boldsymbol{v} \in V$, *the equation of continuity (3) a.e. in* Ω *and the boundary conditions (3)–(8) (on* Γ_i, Γ_w, Γ_- *and* Γ_+*) in the sense of traces.*

3.3 Existence of a weak solution

Lemma 1. *There exists a constant $c_1 > 0$ independent of $\boldsymbol{g}$ and a divergence–free extension $\boldsymbol{g}^* \in H^1(\Omega)^2$ of function $\boldsymbol{g}$ from Γ_i onto Ω such that $\boldsymbol{g}^* = \mathbf{0}$ on Γ_w, $\boldsymbol{g}^*$ satisfies the condition of periodicity*

$$\boldsymbol{g}^*(x_1, x_2 + \tau) \;=\; \boldsymbol{g}^*(x_1, x_2) \qquad \text{for } (x_1, x_2) \in \Gamma_- \tag{10}$$

and the estimate

$$\|\boldsymbol{g}^*\|_1 \;\leq\; c_1 \, \|\boldsymbol{g}\|_{s;\Gamma_i} \,. \tag{11}$$

Now we construct the weak solution $\boldsymbol{u}$ in the form $\boldsymbol{u} = \boldsymbol{g}^* + \boldsymbol{z}$ where $\boldsymbol{z} \in V$ is a new unknown function. Substituting $\boldsymbol{u} = \boldsymbol{g}^* + \boldsymbol{z}$ into equation (9), we get the following problem: *Find a function $\boldsymbol{z} \in V$ such that it satisfies the equation*

$$\boxed{a(\boldsymbol{g}^* + \boldsymbol{z}, \boldsymbol{v}) \;=\; (\boldsymbol{f}, \boldsymbol{v})_0 + b(\boldsymbol{h}, \boldsymbol{v})} \tag{12}$$

for all $\boldsymbol{v} \in V$.

Theorem (on the existence of a weak solution). *There exists* $\varepsilon > 0$ *such that if* $\|\boldsymbol{g}\|_{s;\Gamma_i} < \varepsilon$ *then there exists a solution* $\boldsymbol{u}$ *of the problem defined in Definition 1.*

3.4 Principle of the proof

We use the Galerkin method and we construct approximations $\boldsymbol{z}_n$ in n–dimensional subspaces V_n of V. A fundamental property which guarantees the existence of the approximations is the **coerciveness of the bilinear form a in space V.** Applying successively estimates of the forms a_1, a_2 and b, we can derive the next lemma.

Lemma 2. *There exist positive constants* c_2, c_3 *and* c_4 *such that*

$$\begin{aligned} & a(\boldsymbol{g}^* + \boldsymbol{z}, \boldsymbol{z}) \\ & \quad \geq \|\nabla \boldsymbol{z}\|_0 \Big(\nu \, \|\nabla \boldsymbol{z}\|_0 - \nu \, c_2 \, \|\boldsymbol{g}\|_{s;\Gamma_i} \\ & \qquad - c_3 \, \|\boldsymbol{g}\|^2_{s;\Gamma_i} - c_4 \, \|\boldsymbol{g}\|_{s;\Gamma_i} \, \|\nabla \boldsymbol{z}\|_0 \Big) \end{aligned} \tag{13}$$

for all $\boldsymbol{z} \in V$.

Now the coerciveness of the form a follows from (13) and the assumption on a sufficient smallness of $\|\boldsymbol{g}\|_{s;\Gamma_i}$.

Advantage of the usage of Bernoulli's pressure:

As a part of necessary estimates which lead to (13), we need to estimate the term

$$a_2(\boldsymbol{z}, \boldsymbol{z}, \boldsymbol{v}) \;=\; \int_\Omega \omega(\boldsymbol{z})\, \boldsymbol{z}^\perp \cdot \boldsymbol{v}\; \mathrm{d}\boldsymbol{x}$$

in the case when $\boldsymbol{v} = \boldsymbol{z}$. However, the integrand equals zero a.e. in Ω in this case because $\omega(\boldsymbol{z})\, \boldsymbol{z}^\perp \cdot \boldsymbol{z} = 0$.