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Numerical approaches to parameter estimates in stochastic evolution equations driven by fractional Brownian motion

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Motivation

- Mathematical modelling of nanotechnological processes of creating thin films of materials
  - New Technologies Research Centre
  - possible further research
- Analysis of models based on stochastic partial differential equations driven by fractional Brownian motion
  - parameter estimates
  - general framework: equations in Hilbert spaces

Outline

- Stochastic equations in Hilbert spaces
- Parameter estimates
  - estimates based on ergodicity
  - estimates based on exact variations
- Numerical simulations
  - Linear SDE
  - Parabolic SPDE

Stochastic evolution equations driven by fractional Brownian motion

Stochastic equations in Hilbert spaces

We consider the linear equation

$$\begin{aligned} dX(t) &= AX(t)\,dt + \Phi\,dB^H(t), \\ X(0) &= x_0, \end{aligned} \tag{1}$$

where $(B^H(t), t \geq 0)$ is a standard U-valued cylindrical fractional Brownian motion with Hurst parameter $H \in [1/2, 1)$ and U is a separable Hilbert space, $A : \mathrm{Dom}(A) \to V$, $\mathrm{Dom}(A) \subset V$, A is the infinitesimal generator of a strongly continuous semigroup $(S(t), t \geq 0)$ on the separable Hilbert space V, $\Phi \in \mathcal{L}(U, V)$ and $x_0 \in V$ is in general random.

Solution

A solution $(X^{x_0}(t), t \geq 0)$ is considered in the mild form, i.e. for all $t \in [0, T]$

$$X^{x_0}(t) = S(t)x_0 + \int_0^t S(t-r)\Phi \, dB^H(r). \tag{2}$$

[20] T. E. Duncan, B. Maslowski, and B. Pasik-Duncan, *Fractional Brownian motion and stochastic equations in Hilbert spaces*, Stoch. Dyn. **2** (2002), no. 2, 225–250.

- if there is a $T_0 > 0$ such that

$$\int_0^{T_0} \int_0^{T_0} |S(r)\Phi|_{\mathcal{L}_2(U,V)} |S(s)\Phi|_{\mathcal{L}_2(U,V)} \phi(r-s) \, dr \, ds < \infty, \tag{A1}$$

  then the solution exists as a V-valued process
- if the semigroup is exponentially stable then there exists a Gaussian centred limiting measure μ_∞ for the solution

Strictly stationary solution

A measurable V-valued process $(X(t), t \geq 0)$ is said to be strictly stationary, if for all $k \in \mathbf{N}$ and for all arbitrary positive numbers $t_1, t_2, \ldots, t_k$, the probability distribution of the V^k-valued random variable $(X(t_1 + r), X(t_2 + r), \ldots, X(t_k + r))$ does not depend on $r \geq 0$, i.e.

$$\mathrm{Law}(X(t_1{+}r), X(t_2{+}r), \ldots, X(t_k{+}r)) = \mathrm{Law}(X(t_1), X(t_2), \ldots, X(t_k))$$

for all $t_1, t_2, \ldots, t_k, r \geq 0$

Theorem

If (A1) is satisfied and the semigroup $(S(t), t \geq 0)$ is exponentially stable, then there exists a strictly stationary solution to (1), i.e. there exists $\tilde{x}$, a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$, such that $(X^{\tilde{x}}(t), t \geq 0)$ is a strictly stationary process with $\mathrm{Law}(X^{\tilde{x}}(t)) = \mu_\infty$, $t \geq 0$. In particular $\mathrm{Law}(\tilde{x}) = \mu_\infty$.

Ergodic theorem for arbitrary solution

Theorem

Let (A1) be satisfied and let $(X^{x_0}(t), t \geq 0)$ *be a solution to* (1) *with initial condition* $X(0) = x_0 \in V$, *generally random. Let* $\varphi : \mathbf{R} \to \mathbf{R}$ *be a real function satisfying the following local Lipschitz condition: let there exists a real constant* $K > 0$ *and an integer* $m > 1$ *such that*

$$|\varphi(x) - \varphi(y)| \leq K|x - y|(1 + |x|^m + |y|^m) \tag{3}$$

for all $x, y \in \mathbf{R}$. *Let* $z \in \mathrm{Dom}(A^*)$ *be arbitrary. Then*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \varphi(\langle X^{x_0}(t), z\rangle)\, dt = \int_V \varphi(\langle y, z\rangle)\, \mu_\infty(dy), \quad a.s.\text{-}\mathbb{P}. \tag{4}$$

for all $x_0 \in V$.

Parameter estimates

Parameter estimates: main results

- Parameter estimates based on exact variations
  - possible from one path observation on a finite interval
  - suitable for diffusion estimates
  - applicable also for drift estimates in one-dimensional equation with space-time white noise
- Parameter estimates based on ergodicity
  - consistent results only for $T \to \infty$
  - suitable for drift estimates

Parameters estimates based on ergodicity

Consider the linear equation

$$\begin{aligned} dX(t) &= \alpha\, AX(t)\, dt + \Phi\, dB^H(t), \\ X(0) &= x_0, \end{aligned} \tag{5}$$

where $\alpha > 0$ is a real constant parameter. Obviously the operator αA is the infinitesimal generator of the semigroup $(S(\alpha t), t \geq 0)$ that is also exponentially stable and there is a limiting measure $\mu_\infty^\alpha = \mathcal{N}(0, Q_\infty^\alpha)$, where

$$\begin{aligned} Q_\infty^\alpha &= \int_0^\infty \int_0^\infty S(\alpha u) Q S^*(\alpha v) \phi(u - v)\, du\, dv \\ &= \frac{1}{\alpha^2} \int_0^\infty \int_0^\infty S(u) Q S^*(v) \phi\left(\frac{u}{\alpha} - \frac{v}{\alpha}\right)\, du\, dv \\ &= \frac{1}{\alpha^2} \frac{1}{\alpha^{2H-2}} \int_0^\infty \int_0^\infty S(u) Q S^*(v) \phi(u - v)\, du\, dv = \frac{1}{\alpha^{2H}} Q_\infty^1. \end{aligned}$$

Parameters estimates based on ergodicity

Theorem

Let (A1) be satisfied and let $(X^{x_0}(t), t \geq 0)$ be a V-valued solution to (5). Let $z \in \mathrm{Dom}(A^*)$ be arbitrary and let the limiting measure μ_∞ exists with covariance Q_∞ such that

$$\langle Q_\infty z, z \rangle_V > 0.$$

Define

$$\hat{\alpha}_T := \left(\frac{\langle Q_\infty z, z \rangle_V}{\frac{1}{T} \int_0^T | \langle X^{x_0}(t), z \rangle_V |^2 \, dt} \right)^{\frac{1}{2H}}. \tag{6}$$

Then

$$\lim_{T \to \infty} \hat{\alpha}_T = \alpha, \quad a.s.\text{-}\mathbb{P},$$

for all $x_0 \in V$.

Parameters estimates based on ergodicity

Theorem

Let (A1) be satisfied and let $(X^{x_0}(t), t \geq 0)$ be a V-valued solution to (5) with initial condition $x_0 \in V$ such that $\mathbb{E}|x_0|_V^2 < \infty$. Let the limiting measure μ_∞ exists with covariance Q_∞ such that $\operatorname{Tr} Q_\infty \neq 0$. Define

$$\hat{\alpha}_T := \left(\frac{\operatorname{Tr} Q_\infty}{\frac{1}{T}\mathbb{E}\int_0^T |X^{x_0}(t)|_V^2 \, dt} \right)^{\frac{1}{2H}}. \tag{7}$$

Then

$$\lim_{T\to\infty} \hat{\alpha}_T = \alpha.$$

Parameters estimates based on variations

Theorem

Let $(X^{x_0}(t), t \geq 0)$ *be a* V*-valued solution to* (1). *Fix* $0 < T_1 < T_2$. *Define, for* $j = 0, 1, \ldots, n$, *a time grid by* $t_j = T_1 + j\delta$, *where* $\delta = \frac{1}{n}(T_2 - T_1)$. *Let* $z \in \mathrm{Dom}(A^*)$ *be arbitrary. Then the following limit holds in mean square for all* $x_0 \in V$

$$\lim_{n\to\infty} \sum_{i=0}^{n} \left|\langle X^{x_0}(t_{i+1}), z\rangle_V - \langle X^{x_0}(t_i), z\rangle_V\right|^{1/H} = c_H \left[\langle Qz, z\rangle_V\right]^{1/(2H)} (T_2 - T_1), \quad (8)$$

where

$$c_H = \frac{2^{1/(2H)}}{\sqrt{\pi}} \Gamma\left(\frac{H+1}{2H}\right). \quad (9)$$

Parameters estimates based on variations

In particular, if we denote by

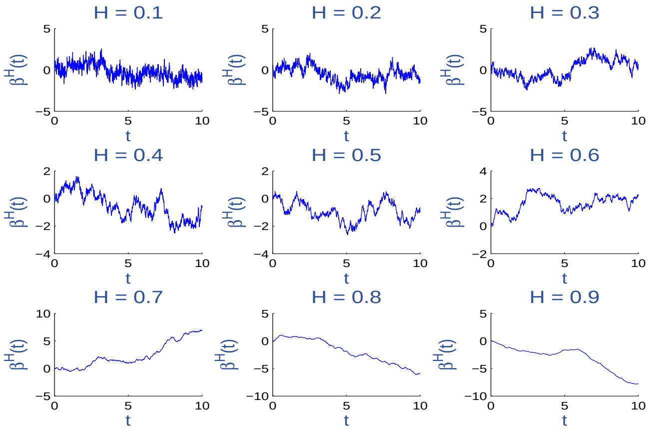
$$\hat{f}_n(z) := \frac{1}{c_H(T_2 - T_1)} \sum_{i=0}^{n} \left| \langle X^{x_0}(t_{i+1}), z \rangle_V - \langle X^{x_0}(t_i), z \rangle_V \right|^{1/H},$$

then

$$\lim_{n \to \infty} \mathbb{E} \left[\hat{f}_n(z) - [\langle Qz, z \rangle_V]^{1/(2H)} \right]^2 = 0. \qquad (10)$$

Numerical simulations

Example: fractional Brownian motion



Nine different sample paths of fractional Brownian motion each with a different value of Hurst parameter H. The roughness of the paths decreases for higher values of H.

Example: Linear SDE

Consider the following one-dimensional linear stochastic differential equation

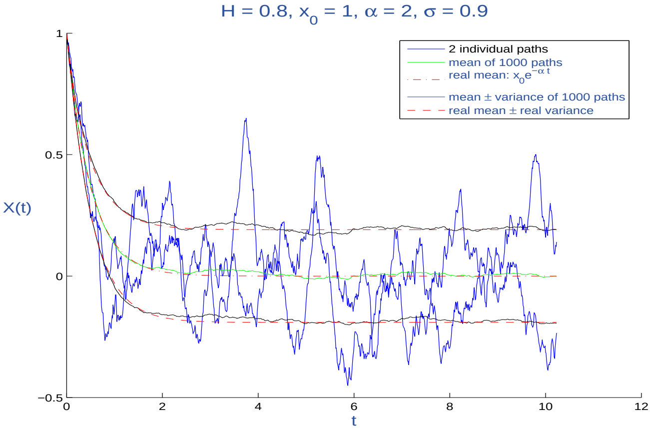
$$\begin{aligned} dX(t) &= -\alpha X(t)\,dt + \sigma\,d\beta^H(t) \\ X(0) &= x_0, \end{aligned} \tag{11}$$

where $\alpha > 0$ and $\sigma > 0$ are real constant parameters and $(\beta^H(t), t \geq 0)$ is a standard fractional Brownian motion with Hurst parameter $H \in (1/2, 1)$.
Modified *Euler-Maruyama method*, explicit scheme:

$$\begin{aligned} Y_0 &= x_0 \\ Y_{j+1} &= Y_j - \alpha Y_j h + \sigma w_j^H, \quad j = 1, \dots, N, \end{aligned} \tag{12}$$

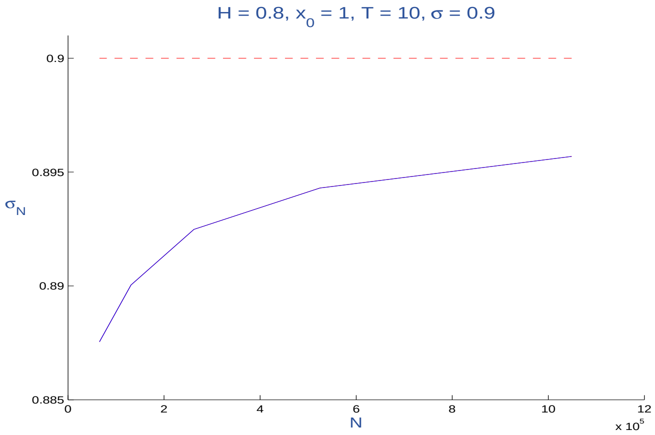
where $w_j^H = \beta^H(t_{j+1}) - \beta^H(t_j)$ is the increment of fractional Brownian motion

Solution: nonzero initial condition



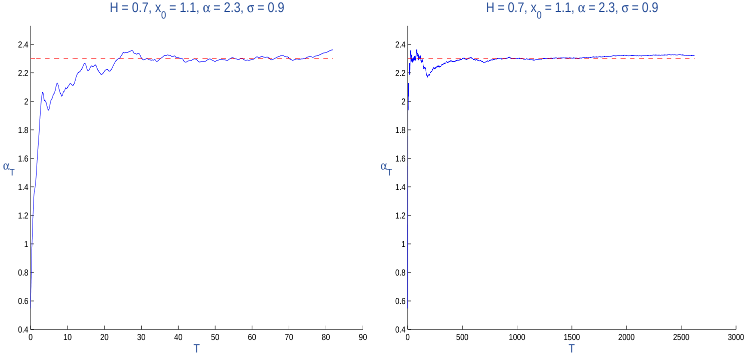
A solution $X(t)$ of stochastic differential equation (11) with nonzero initial condition. Only two individual paths are drawn.

Diffusion estimate $\hat{\sigma}_N$



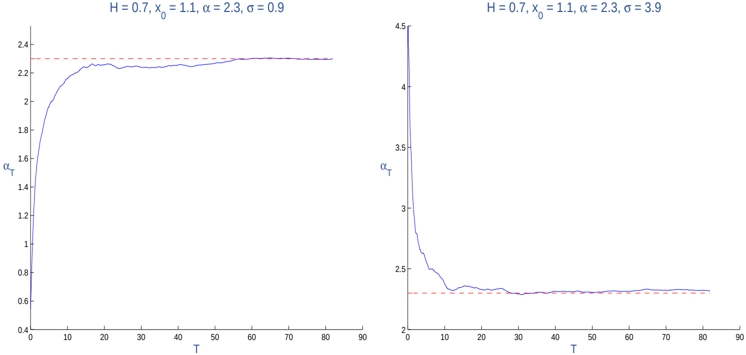
Convergence of σ_N to the true value σ for particular values of x_0, T and H.

Drift estimate $\hat{\alpha}_T$



Convergence of $\hat{\alpha}_T$ computed using 1 path observation to the true value α for particular values of x_0, σ and H (same trajectory viewed in a different time interval).

Drift estimate $\hat{\alpha}_T$



Convergence of $\hat{\alpha}_T$ computed using 50 paths observation to the true value α for particular values of x_0, σ and H.

Example: Linear SPDE

Consider the following initial boundary value problem for linear stochastic heat equation

$$
\begin{aligned}
dX(t,x) &= \alpha\Delta X(t,x)\,dt + \sigma\,dB^H(t), \quad t \geq 0,\ x \in [0,L],\ L > 0 \\
X(0,x) &= x_0(x), \quad x \in [0,L], \\
X(t,0) &= X(t,L) = 0, \quad t \geq 0,
\end{aligned}
\tag{13}
$$

where $\alpha > 0$ and $\sigma > 0$ are real constant parameters, $x_0 \in L^2([0,L])$ and $(B^H(t), t \geq 0)$ is a standard cylindrical fractional Brownian motion with Hurst parameter $H \in (1/2, 1)$.

Finite difference for Laplacian

Space grid $x_i = ik, k = L/M, i = 0, 1, \dots, M$,

$$dX(t, x_i) = \frac{\alpha}{k^2} \left(X(t, x_{i+1}) - 2X(t, x_i) + X(t, x_{i-1}) \right) dt + \sigma \, d\beta_i^H(t),$$

where $\beta_i^H(t)$ are stochastically independent. In matrix form:

$$dX(t) = AX(t) \, dt + \sigma \, dB^H(t),$$

where $X(t)$ is now an $M \times 1$ matrix (vector) with elements $X(t, x_i)$, A is an $M \times M$ matrix and $B^H(t)$ an $M \times 1$ vector of the form

$$A = \frac{\alpha}{k^2} \begin{bmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & 1 & -2 & 1 \\ 0 & \cdots & 0 & 1 & -2 \end{bmatrix}, \quad B^H(t) = \begin{bmatrix} \beta_1^H(t) \\ \beta_2^H(t) \\ \vdots \\ \beta_M^H(t) \end{bmatrix}.$$

Euler-Maruyama method

Implicit scheme:

$$\begin{aligned} Y_0 &= x_0 \\ Y_{j+1} &= Y_j + AY_{j+1}h + \sigma W_j^H, \quad j = 1, \dots, N \end{aligned} \tag{14}$$

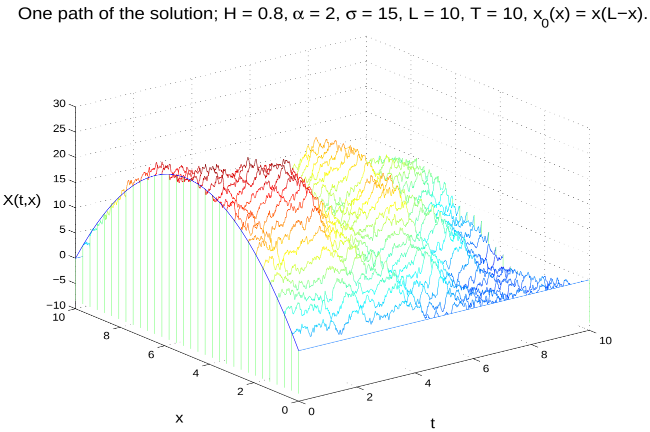
where $W_j^H = B^H(t_{j+1}) - B^H(t_j)$ are the increments of fBm. We calculate Y_{j+1} by solving the following systems of equations

$$(I - Ah)Y_{j+1} = Y_j + \sigma W_j^H, \quad j = 1, \dots, N,$$

where I denotes the identity matrix.

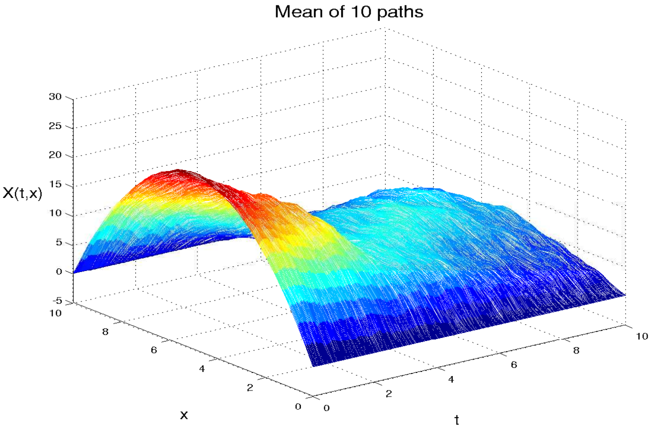
Observation: it is necessary to control some relation between time and space steps. For a deterministic PDE, i.e. when $\sigma = 0$, and an *explicit* scheme the relation is $\alpha \frac{h}{k^2} \leq 1/2$. Here dependance on H?

One path of the solution



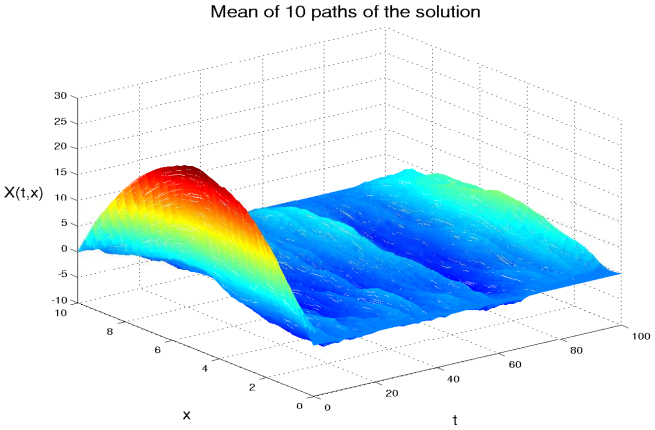
One path solution to (13) with initial condition $x_0(x) = x(L-x)$, $x \in [0, L]$, and particular values of H, α, σ, L and T.

Mean of 10 paths of the solution



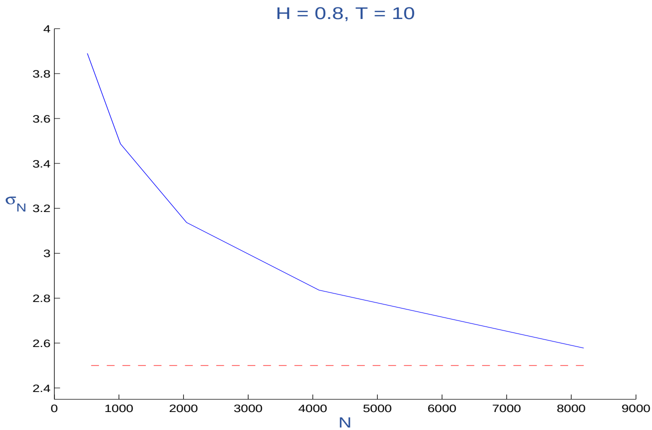
Mean of $P = 10$ paths of the solution.

Solution for large time interval



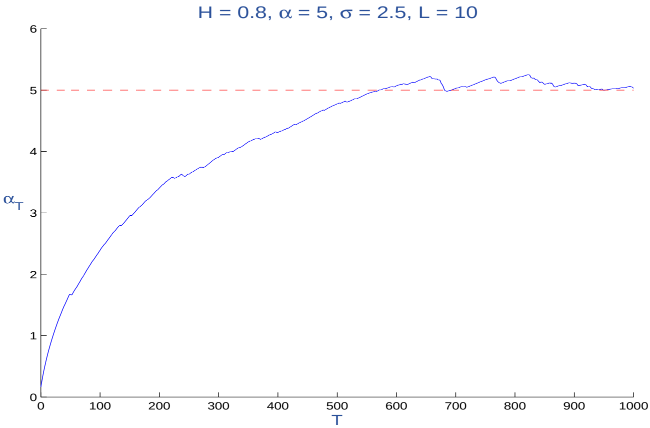
Mean of 10 paths of the solution to (13) with initial condition $x_0(x) = x(L-x)$, $x \in [0, L]$, for large time interval.

Diffusion estimate $\hat{\sigma}_N$



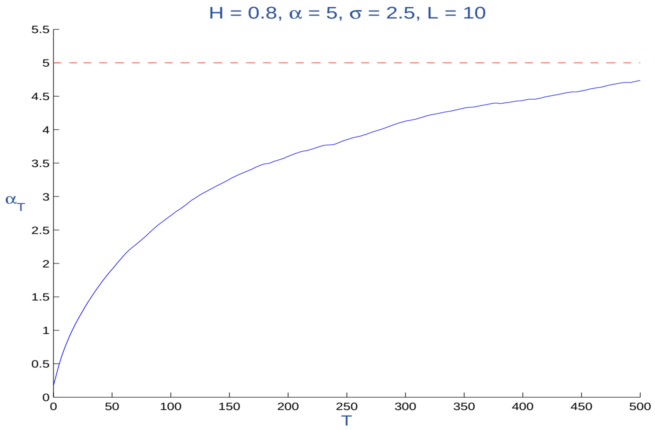
Convergence of $\hat{\sigma}_N$ to the true value σ for particular values of H and T.

Drift estimate $\hat{\alpha}_T$



Convergence of $\hat{\alpha}_T$ computed using 1 path observation to the true value α for particular value of H (σ and L appears in the solution).

Drift estimate $\hat{\alpha}_T$



Convergence of $\hat{\alpha}_T$ computed using 10 paths observation to the true value α for particular values of σ, H and L.

Revision

- ✓ Stochastic equations in Hilbert spaces
- ✓ Parameter estimates
  - estimates based on ergodicity
  - estimates based on exact variations
- ✓ Numerical simulations
  - Linear SDE
  - Parabolic SPDE

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