Periodic Boundary Value Problems for Nonlinear Second Order Differential Equations with Impulses - Part III

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Summary. This paper provides existence results for the nonlinear impulsive periodic boundary value problem

(1.1) $$u'' = f(t, u, u'),$$

(1.2) $$u(t_i+) = \mathrm{J}_i(u(t_i)), \quad u'(t_i+) = \mathrm{M}_i(u'(t_i)), \quad i = 1, 2, \ldots, m,$$

(1.3) $$u(0) = u(T), \quad u'(0) = u'(T),$$

where $f \in \mathrm{Car}([0, T] \times \mathbb{R}^2)$ and J_i, $\mathrm{M}_i \in \mathbb{C}(\mathbb{R})$. The basic assumption is the existence of lower/upper functions σ_1/σ_2 associated with the problem. Here we generalize and extend the existence results of our previous papers.

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0 . Introduction

This paper deals with the solvability of the nonlinear impulsive boundary value problem (1.1)–(1.3). The investigation of this problem was initiated by Hu and Lakshmiknatham in [3]. For the further development, see e.g. [1], [2], [4], [9] and the papers cited therein. We have already studied this problem in [7] and [8] under the assumption that there are lower/upper functions σ_1/σ_2 associated with the problem. In [7] we improved the already known results for the case that σ_1, σ_2 are well-ordered, i.e. $\sigma_1 \le \sigma_2$ on $[0, T]$. On the other hand, in [8] we have delivered the first existence results valid if σ_1, σ_2 are not well-ordered, i.e.

(0.1) $$\sigma_1(\tau) > \sigma_2(\tau) \quad \text{for some} \quad \tau \in [0, T].$$

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The goal of this paper is to generalize the main existence results of [8], where we restricted our attention to impulsive functions M_i, $i = 1, 2, \ldots, m$, fulfilling the conditions

$$y\,\mathrm{M}_i(y) \ge 0 \text{ for } y \in \mathbb{R}, \quad i = 1, 2, \ldots, m. \tag{0.2}$$

Here we prove existence criteria without restriction (0.2).

Throughout the paper we keep the following notation and conventions: For a real valued function u defined a.e. on $[0, T]$, we put

$$\|u\|_\infty = \sup_{t \in [0,T]} |u(t)| \quad \text{and} \quad \|u\|_1 = \int_0^T |u(s)|\,\mathrm{d}s.$$

For a given interval $J \subset \mathbb{R}$, by $\mathbb{C}(J)$ we denote the set of real valued functions which are continuous on J. Furthermore, $\mathbb{C}^1(J)$ is the set of functions having continuous first derivatives on J and $\mathbb{L}(J)$ is the set of functions which are Lebesgue integrable on J.

Let $m \in \mathbb{N}$ and let $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = T$ be a division of the interval $[0, T]$. We denote $\mathrm{D} = \{t_1, t_2, \ldots, t_m\}$ and define $\mathbb{C}^1_{\mathrm{D}}[0, T]$ as the set of functions $u : [0, T] \mapsto \mathbb{R}$ of the form

$$u(t) = \begin{cases} u_{[0]}(t) & \text{if } t \in [0, t_1], \\ u_{[1]}(t) & \text{if } t \in (t_1, t_2], \\ \ldots & \quad \ldots \\ u_{[m]}(t) & \text{if } t \in (t_m, T], \end{cases}$$

where $u_{[i]} \in \mathbb{C}^1[t_i, t_{i+1}]$ for $i = 0, 1, \ldots, m$. Moreover, $\mathbb{AC}^1_{\mathrm{D}}[0, T]$ stands for the set of functions $u \in \mathbb{C}^1_{\mathrm{D}}[0, T]$ having first derivatives absolutely continuous on each subinterval (t_i, t_{i+1}), $i = 0, 1, \ldots, m$. For $u \in \mathbb{C}^1_{\mathrm{D}}[0, T]$ and $i = 1, 2, \ldots, m + 1$ we define

$$u'(t_i) = u'(t_i-) = \lim_{t \to t_i-} u'(t), \quad u'(0) = u'(0+) = \lim_{t \to 0+} u'(t) \tag{0.3}$$

and $\|u\|_{\mathrm{D}} = \|u\|_\infty + \|u'\|_\infty$. Note that the set $\mathbb{C}^1_{\mathrm{D}}[0, T]$ becomes a Banach space when equipped with the norm $\|.\|_{\mathrm{D}}$ and with the usual algebraic operations.

We say that $f : [0, T] \times \mathbb{R}^2 \mapsto \mathbb{R}$ satisfies the *Carathéodory conditions* on $[0, T] \times \mathbb{R}^2$ if (i) for each $x \in \mathbb{R}$ and $y \in \mathbb{R}$ the function $f(., x, y)$ is measurable on $[0, T]$; (ii) for almost every $t \in [0, T]$ the function $f(t, ., .)$ is continuous on $\mathbb{R}^2$; (iii) for each compact set $K \subset \mathbb{R}^2$ there is a function $m_K(t) \in \mathbb{L}[0, T]$ such that $|f(t, x, y)| \le m_K(t)$ holds for a.e. $t \in [0, T]$ and all $(x, y) \in K$. The set of functions satisfying the Carathéodory conditions on $[0, T] \times \mathbb{R}^2$ will be denoted by $\mathrm{Car}([0, T] \times \mathbb{R}^2)$.

Given a Banach space $\mathbb{X}$ and its subset M, let $\mathrm{cl}(M)$ and ∂M denote the closure and the boundary of M, respectively.

Let Ω be an open bounded subset of $\mathbb{X}$. Assume that the operator $\mathrm{F} : \mathrm{cl}(\Omega) \mapsto \mathbb{X}$ is completely continuous and $\mathrm{F}\,u \neq u$ for all $u \in \partial\,\Omega$. Then $\deg(\mathrm{I} - \mathrm{F}, \Omega)$ denotes the *Leray-Schauder topological degree* of $\mathrm{I} - \mathrm{F}$ with respect to Ω, where I is the identity operator on $\mathbb{X}$. For the definition and properties of the degree see e.g. [5].

1 . Formulation of the problem and main assumptions

Here we study the existence of solutions to the problem

$$u'' = f(t, u, u'), \tag{1.1}$$

$$u(t_i+) = \mathrm{J}_i(u(t_i)), \quad u'(t_i+) = \mathrm{M}_i(u'(t_i)), \quad i = 1, 2, \ldots, m, \tag{1.2}$$

$$u(0) = u(T), \quad u'(0) = u'(T), \tag{1.3}$$

where $u'(t_i)$ are understood in the sense of (0.3), $f \in \mathrm{Car}([0, T] \times \mathbb{R}^2)$, $\mathrm{J}_i \in \mathbb{C}(\mathbb{R})$ and $\mathrm{M}_i \in \mathbb{C}(\mathbb{R})$.

1.1. Definition. By a *solution of the problem* (1.1)–(1.3) we understand a function $u \in \mathbb{AC}^1_{\mathrm{D}}[0, T]$ which satisfies the impulsive conditions (1.2), the periodic conditions (1.3) and for a.e. $t \in [0, T]$ fulfils the equation (1.1).

1.2. Definition. A function $\sigma_1 \in \mathbb{AC}^1_{\mathrm{D}}[0, T]$ is called a *lower function of the problem* (1.1)–(1.3) if

$$\sigma_1''(t) \geq f(t, \sigma_1(t), \sigma_1'(t)) \quad \text{for a.e. } t \in [0, T], \tag{1.4}$$

$$\sigma_1(t_i+) = \mathrm{J}_i(\sigma_1(t_i)), \quad \sigma_1'(t_i+) \geq \mathrm{M}_i(\sigma_1'(t_i)), \quad i = 1, 2, \ldots, m, \tag{1.5}$$

$$\sigma_1(0) = \sigma_1(T), \quad \sigma_1'(0) \geq \sigma_1'(T). \tag{1.6}$$

Similarly, a function $\sigma_2 \in \mathbb{AC}^1_{\mathrm{D}}[0, T]$ is an *upper function of the problem* (1.1)–(1.3) if

$$\sigma_2''(t) \leq f(t, \sigma_2(t), \sigma_2'(t)) \quad \text{for a.e. } t \in [0, T], \tag{1.7}$$

$$\sigma_2(t_i+) = \mathrm{J}_i(\sigma_2(t_i)), \quad \sigma_2'(t_i+) \leq \mathrm{M}_i(\sigma_2'(t_i)), \quad i = 1, 2, \ldots, m, \tag{1.8}$$

$$\sigma_2(0) = \sigma_2(T), \quad \sigma_2'(0) \leq \sigma_2'(T). \tag{1.9}$$

1.3. Assumptions. In the paper we work with the following assumptions:

$$(1.10)\qquad \begin{cases} 0 = t_0 < t_1 < \dots < t_m < t_{m+1} = T < \infty,\ \mathrm{D} = \{t_1, t_2, \dots, t_m\}, \\ f \in \mathrm{Car}([0,T] \times \mathbb{R}^2),\ \mathrm{J}_i \in \mathbb{C}(\mathbb{R}),\ \mathrm{M}_i \in \mathbb{C}(\mathbb{R}),\ i = 1, 2, \dots, m; \end{cases}$$

(1.11) σ_1 and σ_2 are respectively lower and upper functions of (1.1)–(1.3);

$$(1.12)\qquad \begin{cases} x > \sigma_1(t_i) \implies \mathrm{J}_i(x) > \mathrm{J}_i(\sigma_1(t_i)), \\ x < \sigma_2(t_i) \implies \mathrm{J}_i(x) < \mathrm{J}_i(\sigma_2(t_i)), \quad i = 1, 2, \dots, m; \end{cases}$$

$$(1.13)\qquad \begin{cases} y \le \sigma_1'(t_i) \implies \mathrm{M}_i(y) \le \mathrm{M}_i(\sigma_1'(t_i)), \\ y \ge \sigma_2'(t_i) \implies \mathrm{M}_i(y) \ge \mathrm{M}_i(\sigma_2'(t_i)), \quad i = 1, 2, \dots, m. \end{cases}$$

1.4. Operator reformulation of (1.1)–(1.3).
Let $G(t,s)$ be the Green function for $u'' = 0,\ u(0) = u(T) = 0$ i.e.

$$(1.14)\qquad G(t,s) = \begin{cases} \dfrac{t\,(s-T)}{T} & \text{if } 0 \le t \le s \le T, \\ \dfrac{s\,(t-T)}{T} & \text{if } 0 \le s < t \le T. \end{cases}$$

Furthermore, we define the operator $\mathrm{F} : \mathbb{C}^1_{\mathrm{D}}[0,T] \mapsto \mathbb{C}^1_{\mathrm{D}}[0,T]$ by

$$(1.15)\qquad (\mathrm{F}\,x)(t) = x(0) + x'(0) - x'(T) + \int_0^T G(t,s)\, f(s, x(s), x'(s))\,\mathrm{d}s$$
$$- \sum_{i=1}^m \frac{\partial G}{\partial s}(t,t_i)\,(\mathrm{J}_i(x(t_i)) - x(t_i)) + \sum_{i=1}^m G(t,t_i)\,(\mathrm{M}_i(x'(t_i)) - x'(t_i)).$$

As in [6, Lemma 3.1], where $m = 1$, we can prove (see Proposition 1.6 below) that F is completely continuous and that a function u is a solution of (1.1)–(1.3) if and only if u is a fixed point of F. To this aim we need the following lemma which extends Lemma 2.1 from [6].

1.5. Lemma. *For each $h \in \mathbb{L}[0,T],\ c, d_i, e_i \in \mathbb{R},\ i = 1, 2, \dots, m,$ there is a unique function $x \in \mathbb{AC}^1_{\mathrm{D}}[0,T]$ fulfilling*

$$(1.16)\qquad \begin{cases} x''(t) = h(t) \ \textit{a.e. on}\ [0,T], \\ x(t_i+) - x(t_i) = d_i,\ x'(t_i+) - x'(t_i) = e_i,\ i = 1, 2, \dots, m, \end{cases}$$

$$(1.17)\qquad x(0) = x(T) = c.$$

This function is given by

$$x(t) = c + \int_0^T G(t,s)\,h(s)\,ds - \sum_{i=1}^{m} \frac{\partial G}{\partial s}(t,t_i)\,d_i + \sum_{i=1}^{m} G(t,t_i)\,e_i \text{ for } t \in [0,T], \tag{1.18}$$

where $G(t,s)$ *is defined by* (1.14)*.*

Proof. It is easy to check that $x \in \mathbb{AC}^1_{\mathrm{D}}[0,T]$ fulfils (1.16) together with $x(0) = c$ if and only if there is $\widetilde{c} \in \mathbb{R}$ such that

$$\begin{aligned} x(t) = c + t\,\widetilde{c} + \sum_{i=1}^{m} \chi_{(t_i,\,T]}(t)\,d_i + \sum_{i=1}^{m} \chi_{(t_i,\,T]}(t)\,(t-t_i)\,e_i \\ + \int_0^t (t-s)\,h(s)\,\mathrm{d}s \text{ for } t \in [0,T], \end{aligned} \tag{1.19}$$

where $\chi_{(t_i,T]}(t) = 1$ if $t \in (t_i,T]$ and $\chi_{(t_i,T]}(t) = 0$ if $t \in \mathbb{R} \setminus (t_i,T]$. Furthermore, $x(T) = c$ if and only if

$$\widetilde{c} = -\sum_{i=1}^{m} \frac{d_i}{T} - \sum_{i=1}^{m} \frac{T-t_i}{T}\,e_i - \int_0^T \frac{T-s}{T}\,h(s)\,\mathrm{d}s. \tag{1.20}$$

Inserting (1.20) into (1.19), we get

$$\begin{aligned} x(t) = \sum_{t_i<t} \frac{t_i\,(t-T)}{T}\,e_i + \sum_{t_i \ge t} \frac{t\,(t_i-T)}{T}\,e_i - \sum_{t_i<t} \frac{(t-T)}{T}\,d_i - \sum_{t_i \ge t} \frac{t}{T}\,d_i \\ + \int_0^t \frac{s\,(t-T)}{T}\,h(s)\,\mathrm{d}s + \int_t^T \frac{t\,(s-T)}{T}\,h(s)\,\mathrm{d}s, \quad t \in [0,T]. \end{aligned}$$

Hence, taking into account (1.14), we conclude that the function x given by (1.18) is the unique solution of (1.16), (1.17) in $\mathbb{AC}^1_{\mathrm{D}}[0,T]$. □

1.6. Proposition. *Assume that* (1.10) *holds. Let the operator* $\mathrm{F} : \mathbb{C}^1_{\mathrm{D}}[0,T] \mapsto \mathbb{C}^1_{\mathrm{D}}[0,T]$ *be defined by* (1.14) *and* (1.15)*. Then* F *is completely continuous and a function* u *is a solution of* (1.1)–(1.3) *if and only if* $u = \mathrm{F}\,u$*.*

Proof. Choose an arbitrary $y \in \mathbb{C}^1_{\mathrm{D}}[0,T]$ and put

$$\begin{cases} h(t) = f(t, y(t), y'(t)) \text{ for a.e. } t \in [0,T], \\ d_i = \mathrm{J}_i(y(t_i)) - y(t_i), \;\; e_i = \mathrm{M}_i(y'(t_i)) - y'(t_i), \quad i = 1, 2, = \ldots, m, \\ c = y(0) + y'(0) - y'(T). \end{cases} \tag{1.21}$$

Then $h \in \mathbb{L}[0,T]$, $c, d_i, e_i \in \mathbb{R}$, $i = 1,2,\dots,m$. By Lemma 1.5, there is a unique $x \in \mathbb{AC}^1_{\mathrm{D}}[0,T]$ fulfilling (1.16), (1.17) and it is given by (1.18). Due to (1.21), we have

$$x(t) = (\mathrm{F}\, y)(t) \text{ for } t \in [0,T].$$

Therefore, $u \in \mathbb{C}^1_{\mathrm{D}}[0,T]$ is a solution to (1.1)–(1.3) if and only if $u = \mathrm{F}\, u$. Define an operator $\mathrm{F}_1 : \mathbb{C}^1_{\mathrm{D}}[0,T] \mapsto \mathbb{C}^1_{\mathrm{D}}[0,T]$ by

$$(\mathrm{F}_1 y)(t) = \int_0^T G(t,s)\, f(s, y(s), y'(s))\, \mathrm{d}s, \quad t \in [0,T].$$

As F_1 is a composition of the Green type operator for the Dirichlet problem $u'' = 0$, $u(0) = u(T) = 0$, and of the superposition operator generated by $f \in \mathrm{Car}([0,T] \times \mathbb{R}^2)$, making use of the Lebesgue Dominated Convergence Theorem and the Arzelà-Ascoli Theorem, we get in a standard way that F_1 is completely continuous. Since J_i, M_i, $i = 1,2,\dots,m$, are continuous, the operator $\mathrm{F}_2 = \mathrm{F} - \mathrm{F}_1$ is continuous, as well. Having in mind that F_2 maps bounded sets onto bounded sets and its values are contained in a $(2m+1)$-dimensional subspace of $\mathbb{C}^1_{\mathrm{D}}[0,T]$, we conclude that the operators F_2 and $\mathrm{F} = \mathrm{F}_1 + \mathrm{F}_2$ are completely continuous. □

In the proof of our main result we will need the next proposition which concerns the case of well-ordered lower/upper functions and which follows from [7, Corollary 3.5].

1.7. Proposition. *Assume that* (1.10) *holds and let* α *and* β *be respectively lower and upper functions of* (1.1)–(1.3) *such that*

(1.22) $\qquad \alpha(t) < \beta(t)$ *for* $t \in [0,T]$ *and* $\alpha(\tau+) < \beta(\tau+)$ *for* $\tau \in \mathrm{D}$,

(1.23) $\qquad \alpha(t_i) < x < \beta(t_i) \implies \mathrm{J}_i(\alpha(t_i)) < \mathrm{J}_i(x) < \mathrm{J}_i(\beta(t_i)), \quad i = 1,2,\dots,m$

and

$$(1.24) \qquad \begin{cases} y \le \alpha'(t_i) \implies \mathrm{M}_i(y) \le \mathrm{M}_i(\alpha'(t_i)), \\ y \ge \beta'(t_i) \implies \mathrm{M}_i(y) \ge \mathrm{M}_i(\beta'(t_i)), \quad i = 1,2,\dots,m. \end{cases}$$

Further, let $h \in \mathbb{L}[0,T]$ *be such that*

(1.25) $\qquad |f(t,x,y)| \le h(t)$ *for a.e.* $t \in [0,T]$ *and all* $(x,y) \in [\alpha(t), \beta(t)] \times \mathbb{R}$

and let the operator F *be defined by* (1.15). *Finally, for* $\gamma \in (0,\infty)$ *denote*

$$(1.26) \qquad \Omega(\alpha,\beta,\gamma) = \{u \in \mathbb{C}^1_{\mathrm{D}}[0,T] : \alpha(t) < u(t) < \beta(t) \text{ for } t \in [0,T],\\ \alpha(\tau+) < u(\tau+) < \beta(\tau+) \text{ for } \tau \in \mathrm{D},\ \|u'\|_\infty < \gamma\}.$$

Then $\deg(\mathrm{I}-\mathrm{F},\Omega(\alpha,\beta,\gamma))=1$ *whenever* $\mathrm{F}\,u\neq u$ *on* $\partial\Omega(\alpha,\beta,\gamma)$ *and*

$$\gamma > \|h\|_1+\frac{\|\alpha\|_\infty+\|\beta\|_\infty}{\Delta}, \quad \textit{where}\ \ \Delta=\min_{i=1,2,\dots,m+1}(t_i-t_{i-1}). \tag{1.27}$$

Proof. Using the Mean Value Theorem, we can show that

$$\|u'\|_\infty \le \|h\|_1+\frac{\|\alpha\|_\infty+\|\beta\|_\infty}{\Delta} \tag{1.28}$$

holds for each $u\in\mathbb{C}^1_{\mathrm{D}}[0,T]$ fulfilling $\alpha(t)<u(t)<\beta(t)$ for $t\in[0,T]$ and $\alpha(\tau+)<u(\tau+)<\beta(\tau+)$ for $\tau\in\mathrm{D}$. Thus, if we denote by c the right-hand side of (1.28), we can follow the proof of [7, Corollary 3.5]. □

2 . A priori estimates

In Section 3 we will need a priori estimates which are contained in Lemmas 2.1–2.3.

2.1. Lemma. *Let* $\rho_1\in(0,\infty)$, $\widetilde{h}\in\mathbb{L}[0,T]$, $\mathrm{M}_i\in\mathbb{C}(\mathbb{R})$, $i=1,2,\dots,m$. *Then there exists* $d\in(\rho_1,\infty)$ *such that the estimate*

$$\|u'\|_\infty < d \tag{2.1}$$

is valid for each $u\in\mathbb{AC}^1_{\mathrm{D}}[0,T]$ *and each* $\widetilde{\mathrm{M}}_i\in\mathbb{C}(\mathbb{R})$, $i=1,2,\dots,m$, *satisfying* (1.3),

$$|u'(\xi_u)|<\rho_1 \quad \textit{for some}\ \ \xi_u\in[0,T], \tag{2.2}$$

$$u'(t_i+)=\widetilde{\mathrm{M}}_i(u'(t_i)), \quad i=1,2,\dots,m, \tag{2.3}$$

$$|u''(t)|<\widetilde{h}(t)\ \textit{for a.e.}\ t\in[0,T] \tag{2.4}$$

and

$$\sup\{|\mathrm{M}_i(y)|:\ |y|<a\}<b \implies \sup\{|\widetilde{\mathrm{M}}_i(y)|:\ |y|<a\}<b \quad \textit{for}\ \ i=1,2,\dots,m,\ a\in(0,\infty),\ b\in(a,\infty). \tag{2.5}$$

Proof. Suppose that $u\in\mathbb{AC}^1_{\mathrm{D}}[0,T]$ and $\widetilde{\mathrm{M}}_i\in\mathbb{C}(\mathbb{R})$, $i=1,2,\dots,m$, satisfy (1.3) and (2.2)–(2.5). Due to (1.3), we can assume that $\xi_u\in(0,T]$, i.e. there is $j\in\{1,2,\dots,m+1\}$ such that $\xi_u\in(t_{j-1},t_j]$. We will distinguish 3 cases: either $j=1$ or $j=m+1$ or $1<j<m+1$.

Let $j=1$. Then, using (2.2) and (2.4), we obtain

$$|u'(t)|<a_1 \text{ on } [0,t_1], \tag{2.6}$$

where $a_1=\rho_1+\|\widetilde{h}\|_1$. Since $\mathrm{M}_1\in\mathbb{C}(\mathbb{R})$, we can find $b_1(a_1)\in(a_1,\infty)$ such that $|\mathrm{M}_1(y)|<b_1(a_1)$ for all $y\in(-a_1,a_1)$.

Hence, in view of (2.3) and (2.5), we have $|u'(t_1+)| < b_1(a_1)$, wherefrom, using (2.4), we deduce that $|u'(t)| < b_1(a_1)+\|\widetilde{h}\|_1$ for $t \in (t_1, t_2]$. Continuing by induction, we get $b_i(a_i) \in (a_i, \infty)$ such that $|u'(t)| < a_{i+1} = b_i(a_i) + \|\widetilde{h}\|_1$ on $(t_i, t_{i+1}]$ for $i = 2, \dots, m$, i.e.

$$\|u'\|_\infty < d := \max\{a_i : i = 1, 2, \dots, m+1\}. \tag{2.7}$$

Assume that $j = m + 1$. Then, using (2.2) and (2.4), we obtain

$$|u'(t)| < a_{m+1} \text{ on } (t_m, T], \tag{2.8}$$

where $a_{m+1} = \rho_1 + \|\widetilde{h}\|_1$. Furthermore, due to (1.3), we have $|u'(0)| < a_{m+1}$ which together with (2.4) yields that (2.6) is true with $a_1 = a_{m+1} + \|\widetilde{h}\|_1$. Now, proceeding as in the case $j = 1$, we show that (2.7) is true also in the case $j = m + 1$.

Assume that $1 < j < m+1$. Then (2.2) and (2.4) yield $|u'(t)| < a_{j+1} = \rho_1 + \|\widetilde{h}\|_1$ on $(t_j, t_{j+1}]$. If $j < m$, then $|u'(t)| < a_{j+2} = b_{j+1}(a_{j+1}) + \|\widetilde{h}\|_1$ on $(t_{j+1}, t_{j+2}]$, where $b_{j+1}(a_{j+1}) > a_{j+1}$. Proceeding by induction we get (2.8) with $a_{m+1} = b_m(a_m) + \|\widetilde{h}\|_1$ and $b_m(a_m) > a_m$, wherefrom (2.7) again follows as in the previous case. □

2.2. Lemma. *Let $\rho_0, d, q \in (0, \infty)$ and $\mathrm{J}_i \in \mathbb{C}(\mathbb{R})$, $i = 1, 2, \dots, m$. Then there exists $c \in (\rho_0, \infty)$ such that the estimate*

$$\|u\|_\infty < c \tag{2.9}$$

is valid for each $u \in \mathbb{C}^1_{\mathrm{D}}[0, T]$ and each $\widetilde{\mathrm{J}}_i \in \mathbb{C}(\mathbb{R})$, $i = 1, 2, \dots, m$, satisfying (1.3), (2.1),

$$u(t_i+) = \widetilde{\mathrm{J}}_i(u(t_i)), \quad i = 1, 2, \dots, m, \tag{2.10}$$

$$|u(\tau_u)| < \rho_0 \quad \textit{for some} \quad \tau_u \in [0, T] \tag{2.11}$$

and

$$\sup\{|\mathrm{J}_i(x)| : |x| < a\} < b \implies \sup\{|\widetilde{\mathrm{J}}_i(x)| : |x| < a\} < b \quad \textit{for} \;\; i = 1, 2, \dots, m, \; a \in (0, \infty), \; b \in (a+q, \infty). \tag{2.12}$$

Proof. We will argue similarly as in the proof of Lemma 2.1. Suppose that $u \in \mathbb{C}^1_{\mathrm{D}}[0, T]$ satisfies (1.3), (2.1), (2.10), (2.11) and that $\widetilde{\mathrm{J}}_i \in \mathbb{C}(\mathbb{R})$, $i = 1, 2, \dots, m$, satisfy (2.12). Due to (1.3) we can assume that $\tau_u \in (0, T]$, i.e. there is $j \in \{1, 2, \dots, m+1\}$ such that $\tau_u \in (t_{j-1}, t_j]$. We will consider three cases: $j = 1$, $j = m + 1$, $1 < j < m + 1$. If $j = 1$, then (2.1) and (2.11) yield $|u(t)| < a_1 = \rho_0 + dT$ on $[0, t_1]$. In particular, $|u(t_1)| < a_1$. Since $\mathrm{J}_1 \in \mathbb{C}(\mathbb{R})$, we can find $b_1(a_1) \in (a_1 + q, \infty)$ such that $|\mathrm{J}_1(x)| < b_1(a_1)$ for all $x \in (-a_1, a_1)$ and consequently, by (2.12), also $|\widetilde{\mathrm{J}}_1(x)| < b_1(a_1)$ for all $x \in (-a_1, a_1)$. Therefore, by (2.1), $|u(t)| <$

$|u(t_1+)| + dT = |\widetilde{\mathrm{J}}_1(u(t_1))| + dT < a_2 = b_1(a_1) + dT$ on $(t_1, t_2]$. Proceeding by induction we get $b_i(a_i) \in (a_i + q, \infty)$ such that $|u(t)| < a_{i+1} = b_i(a_i) + dT$ for $t \in (t_i, t_{i+1}]$ and $i = 2, \dots, m$. As a result, (2.9) is true with $c = \max\{a_i : i = 1, 2, \dots, m+1\}$. Analogously we would proceed in the remaining cases $j = m+1$ or $1 < j < m+1$. ☐

Finally, we will need two estimates for functions u satisfying one of the following conditions:

$$u(s_u) < \sigma_1(s_u) \quad \text{and} \quad u(t_u) > \sigma_2(t_u) \quad \text{for some} \;\; s_u, t_u \in [0, T], \tag{2.13}$$

$$u \ge \sigma_1 \text{ on } [0, T] \quad \text{and} \quad \inf_{t \in [0,T]} |u(t) - \sigma_1(t)| = 0, \tag{2.14}$$

$$u \le \sigma_2 \text{ on } [0, T] \quad \text{and} \quad \inf_{t \in [0,T]} |u(t) - \sigma_2(t)| = 0. \tag{2.15}$$

2.3. Lemma. *Assume that* $\sigma_1,\ \sigma_2 \in \mathbb{AC}^1_{\mathrm{D}}[0, T]$, J_i, M_i, $\widetilde{\mathrm{J}}_i$, $\widetilde{\mathrm{M}}_i \in \mathbb{C}(\mathbb{R})$, $i = 1, 2, \dots, m$, *satisfy* (1.12), (1.13) *and*

$$\begin{cases} x > \sigma_1(t_i) \implies \widetilde{\mathrm{J}}_i(x) > \widetilde{\mathrm{J}}_i(\sigma_1(t_i)) = \mathrm{J}_i(\sigma_1(t_i)), \\ x < \sigma_2(t_i) \implies \widetilde{\mathrm{J}}_i(x) < \widetilde{\mathrm{J}}_i(\sigma_2(t_i)) = \mathrm{J}_i(\sigma_2(t_i)), \quad i = 1, 2, \dots, m \end{cases} \tag{2.16}$$

and

$$\begin{cases} y \le \sigma_1'(t_i) \implies \widetilde{\mathrm{M}}_i(y) \le \mathrm{M}_i(\sigma_1'(t_i)), \\ y \ge \sigma_2'(t_i) \implies \widetilde{\mathrm{M}}_i(y) \ge \mathrm{M}_i(\sigma_2'(t_i)), \quad i = 1, 2, \dots, m. \end{cases} \tag{2.17}$$

Define

$$B = \{u \in \mathbb{C}^1_{\mathrm{D}}[0, T] : u \textit{ satisfies } (1.3), (2.10), (2.3) \textit{ and one of the conditions } (2.13), (2.14), (2.15)\}. \tag{2.18}$$

Then each function $u \in B$ *satisfies*

$$\begin{cases} |u'(\xi_u)| < \rho_1 \quad \textit{for some} \;\; \xi_u \in [0, T], \;\; \textit{where} \\ \rho_1 = \dfrac{2}{t_1} \left(\|\sigma_1\|_\infty + \|\sigma_2\|_\infty \right) + \|\sigma_1'\|_\infty + \|\sigma_2'\|_\infty + 1. \end{cases} \tag{2.19}$$

Proof. • Part 1. Assume that $u \in B$ satisfies (2.13). There are 3 cases to consider:

Case A. If $\min\{\sigma_1(t), \sigma_2(t)\} \le u(t) \le \max\{\sigma_1(t), \sigma_2(t)\}$ for $t \in [0, T]$, then, by the Mean Value Theorem, there is $\xi_u \in (0, t_1)$ such that

$$|u'(\xi_u)| \le \frac{2}{t_1} \left(\|\sigma_1\|_\infty + \|\sigma_2\|_\infty \right). \tag{2.20}$$

Case B. Assume that $u(s) > \sigma_1(s)$ for some $s \in [0,T]$ and denote $v = u - \sigma_1$. Due to (2.13) we have

$$v_* = \inf_{t\in[0,T]} v(t) < 0 \quad \text{and} \quad v^* = \sup_{t\in[0,T]} v(t) > 0. \tag{2.21}$$

We are going to prove that

$$v'(\alpha) = 0 \text{ for some } \alpha \in [0,T] \text{ or } v'(\tau+) = 0 \text{ for some } \tau \in \mathrm{D}\,. \tag{2.22}$$

Suppose, on the contrary, that (2.22) does not hold.

Let $v'(0) > 0$. Then, according to (1.3) and (1.6), $v'(T) > 0$, as well. Due to the assumption that (2.22) does not hold, this together with (1.5) yields that

$$0 < v'(t_m+) = u'(t_m+) - \sigma_1'(t_m+) \le \widetilde{\mathrm{M}}_m(u'(t_m)) - \mathrm{M}_m(\sigma_1'(t_m)),$$

which is by (2.17) possible only if $u'(t_m) > \sigma_1'(t_m)$, i.e. $v'(t_m) > 0$. Continuing in this way on each $(t_i, t_{i+1}]$, $i = 0, 1, \ldots, m-1$, we get

$$v'(t) > 0 \text{ for } t \in [0,T] \quad \text{and} \quad v'(\tau+) > 0 \text{ for } \tau \in \mathrm{D}\,. \tag{2.23}$$

If $v(0) \ge 0$, then $v(t) > 0$ on $(0, t_1]$ due to (2.23). Further, it follows by (1.5), (2.10) and (2.16) that $u(t_1+) > \sigma_1(t_1+)$, i.e. $v(t_1+) > 0$. Continuing by induction we deduce that $v \ge 0$ on $[0,T]$, contrary to (2.21).

If $v(0) < 0$, then by (1.3) and (1.6) we have $v(T) < 0$. Further, by virtue of (2.23) we obtain $v < 0$ on $(t_m, T]$ and, in particular, $v(t_m+) < 0$. So, $\widetilde{\mathrm{J}}_m(u(t_m)) < \mathrm{J}_m(\sigma_1(t_m))$ wherefrom $u(t_m) \le \sigma_1(t_m)$ follows, due to (2.16). Thus, we have $v < 0$ on (t_{m-1}, t_m). Continuing by induction we get $v \le 0$ on $[0,T]$, contrary to (2.21).

Now, assume that $v'(0) < 0$. Then $v'(t_1) < 0$, i.e. $u'(t_1) < \sigma_1'(t_1)$ wherefrom, by (1.5), (1.13) and the assumption that (2.22) does not hold, the inequality $v'(t_1+) = u'(t_1+) - \sigma_1'(t_1+) < 0$ follows. Similarly as in the proof of (2.23) we show that

$$v'(t) < 0 \text{ for } t \in [0,T] \quad \text{and} \quad v'(\tau+) < 0 \text{ for } \tau \in \mathrm{D}\,. \tag{2.24}$$

Now, having (2.24), we consider as above two cases: $v(0) \ge 0$ and $v(0) < 0$, and construct a contradiction by means of analogous arguments.

So we have proved that (2.22) is true, which yields the existence of $\xi_u \in [0,T]$ having the property

$$|u'(\xi_u)| < \|\sigma_1'\|_\infty + 1. \tag{2.25}$$

Case C. If $u(s) < \sigma_2(s)$ for some $s \in [0,T]$, we put $v = u - \sigma_2$ and, using the properties of σ_2 instead of σ_1, we can argue as in Case B and show that there exists $\xi_u \in [0,T]$ such that

$$|u'(\xi_u)| < \|\sigma_2'\|_\infty + 1. \tag{2.26}$$

Taking into account (2.20), (2.25) and (2.26) we conclude that (2.19) is valid for any $u \in B$ fulfilling (2.13).

• PART 2. Let $u \in B$ satisfy (2.14). Then $u \geq \sigma_1$ on $[0,T]$ and either there is $\alpha_u \in [0,T]$ such that $u(\alpha_u) = \sigma_1(\alpha_u)$ or there is $t_j \in \mathrm{D}$ such that $u(t_j+) = \sigma_1(t_j+)$.

CASE A. Let the first possibility occur. If $\alpha_u \in (0,T) \setminus \mathrm{D}$, then necessarily $u'(\alpha_u) = \sigma_1'(\alpha_u)$. Consequently, the estimate (2.25) is valid. If $\alpha_u = 0$, then $\inf\{u(t) - \sigma_1(t) : t \in [0,T]\} = u(0) - \sigma_1(0) = u(T) - \sigma_1(T) = 0$, which, by virtue of (1.3) and (1.6), implies $0 \leq u'(0) - \sigma_1'(0) \leq u'(T) - \sigma_1'(T) \leq 0$, i.e. $u'(0) = \sigma_1'(0)$ and the estimate (2.25) is valid with $\xi_u = 0$. If $\alpha_u = t_j$ for some $t_j \in \mathrm{D}$, then $0 = u(t_j) - \sigma_1(t_j) = u(t_j+) - \sigma_1(t_j+)$. Having in mind that $u \geq \sigma_1$ on $[0,T]$, we get $u'(t_j+) \geq \sigma_1'(t_j+)$ and $u'(t_j) \leq \sigma_1'(t_j)$. On the other hand, with respect to (2.17), the last inequality gives also $\widetilde{\mathrm{M}}_j(u'(t_j)) \leq \mathrm{M}_j(\sigma_1'(t_j))$, which leads to $\sigma_1'(t_j+) = u'(t_j+)$. Thus, (2.25) is fulfilled for some $\xi_u \in (t_j, t_{j+1})$ which is sufficiently close to t_j.

CASE B. Let the second possibility occur, i.e. $u(t_j+) = \sigma_1(t_j+)$ for some $t_j \in \mathrm{D}$. According to (1.5) and (2.10), we have $\widetilde{\mathrm{J}}_j(u(t_j)) = \mathrm{J}_j(\sigma_1(t_j))$. Taking into account (2.16), we see that this can occur only if $u(t_j) \leq \sigma_1(t_j)$. On the other hand, by the assumption (2.14) we have $u \geq \sigma_1$ on $[0,T]$. Hence we conclude that $u(t_j) = \sigma_1(t_j)$ and so, arguing as before, we get (2.25) again.

To summarize: (2.19) holds for any $u \in B$ fulfilling (2.14).

• PART 3. Let $u \in B$ satisfy (2.15). Then using the properties of σ_2 instead of σ_1, we argue analogously to PART 2 and prove that (2.26) is valid for each $u \in B$ which satisfies (2.15). In particular, (2.19) holds for any $u \in B$ fulfilling (2.15). ☐

3 . Main result

Our main result consists in a generalization of [8, Theorem 3.1]. Particularly, we remove the condition (0.2), which was assumed in [8], and prove the following theorem.

3.1. Theorem. *Assume that* (1.10)–(1.13) *and* (0.1) *hold and let* $h \in \mathbb{L}[0,T]$ *be such that*

$$(3.1) \qquad |f(t,x,y)| \leq h(t) \text{ for a.e. } t \in [0,T] \text{ and all } (x,y) \in \mathbb{R}^2.$$

Then the problem (1.1)–(1.3) *has a solution* u *satisfying one of the conditions* (2.13)–(2.15).

Proof. • STEP 1. *We construct a proper auxiliary problem.*

Let σ_1 and σ_2 be respectively lower and upper functions of (1.1)–(1.3) and let

ρ_1 be associated with them as in (2.19). Put

$$\widetilde{h}(t) = 2\,h(t) + 1 \text{ for a.e. } t \in [0,T],$$
$$\widetilde{\rho} = \rho_1 + \sum_{i=1}^{m} \big(|\,\mathrm{M}_i(\sigma_1'(t_i))| + |\,\mathrm{M}_i(\sigma_2'(t_i))| \big).$$

By Lemma 2.1, find $d \in (\widetilde{\rho}, \infty)$ satisfying (2.1). Furthermore, put $\rho_0 = \|\sigma_1\|_\infty + \|\sigma_2\|_\infty + 1$ and

$$q = \frac{T}{m} \sum_{i=1}^{m} \max\{ \max_{|y| \le d+1} |\,\mathrm{M}_i(y)|, d+1 \} \tag{3.2}$$

and, by Lemma 2.2, find $c \in (\rho_0 + q, \infty)$ fulfilling (2.9). In particular, we have

$$c > \|\sigma_1\|_\infty + \|\sigma_2\|_\infty + q + 1, \quad d > \|\sigma_1'\|_\infty + \|\sigma_2'\|_\infty + 1. \tag{3.3}$$

Finally, for a.e. $t \in [0,T]$ and all $x, y \in \mathbb{R}$ and $i = 1, 2, \ldots, m$, define functions

$$\widetilde{f}(t,x,y) = \begin{cases} f(t,x,y) - h(t) - 1 & \text{if } x \le -c-1, \\ f(t,x,y) + (x+c)\,(h(t)+1) & \text{if } -c-1 < x < -c, \\ f(t,x,y) & \text{if } -c \le x \le c, \\ f(t,x,y) + (x-c)\,(h(t)+1) & \text{if } c < x < c+1, \\ f(t,x,y) + h(t) + 1 & \text{if } x \ge c+1, \end{cases} \tag{3.4}$$

$$\widetilde{\mathrm{J}}_i(x) = \begin{cases} x + q & \text{if } x \le -c-1, \\ \mathrm{J}_i(-c)\,(c+1+x) - (x+q)\,(x+c) & \text{if } -c-1 < x < -c, \\ \mathrm{J}_i(x) & \text{if } -c \le x \le c, \\ \mathrm{J}_i(c)\,(c+1-x) + (x-q)\,(x-c) & \text{if } c < x < c+1, \\ x - q & \text{if } x \ge c+1, \end{cases} \tag{3.5}$$

$$\widetilde{\mathrm{M}}_i(y) = \begin{cases} y & \text{if } y \le -d-1, \\ \mathrm{M}_i(-d)\,(d+1+y) - y\,(y+d) & \text{if } -d-1 < y < -d, \\ \mathrm{M}_i(y) & \text{if } -d \le y \le d, \\ \mathrm{M}_i(d)\,(d+1-y) + y\,(y-d) & \text{if } d < y < d+1, \\ y & \text{if } y \ge d+1 \end{cases} \tag{3.6}$$

and consider the auxiliary problem

$$(3.7)\qquad u'' = \widetilde{f}(t,u,u'),\quad (2.10),\quad (2.3),\quad (1.3).$$

Due to (1.10), $\widetilde{f} \in \mathrm{Car}([0,T]\times\mathbb{R})$ and $\widetilde{\mathrm{J}}_i, \widetilde{\mathrm{M}}_i \in \mathbb{C}(\mathbb{R})$ for $i = 1,2,\dots,m$. According to (3.3)–(3.6) the functions σ_1 and σ_2 are respectively lower and upper functions of (3.7). By (3.1) we have

$$(3.8)\qquad |\widetilde{f}(t,x,y)| \le \widetilde{h}(t) \text{ for a.e. } t \in [0,T] \text{ and all } (x,y) \in \mathbb{R}^2$$

and

$$(3.9)\qquad \begin{cases} \widetilde{f}(t,x,y) < 0 & \text{for a.e. } t \in [0,T] \text{ and all } (x,y) \in (-\infty,-c-1]\times\mathbb{R}, \\ \widetilde{f}(t,x,y) > 0 & \text{for a.e. } t \in [0,T] \text{ and all } (x,y) \in [c+1,\infty)\times\mathbb{R}. \end{cases}$$

- Step 2. *We show that $\widetilde{\mathrm{J}}_i$ and $\widetilde{\mathrm{M}}_i$ satisfy the assumptions of Lemmas* 2.1 – 2.3*.* Choose an arbitrary $i \in \{1,2,\dots,m\}$.
  - (i) *Condition* (2.5). Let $a \in (0,\infty)$, $b \in (a,\infty)$ and $M_i^* = \sup\{|\,\mathrm{M}_i(y)| : |y| < a\} < b$. Then, by (3.6), we have $\sup\{|\widetilde{\mathrm{M}}_i(y)| : |y| < a\} \le \max\{a, M_i^*\} < b$.
  - (ii) *Condition* (2.12). Let $a \in (0,\infty)$, $b \in (a+q,\infty)$ and $J_i^* = \sup\{|\,\mathrm{J}_i(x)| : |x| < a\} < b$. Then, by (3.5), we have $\sup\{|\widetilde{\mathrm{J}}_i(x)| : |x| < a\} \le \max\{a+q, J_i^*\} < b$.
  - (iii) *Condition* (2.16). Due to (1.12), (3.3) and (3.5), we see that (2.16) holds if $|x| \le c$. Assume that $x > c$. Then $x > \max\{\sigma_1(t_i), \sigma_2(t_i)\}$ which means that the second condtion in (2.16) need not be considered in this case. Since $|\sigma_1(t_i)| < c$, we have $\widetilde{\mathrm{J}}_i(\sigma_1(t_i)) = \mathrm{J}_i(\sigma_1(t_i))$. Furthermore, due to (3.3), $x - q > \|\sigma_1\|_\infty + \|\sigma_2\|_\infty + 1$. If $x \ge c+1$, then $\widetilde{\mathrm{J}}_i(x) = x - q > \sigma_1(t_i+) = \mathrm{J}_i(\sigma_1(t_i))$. Finally, if $x \in (c, c+1)$, then $\widetilde{\mathrm{J}}_i(x) = \mathrm{J}_i(c)\,(c+1-x) + (x-q)\,(x-c) > \mathrm{J}_i(\sigma_1(t_i))$ because $\mathrm{J}_i(c) > \mathrm{J}_i(\sigma_1(t_i))$ by (1.12). For $x < (\infty, -c)$ we can argue similarly.
  - (iv) *Condition* (2.17). Due to (1.13), (3.3) and (3.6), we see that (2.17) holds for $|y| < d$. Assume that $y > d$. Then $y > \max\{\sigma_1'(t_i), \sigma_2'(t_i)\}$ which means that the first condition in (2.17) need not be considered in this case. Since $d > \widetilde{\rho} > \mathrm{M}_i(\sigma_2'(t_i))$, we have $\widetilde{\mathrm{M}}_i(y) = y > \mathrm{M}_i(\sigma_2'(t_i))$ if $y > d+1$ and $\widetilde{\mathrm{M}}_i(y) = \mathrm{M}_i(d)\,(d+1-y) + y\,(y-d) > \mathrm{M}_i(\sigma_2'(t_i))$ if $y \in (d, d+1)$. Hence the second condition in (2.17) is satisfied for $y \in (d,\infty)$. Similarly we can verify the first condition in (2.17) for $y \in (-\infty, -d)$.
- Step 3. *We construct a well-ordered pair of lower/upper functions for* (3.7)*.* Put

$$(3.10)\qquad A^* = q + \sum_{i=1}^{m} \max_{|x| \le c+1} |\widetilde{\mathrm{J}}_i(x)|$$

and

$$
(3.11)\qquad \begin{cases} \sigma_4(0) = A^* + m\,q, \\ \sigma_4(t) = A^* + (m-i)\,q + \frac{m\,q}{T}\,t \text{ for } t \in (t_i, t_{i+1}],\ i = 0,1,\dots,m, \\ \sigma_3(t) = -\sigma_4(t) \text{ for } t \in [0,T]. \end{cases}
$$

Then $\sigma_3, \sigma_4 \in \mathbb{AC}^1_{\mathrm{D}}[0,T]$ and, by (3.5) and (3.10),

$$
(3.12)\qquad \sigma_3(t) < -A^* < -c-1, \quad \sigma_4(t) > A^* > c+1 \text{ for } t \in [0,T].
$$

In view of (3.2),

$$
(3.13)\qquad \sigma_3'(t) = -\frac{m\,q}{T} \le -(d+1) \quad \text{and} \quad \sigma_4'(t) = \frac{m\,q}{T} \ge d+1 \text{ for } t \in [0,T].
$$

Now, we prove that σ_4 is an upper function of (3.7):
By (3.9) and (3.12), we have

$$
0 = \sigma_4''(t) < \widetilde{f}(t, \sigma_4(t), \sigma_4'(t)) \text{ for a.e. } t \in [0,T].
$$

Furthermore, by (3.5),

$$
\sigma_4(t_i+) = A^* + (m-i)\,q + \frac{m\,q}{T} t_i = \sigma_4(t_i) - q = \widetilde{\mathrm{J}}_i(\sigma_4(t_i)).
$$

By virtue of (3.2) and (3.6), we get

$$
\sigma_4'(t_i+) = \frac{m\,q}{T} = \sigma_4'(t_i) = \widetilde{\mathrm{M}}_i(\sigma_4'(t_i)) \text{ for } i = 1,2,\dots,m.
$$

Finally, $\sigma_4(0) = A^* + m\,q = \sigma_4(T)$ and $\sigma_4'(0) = \frac{m\,q}{T} = \sigma_4'(T)$, i.e. σ_4 is an upper function of (3.7). Since $\sigma_3 = -\sigma_4$, we can see that σ_3 is a lower function of (3.7). Clearly,

$$
(3.14)\qquad \sigma_3 < \sigma_4 \text{ on } [0,T] \quad \text{and} \quad \sigma_3(\tau+) < \sigma_4(\tau+) \text{ for } \tau \in \mathrm{D}.
$$

Having G from (1.15), define an operator $\widetilde{\mathrm{F}} : \mathbb{C}^1_{\mathrm{D}}[0,T] \mapsto \mathbb{C}^1_{\mathrm{D}}[0,T]$ by

$$
(3.15)\qquad \begin{aligned} (\widetilde{\mathrm{F}}u)(t) = {} & u(0) + u'(0) - u'(T) + \int_0^T G(t,s)\,\widetilde{f}(s,u(s),u'(s))\,\mathrm{d}s \\ & - \sum_{i=1}^m \frac{\partial G}{\partial s}(t,t_i)(\widetilde{\mathrm{J}}_i(u(t_i)) - u(t_i)) \\ & + \sum_{i=1}^m G(t,t_i)(\widetilde{\mathrm{M}}_i(u'(t_i)) - u'(t_i)), \quad t \in [0,T]. \end{aligned}
$$

By Proposition 1.6, $\widetilde{\mathrm{F}}$ is completely continuous and u is a solution of (3.7) whenever $\widetilde{\mathrm{F}}u = u$.

- Step 4. *We prove the first a priori estimate for solutions of* (3.7).
  Define

$$\Omega_0 = \{u \in \mathbb{C}^1_{\mathrm{D}}[0,T] : \|u'\|_\infty < C^*,\ \sigma_3 < u < \sigma_4 \text{ on } [0,T], \\ \sigma_3(\tau+) < u(\tau+) < \sigma_4(\tau+) \text{ for } \tau \in \mathrm{D}\}, \tag{3.16}$$

where

$$C^* = 1 + \|\widetilde{h}\|_1 + \frac{\|\sigma_3\|_\infty + \|\sigma_4\|_\infty}{\Delta} \tag{3.17}$$

and Δ is defined in (1.27). We are going to prove that for each solution u of (3.7) the estimate

$$u \in \mathrm{cl}(\Omega_0) \implies u \in \Omega_0 \tag{3.18}$$

is true. To this aim, suppose that u is a solution of (3.7) and $u \in \mathrm{cl}(\Omega_0)$, i.e. $\|u'\|_\infty \le C^*$ and

$$\sigma_3 \le u \le \sigma_4 \text{ on } [0,T]. \tag{3.19}$$

By the Mean Value Theorem, there are $\xi_i \in (t_i, t_{i+1}),\ i = 1, 2, \ldots, m$, such that

$$|u'(\xi_i)| \le \frac{\|\sigma_3\|_\infty + \|\sigma_4\|_\infty}{\Delta}.$$

Hence, by (3.8), we get

$$\|u'\|_\infty < C^*, \tag{3.20}$$

where C^* is defined in (3.17). It remains to show that $\sigma_3 < u < \sigma_4$ on $[0,T]$ and $\sigma_3(\tau+) < u(\tau+) < \sigma_4(\tau+)$ for $\tau \in \mathrm{D}$. Assume the contrary. Then there exists $k \in \{3,4\}$ such that

$$u(\xi) = \sigma_k(\xi) \qquad \text{for some } \ \xi \in [0,T] \tag{3.21}$$

or

$$u(t_i+) = \sigma_k(t_i+) \quad \text{for some } \ t_i \in \mathrm{D}. \tag{3.22}$$

Case A. Let (3.21) hold for $k = 4$.

(i) If $\xi = 0$, then $u(0) = \sigma_4(0) = \sigma_4(T) = u(T) = A^* + q\,m$ which gives, in view of (1.3), (3.13) and (3.19),

$$u'(0) = u'(T) = \frac{m\,q}{T} = \sigma_4'(t) \text{ for } t \in [0,T].$$

Further, due to (3.9) and (3.12), we can find $\delta > 0$ such that $u > c+1$ on $[0,\delta]$ and

$$u'(t) - u'(0) = \int_0^t \widetilde{f}(s, u(s), u'(s))\,\mathrm{d}s > 0 \text{ for } t \in [0,\delta].$$

Hence $u'(t) > u'(0) = \sigma_4'(t)$ on $(0,\delta]$ which implies that $u > \sigma_4$ on $(0,\delta]$, contrary to (3.19).

(ii) If $\xi \in (t_i, t_{i+1})$ for some $t_i \in \mathrm{D}$, then $u'(\xi) = \sigma_4'(\xi) = \frac{mq}{T} = \sigma_4'(t)$ for $t \in [0,T]$ and we reach a contradiction as above.

(iii) If $\xi = t_i \in \mathrm{D}$, then $u(t_i) = \sigma_4(t_i)$ and, by (3.5) and (3.12),

$$u(t_i+) = \sigma_4(t_i+) = \sigma_4(t_i) - q > c + 1 - q > \|\sigma_1\|_\infty + \|\sigma_2\|_\infty.$$

By virtue of (3.19) we have $u'(t_i+) \le \sigma_4'(t_i+)$ and $u'(t_i) \ge \sigma_4'(t_i)$. Now, since the last inequality together with (3.6) and (3.13) yield $u'(t_i+) \ge \sigma_4'(t_i+)$, we get $u'(t_i+) = \sigma_4'(t_i+) = \frac{mq}{T} = \sigma_4'(t)$ for $t \in [0,T]$. Similarly as above, this leads again to a contradiction.

Case B. Let (3.22) hold for $k = 4$, i.e. $u(t_i+) = \sigma_4(t_i+)$. By (3.5) and (3.12), $\widetilde{\mathrm{J}}_i(u(t_i)) = \sigma_4(t_i+) = \sigma_4(t_i) - q > A^* - q$, wherefrom, with respect to (3.10), we get $u(t_i) > c+1$ and hence $\widetilde{\mathrm{J}}_i(u(t_i)) = u(t_i) - q$. Therefore $u(t_i) = \sigma_4(t_i)$ and we can continue as in Case A (iii).

If (3.21) or (3.22) hold for $k = 3$, then we use analogical arguments as in Case A or Case B.

- Step 5. *We prove the second a priori estimate for solutions of* (3.7).
  Define sets

$$\Omega_1 = \{u \in \Omega_0 : u(t) > \sigma_1(t) \text{ for } t \in [0,T],\ u(\tau+) > \sigma_1(\tau+) \text{ for } \tau \in \mathrm{D}\},$$

$$\Omega_2 = \{u \in \Omega_0 : u(t) < \sigma_2(t) \text{ for } t \in [0,T],\ u(\tau+) < \sigma_2(\tau+) \text{ for } \tau \in \mathrm{D}\}$$

and $\widetilde{\Omega} = \Omega_0 \setminus \mathrm{cl}(\Omega_1 \cup \Omega_2)$. Then, by (0.1), $\Omega_1 \cap \Omega_2 = \emptyset$ and

(3.23) $$\widetilde{\Omega} = \{u \in \Omega_0 : u \text{ satisfies } (2.13)\}.$$

Furthermore, with respect to (1.26), (3.16) and (3.11) we have

$$\Omega_0 = \Omega(\sigma_3, \sigma_4, C^*),\ \Omega_1 = \Omega(\sigma_1, \sigma_4, C^*) \quad \text{and} \quad \Omega_2 = \Omega(\sigma_3, \sigma_2, C^*).$$

Consider c from Step 1. We are going to prove that the estimates

(3.24) $$u \in \mathrm{cl}(\widetilde{\Omega}) \implies \|u\|_\infty < c, \quad \|u'\|_\infty < d$$

are valid for each solution u of (3.7). So, assume that u is a solution of (3.7) and $u \in \mathrm{cl}(\widetilde{\Omega})$. Then, due to (3.18), u fulfils one of the conditions (2.13), (2.14), (2.15) and so, by (2.18), $u \in B$. Since we have already proved that (2.16) and (2.17) hold, we can use Lemma 2.3 and get $\xi_u \in [0,T]$ such that (2.19) is true. Further, since $\widetilde{\mathrm{M}}_i$, $i = 1, 2, \ldots, m$, fulfil (2.5) and since (1.3), (2.3) and (3.8) are valid, we can apply Lemma 2.1 to show that u satisfies the estimate (2.1). Finally, by [8, Lemma 2.4], u satisfies (2.11) with ρ_0 defined in STEP 1. Moreover, let us recall that $\widetilde{\mathrm{J}}_i$, $i = 1, 2, \ldots, m$, verify the condition (2.12). Hence, by Lemma 2.2, we have (2.9), i.e. each solution u of (3.7) satisfies (3.24).

- STEP 6. *We prove the existence of a solution to the problem* (1.1)–(1.3).

Consider the operator $\widetilde{\mathrm{F}}$ defined by (3.15). We distinguish two cases: either $\widetilde{\mathrm{F}}$ has a fixed point in $\partial\widetilde{\Omega}$ or it has no fixed point in $\partial\widetilde{\Omega}$.

Assume that $\widetilde{\mathrm{F}}\,u = u$ for some $u \in \partial\widetilde{\Omega}$. Then u is a solution of (3.7) and, with respect to (3.24), we have $\|u\|_\infty < c$, $\|u'\|_\infty < d$, which means, by (3.4)–(3.6), that u is a solution of (1.1)–(1.3). Furthermore, due to (3.18), u satisfies (2.14) or (2.15).

Now, assume that $\widetilde{\mathrm{F}}\,u \neq u$ for all $u \in \partial\widetilde{\Omega}$. Then $\widetilde{\mathrm{F}}u \neq u$ for all $u \in \partial\Omega_0 \cup \partial\Omega_1 \cup \partial\Omega_2$. If we replace f, h, J_i, M_i, $i = 1, 2, \ldots, m$, α, β and γ respectively by $\widetilde{f}$, $\widetilde{h}$, $\widetilde{\mathrm{J}}_i$, $\widetilde{\mathrm{M}}_i$, $i = 1, 2, \ldots, m$, σ_3, σ_4 and C^* in Proposition 1.7, we see that the assumptions (1.22)–(1.25) and (1.27) are satisfied. Thus, by Proposition 1.7, we obtain that

$$\deg(\mathrm{I} - \widetilde{\mathrm{F}}, \Omega(\sigma_3, \sigma_4, C^*)) = \deg(\mathrm{I} - \widetilde{\mathrm{F}}, \Omega_0) = 1. \tag{3.25}$$

Similarly, we can apply Proposition 1.7 to show that

$$\deg(\mathrm{I} - \widetilde{\mathrm{F}}, \Omega(\sigma_1, \sigma_4, C^*)) = \deg(\mathrm{I} - \widetilde{\mathrm{F}}, \Omega_1) = 1 \tag{3.26}$$

and

$$\deg(\mathrm{I} - \widetilde{\mathrm{F}}, \Omega(\sigma_3, \sigma_2, C^*)) = \deg(\mathrm{I} - \widetilde{\mathrm{F}}, \Omega_2) = 1. \tag{3.27}$$

Using the additivity property of the Leray-Schauder topological degree we derive from (3.25)–(3.27) that

$$\deg(\mathrm{I} - \widetilde{\mathrm{F}}, \widetilde{\Omega}) = \deg(\mathrm{I} - \widetilde{\mathrm{F}}, \Omega_0) - \deg(\mathrm{I} - \widetilde{\mathrm{F}}, \Omega_1) - \deg(\mathrm{I} - \widetilde{\mathrm{F}}, \Omega_2) = -1.$$

Therefore, $\widetilde{\mathrm{F}}$ has a fixed point $u \in \widetilde{\Omega}$. By (3.24) we have $\|u\|_\infty < c$ and $\|u'\|_\infty < d$. This together with (3.4)–(3.6) and (3.23) yields that u is a solution to (1.1)–(1.3) fulfilling (2.13). □

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