Dynamical systems method (DSM) for solving operator equations

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Introduction

$$F(u)=0 \quad (1), \quad F:H\to H, \quad \exists y: F(y)=0$$

$$\sup_{u\in B(u_0,R)} \|F^{(j)}(u)\| \le M_j(R), \quad j=0,1,2.$$

$$B(u_0,R):=\{u:\|u-u_0\|\le R\}$$

Well-posed (WP): $\sup_{w\in B(u_0,R)} \|[F'(u)]^{-1}\| \le m(R)$

Ill-posed (IP): not well-posed.

DSM: $\begin{cases} \dot{u} & = \Phi(t, u) \\ u(0) & = u_0 \end{cases}$

$(*)$ $\qquad \exists! u(t)$ on $[0, \infty)$; $\quad \exists u(\infty)$; $\quad F(u(\infty)) = 0$

For what classes of equation (1) can one find Φ such that $(*)$ holds?
How does one choose Φ?

Theorem 1. For *any WP eq. (1)* one can find Φ such that $(*)$ holds and

$$\|u(t) - u(\infty)\| \leq re^{-c_1 t};$$
$$\|F(u(t))\| \leq \|F(u_0)\| e^{-c_1 t}. \quad (**)$$

Here $c_1, r > 0$ are constants.
E.g.,

a) $\Phi = -[F'(u)]^{-1} F(u)$,
b) $\Phi = -[F'(u_0)]^{-1} F(u)$,
c) $\Phi = -T^{-1} A^* F, \quad A := F'(u), \quad T := A^* A$,
d) $F = -A^* F$.

Theorem 2. For *any linear IP eq. (1)*:

$$F(u) = Au - f = 0,$$

where A is a linear, closed, densely defined operator, and equation (1) is solvable, one can find Φ such that $(*)$ holds, convergence

$$u(t) \underset{t\to\infty}{\longrightarrow} y$$

is uniform with respect to u_0, and y is a unique minimal-norm element of the set

$$N := \{u : Au - f = 0\}.$$

E.g., one can take

$$\Phi = -u + T^{-1}_{\varepsilon(t)} A^* f, \quad T = A^* A, \quad T_\varepsilon = T + \varepsilon I,$$

$$0 < \varepsilon(t) \searrow 0, \quad \int^{\infty} \varepsilon(s) ds = \infty.$$

For unbounded A the element f may not belong to $D(A^*)$. In this case, the element $T^{-1}_{\varepsilon(t)} A^* f$, with $\varepsilon(t) > 0$, can be defined by considering the closure of the operator $T^{-1}_{\varepsilon(t)} A^*$ with the domain $D(A^*)$. This operator is closable, its closure is a bounded, defined on all of H operator, and

$$||T^{-1}_{\varepsilon(t)} A^*|| \le \frac{1}{2\sqrt{\varepsilon(t)}}, \quad \varepsilon(t) > 0.$$

It is possible to replace element $T^{-1}_{\varepsilon(t)}A^*f$ by the originally well defined element $A^*Q^{-1}_{\varepsilon(t)}f$, with

$$Q := AA^*.$$

The operator $A^*Q^{-1}_{\varepsilon(t)}$ is a bounded linear operator defined on all of H, and

$$||A^*Q^{-1}_{\varepsilon(t)}|| \leq \frac{1}{2\sqrt{\varepsilon(t)}}, \quad \varepsilon(t) > 0.$$

These assumptions allow one, among other things, to handle differential operators on unbounded domains in the cases when the spectrum of such operators is continuous and contains the point $\lambda = 0$.

Theorem 3. For any eq. (1) with $F' \geq 0$, one can find Φ such that the conclusion of Theorem 2 holds.

E.g., $\Phi = -A^{-1}_{\varepsilon(t)}[F(u) + \varepsilon(t)u]$, $0 < \varepsilon \searrow 0$, $\frac{|\dot{\varepsilon}|}{\varepsilon} \leq \frac{1}{2}$.

Theorem 4. In Theorems 2 and 3 the DSM yields a stable approximation to y in the following sense: if

$$\|f_\delta - f\| \leq \delta,$$

and the data are $\{\delta, f_\delta\}$, then there exists a t_δ such that $\lim_{\delta\to 0} \|u_\delta - y\| = 0$, where

$$u_\delta := u_\delta(t_\delta),$$

and $u_\delta(t)$ solves eq. (2) with f replaced by f_δ.

E.g., $\begin{cases} \dot{u}_\delta & = -u_\delta + T^{-1}_{\varepsilon(t)} A^* f_\delta \\ u_\delta(0) & = u_0, \end{cases}$

where $T := A^*A$, $T_\epsilon = T + \epsilon I$.

A priori and a posteriori stopping rules for finding t_δ are found.

Arbitrary F with $\widetilde{A} := F'(y) \neq 0$

Theorem 5. If $\widetilde{A} \neq 0$, then there exists Φ such that $(*)$ holds for a solution y of eq. (1) .
E.g., $\Phi = -T^{-1}_{\varepsilon(t)}[A^*F(u) + \varepsilon(t)(u - \widetilde{u}_0)]$,
where $\widetilde{u}_0$ is properly chosen, $T := A^*A$, $T_\epsilon = T + \epsilon I$, and
$0 < \varepsilon(t) \searrow 0, \quad \frac{|\dot{\varepsilon}|}{\varepsilon} \leq \frac{1}{2}$.

Spectral assumption

$$F(u) + \varepsilon u = 0 \quad (1),\ F : X \to X,\ X \text{ is a Banach space.}$$

$$(S) \quad \text{Assume:} \|A_\varepsilon^{-1}\| \leq \frac{c}{\varepsilon},\ c = const > 0,$$
$$A := F'(u),\ A_\varepsilon := A + \varepsilon I.$$

Theorem 6. If (S) holds, then eq. $F(u) + \varepsilon u = 0$ can be solved by a DSM.
E.g, $\Phi = -A_\varepsilon^{-1}(F(u) + \varepsilon u)$.

Theorem 7. If (S) holds and $\exists y: \; F(y) = 0$, then one can choose w such that equation

$$F(u_\varepsilon) + \varepsilon(u_\varepsilon - w) = 0$$

is solvable for every $\varepsilon \in (0, \varepsilon_0)$, and $\lim_{\varepsilon \to 0} \|u_\varepsilon - y\| = 0$.

Unbounded F

If $F(u) = Lu + g(u)$, L is linear, closed, densely defined operator, and $\|L^{-1}\| \leq m$,
then equation $F(u) = 0$ is equivalent to

$$u + L^{-1}g(u) = 0. \tag{1'}$$

Theorem 8. Assume that

$$\sup_{u\in B(u_0,R)} \|[I + L^{-1}g'(u)]^{-1}\| \leq m_1(R),$$

and

$$\|u_0 + L^{-1}g(u_0)\| m_1(R) \leq R.$$

Then $(*)$ holds for the problem:

$$\begin{cases} \dot{u} = -[I + L^{-1}g'(u)]^{-1}[u + L^{-1}g(u)], \\ u(0) = u_0. \end{cases}$$

Nonsmooth F

Theorem 9. If F is monotone, i.e.,

$$(F(u) - F(v), u - v) \geq 0,$$

hemicontinuous, $D(F) = H$, and $\exists y: \; F(y) = 0$, then DSM holds for the problem:

$$\begin{cases} \dot{u} = -F(u) - \varepsilon(t)u, \\ u(0) = u_0, \end{cases}$$

where $0 < \varepsilon(t) \searrow 0$, $\varepsilon(t) = \dfrac{c_1}{(c_0 + t)^b}$,
$0 < b < 1$, $c_0, c_1 = const > 0$.

Criteria for surjectivity and homeomorphism

Theorem 10. If

$$\sup_{R>0} \frac{R}{m(R)} = \infty,$$

then eq. $F(u) = f$ is solvable for any $f \in H$.

Theorem 11. If

$$\|[F'(u)]^{-1}\| \leq \psi(\|u\|),$$

where ψ is a continuous positive function, and $\int_0^\infty \frac{ds}{\psi(s)} = \infty$, then F is a global homeomorphism of H onto H.

Construction of convergent iterative processes.

$$u_{n+1} = u_n + h_n\Phi(t_n, u_n), t_{n+1} = t_n + h_n.$$

Theorem 12. *Any WP eq.* $F(u) = 0$ can be solved by a convegent iterative process with $h_n = h = const$ and $\Phi = \Phi(u)$. The process converges at an exponential rate.

Other iterative schemes can be constructed, e.g., Runge-Kutta's type, et al.

Getting rid of the inversion of the derivative $[F'(u)]^{-1}$.

WP:

$$F(u) = 0.$$

$A = F'(u), \;\; T = A^*A, \;\; T_\epsilon = T + \epsilon I.$

$$(2') \qquad \begin{cases} \dot{u} & = -QF(u), \\ \dot{Q} & = -TQ + A^*, \\ u(0) & = u_0, \quad Q(0) = Q_0, \end{cases}$$

Theorem 13. For problem (2') conclusion $(*)$ holds.

IP:

$$(2'') \qquad \begin{cases} \dot{u} & = -Q[A^*F(u) + \varepsilon(t)(u - \widetilde{u}_0)], \\ \dot{Q} & = -T_{\varepsilon(t)}Q + I, \\ u(0) & = u_0, \quad Q(0) = Q_0. \end{cases}$$

Assume: $0 < \varepsilon(t) \searrow 0,\ 0 < \frac{|\dot{\varepsilon}|}{\varepsilon} \le c,\ T(y) \neq 0.$

Theorem 14. For problem $(2'')$ conditions $(*)$ hold.

Theorem 15. If $g(t)$, $\gamma(t)$, $\alpha(t)$ and $\beta(t)$ are nonnegative continuous functions, $g \in C^1[0,\infty)$,

$$\dot{g} \le -\gamma(t)g + \alpha(t)g^2 + \beta(t), \quad t \ge 0,$$

and there exists a $\mu(t) > 0$, $\quad \lim_{t\to\infty} \mu(t) = \infty$,
$\mu \in C^1[0,\infty)$ such that
1) $\alpha \le \frac{\mu}{2}\left(\gamma - \frac{\dot{\mu}}{\mu}\right)$,

$$2)\ \beta \le \frac{1}{2\mu}\left(\gamma - \frac{\dot{\mu}}{\mu}\right), \qquad 3)\ g(0)\mu(0) < 1,$$

then $\exists g(t)$ on $[0,\infty)$ and $0 \le g(t) < \dfrac{1}{\mu(t)}$.

Theorem 16. If $Q(t)$, $G(t)$ and $T(t)$ are linear operator-functions from $[0, \infty) \to H$, where H is a Hilbert space, and

$$\begin{cases} \dot{Q} & = -T(t)Q + G(t), \\ Q(0) & = Q_0, \end{cases}$$

where $(Th, h) \geq \varepsilon(t)\|h\|^2$, $\varepsilon(t) \geq 0$, then, with $a(t) := e^{\int_0^t \varepsilon(s)ds}$, one has:

$$\|Q(t)\| \leq a^{-1}(t)\,\|Q_0\| + a^{-1}(t)\,\int_0^t a(s)\|G(s)\|ds.$$

Proofs.

$$F(u) = 0 \quad (1), \quad \begin{cases} \dot{u} &= \Phi(t,u), \\ u(0) &= u_0. \end{cases} \quad (2)$$

Theorem 17. If

1) $(F'\Phi, F) \leq -c_1\|F\|^2, \ \forall u \in H, \ c_1 = const > 0$

2) $\|\Phi\| \leq c_2\|F\|$,

3) $r \leq R$, where $r := \frac{c_2}{c_1}\|F_0\|$, $F_0 = F(u_0)$,

then $(*)$ and $(**)$ hold, where

$$(**) \qquad \|u(t) - u(\infty)\| \leq re^{-c_1 t}, \quad \|F(u(t))\| \leq \|F_0\|e^{-c_1 t}.$$

Proof. Let $g(t) := \|F(u(t))\|$.
Then
$g\dot{g} = (F'\Phi, F) \le -c_1 g^2$.
Thus

$$g(t) \le g(0)e^{-c_1 t} = \|F_0\|e^{-c_1 t}, \ \|\dot{u}\| \le \|\Phi\| \le c_2\|F_0\|e^{-c_1 t}.$$

So
$\|u(t) - u(\infty)\| \le re^{-c_1 t}$,
$\|u(t) - u(0)\| \le R$.
Theorem 17 is proved. □

Ex. WP

a) $\Phi = -[F'(u)]^{-1} F \Rightarrow c_1 = 1,\ c_2 = m,$ $\boxed{m\|F_0\| \leq R.}$

b) $\Phi = -[F'(u_0)]^{-1} F \Rightarrow c_2 = m, -(F'(u_0) - F'(u_0) + F(u_0)[F'(u)]^{-1} F, F) \leq -\|F\|^2 + mMR\|F\|^2,$
$mM_2 R = \frac{1}{2} \Rightarrow c_1 = \frac{1}{2},\ R = \frac{1}{2mM_2} \quad \|F_0\| 2m \leq \frac{1}{2mM_2},$
$\boxed{4m^2 M_2 \|F_0\| \leq 1.}$

c) $\Phi = -[T]^{-1} A^* F \Rightarrow c_1 = 1,\ c_2 = m^2 M_1,$ $\boxed{m^2 M_1 \|F_0\| \leq R.}$

Linear IP.

$$\begin{cases} \dot{u} & = -u + T^{-1}_{\varepsilon(t)} A^* f, \quad T = A^* A. \\ u(0) & = u_0 \end{cases}$$

$$0 < \varepsilon(t) \searrow 0, \int^{\infty} \varepsilon ds = \infty.$$

$$u = u_0 e^{-t} + \int_0^t e^{-(t-s)} T^{-1}_{\varepsilon(s)} Ty \, ds$$

Lemma 1. $\lim_{t\to\infty} \int_0^t e^{-(t-s)} h(s) ds = h(\infty)$ (if $\exists h(\infty)$.)

Lemma 2. $\lim_{\varepsilon\to\infty} T_{\varepsilon}^{-1} Ty = y$ if $y \perp N(T) = N(A)$. Otherwise the limit is $y - P_{N(T)} y$.

Stopping rules 1.

Assume $\|f_\delta - f\| \leq \delta$. Then

$$\|u_\delta(t_\delta) - y\| \leq \|u_\delta(t_\delta) - u(t_\delta)\| + \|u(t_\delta) - y\|.$$

$$\lim_{t_\delta \to \infty} \|u(t_\delta) - y\| = 0$$

$$\|u_\delta(t_\delta) - u(t_\delta)\| \leq \|\int_0^{t_\delta} e^{-(t_\delta - s)} T^{-1}_{\varepsilon(s)} A^*(f_\delta - f)\|$$

$$\leq \frac{\delta}{2\sqrt{\varepsilon(t_\delta)}}$$

If $\lim_{\delta \to 0} \frac{\delta}{\sqrt{\varepsilon(t_\delta)}} = 0$ and $\lim_{\delta \to 0} t_\delta = \infty$,

then $\lim_{\delta \to 0} \|u_\delta(t_\delta) - y\| = 0$.

Stopping rules 2.

Theorem. (New discrepancy principle) Assume that A is a bounded linear operator in a Hilbert space H, equation $Au = f$ is solvable, y is
its minimal-norm solution, $||f_\delta - f|| \leq \delta$, and $||f_\delta|| > C\delta$, where $C > 1$ is a constant. Then equation $||Au_{\delta,\epsilon} - f_\delta|| = C\delta$ $(*)$ is solvable for ϵ for any fixed $\delta > 0$, where $u_{\delta,\epsilon}$ is any element satisfying inequality $F(u_{\delta,\epsilon}) \leq m + (C^2 - 1 - b)\delta^2$,
$F(u) := ||A(u) - f_\delta||^2 + \epsilon||u||^2$, $m = m(\delta, \epsilon) := inf_u F(u)$,
$b = const > 0$, and $C^2 > 1 + b$.
If $\epsilon = \epsilon(\delta)$ solves $(*)$, and $u_\delta := u_{\delta,\epsilon(\delta)}$, then $\lim_{\delta\to 0} \|u_\delta - y\| = 0$.

Nonlinear operator equations without monotonicity assumptions.

Theorem. If $\widetilde{A} := F'(y) \neq 0$, then for the problem

$$\begin{cases} \dot{u} & = -T^{-1}_{\varepsilon(t)}(A^*F + \varepsilon(u - \widetilde{u}_0)), \\ u(0) & = u_0, \end{cases}$$

conclusions $(*)$ hold, where $\widetilde{u}_0$ is suitably chosen.

Proof.

$$u - y = w, \|w\| = g,\ F(u) - F(y) = Aw + K,$$

$$\|K\| \leq \frac{M_2}{2} g^2,\ u - \widetilde{u}_0 = u - y + y - \widetilde{u}_0$$

$$\dot{w} = -T_\varepsilon^{-1}(A^*Aw + \varepsilon w + A^*K + \varepsilon(y - \widetilde{u}_0))$$

$$= -w - T_\varepsilon^{-1}A^*K - \varepsilon T_\varepsilon^{-1}(y - \widetilde{u}_0)$$

$$\dot{w} = -w - T_\varepsilon^{-1}A^*K - \varepsilon T_\varepsilon^{-1}\widetilde{T}v,\ \|v\| \ll 1;$$

$$\widetilde{T}v = y - \widetilde{u}_0 \text{ if } \widetilde{T} \neq 0.$$

$$g\dot{g} \leq -g^2 + \frac{c_0 g^3}{\sqrt{\varepsilon(t)}} + \varepsilon(T_\varepsilon^{-1} - \widetilde{T}_\varepsilon^{-1} + \widetilde{T}_\varepsilon^{-1})\widetilde{T}v.$$

$$\varepsilon\|\widetilde{T}_\varepsilon^{-1}\widetilde{T}v\| \leq \varepsilon\|v\|,$$

$$\varepsilon\|T_\varepsilon^{-1}(A^*A - \widetilde{A}^*A)\widetilde{T}_\varepsilon^{-1}\widetilde{T}\|\ \|v\|,$$

$$\leq 2M_2M_1g\|v\|;\ 2M_1M_2\|v\| = \frac{1}{2}.$$

Thus

$$\dot{g} \leq -\frac{1}{2}g + \frac{c_0 g^2}{\sqrt{\varepsilon(t)}} + \varepsilon\|v\|; \quad \mu = \frac{\lambda}{\sqrt{\varepsilon(t)}}, \quad \frac{\dot{\mu}}{\mu} = \frac{1}{2}\frac{|\dot{\varepsilon}|}{\varepsilon} \leq \frac{1}{4}.$$

1) $\frac{c_0}{\sqrt{\varepsilon(t)}} \leq \frac{\lambda}{2\sqrt{\varepsilon}}\,\frac{1}{4}; \quad \lambda = 8c_0.$

2) $\varepsilon(t)\|v\| \leq \frac{\sqrt{\varepsilon(t)}}{2\lambda}\,\frac{1}{4}; \qquad 8\lambda\|v\|\sqrt{\varepsilon(t)} \leq 1$

3) $g(0)\frac{\lambda}{\sqrt{\varepsilon(0)}} < 1$

If $\varepsilon(0) > 8g(0)c_0$, then condition 3) holds.
If $\|v\| < \frac{1}{8\lambda\sqrt{\varepsilon(0)}}$, then 2) holds.
Theorem is proved. $\square$

Getting rid of the inversion of the derivative operator.

$F(u) = 0 \quad (1) \qquad \|[F'(u)]^{-1}\| \leq m, \quad F(y) = 0.$

Theorem.

$$\begin{cases} \dot{u} & = -QF, \qquad\qquad (2) \\ \dot{Q} & = -TQ + A^*, \quad u(0) = u_0, \quad Q(0) = Q_0. \end{cases}$$

If u_0 and Q_0 are properly chosen then $(*)$ holds.

Proof. $T \geq \varepsilon > 0$, $\varepsilon = const$. Thus

$$\|Q(t)\| \leq \|Q_0\|e^{\varepsilon t} + e^{-\varepsilon t}\int_0^t e^{\varepsilon s} M_1 ds$$
$$\leq \|Q_0\| + \frac{M_1}{\varepsilon} := c_0.$$

$$u - y = w, \|w\| = g(t);\ \ F(u) - F(y) = \widetilde{A}w + K,$$

$$\widetilde{A} = F'(y), \|K\| \leq \frac{M_2}{2}g^2,$$

$$\dot{w} = -w + w - Q(F(u) - F(y)) = -w + \Lambda w - QK,$$
$$\Lambda = I - Q\widetilde{A}.$$

$$g\dot{g} \leq -g^2 + (\Lambda w, w) + c_1 g^3, \qquad c_1 = \frac{c_0 M_2}{2},$$

Lemma.

$|(\Lambda w, w)| \le q\|w\|^2, \ 0 < q < 1.$

$$\dot{g} \le -\gamma g + c_1 g^2, \ 0 < \gamma := 1 - q < 1.$$

Thus $g(t) \le c_2 e^{-\gamma t}, \ c_2 = \dfrac{g(0)}{1 - c_1 g(0)}$, if $c_1 g(0) < 1$.

Theorem is proved. $\square$

Proof of the Lemma.

$$\dot{\Lambda} = -\dot{Q}\widetilde{A} = TQ\widetilde{A} - A^*\widetilde{A} = -T\Lambda + A^*(A\widetilde{A}).$$

$$\|\Lambda\| \leq \|\Lambda_0\|e^{-\varepsilon t} + e^{-\varepsilon t}\int_0^t e^{-\varepsilon t} M_1 M_2 c_2 e^{-\gamma s} ds$$

$$\leq \|\Lambda_0\| + c_3\|u_0 - y\| < q < 1.$$

The Lemma is proved. □

(1), $$F(u) = 0$$

(2), $$u_{n+1} = u_n + h\Phi(u_n)$$

there exists y such that $F(y) = 0$, and

$$\|\Phi(u) - \Phi(v)\| \leq L_2\|u - v\|$$

Theorem. If

a) $(F'\Phi, F) \le -c_1\|F\|^2$,

b) $\|\Phi\| \le c_2\|F\|$, and

c) $r \le R$,

where $r = \frac{c_2}{c_1}\|F_0\|$ and $F_0 = F(u_0)$, then
$\|u_n - y\| \le re^{-chn}$, $\|F(u_n)\| \le \|F_0\|e^{-chn}$,
where $0 < c < c_1$.

Proof.

Let $\begin{cases} \dot{w}_{n+1}(t) & = \Phi(w_{n+1}) \\ w_{n+1}(t_n) & = u_n, \;\; t_n = hn. \end{cases}$

Then

$$\begin{aligned}
\|w_{n+1}(t) - y\| &\leq \frac{c_2}{c_1} \|F_n\| e^{-c_1(t-t_n)} \\
&\leq re^{-chn - c_1(t-t_n)}, \quad t > hn, \\
\|u_{n+1} - y\| &\leq \|u_{n+1} - w_{n+1}\| + \|w_{n+1} - y\|, \\
\|u_{n+1} - w_{n+1}\| &\leq \int_{t_n}^{t_n+h} \|\Phi(u_n) - \Phi(w_{n+1}(s))\| ds \\
&\leq L_1 \int_{t_n}^{t_n+h} \|u_n - w_{n+1}(s)\| ds \\
&\leq L_1 h \int_{t_n}^{t_n+h} \|\Phi(w_{n+1}(s))\| ds \\
&\leq L_1 h c_2 \int_{t_n}^{t_n+1} \|F(w_{n+1}(s))\| ds \\
&\leq L_1 h c_2 \|F(u_n))\| \leq L_1 c_2 h^2 \|F_0\| e^{-chn} \\
&= L_1 c_1 h^2 re^{-chn}.
\end{aligned}$$

Thus $\|u_{n+1} - y\| \le re^{-chn}(e^{-c_1 h} + L_1 c_1 h^2) \le re^{-ch(n+1)}$ provided that
$c < c_1$ and h is such that $e^{-c_1 h} + L_1 c_1 h^2 < e^{ch}$.

$$\begin{aligned}
\|F(u_{n+1})\| &\le \|F(u_{n+1}) - F(w_{n+1}(t_{n+1}))\| \\
&\quad + \|F(w_{n+1})(t_{n+1}))\| \\
\|F(w_{n+1}(t_{n+1}))\| &\le \|F(u_n)\|e^{-c_1 h} \le \|F_0\|e^{chn - c_1 h} \\
\|F(u_{n+1}) - F(w_{n+1}(t_{n+1}))\| &\le M_1 L_1 c_1 h^2 re^{-chn} \\
&= \|F_0\|e^{-chn} M_1 L_1 c_2 h^2.
\end{aligned}$$

Thus

$$\begin{aligned}
\|F(u_{n+1})\| &\le \|F_0\|e^{-chn}(e^{c_1 h} + M_1 L_1 c_2 h^2) \\
&\le \|F_0\|e^{-ch(n+1)}.
\end{aligned}$$

Theorem is proved. □

Lemma. *Let $f(t, w)$, $g(t, u)$ be continuous in the region $[0, T) \times D$ $(D \subset R,\ T \leq \infty)$ and $f(t, w) \leq g(t, u)$ if $w \leq u$, $t \in (0, T)$, $w, u \in D$. Assume that $g(t, u)$ is such that the Cauchy problem*

$$\dot{u} = g(t, u), \quad u(0) = u_0, \quad u_0 \in D,$$

has a unique solution. If

$$\dot{w} \leq f(t, w), \quad w(0) = w_0 \leq u_0, \quad w_0 \in D,$$

then $u(t) \geq w(t)$ for all t for which $u(t)$ and $w(t)$ are defined.

Let g be the function from Theorem 15. Define the new function w by the formula:

$$w(t) := g(t)e^{\int_{t_0}^{t} \gamma(s)ds}.$$

Then

$$\dot{w}(t) \leq a(t)w^2(t) + b(t), \quad w(t_0) = g(t_0),$$

where

$$a(t) = \sigma(t)e^{-\int_{t_0}^{t} \gamma(s)ds}, \quad b(t) = \beta(t)e^{\int_{t_0}^{t} \gamma(s)ds}.$$

Consider the equation:

$$\dot{u}(t) = \frac{\dot{f}(t)}{G(t)} u^2(t) - \frac{\dot{G}(t)}{f(t)}. \qquad (1)$$

One can check by a direct calculation that the the solution to this equation is given by the following formula

$$u(t) = -\frac{G(t)}{f(t)} + \left[f^2(t) \left(C - \int_{t_0}^{t} \frac{\dot{f}(s)}{G(s) f^2(s)} ds \right) \right]^{-1}, \qquad (2)$$

where C is a constant. If $u(0) = u_0$, then $C = \frac{1}{u_0 f^2(0) + G(0) f(0)}$. Define f and G as follows:

$$f(t) := \mu^{\frac{1}{2}}(t) e^{-\frac{1}{2}\int_{t_0}^{t} \gamma(s) ds}, \quad G(t) := -\mu^{-\frac{1}{2}}(t) e^{\frac{1}{2}\int_{t_0}^{t} \gamma(s) ds},$$

and consider the Cauchy problem for equation (1) with the initial condition $u(t_0) = g(t_0)$.

Then C in (2) can be calculated:

$$C = \frac{1}{\mu(t_0)g(t_0) - 1}.$$

One gets

$$a(t) \leq \frac{\dot{f}(t)}{G(t)}, \quad b(t) \leq -\frac{\dot{G}(t)}{f(t)}.$$

Since $fG = -1$ one has:

$$\int_{t_0}^{t} \frac{\dot{f}(s)}{G(s)f^2(s)} ds = -\int_{t_0}^{t} \frac{\dot{f}(s)}{f(s)} ds = \frac{1}{2}\int_{t_0}^{t} \left(\gamma(s) - \frac{\dot{\mu}(s)}{\mu(s)}\right) ds.$$

Thus

$$u(t) = \frac{e^{\int_{t_0}^{t} \gamma(s)ds}}{\mu(t)} [1 - \nu]$$

where

$$\nu = \left(\frac{1}{1 - \mu(t_0)g(t_0)} + \frac{1}{2} \int_{t_0}^{t} \left(\gamma(s) - \frac{\dot{\mu}(s)}{\mu(s)} \right) ds \right)^{-1}. \qquad (3)$$

It follows from conditions of Theorem 15 that the solution to problem (1) exists for all $t \in [0, \infty)$ and the following inequality holds with $\nu(t) > 0$:

$$1 > 1 - \nu(t) \geq \mu(t_0)g(t_0).$$

From Lemma and from formula (3) one gets:

$$g(t)e^{\int_{t_0}^{t}\gamma(s)ds} := w(t) \le u(t) = \frac{1-\nu(t)}{\mu(t)}e^{\int_{t_0}^{t}\gamma(s)ds}$$
$$< \frac{1}{\mu(t)}e^{\int_{t_0}^{t}\gamma(s)ds}.$$

Thus, Theorem 15 is proved. □

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