On a traffic problem

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Outline:

- **Follow-the-Leader** model
- **Bifurcation analysis**
  Gasser, Sirito, Werner: Physica D 197 (2004)
- **How to overtake?**
  in the Follow-the-Leader model
- **On pattern formation**
  long-time behaviour

follow-the-leader model:

Problem 1

$$\begin{aligned} x_i' &= y_i\,, \\ y_i' &= V(x_{i+1} - x_i) - y_i\,, \quad x_{N+1} = x_1 + L \end{aligned}$$

$i = 1, \ldots, N$

L ... parameter (length of the roundabout)
N ... # of cars

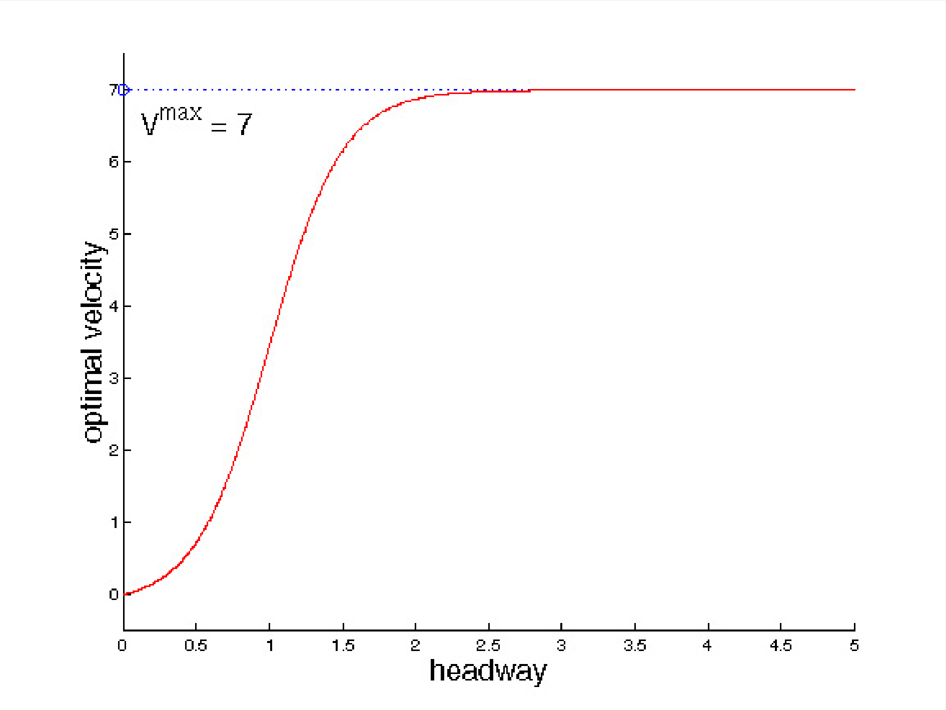
$r \mapsto V(r)$ ... *optimal velocity function*
e.g.

$$V(r) = V^{max} \frac{\tanh\left(a(r-1)\right) + \tanh(a)}{1 + \tanh(a)}$$

Definition 1 $h_i \equiv x_{i+1} - x_i$, *i = 1,. . . ,N*
... the i-th headway

optimal velocity function example:

$V^{max} = 7$, $a = 2$



Model interpretation

Problem 1 $\equiv$ a system ODE's

Solving *initial value problem* yields

$$x_i(t)\,, \quad y_i(t)$$

... the position/velocity of the i-th car at time t

State space: $\mathbb{R}^N \times \mathbb{R}^N$, $x \in \mathbb{R}^N$, $y \in \mathbb{R}^N$

Initial condition: $x^0 \in \mathbb{R}^N$, $y^0 \in \mathbb{R}^N$
$s \in \mathbb{R}$ ... an arbitrary phase shift,

$$s \le x_1^0 \le x_2^0 \le \cdots \le x_{N-1}^0 \le x_N^0 \le L + s$$

Visualization: Transforming $x(t) = (x_1(t), \dots, x_N(t))$ to polar coordinates:

$$x_i(t) \longrightarrow \frac{L}{2\pi}\, e^{\frac{2\pi}{L} x_i(t)}\,, \quad i = 1, \dots, N$$

Model interpretation: Example

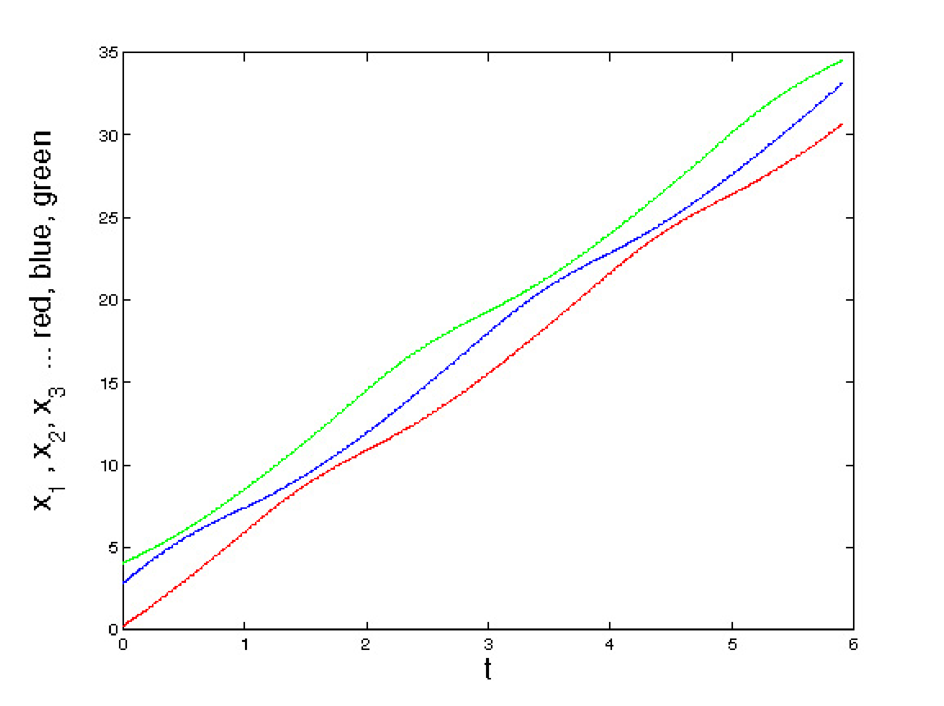
- parameter setting:

$L = 4.9383$, $N = 3$, $V^{max} = 7$, $a = 2$

- the initial condition:

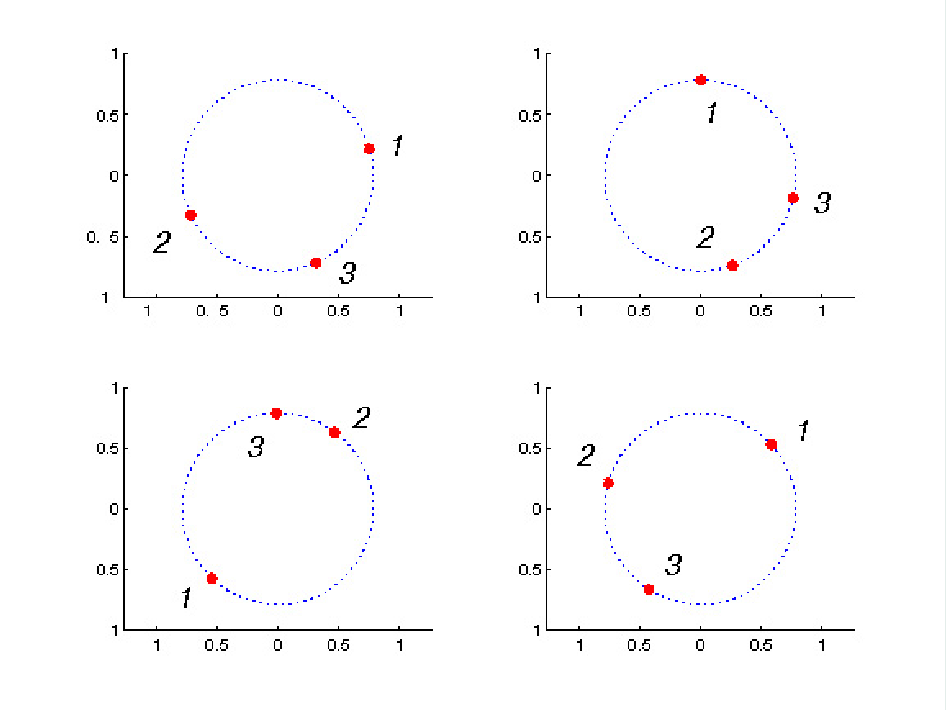
$x^0 = [0.2243; 2.8069; 4.0269]$

$x^0 \longmapsto x(t)$, $t \geq 0$



Model interpretation ... continued

Snapshots in polar coordinates

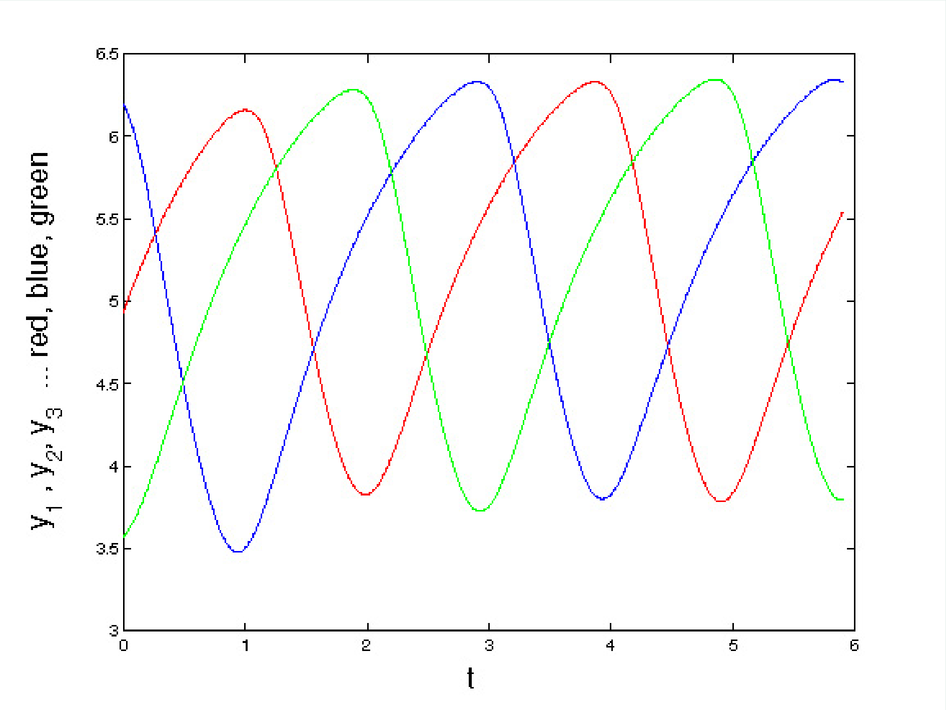


at time $t = 0, 0.1960, 0.5346, 0.9360$

Model interpretation ... continued

- $y^0 = [4.9341; 6.1946; 3.5676]$

$y^0 \longmapsto y(t),\ t \geq 0$



Quasi steady state

- parameter setting:

$L = 4.9383$, $N = 3$, $V^{max} = 7$, $a = 2$

- initial condition:

$x^0 \equiv [s; s + L/N; s + 2L/N] = [0.9877; 2.6338; 4.2799]$
s ... an arbitrary *phase shift*, e.g. $s = 0.9877$
$y^0 \equiv [c; c; c] = [6.5; 6.5; 6.5]$, where $c \equiv V(L/N)$

Evolution of the particular initial condition:

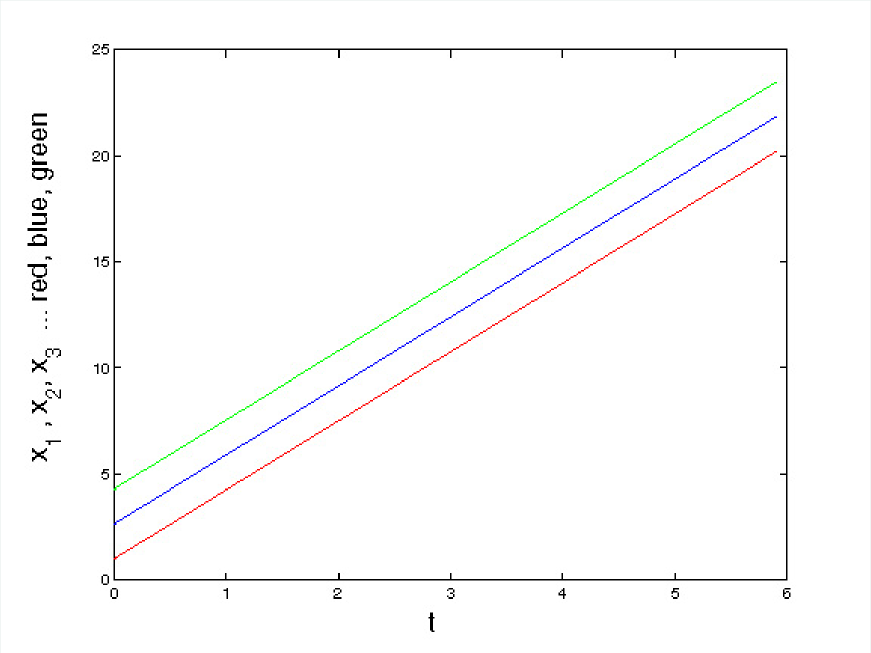
$$x^0 \longmapsto x(t) \equiv [s + ct; s + L/N + ct; s + 2L/N + ct]$$

$$y^0 \longmapsto y(t) \equiv [c; c; c]$$

as $t \geq 0$

Quasi steady state … continued

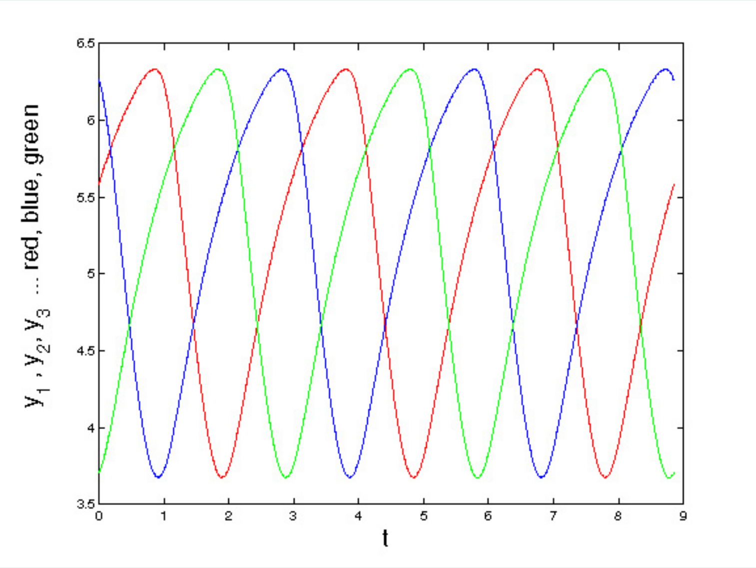
$$x^0 \longmapsto x(t) \equiv [s+ct; s+L/N+ct; s+2L/N+ct]$$
$$x^0 \equiv [s; s+L/N; s+2L/N] = [0.9877; 2.6338; 4.2799]$$



up to a phase shift s

Periodic solutions: cycles

Example: $L = 4.9383$, $N = 3$, $V^{max} = 7$, $a = 2$



initial condition: $[x^0, y^0] \in \mathbb{R}^6$

$[x^0, y^0] \longmapsto [x(t), y(t)]$, $t \geq 0$

... T-**periodic**, $T = 2.9504$

$y^0 = [5.5810; 6.2594; 3.7049]$

Formulation in headway and velocity components

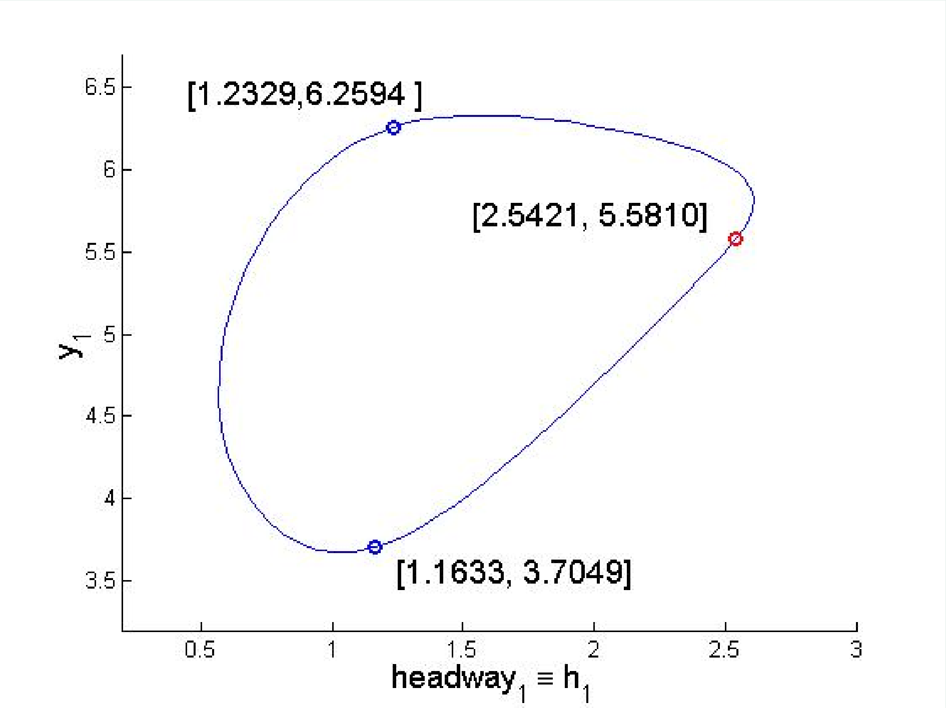
Problem 2

$$\begin{aligned} h_i' &= y_{i+1} - y_i \,, \\ y_i' &= V(h_i) - y_i \,, \\ y_N' &= V(L - \sum_{k=1}^{N-1} h_k) - y_N \end{aligned}$$

$i = 1, \ldots, N-1$

L ... parameter (length of the roundabout)
N ... # of cars

Cycles ... continued

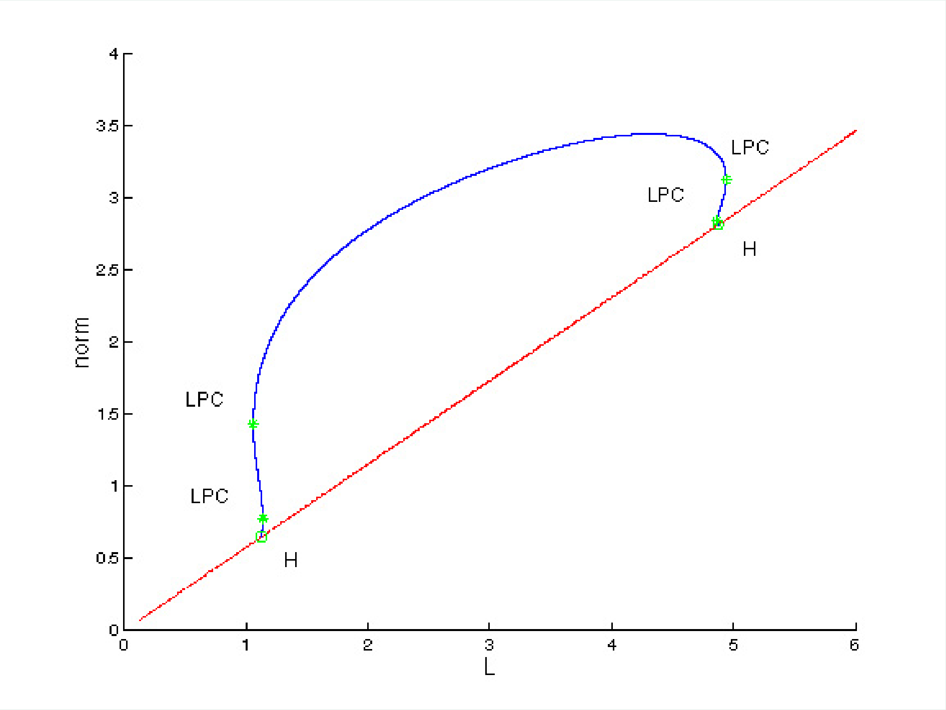


identical copy in $[h_2, y_2]$ and $[h_3, y_3]$ plane

[2.5421, 5.5810] and [1.2329, 6.2594] and [1.1633, 1.1634]
... initial points

Bifurcation analysis:

Gasser, Sirito, Werner: Physica D 197 (2004)



branches: red/blue

$L \mapsto [L/N, L/N, L/N, V(L/N), V(L/N), V(L/N)]$

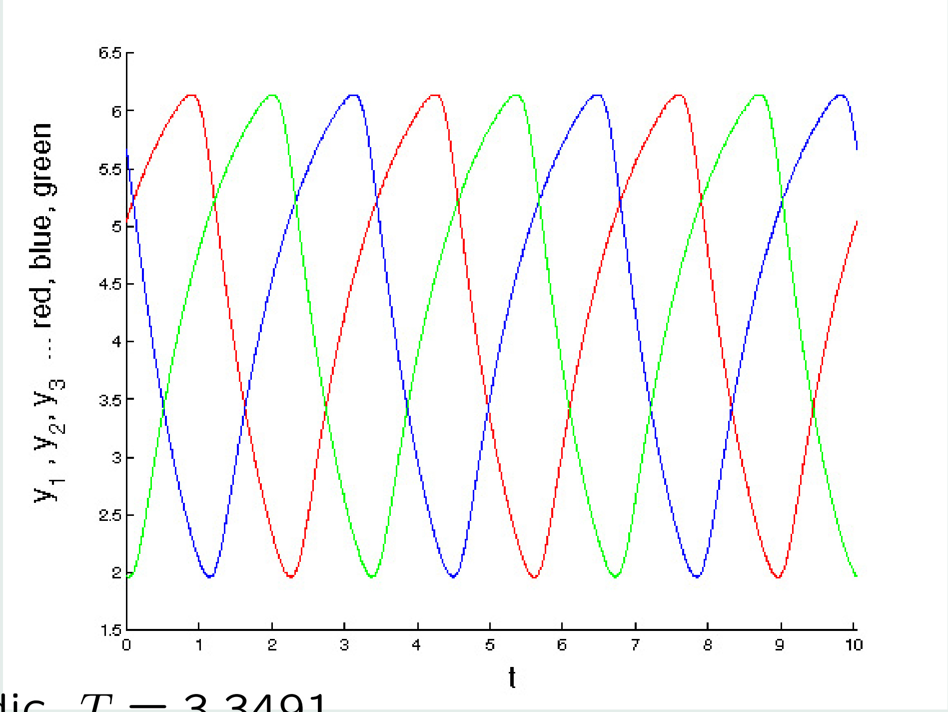
$\in \mathbb{R}^{2N}$ ... steady states

$L \mapsto C^1(S_L, \mathbb{R}^{2N})$ ... cycles

Non physical cycle: Example

• parameter setting:
$L = 4.5628$, $N = 3$, $V^{max} = 7$, $a = 2$

$[x^0, y^0] \longmapsto [x(t), y(t)]$, $t \geq 0$
$y^0 = [5.0389; 5.6683; 1.9686]$



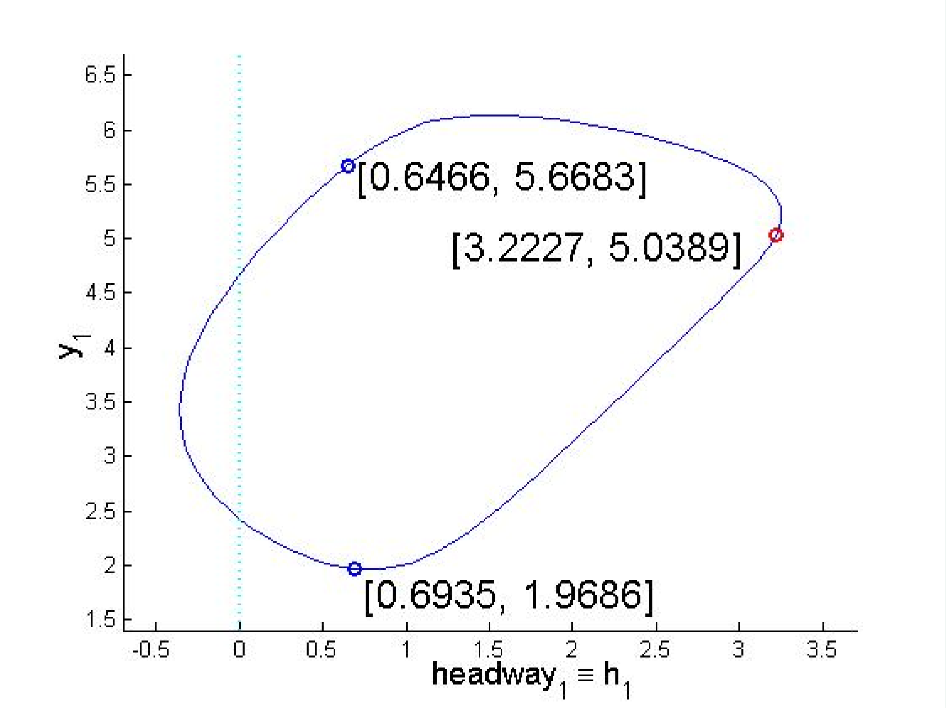
... T-periodic, $T = 3.3491$

Cycle:

$$[h^0, y^0] \longmapsto [h(t), y(t)],\ t \geq 0$$

$h^0 = [3.2227; 0.6466; 0.6935]$

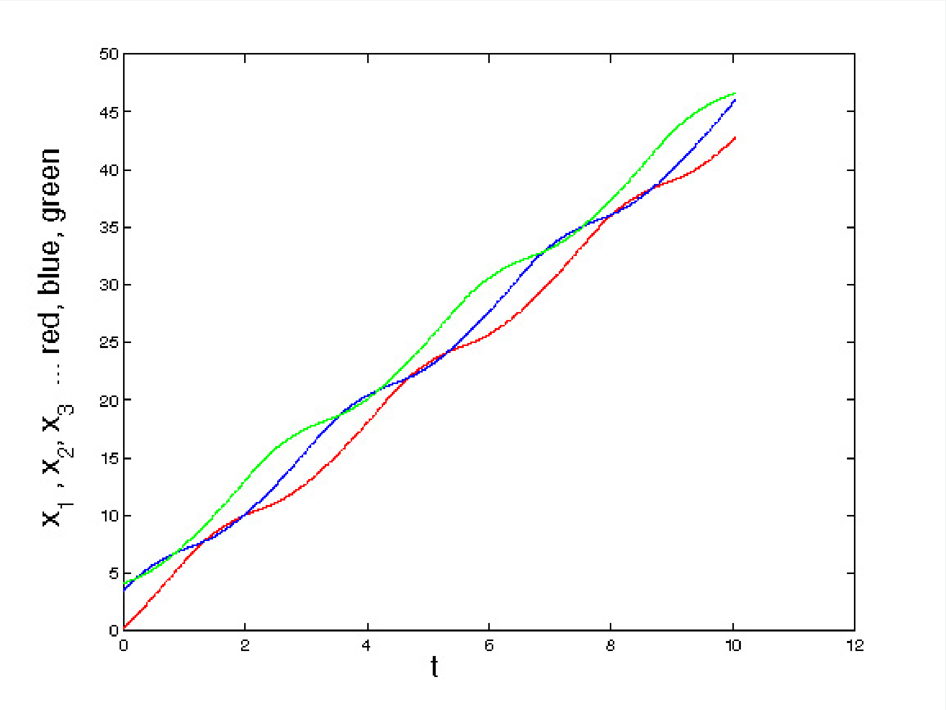
$y^0 = [5.0389; 5.6683; 1.9686]$



+ identical copies in $[h_2, y_2]$ and $[h_3, y_3]$ plane

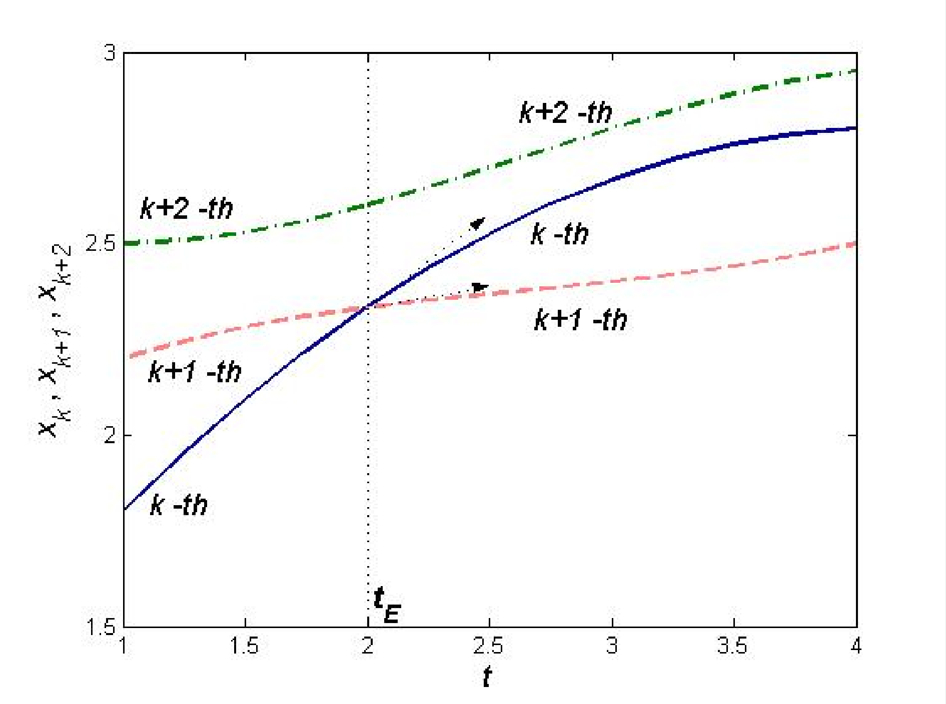
Consequences:

$$[x^0, y^0] \longmapsto [x(t), y(t)],\ t \geq 0$$
$$x^0 = [0.2373; 3.4600; 4.1065]$$



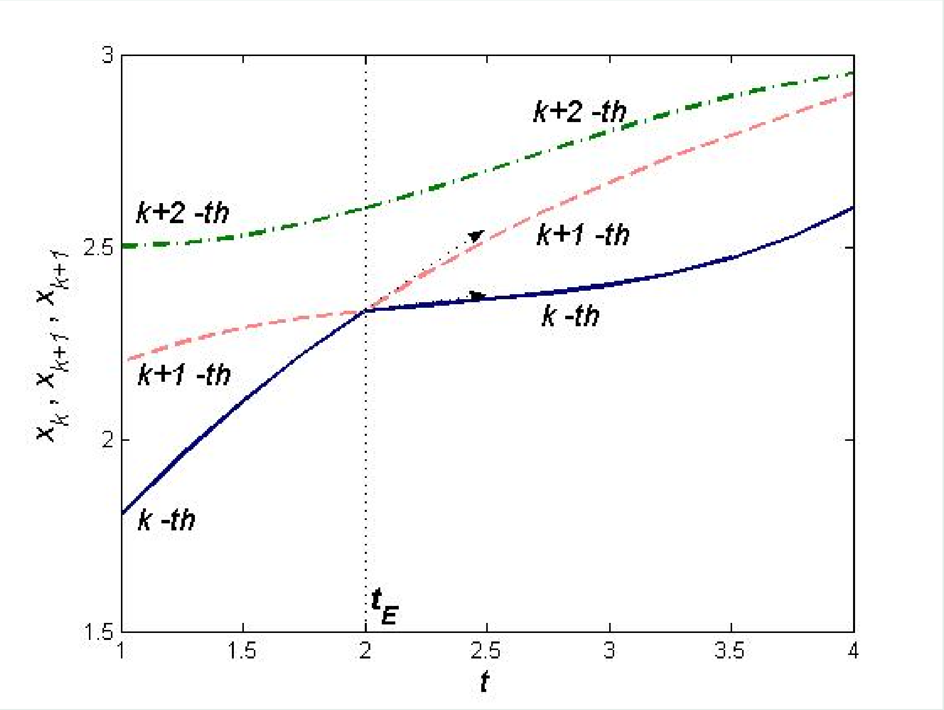
Overtaking Model

via a piecewise smooth dynamical system



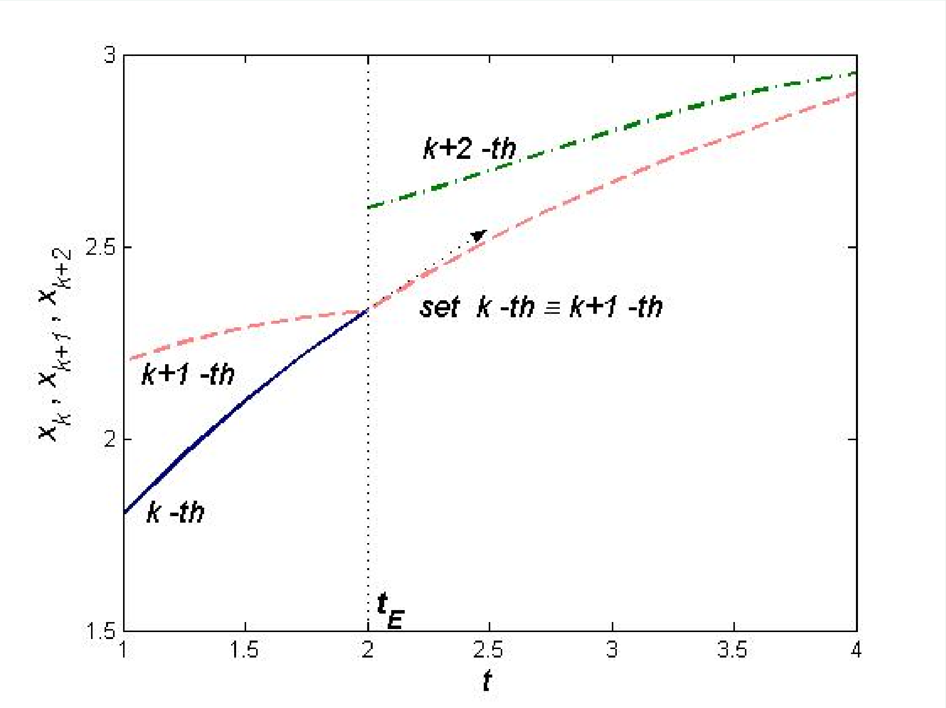
A sketch of three trajectories of the flow

Overtaking Model ... continued



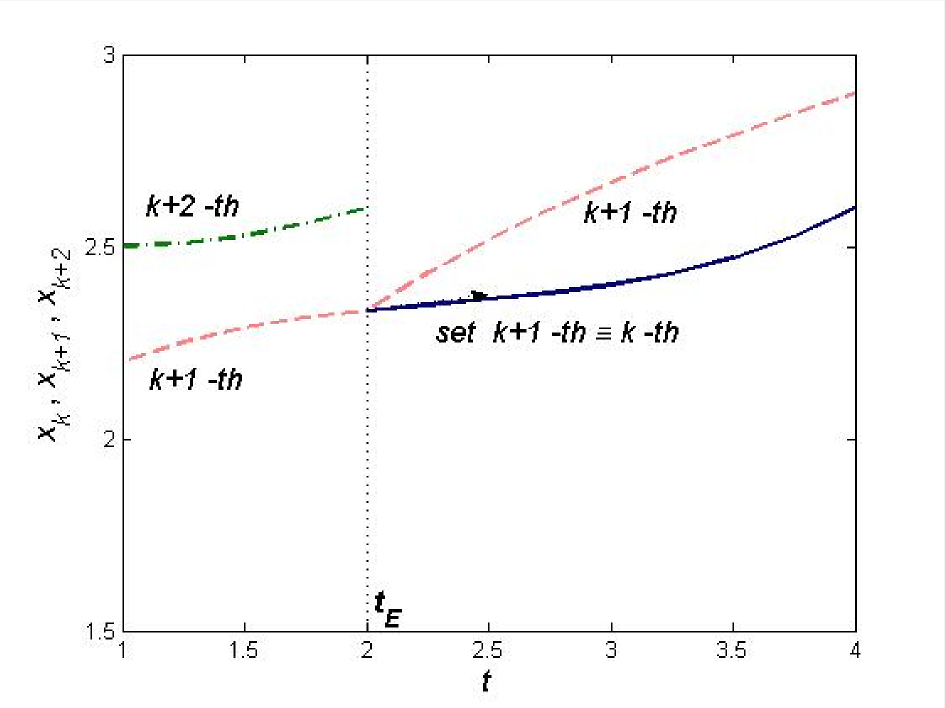
The trajectories after imposing the swap of the initial condition at $t = t_E$.

Overtaking Model ... continued



Trajectory of the k-th car.
The headway is discontinuous at t_E.

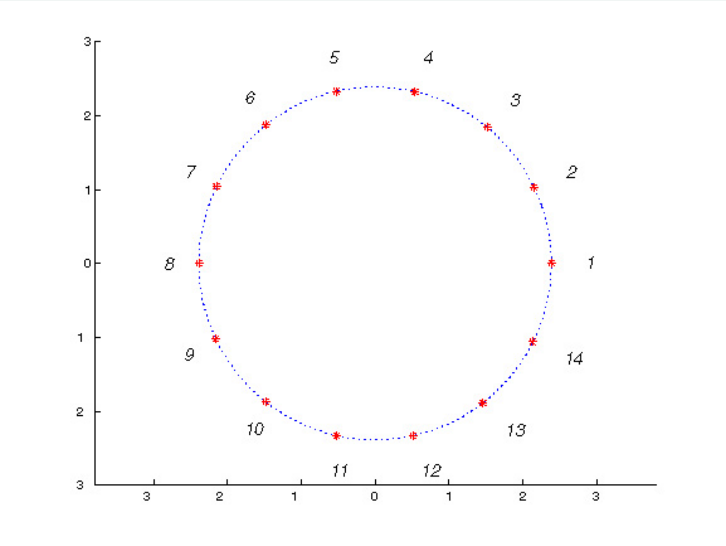
Overtaking Model ... continued



Trajectory of the $k + 1$-th car.
The headway is discontinuous at t_E.

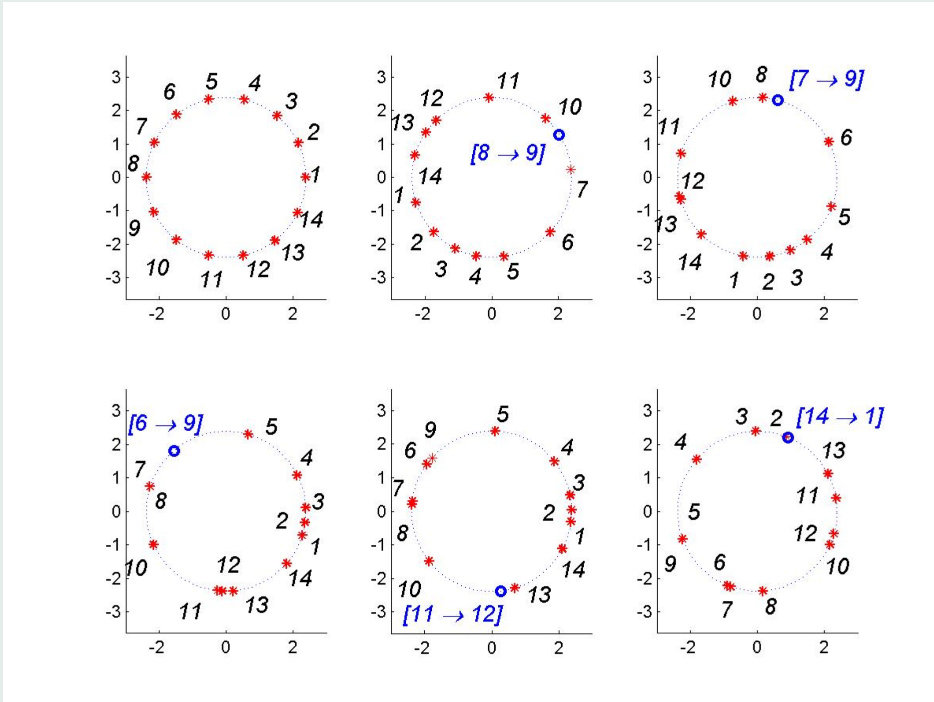
Numerical tests

• parameter setting:
$L = 15$, $N = 14$, $V^{max} = 37$, $a = 2$
• initial condition:
$[x^0, y^0] \in \mathbb{R}^{14} \times \mathbb{R}^{14}$
close to the **unstable** steady state



Computing the model evolution
$[x^0, y^0] \longmapsto [x(t), y(t)]$, $t \in [0, 3]$

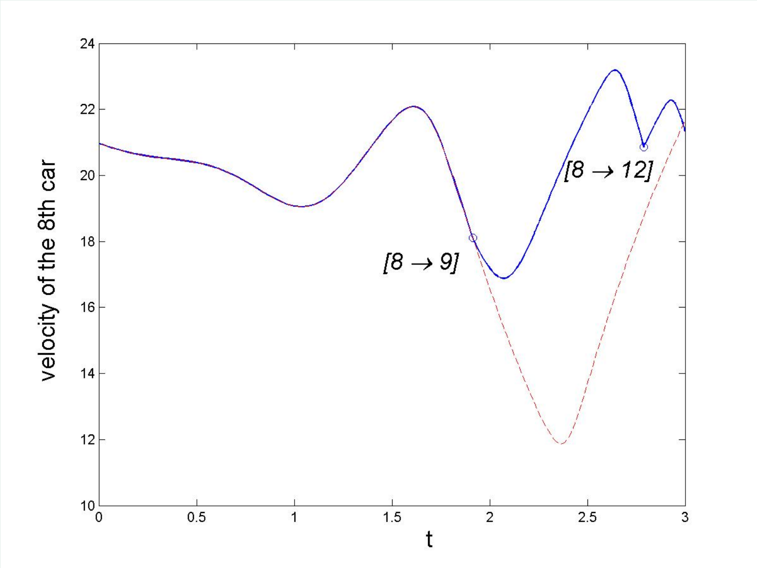
Numerical tests ... continued



Events:
car No 8 overtakes car No 9 at time 1.9136
car No 7 overtakes car No 9 at time 2.0426
car No 6 overtakes car No 9 at time 2.2294
car No 11 overtakes car No 12 at time 2.4605
car No 14 overtakes car No 1 at time 2.2546
... etc. # Events = 18.

Numerical tests ... continued

Velocity of a selected car: car No 8
$[x^0, y^0] \longmapsto y_8(t),\ t \in [0,3]$
blue ... via **Overtaking Model**
red ... via the original "smooth" model (1)



Car No 8 overtakes car No 9, No 12

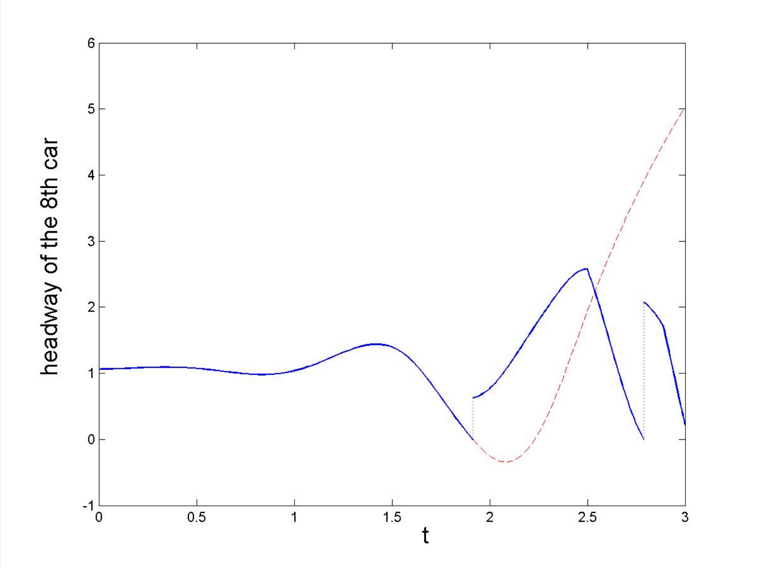
Numerical tests ... continued

Headway of a selected car: car No 8

$[x^0, y^0] \longmapsto h_8(t)$, $t \in [0, 3]$

blue ... via **Overtaking Model**

red ... via the original "smooth" model (1)



Asymptotic properties
of the Overtaking Model:

$[x^0, y^0] \longmapsto [x(t), y(t)]$ as $t \to \infty$

... $\exists$ **invariant objects**
... experimental evidence (only)

formally,

$$[x(t), y(t)] = \Pi(t, [x^0, y^0])\,, \quad t \geq 0$$

? (semi) flow on $\mathbb{R}^N \times \mathbb{R}^N$

... ω-**limit sets**
... **dynamical simulation**

Example:

$L = 3.6998$, $N = 3$, $V^{max} = 7$, $a = 2$

$[x^0, y^0] \longmapsto [x(t), y(t)]$ as $t \to \infty$

via the Overtaking Model

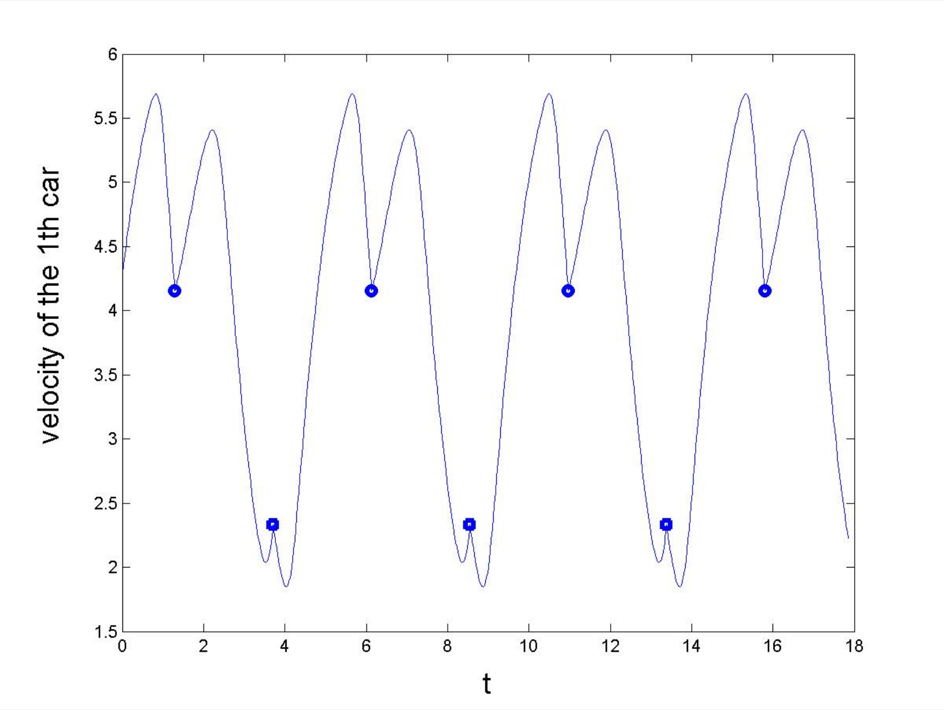
$x^0 = [0.0314; 3.3573; 3.6628]$,

$y^0 = [4.6465; 5.1303; 1.4956]$

Transition time: $3.5714 * 50$,

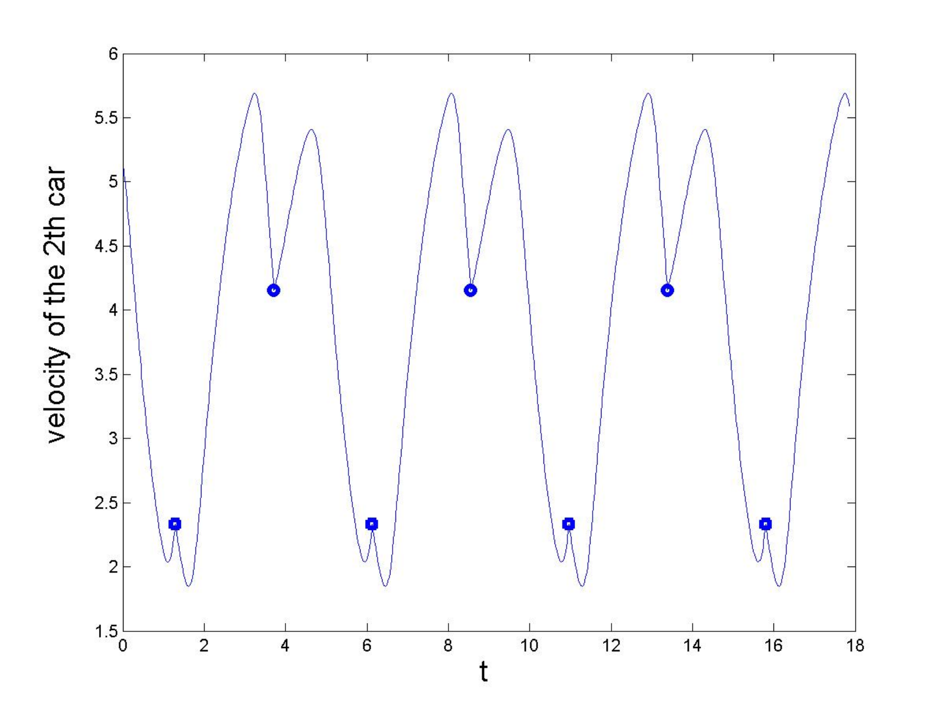
3.5714 ... the period of the "smooth" cycle due to model (1)

Velocity $y_1(t)$ of the car No 1 vs time t



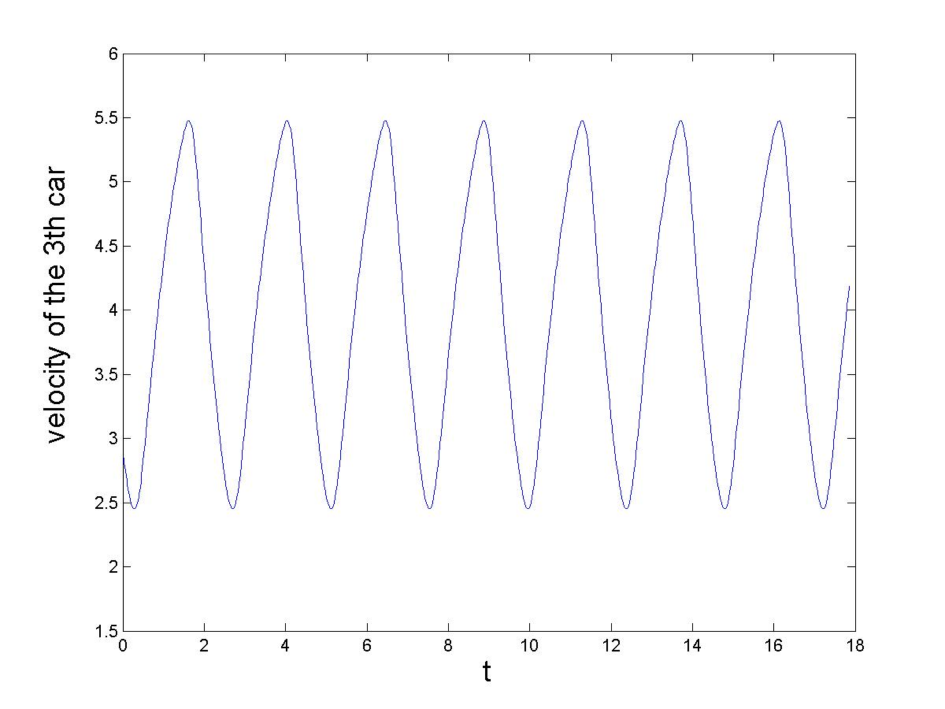
... the waveform with period $T = 4.8525$

Velocity $y_2(t)$ of the car No 2 vs time t



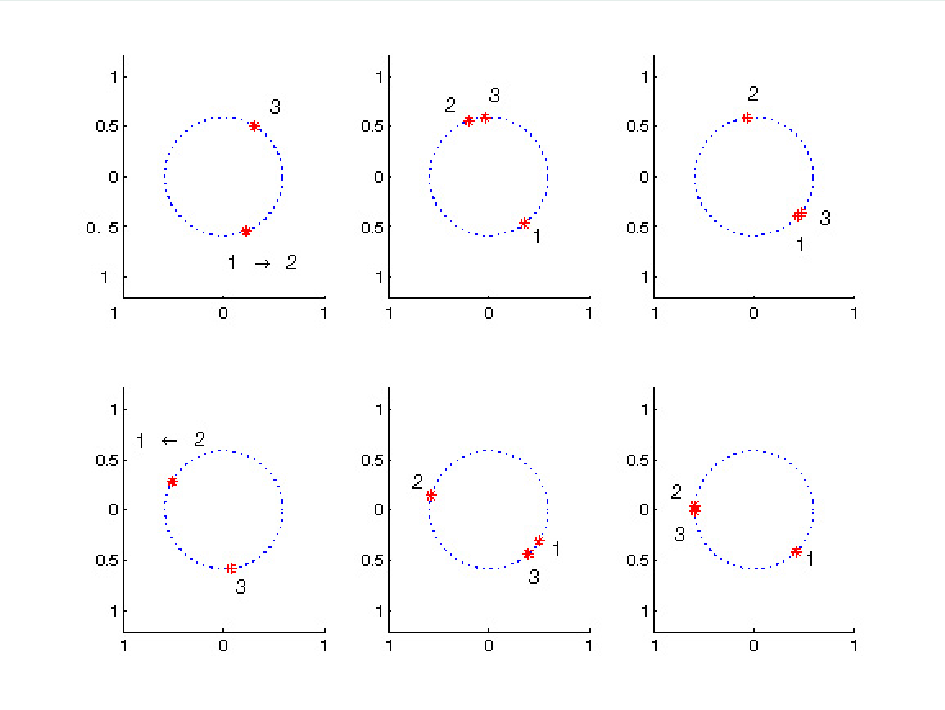
... the same T-periodic waveform,
$T/2$ out of phase.

Velocity $y_3(t)$ of the car No 3 vs time t



... the waveform with period $T/2$ **!**
No 3 orbits **without any interference**
with No 1 and No 2.

Dynamical simulation over one period:



Snapshots at the time
$t = 1.2951$, $t + T/6$, $t + 2T/6$
$t + 3T/6$, $t + 4T/6$, $t + 5T/6$.

At time t, car No 1 overtakes car No 2,
At time $t + T/2$, car No 2 overtakes car No 1

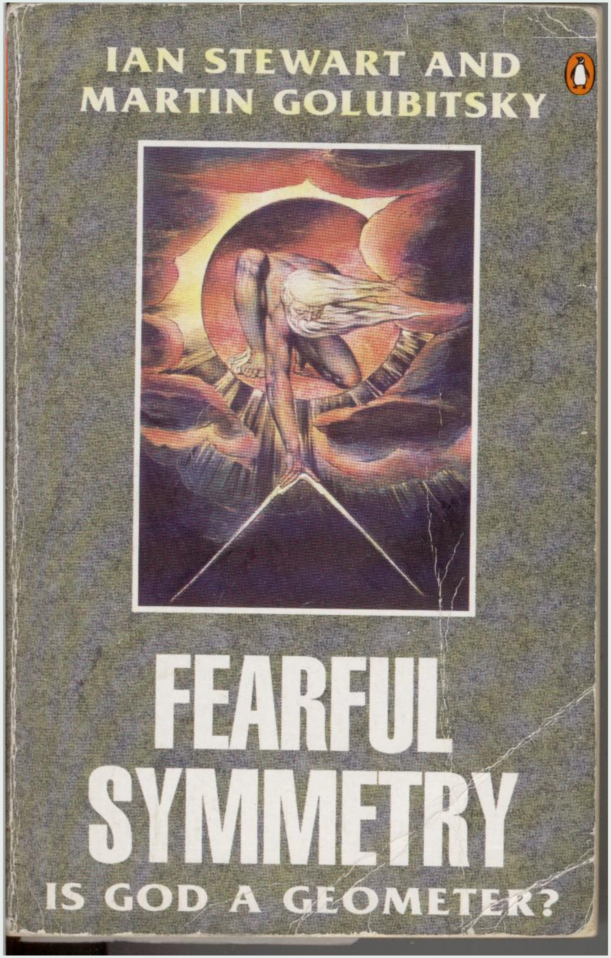
Observation: Oscillatory Patterns

attributes:

- periodicity ... in time
- symmetry ... in space

ad: **Follow-the-Leader** model:
... *rotating wave*
i.e., three **identical** wave forms, $2T/3$ out of phase

ad: **Overtaking Model**:
... 5 different oscillatory patterns


IAN STEWART AND
MARTIN GOLUBITSKY
FEARFUL
SYMMETRY
IS GOD A GEOMETER?

Investigation of animal gait
from *group theoretic* point if view

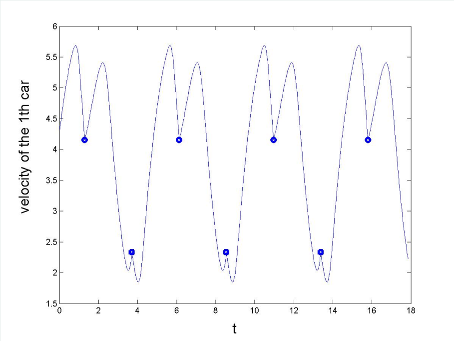
walk, trot, pace(rack), canter, transverse gallop,
rotary gallop, bound, pronk, ...

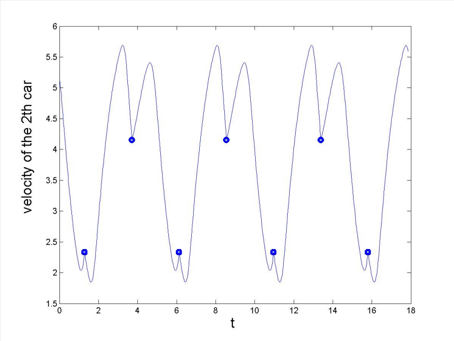
analogy: Hopf bifurcation in a ring
of coupled oscillators with a $\mathbf{D_N}$-symmetry

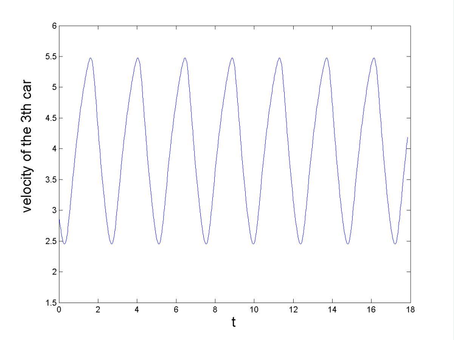
a three-legged dog ? ... prediction:

- **discrete rotating wave**
- reflectionally symmetric oscillations
- **phase-shifted reflectionally symmetric oscillations**

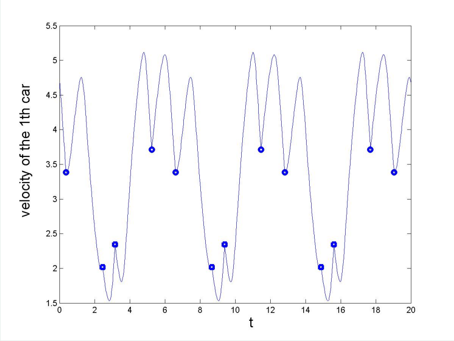
"The three-legged dog":

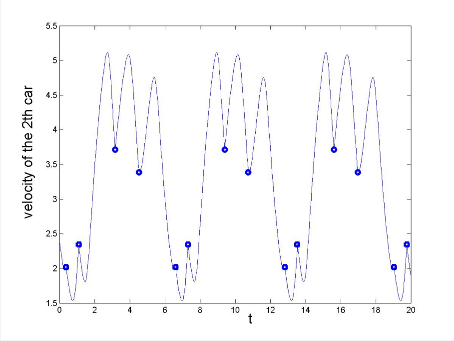


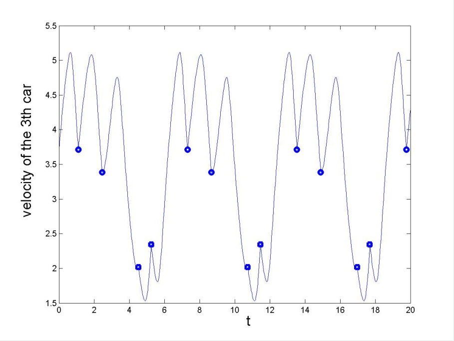




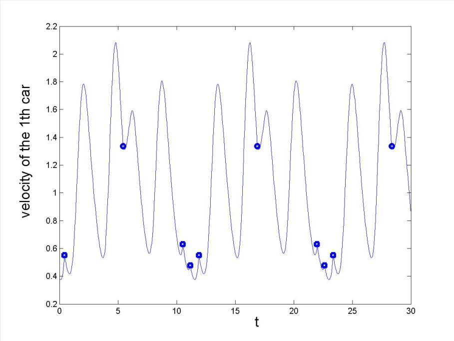
Phase-shifted reflectionally symmetric oscillations

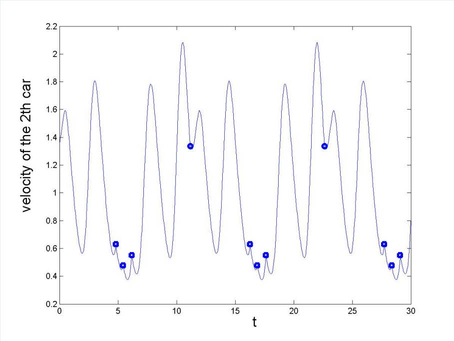


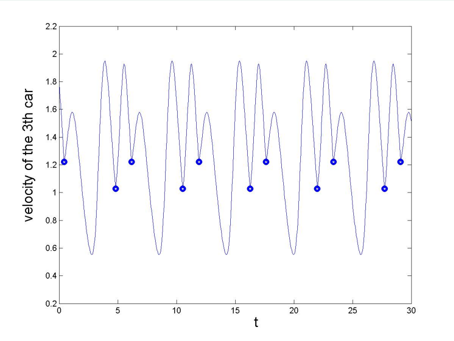




Rotating wave







Phase-shifted reflectionally symmetric oscillations
conjugate

How to recognize a pattern?

symbolic dynamical notions

event map ... ad: rotating wave example

$$G_E = \{[1 \to 2]\,, [3 \to 2]\,, [3 \to 1]\,, [2 \to 1]\,, [2 \to 3]\,, [1 \to 3]\,, \text{etc.}\}\ .$$

$\Longrightarrow$ overtaking pattern

event matrix

$$M_E = \left[\begin{array}{ccccccccccccc} \overline{2} & \emptyset & \underline{3} & \underline{2} & \emptyset & \overline{3} & \overline{2} & \emptyset & \underline{3} & \underline{2} & \emptyset & \overline{3} & \ldots \\ \underline{1} & \underline{3} & \emptyset & \overline{1} & \overline{3} & \emptyset & \underline{1} & \underline{3} & \emptyset & \overline{1} & \overline{3} & \emptyset & \ldots \\ \emptyset & \overline{2} & \overline{1} & \emptyset & \underline{2} & \underline{1} & \emptyset & \overline{2} & \overline{1} & \emptyset & \underline{2} & \underline{1} & \ldots \end{array}\right]$$

$$M_E = \left[\begin{array}{rrrrrrrrrrrrr} 2 & 0 & -3 & -2 & 0 & 3 & 2 & 0 & -3 & -2 & 0 & 3 & \ldots \\ -1 & -3 & 0 & 1 & 3 & 0 & -1 & -3 & 0 & 1 & 3 & 0 & \ldots \\ 0 & 2 & 1 & 0 & -2 & -1 & 0 & 2 & 1 & 0 & -2 & -1 & \ldots \end{array}\right]$$

state space: the set of all event maps

$$\mathbf{D_3} = \{\mathbf{Id}\,,\, \mathbf{Flip}\,,\, \mathbf{Rot}\,,\, \mathbf{Rot} \circ \mathbf{Rot}\,,\, \mathbf{Flip} \circ \mathbf{Rot}\,,\, \mathbf{Flip} \circ \mathbf{Rot} \circ \mathbf{Rot}\}$$

... a group of symmetries of an equilateral triangle

examples of actions:

$$\begin{aligned} G_E &= \{[1 \to 2]\,,[3 \to 2]\,,[3 \to 1]\,,[2 \to 1]\,,[2 \to 3]\,,[1 \to 3]\,,\text{etc.}\} \\ \mathbf{Rot}(G_E) &= \{[2 \to 3]\,,[1 \to 3]\,,[1 \to 2]\,,[3 \to 2]\,,[3 \to 1]\,,[2 \to 1]\,,\text{etc.}\} \\ \mathbf{Flip}(G_E) &= \{[2 \to 1]\,,[3 \to 1]\,,[3 \to 2]\,,[1 \to 2]\,,[1 \to 3]\,,[2 \to 3]\,,\text{etc.}\} \end{aligned}$$

event period of G_E: $p_E = 6$

temporal symmetries: ... **event shift**

$$\begin{aligned} G_E &= \{[1 \to 2]\,,[3 \to 2]\,,[3 \to 1]\,,[2 \to 1]\,,[2 \to 3]\,,[1 \to 3]\,,\text{etc.}\} \\ \mathbf{S}(G_E, 1) &= \{[3 \to 2]\,,[3 \to 1]\,,[2 \to 1]\,,[2 \to 3]\,,[1 \to 3]\,,[1 \to 2]\,,\text{etc.}\} \\ \mathbf{S}(G_E, 2p_E/3) &= \{[2 \to 3]\,,[1 \to 3]\,,[1 \to 2]\,,[3 \to 2]\,,[3 \to 1]\,,[2 \to 1]\,,\text{etc.}\} \end{aligned}$$

The Spatial-Temporal Symmetries

of a periodic solution

Idea: the spatial action of $\gamma \in \mathbf{D_3}$ on the state space may be exactly compensated by a proper event shift

Let p_E be divisible by 3:

$$\begin{aligned} \mathbf{Rot}(G_E) &= \mathbf{S}(G_E, 2p_E/3) \\ \mathbf{Rot}(G_E) &= \mathbf{S}(G_E, p_E/3) \end{aligned}$$

... rotating wave

Let p_E be divisible by 2:

$$\begin{aligned} \mathbf{Flip}(G_E) &= \mathbf{S}(G_E, p_E/2) \\ \mathbf{Flip} \circ \mathbf{Rot}(G_E) &= \mathbf{S}(G_E, p_E/2) \\ \mathbf{Flip} \circ \mathbf{Rot} \circ \mathbf{Rot}(G_E) &= \mathbf{S}(G_E, p_E/2) \end{aligned}$$

... phase-shifted reflectionally symmetric oscillations

Conclusions

- Considered: **Follow-the-Leader** model of a circular road.
  $\Longrightarrow$ overtaking prohibited

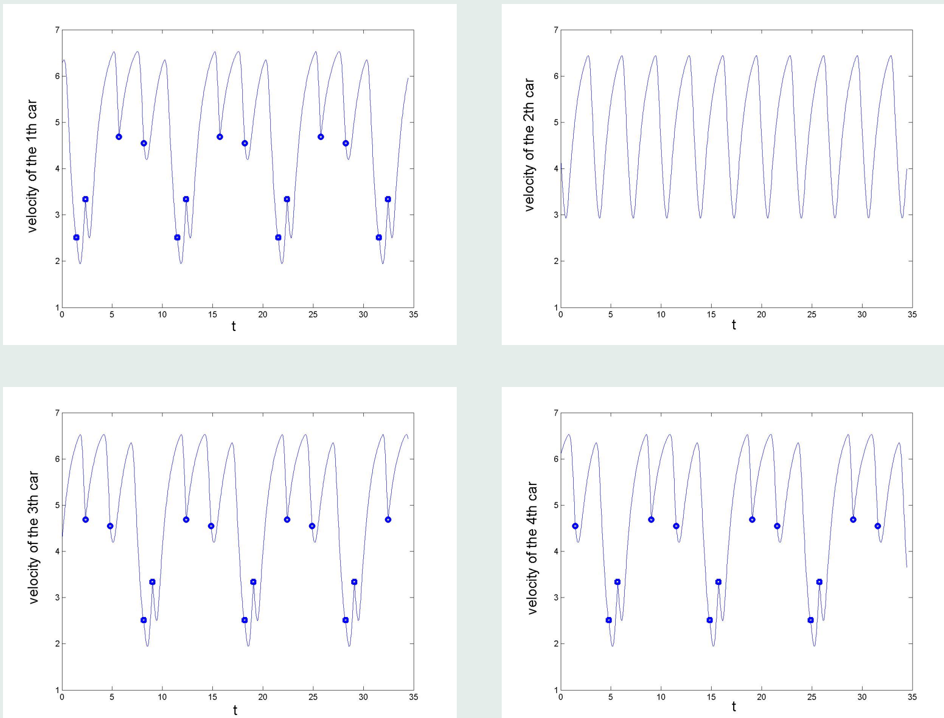
- We learned how to **simulate overtaking** in Follow-the-Leader model.

  Mathematically: *Filipov system* i.e., discontinuous righthand sides.

- Long-time behaviour? Case study $N = 3$:
  - **Spatial-temporal symmetries** of periodic solutions.
  - Patterns detected: **rotating waves**,
    phase-shifted reflectionally symmetric oscillations.
  - Idea: check for "landmarks" rather then for trajectories.

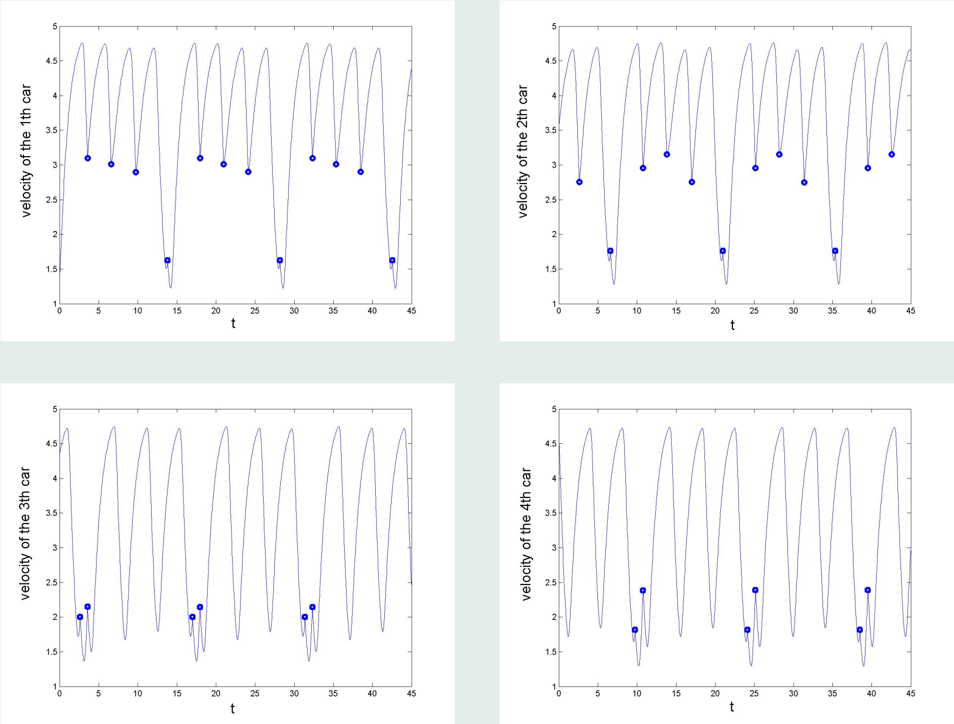
$N = 4$, pattern:

$$\begin{matrix} A & B \\ A + \frac{1}{3} & A + \frac{2}{3} \end{matrix}$$



$N = 4$, pattern:

$$\begin{matrix} A & A+\frac{1}{2} \\ B & B+\frac{1}{2} \end{matrix}$$



$\approx$ **asymmetric bound**