# Coloring graphs from lists with bounded size of their union

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**ABSTRACT**

*A graph $G$ is $k$-choosable if its vertices can be colored from any lists $L(v)$ of colors with $|L(v)| \geq k$ for all $v \in V(G)$. A graph $G$ is said to be $(k,\ell)$-choosable if its vertices can be colored from any lists $L(v)$ with $|L(v)| \geq k$, for all $v \in V(G)$, and with $|\bigcup_{v \in V(G)} L(v)| \leq \ell$. For each $3 \leq k \leq \ell$, we construct a graph $G$ which is $(k,\ell)$-choosable but not $(k,\ell+1)$-choosable. On the other hand, it is proven that each $(k,2k-1)$-choosable graph $G$ is $O(k \cdot \ln k \cdot 2^{4k})$-choosable.* © *??? John Wiley & Sons, Inc.*

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## 1. INTRODUCTION

An undirected simple graph $G = (V, E)$ is said to be *$k$-choosable* if for every list assignment $L(v)$ of lists of at least $k$ colors to each vertex $v \in V(G)$, there is a proper vertex coloring $c$ with $c(v) \in L(v)$. The smallest number $k$ for which a graph $G$ is $k$-choosable is also called the *list chromatic number* or the *choice number* of $G$. A list assignment $L$ with $|L(v)| \geq k$ is called a *$k$-list assignment* and a list assignment $L$ from which a graph $G$ cannot be colored is called *bad*. Hence a graph is not $k$-choosable if and only if there is a bad $k$-list assignment for it. The concepts of choosability and list coloring were introduced independently by Erdős, Rubin and Taylor [3] and Vizing [6]. Since then, they have received a considerable amount of attention as witnessed by several recent surveys on the subject [2, 4, 5, 7].

By the definition, the list chromatic number of a graph $G$ is at least its chromatic number and there are graphs for which the inequality is strict. There are even bipartite (2-colorable) graphs whose list chromatic number is arbitrarily large. In fact, Erdős, Rubin and Taylor [3] showed that the list chromatic number of the complete bipartite graph $K_{n,n}$ is $(1 + o(1)) \cdot \log_2 n$. Actually, they proved the following:

**Proposition 1.** For the complete bipartite graph $K_{n,n}$ with $n \geq \binom{2k-1}{k}$, there is a bad $k$-list assignment $L$ such that the size of the union of all the lists $L(v)$ is $2k - 1$.

On the other hand, it is easy to see that each bipartite graph can be colored from lists of sizes $k$ whose union is at most $2k - 2$ for $k \geq 2$. Indeed, consider a bipartite graph $G$ with parts $V_1$ and $V_2$ and a $k$-list assignment $L$ such that the union $L_0$ of all the lists $L(v)$, $v \in V(G)$, has size at most $2k - 2$. Partition the set $L_0$ to two non-empty disjoint sets $A$ and $B$ such that $|A|, |B| \leq k - 1$. Observe that each list $L(v)$ has a non-empty intersection with both sets $A$ and $B$ as otherwise $|L(v)| \leq k - 1$. Color every vertex $v$ of $V_1$ by a color from $L(v) \cap A$ and every vertex $v$ of $V_2$ by a color from $L(v) \cap B$. Since the vertices of $V_1$ are colored only with the colors from the set $A$ and the vertices of $V_2$ only with the colors from the set $B$, the obtained coloring is a proper coloring of $G$.

This motivates the following problem which we address in this paper:

**Problem 1.** Given a number $k$, does there exist a number $\ell_k$ such that if a graph $G$ can be colored from any lists whose sizes are at least $k$ and whose union has size at most $\ell_k$, then the graph $G$ is $k$-choosable?

In order to make our presentation clearer, we introduce the following two definitions: a list assignment $L(v)$ of colors to the vertices of a graph $G$ is said to be an *$(k, \ell)$-list assignment* if $|L(v)| \geq k$ for each $v \in V(G)$ and $|\bigcup_{v \in V(G)} L(v)| \leq \ell$. A graph $G$ is *$(k, \ell)$-choosable* if there is a proper vertex coloring $c$ with $c(v) \in L(v)$ for every $(k, \ell)$-list assignment $L(v)$. Note that if $G$ is $(k, \ell)$-choosable, then $G$ is also $(k, \ell')$-choosable for every $\ell'$, $k \leq \ell' \leq \ell$. Clearly, a graph $G$ is $k$-colorable if and only if it is $(k, k)$-choosable and $G$ is $k$-choosable if and only if it is $(k, \ell)$-choosable for all $\ell \geq k$. Let us remark that

the class of $(k,\ell)$-choosable graphs remains the same if we require that the lists $L(v)$ have sizes exactly $k$.

Proposition 1 can now be reformulated to our new notation as follows:
*The complete bipartite graph $K_{n,n}$ with $n \geq \binom{2k-1}{k}$ is not $(k, 2k-1)$-choosable.*

Similarly, Problem 1 can be reformulated as follows:
*Given a number $k$, does there exist a number $\ell_k$ such that a graph $G$ is $k$-choosable if it is $(k,\ell_k)$-choosable?*

As noted above, each bipartite graph is $(k, 2k-2)$-choosable for all $k \geq 2$ and hence if such a number $\ell_k$ exists, it must be at least $2k-1$.

In this paper, we completely resolve Problem 1 and show that the answer is mostly negative. Namely, we show that a graph $G$ is 2-choosable if it is $(2,4)$-choosable, i.e., the number $\ell_2$ exists and it is equal to 4 (Proposition 2). We further construct, for each $k$ and $\ell$, $3 \leq k \leq \ell$, a (small) graph $G$ which is $(k,\ell)$-choosable but which is not $(k,\ell+1)$-choosable (Theorem 1). This shows that the number $\ell_k$ does not exist for any $k \geq 3$. For the sake of completeness, we remark that a graph $G$ is 1-choosable if and only if it has no edges. Hence, the number $\ell_1$ also exists and is equal to 1.

In addition, we consider the following question suggested to us by Tomasz Łuczak. First, let us reformulate the statement of Theorem 1 as follows:
*For each $k \geq 3$ and $\ell \geq k$, the fact that $G$ is $(k,\ell)$-choosable does not imply that $G$ is $k$-choosable.*

This leads to the problem whether at least the following holds:

**Problem 2.** Is it true that for some $k \geq 3$ and $\ell \geq k$ there exists a number $\mathcal{K}_{k,\ell}$ such that each $(k,\ell)$-choosable graph $G$ is $\mathcal{K}_{k,\ell}$-choosable?

Clearly, if such number $\mathcal{K}_{k,\ell}$ exists, the number $\mathcal{K}_{k,\ell}$ is at least $k+1$ by Theorem 1 and $\ell \geq 2k-1$. The latter follows from Proposition 1 and the fact that each bipartite graph is $(k, 2k-2)$-choosable. We focus on the case $\ell = 2k-1$ and show the following: each $(k, 2k-1)$-choosable graph $G$ is $O(k \cdot \ln k \cdot 2^{4k})$-choosable (Theorem 2). In particular, the number $\mathcal{K}_{k,\ell}$ exists iff $\ell \geq 2k-1$. Note also that the sequence of the (least) numbers $\mathcal{K}_{k,\ell}$ is non-increasing for a fixed $k$.

## 2. 2-CHOOSABLE GRAPHS

In this section, we prove that the number $\ell_2$ as defined in the Introduction is equal to 4. Since the complete bipartite graph $K_{2,4}$ is $(2,3)$-choosable but not 2-choosable, the number $\ell_2$ must be at least 4. The upper bound follows from the next theorem:

**Proposition 2.** Each $(2,4)$-choosable graph is 2-choosable.

*Proof.* It was shown in [3] that if a graph $G$ is not 2-choosable, then it contains one of the following subgraphs:

- an odd cycle,

- two vertex-disjoint (even) cycles joined by a path,
- two (even) cycles sharing a single vertex,
- two vertices joined by three vertex-disjoint paths (one of them might be an edge) such that at least two paths have two or more internal vertices, or
- two vertices joined by four vertex-disjoint paths.

It is easy to see that each of these graphs is not $(2,4)$-choosable. Actually, all of them except for the graph $K_{2,4}$ (covered by the last case) are not even $(2,3)$-choosable.

Hence, if a graph $G$ is $(2,4)$-choosable, $G$ cannot contain any of the above graphs as a subgraph and it must be 2-choosable. ∎

## 3. THE NEGATIVE RESULT

In this section, we construct graphs which are $(k,\ell)$-choosable but not $(k,\ell+1)$-choosable for all $3 \leq k \leq \ell$. As the first and most important step of our construction, we prove the following version with precolored vertices:

**Lemma 1.** For every $k \geq 3$, $t \geq 2$, there exists a graph $H_k^t = (V,E)$ with a set $T$ of $t$ distinguished vertices, $T = \{v_1, \ldots, v_t\} \subseteq V$, which has the properties (1) and (2) below. Let $W$ denote the set $V \setminus T$.

(1) For any precoloring $c_0$ of the vertices $T$ by $t$ distinct colors and any set $A = \{\alpha_1, \alpha_2, \ldots, \alpha_{\max\{2,k-t\}}\}$ of $\max\{2, k-t\}$ of additional colors (not used by $c_0$), there exists a $(k, \max\{t+2, k\})$-list assignment $L$ for vertices of $W$ such that $c_0$ cannot be extended to a proper coloring $c$ of the whole graph with $c(w) \in L(w)$ for $w \in W$. Moreover, the lists $L(w)$ contain only the $t$ colors used by $c_0$ and the colors from $A$.

(2) Any precoloring $c_0$ of the vertices $T$ by at most $t-1$ distinct colors can be extended to a proper coloring $c$ of the graph $H_k^t$ with $c(w) \in L(w)$, $w \in W$, for any $k$-list assignment $L$ for vertices of $W$.

Moreover, the order of the graph $H_k^t$ is at most $O(t^2/k + k)$.

*Proof.* In the proof, we distinguish three cases.

**The first case: $t \leq k-1$.** We construct the graph $H_k^t$ as follows: the set $W$ consists of $k+1-t$ vertices $w_1, \ldots, w_{k+1-t}$ and edges of the graph $H_k^t$ are all edges $v_i w_j$ for $1 \leq i \leq t$ and $1 \leq j \leq k+1-t$ and all edges $w_i w_j$ for $1 \leq i < j \leq k+1-t$.

We now verify that the graph $H_k^t$ has the desired properties:

(1) Let us consider a precoloring $c_0$ of the vertices of $T$ by $t$ distinct colors and, for all vertices $w_j \in W$, set the lists $L(w_j)$ to the set $\{c_0(v_1), \ldots, c_0(v_t), \alpha_1, \ldots, \alpha_{k-t}\}$ of $k$ colors. The vertices of $W$ can be colored only by colors which are different from the colors $c_0(v_1), \ldots, c_0(v_t)$, i.e., only by the $k-t$ colors $\alpha_1, \ldots, \alpha_{k-t}$. Since $W$ induces a clique on $k-t+1$ vertices, this is impossible and the precoloring $c_0$ cannot be extended to a proper coloring $c$ with $c(w_j) \in L(w_j)$ for all $w_j \in W$.

(2) If $c_0$ is a precoloring of the vertices of $T$ by at most $t-1$ distinct colors and $L$ a list assignment with $|L(w_j)| = k$ for all $w_j \in W$, then $c_0$ can be extended to a coloring $c$ of the whole graph $H_k^t$. Simply remove the (at most $t-1$) colors used by a precoloring $c_0$ from the lists $L$. The vertices of $W$ form a complete graph (of order $k+1-t$) and each of them has a list of size at least $k-(t-1) = k+1-t$. Hence, this complete graph can be properly colored from the shortened lists and such a coloring is an extension of the precoloring $c_0$.

**The second case: $k = 3$ and $t \geq 3$.** We construct the graph $H_3^t$ as follows: let $X$ be an odd-cardinality set of vertices and connect each vertex of $X$ to exactly two nodes $v_i, v_{i'}$, chosen so that each pair of vertices $v_i$ and $v_{i'}$, $1 \leq i < i' \leq t$, is connected to at least one vertex $x \in X$. Note that the set $X$ can be chosen to have size $\binom{t}{2}$ or $\binom{t}{2}+1$ (a single pair can be used twice to get $|X|$ odd). Add an odd cycle $C$ of length $|X|$ on a new set of vertices and form a perfect matching between the vertices of $C$ and $X$. Take now $W = X \cup C$ and let $E$ be the edges described above.

Again, we verify that the graph $H_{3,\ell}^t$ has the two claimed properties:

(1) For each vertex $x \in X$ whose neighbors are vertices $v_i$ and $v_{i'}$, set its list $L(x) = \{c_0(v_i), c_0(v_{i'}), \alpha_1\}$. For each vertex $w \in C$, set its list $L(w) = \{\alpha_1, c_0(v_1), c_0(v_2)\}$. A proper extension of the precoloring $c_0$ must color all the vertices of $X$ by the color $\alpha_1$. Then the odd cycle $C$ must be colored using only the colors $c_0(v_1)$ and $c_0(v_2)$ which is impossible.

(2) Consider now a precoloring $c_0$ of the vertices of $T$ using at most $t-1$ colors and a list assignment $L$ with $|L(w)| = k = 3$ for all $w \in W$. Since the precoloring $c_0$ uses less than $t$ colors, there exists a vertex $x_0 \in X$ such that its two neighbors $v_i$ and $v_{i'}$ have the same color. Color the vertices of $X \setminus \{x_0\}$ from their lists: this is possible since each of them has two precolored neighbors and its list is of size three. Let $w_0 \in C$ be the neighbor of the vertex $x_0$ in the cycle $C$. For each vertex $w \in C \setminus \{w_0\}$, remove from its list the color of its single neighbor in $X$ (if the color is contained in the list $L(w)$). This leaves lists of at least two colors available for each $w \in C \setminus \{w_0\}$ and a list of three colors for the vertex $w_0$. Any such lists allow a proper coloring of the cycle $C$. Finally, color the vertex $x_0$. Note that the two neighbors of $x_0$ in $T$ have the same color and hence it is possible to color $x_0$ by a color from its list.

**The third case: $k \geq 4$ and $t \geq k$.** The structure of the graph $H_k^t$ is the following: $W$ is an odd set of vertices, each of them connected to exactly $k-2$ nodes from $\{v_1, \ldots, v_t\}$. The edges are chosen so that each pair of vertices $v_i$ and $v_{i'}$, $1 \leq i < i' \leq t$, has a common neighbor in $W$. In addition to these edges, the graph $H_k^t$ contains an odd cycle formed on the vertices $W$.

A straightforward way to construct such a graph of order $O(t^2/k)$ is to partition the vertices $v_1, \ldots, v_t$ into $\lceil t/(k-3) \rceil$ sets $T_j$ with $|T_j| \leq k-3$ and for each $1 \leq i \leq t$ and $1 \leq j \leq \lceil t/(k-3) \rceil$, create a vertex $w \in W$ whose neighbors are the vertex $v_i$ and all the vertices of $T_j$ (and, if necessary, some other vertices of $T$ so that $w$ has exactly $k-2$ neighbors in $T$). Possibly add one vertex to get $|W|$ odd.

We verify that the constructed graph $H_k^t$ has the two desired properties:

(1) For a vertex $w \in W$ with neighbors $v_{i_1}, \ldots, v_{i_{k-2}}$, we set the list $L(w)$ to be $\{c_0(v_{i_1}), \ldots, c_0(v_{i_{k-2}}), \alpha_1, \alpha_2\}$. In any proper coloring from lists $L$ which extends the precoloring $c_0$, all the vertices in $W$ must be colored by the colors $\alpha_1$ and $\alpha_2$ which is impossible as the vertices $W$ induce an odd cycle in $H_k^t$.

(2) Consider now a precoloring $c_0$ of the vertices of $T$ using at most $t-1$ colors and a list assignment $L$ with $|L(w)| = k$ for all $w \in W$. Since the precoloring $c_0$ uses at most $t-1$ colors and each pair of vertices of $T$ is contained in at least one set $T_j$, there exists a vertex $w_0 \in W$ such that its $k-2$ neighbors in $T$ have at most $k-3$ distinct colors. Thus the vertex $w_0$ has at least three colors not conflicting with the given precoloring $c_0$ in its list $L(w_0)$ and each vertex $w \in W \setminus \{w_0\}$ has at least two non-conflicting colors in its list $L(w)$. Since the vertices of $W$ induce a cycle, they can be properly colored by non-conflicting colors. Such a coloring is an extension of the precoloring $c_0$.

∎

We can now prove the main theorem of this section:

**Theorem 1.** For each $\ell \geq k \geq 3$, there exists a $(k,\ell)$-choosable graph $G$ which is not $(k,\ell+1)$-choosable. Moreover, the order of $G$ is at most $O(\ell^2)$ (the constant is independent of $k$).

*Proof.* Let $t = \ell + 2 - k \geq 2$. The graph $G$ consists of a vertex $r$, $k$ cliques $S_i$ $(i = 1, \ldots, k)$ on $k-1$ vertices each, and $k$ copies of the graph $H_k^t$ from Lemma 1. Let $T_i$ denote the $t$ distinguished vertices of the $i$th copy of the graph $H_k^t$. In addition to the edges of the graphs $S_i$ and $H_k^t$, the graph $G$ contains, for each $i = 1, \ldots, k$, all edges between $r$ and $S_i$ and all edges between $S_i$ and $T_i$. Since $G$ consists of $k$ cliques of order $k-1$ and $k$ copies of the graph $H_k^t$ whose order is at most $O(t^2/k + k)$, the order of $G$ is at most $O(k^2 + t^2) = O(\ell^2)$.

First, we show that $G$ is not $(k,\ell+1)$-choosable. Consider the following list assignment $L$: the vertex $r$ and the vertices of cliques $S_i$ have the same list $\{1, \ldots, k\}$. For an integer $i = 1, \ldots, k$, associate to each vertex $v \in T_i$ one of the $t$ colors $i, k+1, k+2, \ldots, \ell+1$ (each color to a single vertex of $T_i$). The list of the vertex $v$ then consists of the colors $\{1, \ldots, k\} \setminus \{i\}$ and the color associated to it. The remaining vertices contained in copies of $H_k^t$ have lists like in the proof of Lemma 1(1) for $A \subseteq \{1, 2, 3, \ldots, k\} \setminus \{i\}$ and a precoloring $c_0$ where each vertex of $T_i$ is precolored by its associated color.

Suppose that $G$ can be properly colored from the lists described above. Let $i$ be the color of $r$. Then the $k-1$ vertices of the clique $S_i$ are colored by all the colors $\{1, \ldots, k\} \setminus \{i\}$. Thus each vertex $v \in T_i$ is colored by its associated color. By Lemma 1 and the choice of the lists in the copies of the graph $H_k^t$, the coloring cannot be completed.

Second, we show that the graph $G$ is $(k,\ell)$-choosable. Fix a $(k,\ell)$-list assignment $L$. First color the vertex $r$, then the vertices of cliques $S_i$ and then the vertices contained in the sets $T_i$. Each vertex has at most $k-1$ of its neighbors colored before it, thus each vertex can be colored from its list. Since the number of vertices contained in $S_i \cup T_i$ is

$(k-1)+t=\ell+1$, two of the vertices of $S_i \cup T_i$ have the same color for each $i$ (recall that $L$ is a $(k,\ell)$-list assignment). However, the vertices of $S_i$ form a clique and each of them is adjacent to all the vertices of $T_i$. Thus the two vertices colored by the same color must be in $T_i$. By Lemma 1(2), the precoloring of the vertices can be extended to the whole copy of the graph $H_k^t$. Hence, the graph $G$ is $(k,\ell)$-choosable. ∎

## 4. THE POSITIVE RESULT

In contrast with the negative result of the previous section that $(k,\ell)$-choosability does not imply $k$-choosability for any $\ell$, we show in this section that if a graph $G$ is $(k,2k-1)$-choosable graph, then it is also $O(k \cdot \ln k \cdot 2^{4k})$-choosable. We follow ideas of a proof from [1] that a graph with minimum degree at least $\Theta(k^4 \cdot 2^{2k})$ is not $(k,k^2)$-choosable.

**Theorem 2.** For all $k \geq 2$, each $(k,2k-1)$-choosable graph is $O(k \cdot \ln k \cdot 2^{4k})$-choosable.

*Proof.* Fix an integer $k$ and let $D = (k+1) \cdot \ln 2 \cdot \ln k \cdot 2^{4k+1}$. We show that if a graph $G$ is $(k,2k-1)$-choosable, then $G$ is also $D$-choosable. Assume for the sake of contradiction that $G$ is a graph of the smallest order $n$ which is $(k,2k-1)$-choosable but not $D$-choosable. Since $G$ has the smallest possible order, its minimum degree is at least $D$.

For a contradiction, the graph $G$ is shown not to be $(k,2k-1)$-choosable using a probabilistic argument. We first choose a small random subset of vertices denoted by $A$ and assign each vertex of $A$ as its list a random subset of $\{1,\ldots,2k-1\}$ of size $k$. Then, as shown in the sequel, with a positive probability, at least $n/2$ vertices $v$ have the following property: the lists of the neighbors of $v$ that are in $A$ are all $\binom{2k-1}{k}$ subsets of $\{1,\ldots,2k-1\}$ of size $k$. We fix such a set $A$ with such a list assignment and show that the remaining vertices cannot be colored from random lists with positive probability. We note that in the rest of the argument we do not use the assumption that $G$ is not $D$-choosable; we only use the fact that all the vertices have degree at least $D$.

Let $p = 1/(\ln k \cdot 2^{2k+1})$. Note that $p < 1/8$. Let $A$ be a subset of vertices of the graph $G$ where each vertex of $G$ is chosen to be in $A$ randomly and independently with the probability $p$. Since the expected size of $A$ is $pn$, the probability that the size of $A$ is larger than $2pn$ is at most $1/2$ by Markov's inequality. Let $\Sigma$ be the set consisting of all $k$-element subsets of the set $\{1,\ldots,2k-1\}$. Note that $|\Sigma| = \binom{2k-1}{k} < 2^{2k-1}$. Assign independently to each vertex of $A$ a list of colors equal to a random set from $\Sigma$. Let $L_A$ be the resulting $(k,2k-1)$-list assignment for vertices of $A$. We say that a vertex $v$ of $G$ is *good* if $v \notin A$ and for each set $S \in \Sigma$, there is a neighbor $v'$ of $v$ in $G$ such that $v' \in A$ and $L_A(v') = S$.

The probability that a vertex $v \in V(G) \setminus A$ is not good is the probability that no neighbor of $v$ in $A$ has its list equal to a set $S$ for some $S \in \Sigma$. The probability that none of at least $D$ neighbors of $v$ both is in $A$ and has a list equal to a fixed set $S \in \Sigma$

is at most

$$\left(1-\frac{p}{|\Sigma|}\right)^D < \left(1-\frac{p}{2^{2k-1}}\right)^D < e^{-\frac{pD}{2^{2k-1}}} = e^{-2(k+1)\cdot\ln 2} = 2^{-2k-2}.$$

Hence the probability that there is a set $S \in \Sigma$ such that no neighbor of $v$ in $A$ has a list equal to $S$ is at most $|\Sigma| \cdot 2^{-2k-2} < 2^{2k-1} \cdot 2^{-2k-2} = 1/8$.

The probability that a given vertex $v$ is not good is at most $1/8$ from the previous calculation plus the probability that $v$ is in $A$, which is $p$. Hence the probability that a vertex $v$ is not good is at most $1/8 + p < 1/4$. It follows by Markov's inequality that the number of vertices which are not good is at most $n/2$ with probability strictly larger than $1/2$. Thus there exists a set $A \subseteq V(G)$ and a $(k, 2k-1)$-list assignment $L_A$ such that $|A| \leq 2pn$ and the number of good vertices is at least $n/2$. Fix such a set $A$ and a list assignment $L_A$ for the rest of the proof. Let $B$ be the set of good vertices.

Fix a coloring $c$ of vertices of $A$ from their lists and assign each vertex of $B$ a list from the set $\Sigma$ randomly and independently. Let $L_B$ be the resulting $(k, 2k-1)$-list assignment for vertices of $B$. Consider a vertex $v \in B$. Since for each set $S \in \Sigma$, there is a neighbor $v' \in A$ of the vertex $v$ with $L_A(v') = S$, the coloring $c$ assigns neighbors of the vertex $v$ at least $k$ different colors. Hence, the probability that $v$ cannot be colored from its list $L_B(v)$ is at least $1/|\Sigma|$. Since the choice of sets $L_B(v)$ for vertices $v \in B$ was independent, the probability that all vertices of $B$ can be colored from their lists for a fixed coloring $c$ is at most:

$$\left(1-\frac{1}{|\Sigma|}\right)^{\frac{n}{2}} < \left(1-\frac{1}{2^{2k-1}}\right)^{\frac{n}{2}} < e^{-\frac{n}{2^{2k}}}$$

Hence, the probability that for a random list assignment $L_B$, there is a coloring $c$ of vertices of $A$ from their lists which can be extended to a coloring of vertices of $B$ from $L_B$ is at most the product of the number of such colorings $c$ and the probability that a single one can be extended. This probability is strictly smaller than

$$k^{|A|} \cdot e^{-\frac{n}{2^{2k}}} \leq k^{2pn} \cdot e^{-\frac{n}{2^{2k}}} = e^{\frac{n}{2^{2k}}} \cdot e^{-\frac{n}{2^{2k}}} = 1$$

Thus, with a positive probability, there exists a list assignment $L_B$ such that no coloring $c$ of vertices of $A$ from their lists can be extended to a coloring of all vertices of $B$ from the list assignment $L_B$. Fix such a list assignment $L_B$. Complete now the $(k, 2k-1)$-list assignments $L_A$ and $L_B$ to a $(k, 2k-1)$-list assignment $L$ for all vertices of the graph $G$ by choosing lists for vertices of $V(G) \setminus (A \cup B)$ arbitrarily from the set $\Sigma$. Such a list assignment $L$ is a bad $(k, 2k-1)$-list assignment for the graph $G$ (the vertices of $A$ and $B$ cannot be colored from their lists) which contradicts the assumption that the graph $G$ is $(k, 2k-1)$-choosable. ∎

Let us remark that the bound $O(k \cdot \ln k \cdot 2^{4k})$ shown in Theorem 2 is not the best possible. It can be improved by a more careful argument to $O(k \cdot 2^{4k})$ (and even slightly more), but we do not know whether a subexponential bound can be proven.

## 5. CONCLUSION

We have proved that in order to show that a graph $G$ is not $k$-choosable it is not enough to restrict one's attention to lists whose union is bounded by some function of $k$. We also settled Problem 2 by showing that the number $\mathcal{K}_{k,\ell}$ exists if and only if $\ell \geq 2k-1$. In particular, each $(k, 2k-1)$-choosable graph $G$ is $O(k \cdot \ln k \cdot 2^{4k})$-choosable, and for each $k \geq 2$ and $\mathcal{K} \geq 2$, there exists a (bipartite) graph $G$ which is $(k, 2k-2)$-choosable but which is not $\mathcal{K}$-choosable. However, it would be interesting to investigate whether for a sufficiently large $\ell$, the number $\mathcal{K}_{k,\ell}$ can be equal to $k+1$:

**Problem 3.** Is it true that for each $k \geq 3$ there exists a number $\ell$ such that each $(k, \ell)$-choosable graph $G$ is $(k+1)$-choosable? If so, what is the least number $\ell$ with this property?

Another problem is to determine the least integer $\mathcal{K}_{k,\ell}$ with the property that each $(k, \ell)$-choosable graph is $\mathcal{K}_{k,\ell}$-choosable. In particular, to determine whether $\mathcal{K}_{k,2k-1}$ must grow exponentially in $k$.

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