Solution of David Gale's Lion and Man Problem

Jiří Sgall*

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1 Introduction

The common setting of so-called "lion and man" problems is the following. A man and a lion are moving within a given area (usually a subset of a plane). The lion wins if he catches the man. The man wins if he can keep escaping for infinite time. Probably the first version is the one with both time and space continuous, attributed to Rado and studied, e.g., by Littlewood [3, page 135] and Croft [1].

In the version we analyze, time is discrete and space is continuous. This version is attributed to Gale; it was stated as an open problem by Guy [2, Problem 31]. In this problem, a man and a lion are moving within the non-negative quadrant of the plane. In each round, first the man moves to any point in Euclidean distance at most 1 from his current position, then the lion moves to any point in Euclidean distance at most 1 from his current position. The lion wins if he moves to the current position of the man. The man wins if he can keep escaping for an infinite number of rounds.

Let $L_0 = [x_0, y_0]$ and $M_0 = [x'_0, y'_0]$ be the initial positions and coordinates of the lion and the man, respectively; let $M_0 \neq L_0$. If either $x'_0 \geq x_0$ or $y'_0 \geq y_0$, then it is easy to see that the man escapes, by always moving a distance 1 horizontally or vertically away from the origin.

We prove that, as conjectured, in the remaining case when both $x'_0 < x_0$ and $y'_0 < y_0$ the lion catches the man in a finite number of moves. In the special case when both the lion and the man are on the diagonal, the bound is quadratic in the distance of the lion from the origin. If the position of the man is close to one coordinate of the lion, there is an additional factor quadratic in the inverse of

*Mathematical Institute, AS CR, Žitná 25, CZ-11567 Praha 1, Czech Republic; and Dept. of Applied Mathematics, Faculty of Mathematics and Physics, Charles University, Praha. E-mail `sgall@math.cas.cz`, `http://www.math.cas.cz/~sgall`. Partially supported by grant A1019901 of GA AV ČR and postdoctoral grant 201/97/P038 of GA ČR.

that distance. In addition, we prove that these bounds on the number of steps are almost optimal.

Throughout the paper $\alpha_0 = (y_0 - y'_0)/(x_0 - x'_0)$ denotes the initial slope of the line connecting the lion and the man. The current coordinates of the lion and the man during the game are denoted (x, y) and (x', y'), respectively. Finally, $\alpha = (y - y')/(x - x')$ is the current slope.

2 Lion's strategy

At the beginning, the lion finds a point C on the line M_0L_0 such that L_0 is inside the segment M_0C and the circle with center C and radius $|CL_0|$ (i.e., passing through L_0) intersects both axes; among all such points we choose the one closest to the origin. Such a point C exists, since $x'_0 < x_0$ and $y'_0 < y_0$. The point C does not move during the whole game.

Let M and L denote the positions of the man and the lion, respectively, before a move. The algorithm maintains the following invariants: (i) M has both coordinates strictly smaller than L, (ii) L is inside the segment MC, and (iii) $|CL| \geq |CL_0|$.

Let M' denote the point which the man moves to. If $|M'L| \leq 1$, the lion moves to M' and wins. Otherwise the lion moves to a point L' on the line $M'C$ such that $|L'L| = 1$; of the two such points he chooses the one with larger distance from C. Such L' always exists, since L is between M and C.

Proof of efficiency

First we analyze one move of the game. We prove that the invariant of the algorithm is maintained and that the distance of the lion from C increases. The last claim enables us to prove a bound on the number of moves needed.

Lemma 1 *Let $r = |LC|$ and $r' = |L'C|$. If the lion does not catch the man in the current move then (i) M' has both coordinates strictly smaller than C, (ii) L' is inside the segment $M'C$, and (iii) $r'^2 \geq 1 + r^2$. (Note that (iii) also implies $r' \geq r \geq |CL_0|$.)*

Proof. If M' is on the line MC, the claims are trivial. So suppose M' is not on the line MC. Let Y be a point on the line $M'C$ such that $YL \perp LC$. Let X be a point on the line YL such that $|L'Y| = |XY|$ and Y is inside the segment XL (see Figure 1).

First we claim that Y is inside the segment $M'C$. If not, the angle between segments LM and LM' is at least $\pi/2$, thus $|LM'| \leq |MM'| \leq 1$, and the lion catches the man.

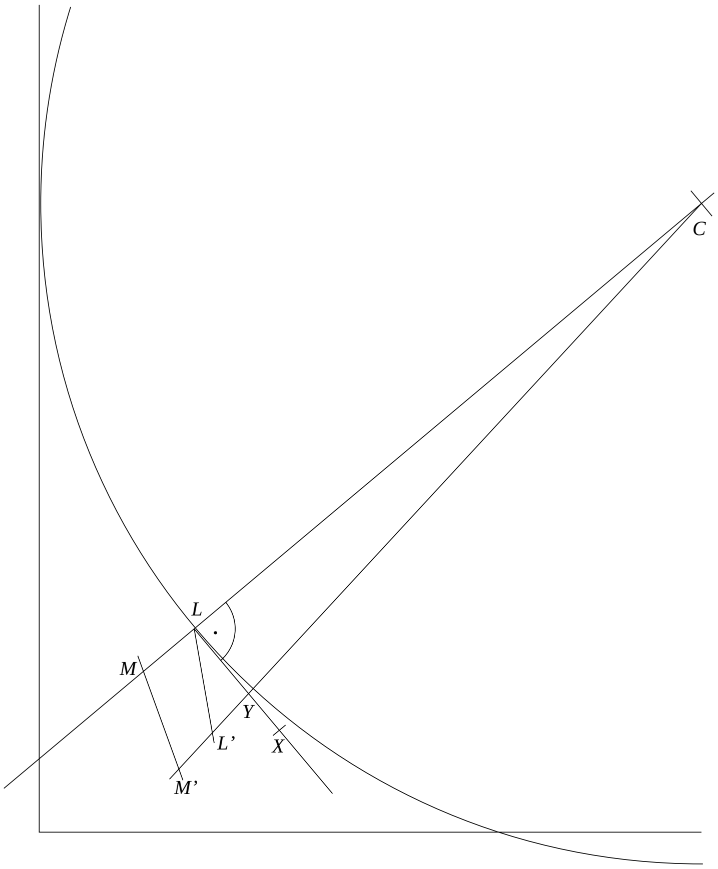

Figure 1: Analysis of one move

Since $|CL| \geq |CL_0|$, all points of the line LY in the non-negative quadrant have both coordinates strictly smaller than C. This holds in particular for Y, thus also for M', and (i) holds.

Next we claim that Y is inside the segment $L'C$. Since L is inside MC, Y is inside $M'C$, and $YL \perp LC$, it follows that $|YL| < |MM'| \leq 1$; therefore Y is inbetween the two points of distance 1 from L on the line $L'C$, and L' is chosen as the one farther from C. Last, M' cannot be in the segment $L'Y$, since then $|M'L| \leq 1$ and the lion wins. It follows that L' is inside the segment $M'C$ and (ii) holds.

Since Y is inside the segment $L'C$, using two triangle inequalities and Pythagoras' theorem, we obtain (iii):

$$r' = |L'C| = |L'Y| + |YC| = |XY| + |YC| \geq |XC|,$$

$$|XL| = |XY| + |YL| = |L'Y| + |YL| \geq |L'L| = 1, \text{ and}$$

$$r'^2 \geq |XC|^2 = |XL|^2 + |LC|^2 \geq r^2 + 1.$$

■

Theorem 2 *Let (x_0, y_0) and (x'_0, y'_0) be the initial positions of the lion and the man, respectively and let $\alpha_0 = (y_0 - y'_0)/(x_0 - x'_0)$. Suppose that $x'_0 < x_0$ and $y'_0 < y_0$. Then, using the strategy described above, the lion catches the man in the number of moves bounded by*

$$\max\{(x_0 + y_0(\alpha_0 + \sqrt{1+\alpha_0^2}))^2, (y_0 + x_0(\alpha_0^{-1} + \sqrt{1+\alpha_0^{-2}}))^2\}$$
$$= O(x_0^2 + y_0^2 + x_0^2\alpha_0^{-2} + y_0^2\alpha_0^2).$$

Proof. First we compute the coordinates of the point $C = [x_C, y_C]$. Since it is on the line M_0L_0 and L_0 is between M_0 and C, we have, for some $t \geq 0$,

$$\begin{aligned} x_C &= x_0 + t, \\ y_C &= y_0 + \alpha_0 t. \end{aligned}$$

The point C is chosen such that $|CL_0| = \max\{x_C, y_C\}$. We analyze the case $|CL_0| = y_C$, the case $|CL| = x_C$ is symmetric. In this case we have

$$(y_0 + \alpha_0 t)^2 = y_C^2 = |CL|^2 = t^2(1 + \alpha_0^2)$$

Solving the quadratic equation for t yields

$$t = y_0(\alpha_0 + \sqrt{1+\alpha_0^2}).$$

Since $|CL|^2$ increases by 1 in each step and $|CL_0| = y_C$, the number of moves is at most

$$|CO|^2 - |CL_0|^2 = x_C^2 + y_C^2 - |CL_0|^2 = x_C^2 = (x_0 + t)^2$$
$$= (x_0 + y_0(\alpha_0 + \sqrt{1+\alpha_0^2}))^2 = O(x_0^2 + y_0^2\alpha_0^2).$$

In the symmetric case, the same calculation shows that the number of moves is at most $(y_0 + x_0(\alpha_0^{-1} + \sqrt{1+\alpha_0^{-2}}))^2 = O(y_0^2 + x_0^2\alpha_0^{-2})$. Thus the number of moves is at most the maximum of the two quantities. ■

In the special case when both the man and the lion are on the diagonal, our bound is quadratic in the distance of the lion from the origin, namely $(2 + \sqrt{2})^2 x_0^2 < 11.7 \cdot x_0^2$.

If in the initial configuration one of the coordinates of the man approaches that of the lion, then C moves to infinity; asymptotically the number of moves is quadratic in α_0 or $1/\alpha_0$. This corresponds to the intuition that if the man moves straight up and the y-coordinate of the lion is almost equal to that of man, then the lion must move also almost straight up, decreasing the x-coordinate only very slightly in each move and thus taking long time to catch the man. In the next section we show that our lion's strategy is never far from optimal.

3 Man's Strategy

In this section we prove that the lion's strategy presented above is almost optimal. To do this we present a strategy for the man that enables him to survive for the appropriate number of steps. We start by a simple strategy which is sufficient to prove that the optimal number of steps is quadratic if both players start on the diagonal. Then we refine the strategy for the case when the slope α is small or large.

In the simple strategy, the man tries to keep large the product $x'y'$, i.e., the area of the rectangle between his position and the origin. In each move, he moves distance 1 in a direction perpendicular to the line connecting him and the lion, and of the two directions he chooses the one that has the invariant $x'y'$ larger. Such a move is legal as long as the invariant is non-negative, and it is clear that the lion cannot catch the man after the move.

Next we bound the number of steps. Let $a, b > 0$ be such that (a, b) is a unit vector parallel with the line from man to lion, i.e., $\alpha = b/a$. The man moves to one of the points $(x' + b, y' - a)$ and $(x' - b, y' + a)$. The sum of the invariants for the two possible new positions is $(x' + b)(y' - a) + (x' + b)(y' - a) = 2(x'y' - ab)$. Consequently, for one of the new positions the invariant is at least $x'y' - ab$. Since (a, b) is a unit vector, we have $ab = \alpha/(1 + \alpha^2) \leq 1/2$, where equality is achieved for $\alpha = 1$. It follows that using this strategy, the man keeps escaping for at least $2x'y'$ moves.

This bound is fairly tight if the man and the lion start on the diagonal close to each other; as they get closer, the ratio between the upper and lower bounds approaches $3 + 2\sqrt{2} < 6$.

If the initial positions of the man and the lion are very different, it is important to know if the correct bound is quadratic in x_0^2 or only in $x_0'^2$, which may be significantly smaller. The following easy argument shows that for any starting positions on the diagonal, the lion's initial coordinate is more relevant: The man first moves to a point on the diagonal with coordinates at least $x/2 - 1$, where (x, x) is the starting position of the lion, and then follows the simple strategy above. This gives a lower bound of $\Omega(x^2)$ for any starting positions on the diagonal.

If the initial positions are not on a diagonal, the upper and lower bounds may differ significantly, even when the man first moves to a point advantageous for him. For example, assume that the man starts at the origin and the lion starts at point $(x_0, 1)$. Then $\alpha_0 = 1/x_0$ and the upper bound is x_0^4. To use the simple lower bound, the man first moves away from the origin to a point with coordinates approximately $(x_0/4, x_0^2/2)$ (he chooses a point with integral distance from the origin); it is easy to verify that this is safe. Now the simple lower bound gives $\Omega(x_0^3)$, still a factor of x_0 away from the upper bound. To decrease this gap, an additional improvement is needed.

A refined strategy

The simple bound above is asymptotically tight as soon as α is bounded by a positive constant both from below and from above. In this section we improve the bound for the remaining cases. We assume that $\alpha_0 > 1$; the case of $\alpha_0 < 1$ is symmetric. For $\alpha_0 > 1$, the upper bound is $O(x_0^2 + \alpha_0^2 y_0^2)$.

The key observation is that in the analysis of the simple strategy, if α is large, the area $x'y'$ actually decreases only by approximately $1/\alpha$. If we could obtain a lower bound of $\alpha x'y'$, then after moving the man first to an optimal safe starting point, this would be sufficient to conclude an asymptotically tight lower bound.

However, the lion can decrease the value of α during the game. In fact, suppose that the man starts at the position $(\alpha_0, 1)$, for a large α_0, using the simple strategy above, and at the same time the lion starts very close, at the slope α_0 above the man, and keeps decreasing α in the optimal way. Then it can be shown that the trajectory of the man is dominated by the curve $y = \alpha_0/x$; in $O(\alpha_0)$ steps it reaches a point close to $(\sqrt{\alpha_0}, \sqrt{\alpha_0})$ from which the lion can catch the man in $O(\alpha_0)$ steps. The total number of steps is only $O(\alpha_0)$, while the upper bound is $O(\alpha_0^2)$ in this case.

We modify the man's strategy to achieve a bound of $\Omega((x_0^2 + \alpha_0^2 y_0^2)/\alpha_0^{\varepsilon})$, for any constant $\varepsilon > 0$. This shows that the lion's strategy is asymptotically almost optimal. For the example of the man starting at the origin and the lion starting at $(x_0, 1)$, the bound is $\Omega(x_0^{4-\varepsilon})$, improving the simple lower bound of $\Omega(x_0^3)$ and almost matching the upper bound of $O(x_0^4)$. (However, the constant hidden in the asymptotic notation depends on ε, so we cannot claim a matching bound.)

The strategy works in a constant number of phases, their number increases as the desired ε approaches 0. In each phase, the man first moves to a safe point on the line satisfying $x = \alpha y$, for the current value of α, as far from the origin as possible. Then he follows the simple strategy modified as if the origin was at the point $(z, 0)$, for some $z > 0$. If α stays within the same order of magnitude or larger, by the analysis of the simple strategy we obtain the desired lower bound. If α decreases by a significant factor, we start a new phase; choosing appropriate z above guarantees that the coordinates of the man are still large enough.

Formal description of the strategy

Let $\alpha_i = \alpha_0^{1-i\varepsilon}$. Let M and L denote the current position of the man and the lion, respectively.

(i) The man moves (in several steps) to a point $M' = (x', y')$ satisfying (i) $x' = \alpha y'$ (with the current α), (ii) the distance $|M'M|$ is integral, and

(iii) $|M'M| < |M'L|$; among all points satisfying (i)-(iii) choose the one maximizing $|M'O|$. Set $x_i = x'$ and $y_i = y'$.

(ii) If $\alpha < \alpha_i$ and $i < 1/\epsilon$, we increase i by 1 and go to step (i) (start a new phase).

Otherwise the man moves to a point $M' = (x', y')$ such that $M'M$ is perpendicular to ML and $|M'M| = 1$; among the two such points the one with smaller value of $(x' - x_i/2)y'$ is chosen. If this invariant would become negative, the game ends (the man gives up).

Analysis of the strategy

Lemma 3 *Let (x_0, y_0) be the initial position of the lion and α_0 the initial slope. Then $x_1 \geq (x_0 + \alpha_0 y_0)/2 - 1 = \Omega(x_0 + \alpha y_0)$.*

Proof. A straightforward calculation demonstrates that the point $X = (a, b) = ((x_0 + \alpha_0 y_0)/2, (x_0/\alpha_0 + y_0)/2)$ satisfies $a = \alpha_0 b$ and $|XM| \leq |XL|$. (Geometrically, X is on a line which is perpendicular to LM and has the same distance from L as from O.) A point satisfying all conditions (i)-(iii) in the first phase of the algorithm, i.e., additionally at an integral distance from M, can be found in distance at most 1 from X. ■

The previous lemma also implies that now it is sufficient to prove a lower bound of $\Omega(x_1^2/\alpha_0^\varepsilon)$.

Lemma 4 *If the algorithm reaches phase $i > 1$ then $x_i \geq x_{i-1}/4 - 1$.*

Proof. Before the initial move in phase i, let $M = (x'', y'')$. We have $x'' \geq x_{i-1}/2$, as the invariant from the previous phase in non-negative. Now, without the integrality requirement, the man could safely move to any point on the line p through M perpendicular to ML. The point (a, b) on this line satisfying $a = \alpha b$ is the midpoint between the intersection of p and coordinate axes, therefore it satisfies $a \geq x''/2 \geq x_{i-1}/4$. Similar to Lemma 3, adding the integrality requirement can decrease the coordinate by at most 1. ■

Lemma 5 *If the game ends (i.e., the man gives up) during phase i, then the number of steps in phase i is at least $x_i^2/\alpha_0^\varepsilon$.*

Proof. During the whole phase i the value of α is at least α_i, if $i < 1/\varepsilon$. Therefore, similarly as in the analysis of the simple strategy, the invariant decreases by at most $1/\alpha_i$ in each step. If $i \geq 1/\varepsilon$, by the definition $\alpha_k \leq 1$, and, regardless of α, the invariant again decreases by at most $1/2 \leq 1/\alpha_k$ in each step. In phase i, the invariant $(x' - x_i/2)y'$ starts at value at least $x_i y_i/2 = x_i^2/(2\alpha_{i-1})$, as the value of α is at most α_{i-1} when we start phase i. The game can only end up when the invariant would become negative, therefore the lower bound on the number of steps is $\alpha_i x_i^2/(2\alpha_{i-1}) \geq x_i^2/(2\alpha_0^\varepsilon)$. ■

Theorem 6 *Let* (x_0, y_0) *and* (x_0', y_0') *be the initial positions of the lion and the man, respectively, and let* $\alpha_0 = (y_0 - y_0')/(x_0 - x_0')$*. For every* $\varepsilon > 0$ *there exist a strategy for the man which guarantees that the length of the game is at least*

$$\Omega\left(\frac{x_0^2 + y_0^2 + x_0^2\alpha_0^{-2} + y_0^2\alpha_0^2}{\alpha_0^\varepsilon + \alpha_0^{-\varepsilon}}\right).$$

Proof. Let $k = \lceil 1/\varepsilon \rceil$. We prove the bound for $\alpha \geq 1$, the other case is symmetric. For the asymptotic result it is sufficient to consider the case when $x_1 \geq 8^k$. Lemma 4 then implies by induction that for any $i \leq k$, $x_i \geq x_1/8^{i-1} = \Omega(x_1)$, since k is a constant. The theorem now follows from the Lemma 5 applied to the phase during which the game ends and the bounds in Lemma 3 and Lemma 4. ■

4 Generalizations

It is easy to see that the lion's strategy above generalizes if the man and the lion move within any wedge (an angle strictly smaller than π). Also, it generalizes to higher dimensions; there the playing area can be any convex cone. In either case, the man wins if the halfplane (or the halfspace, in higher dimension) of points with smaller distance to M_0 than to L_0 has an unbounded intersection with the playing area. Otherwise the lion catches the man in a bounded number of steps.

We do not give explicit bounds on the number of steps. However, we note that if the position of the man and lion are both on a given line starting at the origin, then both the lower and upper bounds are quadratic in the distance of the lion from the origin. In particular, the exponent in the bound does not increase with the dimension. The intuition is that we can restrict our attention to the "worst" plane containing that line. This obviously leads to a quadratic lower bound. The upper bound is quadratic in the radius of the ball used, and the radius is proportional to the initial distance to the origin.

5 Conclusions

We gave a simple and (in retrospect) natural strategy for the lion to win the game. Furthermore, we showed that it is very close to being optimal: If both players start on the diagonal close to each other, the gap is smaller than a factor of 6. If the slope of the line connecting the man and the lion is bounded away from vertical or horizontal, the bound is asymptotically tight. Finally, in the general case the gap is very small.

It remains an open question to remove the remaining small gap of α^ε between the bounds. We conjecture that the upper bound, i.e., the lion's strategy is closer to the truth. As noted above, the simple strategy of man is not sufficient to close the gap.

As an interesting special case, note that if the man starts at the origin and the lion starts at $(x, 1)$, the number of steps necessary and sufficient to catch the lion is (close to) x^4.

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