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ABSTRACT

Suppose we have a tournament with edges labelled so that the edges incident with any vertex have at most k distinct labels (and no vertex has outdegree 0). Let m be the minimal size of a subset of labels such that for any vertex there exists an outgoing edge labelled by one of the labels in the subset. It was known that $m \leq \binom{k+1}{2}$ for any tournament. We show that this bound is almost best possible, by a probabilistic construction of tournaments with $m = \Omega(k^2/\log k)$. We give explicit tournaments with $m = k^{2-o(1)}$. If the number of vertices is bounded by $N < 2^k$, we have a better upper bound of $m = O(k \log N)$, which is again almost optimal. We also consider a relaxation of this problem in which instead of the size of a subset of labels we minimize the total weight of a fractional set with analogous properties. In that case the optimal bound is $2k-1$.

1. INTRODUCTION AND PRELIMINARIES

We present a solution of a problem about labelled tournaments stated by Avi Wigderson at the NSF/CBMS Conference in Oregon in 1985.

1.1. The problem

A tournament is a directed graph which for each pair of vertices v and v', $v \neq v'$ contains exactly one of the edges (v, v') and (v', v). In our problem the edges are labelled. We assume that there is no vertex of outdegree 0.

* E-mail: `sgallj@mbox.cesnet.cz`. Partially supported by grants A119107 and A1019602 of AV ČR. Part of this work was done at School of Computer Science, Carnegie-Mellon University, Pittsburgh, PA, U.S.A.

Journal of Graph Theory Vol. (Volume Number), 1–9 ((Year))
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We can think of this as a representation for a real tournament, an edge (v, v') then means that the player v defeated the player v', and the label denotes the day on which the match took place.

Call a subset S of labels of a tournament T a *label cover* if for every vertex v there exists an outgoing edge labelled by some label from S. Let $k = k(T)$ be the maximal number of distinct labels on edges incident with a vertex. Let $m = m(T)$ be the minimal size of a label cover of T. Informally, k is the maximum number of days on which a player plays, and m is the minimal size of a subset of days such that every player wins at least once on a day from the subset.

We are interested in the dependency of m on k; more precisely, we want to maximize m over all tournaments with given k. In particular, Avi Wigderson asked whether m is always linear in k.

It is easy to construct a tournament with $m = 2k - 1$. Take any tournament with $2k - 1$ players in which each of them wins exactly half of the matches (such tournaments are easily obtained by decomposing the complete graph into edge-disjoint cycles). Each player plays all the matches in which he wins on a single day, but this day is different for every two players. This determines the labelling uniquely, and clearly only the set of all labels satisfies the condition.

It is not immediately clear why m should be bounded if there is no restriction on the total number of players and days. But an easy induction shows that $m \leq \binom{k+1}{2}$ for any tournament. Pick a player p and put all k days on which he plays in the set of chosen days. Now consider the set P of players who do not win on any of these days. A match between any two players from P takes place on a day when the player p does not play, since otherwise the winning player is not in P. The restriction of the tournament to P has the number of playing days of each player bounded by $k - 1$, since each player plays with P, and this day is not in the restricted tournament. The proof is finished by using the induction assumption for the restricted tournament.

1.2. Notation

We use the usual notation for the asymptotic growth of functions, mostly for functions of k. Thus, $f = o(g)$ means $|f(k)/g(k)| \to 0$ as $k \to \infty$, $f = O(g)$ means that $|f(k)| \leq c|g(k)|$ for some constant c and any sufficiently large k, $f = \Omega(g)$ is equivalent to $g = O(f)$, $f = \omega(g)$ is equivalent to $g = o(f)$, and finally $f = \Theta(g)$ means that both $f = O(g)$ and $f = \Omega(g)$. When we claim that certain tournaments are given explicitly, we mean that there is an algorithm running in time polynomial in k which given an edge outputs its orientation and its label. Since the tournaments are generally exponential in k, this is the strongest possible condition, in particular stronger than the usual condition that we can generate the graphs in time polynomial in the number of nodes.

1.3. Our results

We show that the previous upper bound of $m \leq \binom{k+1}{2}$ is almost best possible, in particular we answer Wigderson's question negatively. We prove that there exist tournaments with $m = \Omega(k^2/\log k)$, which is optimal up to the factor of $O(\log k)$. The proof uses expanders as a tool for constructing the tournament. Using the explicit construction of almost optimal expanders from [3], we give explicit tournaments with $m = k^{2-o(1)}$.

The size of these tournaments is exponential in k. This again brings up the question of whether and to which extent m depends on the order (the number of vertices) of the tournaments. If we restrict the number of vertices to $N \leq 2^k$, we can prove existence of a tournament with $m = \Omega(k \log N/\log k)$, and construct explicitly a tournament with $m = k^{1-o(1)} \log N$. It turns out that this is almost optimal, because every tournament with N vertices has $m = O(k \log N)$. This shows that after all the number of vertices has some influence on m. In particular, the number of vertices of a tournament with m close to $\Theta(k^2)$ has to be large.

The bound of $m = O(k \log N)$ is proved indirectly using a relaxation of this problem, which is interesting in its own right. In the relaxation we replace label covers in the definition of m by fractional covers. We are now looking for a weight function on labels f, such that $f(l) \geq 0$ for every label l, and for every vertex $v \in V$, $\sum f(l)$ over all labels of outgouing edges from v is at least one, i.e., the total weight of days when a player wins is at least 1. Let $M = \min_f \sum_l f(l)$ be the minimal total weight over all legal weight functions. Using the duality in linear programming we show that $M \leq 2k - 1$. This bound is optimal, because the previous example of a tournament with $m = 2k - 1$ has a single losing day for each player, and hence also satisfies $M = 2k - 1$.

1.4. An equivalent covering problem

The problem can be stated equivalently as an extremal covering problem without any reference to the tournaments, if we are interested only in the asymptotic dependence of m on k.

Let us recall that a (vertex) cover of a set system $\mathcal{S}$ is any set intersecting all elements in $\mathcal{S}$; we call the minimal size of a cover of $\mathcal{S}$ its *cover number*. Given a labelled tournament T, let A_v, resp. B_v, denote the set of labels of all edges incident with a vertex v, resp. of all outgoing edges. Clearly the size of sets A_v and B_v is bounded by $k(T)$ and the cover number of the system $\mathcal{S} = \{A_v, B_v \mid v \in T\}$ is $m(T)$ (since $A_v \supseteq B_v$, the sets A_v are irrelevant for the cover). Moreover, the sets satisfy the key condition

$$\forall v, v' \in T(A_v \cap B_{v'} \neq \emptyset \vee A_{v'} \cap B_v \neq \emptyset), \tag{1.1}$$

since every edge in the tournament has a label and an orientation consistent with the sets.

Conversely, suppose we have a set system $\mathcal{S} = \{A_v, B_v \mid v \in V\}$ satisfying (1.1), such that the maximal size of a set is $k(\mathcal{S})$, and the cover number of $\mathcal{S}$ is $m(\mathcal{S})$ (note that the condition (1.1) for $v = v'$ guarantees that the sets are nonempty). Then we can construct a tournament T with $m(T) \geq m(\mathcal{S})/2$ and $k(T) \leq 2k(\mathcal{S})$ as follows. One of the set systems $\{A_v \mid v \in V\}$, $\{B_v \mid v \in V\}$ has a cover number at least $m(\mathcal{S})/2$; w.l.o.g. we assume this is true for the system $\{B_v \mid v \in V\}$. The tournament has vertices V and the edges are labelled and oriented so that for each vertex v the labels of all outgoing edges are in B_v and the labels of all incoming edges are in A_v; the condition (1.1) guarantees that each edge can be oriented and labelled consistently. Clearly $k(T)$ and $m(T)$ now satisfy the conditions stated above, and also the number of vertices corresponds closely to the number of sets. (A small technical detail is that the tournament can have one vertex v with outdegree 0. We fix this by including two additional vertices v_1, v_2 so that for each $v' \in V - \{v\}$ the edges (v', v_1) and (v', v_2) are included and labelled by the same label as (v', v), and the edges (v, v_1), (v_1, v_2), and (v_2, v) are included and labelled by some label $l \in B_v$.[1])

Hence existence of a tournament with a given asymptotic relation of m, k, and the number of vertices it is equivalent to existence of a system of sets satisfying (1.1) with the same asymptotic relation of m, k, and the number of sets.

2. THE CONSTRUCTION

Before we give the construction, we review the necessary background about expanders.

Definition [2, 3]. An undirected graph on n vertices is an α-*expander*, if any two disjoint sets of vertices, each containing at least α vertices, are joined by an edge.

We need expanders with small maximal degree. Clearly an n^δ-expander must have maximal degree at least $n^{1-\delta}$. It is not too hard to show that random regular graphs of degree $n^{1-\delta} \log n$ are n^δ-expanders. Pippenger [2] proves this for random graphs if each edge is chosen independently with

[1]Note that this construction also shows that the assumption that no vertex has outdegree 0 is only technical.

the appropriate probability; the case of random regular graphs is similar. Recent work of Wigderson and Zuckerman [3] gives an explicit polynomial time construction of n^{δ}-expanders with maximal degree $n^{1-\delta+o(1)}$ for any constant $0 < \delta < 1$, which is almost optimal.

For our construction we need k-expanders with maximal degree k with as many vertices as possible. The above probabilistic argument guarantees that such expanders with $\Omega(k^2/\log k)$ vertices exist, while the construction from [3] gives such expanders with $k^{2-o(1)}$ vertices.

Now we are ready to prove our main theorem. To avoid confusion, we point out that we use the vertices of an expander as the labels of the constructed tournament, while the vertices of the tournament are subsets of the set of labels, i.e., subsets of the set of vertices of the expander.

Theorem 1. For each k, there exists a tournament such that for each vertex v the number of distinct labels on edges incident with v is at most k, and the minimal size m of a label cover satisfies $m = \Omega(k^2/\log k)$. There exist such tournaments explicitly given and satisfying $m = k^{2-o(1)}$.

Proof. We construct a system of nonempty sets $\{A_v, B_v \mid v \in V\}$ satisfying (1.1), with maximal size of sets $2k-1$ and the required cover number m for $\{B_v \mid v \in V\}$. By the transformation from Section 1.4., this is sufficient as we are only interested in the asymptotic behavior of m.

Let (L, E) be a k-expander with maximal degree k. The sets of the system (and hence also the vertices of the tournament) are indexed by $V = [L]^k$, the set of all k-element subsets of L.

We put $B_v = v$ for every $v \in V$. It follows that the cover number of $\{B_v \mid v \in V\}$ is $m = |L| - k + 1$, as otherwise one $v \in V$ is missed. Using the expanders mentioned above we get the desired bounds on m both for the existential proof and for the explicit construction.

Now we construct the sets A_v. Fix a selection function $g : V \to L$ such that $g(v) \in v$. For a set $L' \subseteq L$, we denote by $N(L')$ the neighborhood of the set L' in the expander (L, E), including the elements of L'. Define $A_v = (L - N(v)) \cup N(\{g(v)\})$.

By the definition of k-expander, $|L - N(v)| < k$. From the bound on the degree of the expander it follows that $|N(\{g(v)\})| \leq k$. Therefore $|A_v| \leq 2k - 1$.

It remains to verify the condition (1.1). Suppose that for some $v, v' \in V$ the first part of the condition is false, i.e., $A_v \cap v' = A_v \cap B_{v'} = \emptyset$. Because $g(v') \in v'$, it follows that $g(v') \notin A_v$. From the definition of A_v we get $g(v') \notin L - N(v)$ and hence $\{g(v')\} \cap N(v) \neq \emptyset$. Therefore $\emptyset \neq N(\{g(v')\}) \cap v \subseteq A_{v'} \cap v = A_{v'} \cap B_v$, and the second part of the condition (1.1) is satisfied. ∎

In the construction above, the order of the tournament is very large, it has $\binom{k^2/\log k}{k} = 2^{\Theta(k \log k)}$ vertices. The following lemma enables us to decrease the number of vertices to 2^k, and also to explore the case when the number of vertices is bounded.

Lemma 1. For any $\varepsilon < 1/2$, any n satisfying both $n = \omega(k)$ and $n = 2^{o(k)}$, and any sufficiently large k there exists a system of 2^k sets of size k chosen from a universe of n elements whose cover number is at least εn.

Proof. By a straightforward probabilistic argument. Pick 2^k sets independently at random with uniform distribution. The probability that a fixed set of size εn and a random set of size k are disjoint is

$$\binom{n(1-\varepsilon)}{k} \Big/ \binom{n}{k} \geq (2-\varepsilon')^{-k}$$

for some $\varepsilon' > 0$ and sufficiently large k. Thus the probability that a fixed set of size εn is a cover for 2^k random sets of size k is at most

$$(1-(2-\varepsilon')^{-k})^{2^k} \leq (e^{-(2-\varepsilon')^{-k}})^{2^k} = e^{-(1-\varepsilon'/2)^{-k}} < 2^{-n}$$

for large k. Thus with non-zero probability there is no cover of size εn. ∎

The set of vertices of the tournament constructed in Theorem 1 was the system $[L]^k$ of all k-element subsets of L. Now take the subtournament induced by vertices represented by the elements of the system constructed in Lemma 1. This tournament has only 2^k vertices and the asymptotic bound on m is the same, $m = \Omega(k^2/\log k)$.

In the next theorem we generalize this construction for the case in which the number of vertices of the tournament is bounded. As we prove in Theorem 4, this bound is almost optimal.

Theorem 2. For every k and N satisfying $k \leq N \leq 2^k$, there exists a tournament with N vertices such that for each vertex v the number of distinct labels on edges incident with v is at most k, and the minimal size m of a label cover satisfies $m = \Omega(k \log N/\log k)$. There exist such tournaments explicitly given and satisfying $m = \Omega(k^{1-o(1)} \log N)$.

Proof. If $\log N = O(\log k)$, the statement of the theorem is trivial. Otherwise we again choose a suitable subset of vertices of the tournament constructed in Theorem 1.

For the randomized construction, we can choose random N vertices, and a similar computation as in Lemma 1 works. Alternatively, we can use the statement of Lemma 1 directly as follows. Partition the set L (the vertices of an expander) into equivalence classes of size $k/\log N$; there are $\Theta(k \log N/\log k)$ of them. We will choose our system so that each set is a union of some $\log N$ equivalence classes. This means that we can take a representative of each class and work with the set of representatives. By Lemma 1 there exists a system of $2^{\log N} = N$ of sets of $\log N$ representatives with the cover number $\Omega(k \log N/\log k)$. The cover number does not change if we replace each representative by the whole equivalence class. We use this system of sets as the vertices of the induced tournament.

For the explicit construction, pick α such that $\binom{|L|\alpha/k}{\alpha} = N$, where L is the set of vertices of an explicitly constructed expander; a calculation shows that $\alpha = \Theta(\log N/\log k)$. Partition the set L into equivalence classes of size k/α; there are $|L|\alpha/k = k^{1-o(1)}\log N/\log k = k^{1-o(1)}\log N$ equivalence classes. Take as the vertices of the tournament all unions of exactly α equivalence classes, there are N of them by our choice of α. Every cover has to contain an element of all but α equivalence classes, which is $k^{1-o(1)}\log N$ elements. ∎

3. THE RELAXATION

As in Section 1.4., A_v and B_v denote the set of labels of all edges incident with v and the set of labels of all outgoing edges, respectively. Now $k = \max_v |A_v|$ is the maximal number of distinct labels on edges incident with a vertex. In this section it is especially useful to view m as the cover number of the set system $\{B_v \mid v \in V\}$.

Let f be a function from the labels L to nonnegative reals. For $L' \subseteq L$ we define $f(L') = \sum_{l \in L'} f(l)$. We say that f is a *fractional cover* if it satisfies $f(B_v) \geq 1$ for all vertices v. Define M to be the minimal total weight $f(L)$ over all fractional covers.

Theorem 3. For every labelled tournament with no vertex of outdegree 0, there exists a fractional cover f with the total weight $f(L) \leq 2k-1$, where $k = \max_{v\in V}|B_v|$ and L is the set of labels.

Proof. Let $\mathbf{A}$ be a 0-1 matrix coding the sets B_v, $v \in V$, as follows. The rows are indexed by the vertices of the tournament and the columns are indexed by the labels of the edges. The entry $A_{v,l}$ is 1 if $l \in B_v$, and it is 0 otherwise.

Let $\bar{\mathbf{x}}$ be an optimal solution of the linear program

$$\text{minimize } \mathbf{x}\cdot\mathbf{1} \text{ subject to } \mathbf{A}\mathbf{x} \geq \mathbf{1},\ \mathbf{x} \geq \mathbf{0} \tag{3.1}$$

The vector $\bar{\mathbf{x}}$ is indexed by the labels, and it is easy to check that the feasible solutions correspond to the fractional covers and that the objective function is the total weight of the set. Therefore $\bar{\mathbf{x}}\cdot\mathbf{1}$ is equal to M, the minimal weight of a fractional cover. We have to prove that $\bar{\mathbf{x}}\cdot\mathbf{1} \leq 2k-1$.

Now consider the dual linear program

$$\text{maximize } \mathbf{w}\cdot\mathbf{1} \text{ subject to } \mathbf{A}^T\mathbf{w} \leq \mathbf{1}, \mathbf{w} \geq \mathbf{0} \tag{3.2}$$

and its optimal solution $\bar{\mathbf{w}}$. The vector $\bar{\mathbf{w}}$ is indexed by the vertices of the tournament. For $V' \subseteq V$ we define $w(V') = \sum_{v \in V'} \bar{\mathbf{w}}_v$. By the duality we know that $w(V) = \bar{\mathbf{w}} \cdot \mathbf{1} = \bar{\mathbf{x}} \cdot \mathbf{1} = M$.

The inequality in (3.2) says that for each label l, $w(\{v \mid l \in B_v\}) \leq 1$, and therefore for every set of labels $L' \subseteq L$ we have

$$w(\{v \mid L' \cap B_v \neq \emptyset\}) \leq |L'| \tag{3.3}$$

The function w can be thought of as a weight function on the players with the property that the weight of any set of players with a small cover is small.

We know that for every $v \in V$, $|A_v - B_v| \leq k - 1$, because $B_v \neq \emptyset$. Therefore from (3.3) it follows for every vertex $v \in V$ that

$$w(\{v' \mid (A_v - B_v) \cap B_{v'} \neq \emptyset\}) \leq k - 1$$

Because there is an edge between v and every v', we know that if $(A_v - B_v) \cap B_{v'} = \emptyset$, then $B_v \cap A_{v'} \neq \emptyset$, and hence we get for every vertex $v \in V$

$$w(\{v' \mid B_v \cap A_{v'} \neq \emptyset\}) \geq w(V) - (k - 1) = M - k + 1$$

It follows that for every $v \in V$

$$\sum_{l \in B_v} w(\{v' \mid l \in A_{v'}\}) \geq w(\{v' \mid B_v \cap A_{v'} \neq \emptyset\}) \geq M - k + 1 \tag{3.4}$$

Define a vector $\mathbf{y}$ indexed by the edge labels by $\mathbf{y}_l = w(\{v \mid l \in A_v\})$. The condition (3.4) can be expressed in terms of $\mathbf{y}$ and $\mathbf{A}$ as

$$\mathbf{A}\mathbf{y} \geq \begin{pmatrix} M-k+1 \\ M-k+1 \\ \vdots \\ M-k+1 \end{pmatrix}$$

which is just a scaled inequality from the linear program (3.1). We have $\mathbf{y} \geq \mathbf{0}$ by the definition of $\mathbf{y}$, hence $\mathbf{y}$ satisfies the constraints of the scaled linear program, and the cost function has to be larger than the scaled optimum. Thus

$$\mathbf{y} \cdot \mathbf{1} \geq M(M + 1 - k) \tag{3.5}$$

The definition of $\mathbf{y}$ and the fact that $|A_v| \leq k$ imply that

$$\mathbf{y} \cdot \mathbf{1} \ = \sum_{v \in \bar{V}} \sum_{l \in A_v} w(v) \leq kw(V) \leq kM$$

This together with (3.5) proves that $kM \geq M(M+1-k)$ and therefore $M \leq 2k-1$. ∎

The next theorem improves the upper bound of $O(k^2)$ for the minimal size of the (non-fractional) cover in the case when the number of vertices is small. It shows that the lower bound in Theorem 2 is almost optimal. It also follows that the tournaments constructed in Theorem 1 have to be large.

Theorem 4. For every labelled tournament with no vertex of outdegree 0, the minimal size m of a label cover is at most $m = O(k \log |V|)$, where $k = \max_{v \in V} |B_v|$.

Proof. By the result of Lovász on fractional covers [1], the ratio of the cover number of a set system to the minimal weight of a fractional cover is at most $1 + \log d$, where d is the maximum degree of a point (i.e., the number of sets in the system containing that point). Obviously, $d \leq |V|$ in our case. Therefore, using Theorem 3, the minimal cover has size at most $(2k-1)(1+\log|V|) = O(k \log |V|)$. ∎

ACKNOWLEDGMENTS

I am grateful to Avi Wigderson, who improved the original construction by using expanders (thus making the explicit constructions possible) and by making the set of vertices smaller, proved Theorem 4 and pointed out the possibility of proving a matching lower bound. I would also like to thank Steven Rudich, Pavel Pudlák, and referees for useful comments.

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