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On the Construction of Nonconstant Lower and Upper Functions to Second Order Nonlinear Periodic Boundary Value Problems

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Summary. In this paper we present effective conditions ensuring the existence of lower and upper functions for the periodic boundary value problem $u'' = f(t,u)$, $u(0) = u(2\pi)$, $u'(0) = u'(2\pi)$. They are constructed as solutions of some related generalized linear problems and they need not be constant. As applications, two new results concerning singular periodic boundary value problems for nonlinear Duffing equations of both attractive and repulsive type are delivered.

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1 . Introduction

Theorems about the existence of solutions of boundary value problems for ordinary differential equations often suppose the existence of lower and upper functions to the studied problem. We can decide whether the problem has constant lower and upper functions (see e.g. [3], [6]) and to find them if they exist. In general, however, it is easy neither to find lower and upper functions which need not be constant nor

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to prove their existence which can make difficult the application of such theorems. One possibility how to get nonconstant lower and upper functions to the periodic boundary value problem

$$u'' = f(t, u), \quad u(0) = u(2\pi), \quad u'(0) = u'(2\pi) \tag{1.1}$$

is shown in this paper. We make use of fairly general definitions of these notions introduced in [11] and we construct them as solutions of generalized periodic boundary value problems for linear differential equations in sections 3 and 4. (Essentially they are solutions of linear generalized differential equations in the sense of J. Kurzweil, cf. e.g. [5], [14], [15] and [16].) This together with the existence results presented in [11] enable us to get some new effective existence criteria for the problem (1.1). In particular, in Section 5 we give two simple applications to singular problems of the Lazer-Solimini type (cf. [9]).

Throughout the paper we assume: $f : [0, 2\pi] \times \mathbb{R} \mapsto \mathbb{R}$ fulfils the Carathéodory conditions on $[0, 2\pi] \times \mathbb{R}$, i.e. f has the following properties: (i) for each $x \in \mathbb{R}$ the function $f(., x)$ is measurable on $[0, 2\pi]$; (ii) for almost every $t \in [0, 2\pi]$ the function $f(t, .)$ is continuous on $\mathbb{R}$; (iii) for each compact set $\mathrm{K} \subset \mathbb{R}$ the function $m_{\mathrm{K}}(t) = \sup_{x \in \mathrm{K}} |f(t, x)|$ is Lebesgue integrable on $[0, 2\pi]$.

The set of functions $f : [0, 2\pi] \times \mathbb{R} \mapsto \mathbb{R}$ satisfying the Carathéodory conditions on $[0, 2\pi] \times \mathbb{R}$ is denoted by $\mathrm{Car}([0, 2\pi] \times \mathbb{R})$. Furthermore, we keep the following notation:

As usual, for a given subinterval J of $\mathbb{R}$ (possibly unbounded) $\mathbb{C}(J)$ denotes the set of functions continuous on J. Furthermore, $\mathbb{L}[0, 2\pi]$ stands for the set of functions Lebesgue integrable on $[0, 2\pi]$, $\mathbb{L}_2[0, 2\pi]$ is the set of functions square Lebesgue integrable on $[0, 2\pi]$, $\mathbb{AC}[0, 2\pi]$ denotes the set of functions absolutely continuous on $[0, 2\pi]$ and $\mathbb{BV}[0, 2\pi]$ is the set of functions of bounded variation on $[0, 2\pi]$. For $x \in \mathbb{C}[0, 2\pi]$, $y \in \mathbb{L}[0, 2\pi]$ and $z \in \mathbb{L}_2[0, 2\pi]$ we denote

$$\|x\|_{\mathbb{C}} = \sup_{t \in [0, 2\pi]} |x(t)|, \quad \overline{y} = \frac{1}{2\pi} \int_0^{2\pi} y(s)\,\mathrm{d}s,$$

$$\|y\|_{\mathbb{L}} = \int_0^{2\pi} |y(t)|\,\mathrm{d}t \quad \text{and} \quad \|z\|_{\mathbb{L}_2} = \Big(\int_0^{2\pi} z^2(t)\,\mathrm{d}t \Big)^{\frac{1}{2}}.$$

If $x \in \mathbb{BV}[0, 2\pi]$, $s \in (0, 2\pi]$ and $t \in [0, 2\pi)$, then the symbols $x(s-)$, $x(t+)$ and $\Delta^+ x(t)$ are respectively defined by

$$x(s-) = \lim_{\tau \to s-} x(\tau), \quad x(t+) = \lim_{\tau \to t+} x(\tau) \text{ and } \Delta^+ x(t) = x(t+) - x(t)$$

and x^{ac} and x^{sing} stand for the absolutely continuous part of x and the singular part of x, respectively. We suppose $x^{\mathrm{sing}}(0) = 0$.

Furthermore, $\mathbb{L}^n[0,2\pi]$ and $\mathbb{L}^{n\times n}[0,2\pi]$ are respectively the sets of column n-vector valued and of $n\times n$-matrix valued functions with elements from $\mathbb{L}[0,2\pi]$, $\mathbb{AC}^n[0,2\pi]$ and $\mathbb{AC}^{n\times n}[0,2\pi]$ are respectively the sets of n-vector valued and of $n\times n$-matrix valued functions whose elements are absolutely continuous on $[0,2\pi]$ and $\mathbb{BV}^n[0,2\pi]$ is the set of n-vector valued functions whose elements have a bounded variation on $[0,2\pi]$.

Finally, for a subset M of $\mathbb{R}$, χ_M denotes the characteristic function of M ($\chi_M(t)=1$ for $t\in M$, $\chi_M(t)=0$ for $t\in\mathbb{R}\backslash M$) and for a given function $\beta\in\mathbb{L}[0,2\pi]$, β^+ denotes its nonnegative part ($\beta^+(t)=\max\{\beta(t),0\}$ for a.e. $t\in[0,2\pi]$) and β^- stands for its nonpositive part ($\beta^-(t)=\max\{-\beta(t),0\}$ for a.e. $t\in[0,2\pi]$).

By a *solution of* (1.1) we understand a function $u:[0,2\pi]\mapsto\mathbb{R}$ such that $u'\in\mathbb{AC}[0,2\pi]$, $u(0)=u(2\pi)$, $u'(0)=u'(2\pi)$ and

$$u''(t)=f(t,u(t)) \quad \text{for a.e.} \quad t\in[0,2\pi].$$

1.1. Definition. Functions $(\sigma_1,\rho_1)\in\mathbb{AC}[0,2\pi]\times\mathbb{BV}[0,2\pi]$ are said to be *lower functions of the problem* (1.1), if the singular part ρ_1^{sing} of ρ_1 is nondecreasing on $[0,2\pi]$,

$$\sigma_1'(t)=\rho_1(t),\quad \rho_1'(t)\ge f(t,\sigma_1(t)) \quad \text{for a.e.} \quad t\in[0,2\pi]$$

and

$$\sigma_1(0)=\sigma_1(2\pi),\quad \rho_1(0+)\ge\rho_1(2\pi-). \tag{1.2}$$

Similarly, functions $(\sigma_2,\rho_2)\in\mathbb{AC}[0,2\pi]\times\mathbb{BV}[0,2\pi]$ are said to be *upper functions of the problem* (1.1), if the singular part ρ_2^{sing} of ρ_2 is nonincreasing on $[0,2\pi]$,

$$\sigma_2'(t)=\rho_2(t),\quad \rho_2'(t)\le f(t,\sigma_2(t)) \quad \text{for a.e.} \quad t\in[0,2\pi]$$

and

$$\sigma_2(0)=\sigma_2(2\pi),\quad \rho_2(0+)\le\rho_2(2\pi-). \tag{1.3}$$

1.2. Remark. Let us note that in virtue of a monotonicity of singular parts of ρ_1 and ρ_2 we get equivalent definition of lower and upper functions of (1.1) if we replace the boundary conditions (1.2) and (1.3) respectively by $\sigma_1(0)=\sigma_1(2\pi)$, $\rho_1(0)=\rho_1(2\pi)$ and $\sigma_2(0)=\sigma_2(2\pi)$, $\rho_2(0)=\rho_2(2\pi)$.

The existence results in Section 5 are based on the following theorem which is contained in [11, Theorems 4.1 and 4.2].

1.3. Theorem. *Let* (σ_1,ρ_1) *and* (σ_2,ρ_2) *be respectively lower and upper functions of the problem* (1.1).

(I) *Suppose* $\sigma_1(t) \le \sigma_2(t)$ *on* $[0,2\pi]$. *Then there is a solution* u *of the problem* (1.1) *such that* $\sigma_1(t) \le u(t) \le \sigma_2(t)$ *on* $[0,2\pi]$.

(II) *Suppose* $\sigma_1(t) \ge \sigma_2(t)$ *on* $[0,2\pi]$ *and there is* $m \in \mathbb{L}[0,2\pi]$ *such that*

$$f(t,x) \ge m(t) \ (or \ \ f(t,x) \le m(t)) \ \ \text{for a.e. } t \in [0,2\pi] \ \ \text{and all} \ \ x \in \mathbb{R}.$$

Then there is a solution u *of the problem* (1.1) *such that* $\|u'\|_{\mathbb{C}} \le \|m\|_{\mathbb{L}}$ *and*

$$\sigma_2(t_u) \le u(t_u) \le \sigma_1(t_u) \quad \text{for some} \ \ t_u \in [0,2\pi].$$

2 . Periodic solutions of certain generalized linear differential problems

In this section we want to show that if for a.e. $t \in [0,2\pi]$ and all $x \in I_t$, where I_t is a subinterval of $\mathbb{R}$, the function f fulfils a condition of the form

$$f(t,x) \le \omega x + \beta(t) \tag{2.1}$$

or

$$f(t,x) \ge \omega x + \beta(t), \tag{2.2}$$

where $\omega \in \mathbb{R}$ and $\beta \in \mathbb{L}[0,2\pi]$ are given, then it is possible to construct lower or upper functions for the problem (1.1), respectively.

It is known that if $\omega \ne -k^2$ for all $k \in \mathbb{N} \cup \{0\}$, then the problem

$$\sigma' = \rho, \quad \rho' = \omega\sigma + \beta(t) \quad \text{a.e. on} \ \ [0,2\pi], \tag{2.3}$$

$$\sigma(0) = \sigma(2\pi), \quad \rho(0) = \rho(2\pi) \tag{2.4}$$

possesses a unique solution $(\sigma,\rho) \in \mathbb{AC}[0,2\pi] \times \mathbb{AC}[0,2\pi]$ for any $\beta \in \mathbb{L}[0,2\pi]$. Consequently, if we have in addition

$$\sigma(t) \in I_t \quad \text{for all} \ \ t \in [0,2\pi], \tag{2.5}$$

then the functions (σ,ρ) are lower or upper functions of (1.1) (according to whether (2.1) or (2.2) is satisfied). In general the relation (2.5) need not be true, of course. However, if we admit a more general notion of a solution to the linear problem (2.3), (2.4) and if the intervals I_t of validity of (2.1) or (2.2) are large enough, we can always use the problem (2.3), (2.4) for a construction of lower or upper functions for (1.1).

To show this, let us first consider a linear differential system on $[0,2\pi]$

$$\xi' = P(t)\xi + q(t), \tag{2.6}$$

where $P \in \mathbb{L}^{n\times n}[0,2\pi]$ and $q \in \mathbb{L}^n[0,2\pi]$. By a solution of (2.6) on $[0,2\pi]$ we mean a function $\xi \in \mathbb{AC}^n[0,2\pi]$ satisfying (2.6) a.e. on $[0,2\pi]$. The corresponding normalized fundamental matrix solution of the system

$$\xi' = P(t)\xi \tag{2.7}$$

is denoted by Φ, i.e. $\Phi \in \mathbb{AC}^{n\times n}[0,2\pi]$ and

$$\Phi(t) = \mathrm{I} + \int_0^t P(s)\Phi(s)\mathrm{d}s \ \text{on} \ [0,2\pi].$$

Its inverse matrix $\Phi^{-1}(t)$ is defined for any $t \in [0,2\pi]$, $\Phi^{-1} \in \mathbb{AC}^{n\times n}[0,2\pi]$ and if

$$\det\left(\Phi^{-1}(2\pi) - \mathrm{I}\right) \neq 0 \tag{2.8}$$

holds, then for any $q \in \mathbb{L}^n[0,2\pi]$ there is a unique solution $\xi \in \mathbb{AC}^n[0,2\pi]$ of (2.6) on $[0,2\pi]$ such that

$$\xi(0) = \xi(2\pi). \tag{2.9}$$

This solution can be written in the form

$$\xi(t) = \int_0^{2\pi} G(t,s)q(s)\mathrm{d}s \ \text{on} \ [0,2\pi],$$

where

$$G(t,s) = \Phi(t)\left\{\begin{array}{cl} \left(\Phi^{-1}(2\pi) - \mathrm{I}\right)^{-1} & \text{for } \ t \le s, \\ \mathrm{I} + \left(\Phi^{-1}(2\pi) - \mathrm{I}\right)^{-1} & \text{for } \ s < t \end{array}\right\}\Phi^{-1}(s) \tag{2.10}$$

is the Green function of the problem (2.7), (2.9).

2.1. Definition. Let $\tau \in [0,2\pi)$ and $d \in \mathbb{R}^n$ be given. By a solution of the problem (2.6), (2.9),

$$\Delta^+\xi(\tau) = d \tag{2.11}$$

we mean a function $\xi \in \mathbb{BV}^n[0,2\pi]$ such that the relations (2.9) and

$$\xi'(t) = P(t)\xi(t) + q(t) \quad \text{a.e. on } [0,2\pi] \tag{2.12}$$

are satisfied and $\xi - d\,\chi_{(\tau,2\pi]} \in \mathbb{AC}^n[0,2\pi]$.

2.2. Proposition. *Assume* (2.8). *Then for any* $\tau \in [0,2\pi)$, *any* $d \in \mathbb{R}^n$ *and any* $q \in \mathbb{L}^n[0,2\pi]$, *the problem* (2.6), (2.9), (2.11) *possesses a unique solution* ξ *and this solution is given by*

$$\xi(t) = G(t,\tau)\, d + \int_0^{2\pi} G(t,s)q(s)\,\mathrm{d}s \;\; on \;\; [0,2\pi], \tag{2.13}$$

where G *is defined by* (2.10).

Proof. For any $c \in \mathbb{R}^n$, the functions

$$x(t) = \Phi(t)\, c + \Phi(t)\int_0^t \Phi^{-1}(s)q(s)\,\mathrm{d}s, \quad t \in [0,2\pi],$$

and

$$y(t) = \Phi(t)\Phi^{-1}(2\pi)\, c - \Phi(t)\int_t^{2\pi} \Phi^{-1}(s)q(s)\,\mathrm{d}s \quad t \in [0,2\pi],$$

are the unique solutions of (2.6) on $[0,2\pi]$, such that $x(0) = c$ and $y(2\pi) = c$, respectively. Define $\xi(t) = x(t)$ for $0 \le t \le \tau$ and $\xi(t) = y(t)$ for $\tau < t \le 2\pi$. Then $\xi \in \mathbb{BV}^n[0,2\pi]$ fulfils (2.9), (2.12) and

$$\Delta^+\xi(\tau) = \Phi(\tau)\left(\Phi^{-1}(2\pi) - \mathrm{I}\right) c - \Phi(\tau)\int_0^{2\pi} \Phi^{-1}(s)q(s)\,\mathrm{d}s.$$

Consequently, if we put

$$c = M^{-1}\Big(\Phi^{-1}(\tau)\, d + \int_0^{2\pi} \Phi^{-1}(s)q(s)\,\mathrm{d}s\Big),$$

where $M = \Phi^{-1}(2\pi) - \mathrm{I}$, then ξ verifies (2.11). Moreover, $\xi(t) - d\chi_{(\tau,2\pi]}(t) = x(t)$ holds on $[0,2\pi]$ and hence $\xi - d\chi_{(\tau,2\pi]} \in \mathbb{AC}^n[0,2\pi]$. Finally, using the relation $\Phi^{-1}(2\pi)M^{-1} = \mathrm{I} + M^{-1}$, we get

$$\xi(t) = \Phi(t)\Big(M^{-1}\Big(\Phi^{-1}(\tau)\, d + \int_0^{2\pi} \Phi^{-1}(s)q(s)\,\mathrm{d}s\Big) + \int_0^t \Phi^{-1}(s)q(s)\,\mathrm{d}s\Big)$$

for $0 \le t \le \tau$ and

$$\xi(t) = \Phi(t)\Big((\mathrm{I} + M^{-1})\Big(\Phi^{-1}(\tau)\, d + \int_0^{2\pi} \Phi^{-1}(s)q(s)\,\mathrm{d}s\Big) - \int_t^{2\pi} \Phi^{-1}(s)q(s)\,\mathrm{d}s\Big)$$

for $\tau < t \le 2\pi$, wherefrom the representation (2.13) of ξ follows. □

2.3. Remark. Clearly, for any solution ξ of (2.6), (2.9), (2.11) we have $\xi^{\mathrm{ac}} = \xi - d\chi_{(\tau,2\pi]}$, $\xi^{\mathrm{sing}} = d\chi_{(\tau,2\pi]}$ and ξ is left-continuous on $(0,2\pi]$.

2.4. Remark. The problem (2.6), (2.9), (2.11) can be rewritten as the integral equation

$$\xi(t) = \xi(0) + \int_0^t P(s)\xi(s)\mathrm{d}s + h(t) - h(0), \quad t \in [0, 2\pi],$$

where

$$h(t) = d\,\chi_{(\tau,2\pi]}(t) + \int_0^t q(s)\mathrm{d}s \text{ on } [0, 2\pi].$$

This equation is a very special case of generalized differential equations introduced by J. Kurzweil in [5].

Now, we shall apply Proposition 2.2 on the problem (2.3), (2.4) generalized in the sense of Definition 2.1. In the case $\omega = \alpha^2$, $\alpha > 0$, we get the following result:

2.5. Corollary. *Let $\alpha > 0$. Then for any $\tau \in [0, 2\pi)$, any $\delta \in \mathbb{R}$ and any $\beta \in \mathbb{L}[0, 2\pi]$, the problem*

$$\sigma' = \rho, \quad \rho' = \alpha^2\sigma + \beta(t), \tag{2.14}$$

$$\sigma(0) = \sigma(2\pi),\ \rho(0) = \rho(2\pi),\ \Delta^+\sigma(\tau) = 0,\ \Delta^+\rho(\tau) = \delta$$

possesses a unique solution (σ, ρ). Moreover, $\sigma \in \mathbb{AC}[0, 2\pi]$, $\rho^{sing} = \delta\chi_{(\tau,2\pi]}$ and

$$\sigma(t) = g(t,\tau)\delta + \int_0^{2\pi} g(t,s)\beta(s)\mathrm{d}s, \text{ on } [0, 2\pi], \tag{2.15}$$

where

$$g(t,s) = -\begin{cases} \dfrac{\cosh(\alpha(\pi + t - s))}{2\,\alpha\,\sinh(\alpha\pi)} & \text{if } 0 \le t \le s \le 2\pi, \\[2ex] \dfrac{\cosh(\alpha(\pi + s - t))}{2\,\alpha\,\sinh(\alpha\pi)} & \text{if } 0 \le s < t \le 2\pi. \end{cases} \tag{2.16}$$

Proof. The fundamental matrix solution Φ of the corresponding homogeneous system $\sigma' = \rho$, $\rho' = \alpha^2\sigma$, is given by

$$\Phi(t) = \begin{pmatrix} \cosh(\alpha t) & \dfrac{\sinh(\alpha t)}{\alpha} \\ \alpha\sinh(\alpha t) & \cosh(\alpha t) \end{pmatrix} \text{ on } [0, 2\pi]$$

and $\det\Big(\Phi^{-1}(2\pi) - \mathrm{I}\Big) = -4\sinh(\alpha\pi) \neq 0$. Thus, we can apply Proposition 2.2 to the problem (2.6), (2.9), (2.11) with

$$\xi(t) = \begin{pmatrix} \sigma \\ \rho \end{pmatrix}, \quad q(t) = \begin{pmatrix} 0 \\ \beta \end{pmatrix} \quad \text{and} \quad d = \begin{pmatrix} 0 \\ \delta \end{pmatrix},$$

to obtain that the problem (2.14) possesses a unique solution (σ,ρ). Since, in particular, $\Delta^+\sigma(\tau)=0$ and $\Delta^+\rho(\tau)=\delta$, it follows from Definition 2.1 that $\sigma\in\mathbb{AC}[0,2\pi]$ and $\rho-\delta\chi_{(\tau,2\pi]}\in\mathbb{AC}[0,2\pi]$ (i.e. $\rho^{\text{sing}}=\delta\chi_{(\tau,2\pi]}$). Furthermore, inserting for Φ into (2.10), we get

$$G(t,s)=\begin{cases}\begin{pmatrix}-\dfrac{\sinh(\alpha(\pi+t-s))}{2\sinh(\alpha\pi)} & -\dfrac{\cosh(\alpha(\pi+t-s))}{2\alpha\sinh(\alpha\pi)}\\[2mm] -\dfrac{\alpha\cosh(\alpha(\pi+t-s))}{2\sinh(\alpha\pi)} & -\dfrac{\sinh(\alpha(\pi+t-s))}{2\sinh(\alpha\pi)}\end{pmatrix} & \\ \qquad\qquad\text{if } \ 0\le t\le s\le 2\pi, & \\ \begin{pmatrix}\dfrac{\sinh(\alpha(\pi+s-t))}{2\sinh(\alpha\pi)} & -\dfrac{\cosh(\alpha(\pi+s-t))}{2\alpha\sinh(\alpha\pi)}\\[2mm] -\dfrac{\alpha\cosh(\alpha(\pi+s-t))}{2\sinh(\alpha\pi)} & \dfrac{\sinh(\alpha(\pi+s-t))}{2\sinh(\alpha\pi)}\end{pmatrix} & \\ \qquad\qquad\text{if } \ 0\le s< t\le 2\pi, & \end{cases}$$

which implies that σ has the form (2.15), where g is defined in (2.16). $\square$

2.6. Remark. We can easily verify that for any $\alpha\in(0,\infty)$, the Green function g from (2.16) satisfies the estimates

$$-\frac{\cosh(\alpha\pi)}{2\alpha\sinh(\alpha\pi)}\le g(t,s)\le-\frac{1}{2\alpha\sinh(\alpha\pi)}<0 \text{ on } \ [0,2\pi]\times[0,2\pi]. \tag{2.17}$$

The next result concerns the case $\omega=-\alpha^2$, $\alpha>0$.

2.7. Corollary. *Let $\alpha>0$ and $\alpha\ne k$ for all $k\in\mathbb{N}$. Then for any $\tau\in[0,2\pi)$, any $\delta\in\mathbb{R}$ and any $\beta\in\mathbb{L}[0,2\pi]$, the problem*

$$\sigma'=\rho,\quad \rho'=-\alpha^2\sigma+\beta(t), \tag{2.18}$$

$$\sigma(0)=\sigma(2\pi),\ \rho(0)=\rho(2\pi),\ \Delta^+\sigma(\tau)=0,\ \Delta^+\rho(\tau)=\delta$$

possesses a unique solution (σ,ρ). Moreover, $\sigma\in\mathbb{AC}[0,2\pi]$, $\rho^{sing}=\delta\chi_{(\tau,2\pi]}$ and σ has the form (2.15), where

$$g(t,s)=\begin{cases}\dfrac{\cos(\alpha(\pi+t-s))}{2\,\alpha\,\sin(\alpha\pi)} & \textit{if } \ 0\le t\le s\le 2\pi,\\[2mm] \dfrac{\cos(\alpha(\pi+s-t))}{2\,\alpha\,\sin(\alpha\pi)} & \textit{if } \ 0\le s<t\le 2\pi.\end{cases} \tag{2.19}$$

Proof. Substituting Φ in (2.10) by

$$\Phi(t) = \begin{pmatrix} \cos(\alpha t) & \dfrac{\sin(\alpha t)}{\alpha} \\ -\alpha\,\sin(\alpha t) & \cos(\alpha t) \end{pmatrix}, \quad t \in [0, 2\pi],$$

we get

$$G(t,s) = \begin{cases} \begin{pmatrix} -\dfrac{\sin(\alpha(\pi+t-s))}{2\sin(\alpha\pi)} & \dfrac{\cos(\alpha(\pi+t-s))}{2\alpha\sin(\alpha\pi)} \\ -\alpha\,\dfrac{\cos(\alpha(\pi+t-s))}{2\sin(\alpha\pi)} & -\dfrac{\sin(\alpha(\pi+t-s))}{2\sin(\alpha\pi)} \end{pmatrix} & \text{if } 0 \le t \le s \le 2\pi, \\ \begin{pmatrix} \dfrac{\sin(\alpha(\pi+s-t))}{2\sin(\alpha\pi)} & \dfrac{\cos(\alpha(\pi+s-t))}{2\alpha\sin(\alpha\pi)} \\ -\dfrac{\alpha\,\cos(\alpha(\pi+s-t))}{2\sin(\alpha\pi)} & \dfrac{\sin(\alpha(\pi+s-t))}{2\sin(\alpha\pi)} \end{pmatrix} & \text{if } 0 \le s < t \le 2\pi \end{cases}$$

and since under our assumptions we have $\det\Big(\Phi^{-1}(2\pi) - \mathrm{I}\Big) = 4\sin^2(\alpha\pi) \neq 0$, the proof follows from Proposition 2.2 similarly as the proof of Corollary 2.5. □

2.8. Remark. Let us notice that for any $\alpha \in (0, \frac{1}{2}]$, the Green function g from (2.19) satisfies the estimates

$$0 \le \frac{\cos(\alpha\pi)}{2\alpha\sin(\alpha\pi)} \le g(t,s) \le \frac{1}{2\alpha\sin(\alpha\pi)} \text{ on } [0,2\pi]\times[0,2\pi]. \tag{2.20}$$

If $\omega = 0$, the system (2.3) becomes

$$\sigma' = \rho, \quad \rho' = \beta(t) \tag{2.21}$$

and the corresponding fundamental matrix solution Φ is defined by

$$\Phi(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \text{ on } [0, 2\pi].$$

Consequently, $\det\Big(\Phi^{-1}(2\pi) - \mathrm{I}\Big) = 0$. Hence, the assumptions of Proposition 2.2 are not satisfied and instead of the generalized periodic boundary value problem we have to deal with a certain related problem with conditions of the mixed type

$$\sigma(0) = \sigma(2\pi) = c_1,\ \rho(0) = \rho(2\pi),\ \Delta^+\rho(\tau) = -2\pi\overline{\beta}, \tag{2.22}$$

where c_1 may be an arbitrary real number.

2.9. Definition. Let $\tau \in [0, 2\pi)$, $c_1 \in \mathbb{R}$ and $\beta \in \mathbb{L}[0, 2\pi]$ be given. By a solution of the problem (2.21), (2.22) we mean a couple of functions $(\sigma, \rho) \in \mathbb{AC}[0, 2\pi] \times \mathbb{BV}[0, 2\pi]$ satisfying (2.22) and

$$\sigma'(t) = \rho(t), \quad \rho'(t) = \beta(t) \quad \text{a.e. on } [0, 2\pi] \tag{2.23}$$

and such that $\rho + 2\pi\overline{\beta}\chi_{(\tau,2\pi]} \in \mathbb{AC}[0, 2\pi]$.

2.10. Proposition. *Let $c_1 \in \mathbb{R}$, $\tau \in [0, 2\pi)$ and $\beta \in \mathbb{L}[0, 2\pi]$. Then the problem (2.21), (2.22) possesses a unique solution (σ, ρ). Moreover, $\rho^{sing} = -2\pi\overline{\beta}\chi_{(\tau,2\pi]}$ and σ is given by*

$$\sigma(t) = c_1 - g(t, \tau)(2\pi\overline{\beta}) + \int_0^{2\pi} g(t, s)\beta(s)\,\mathrm{d}s \ \text{ on } \ [0, 2\pi], \tag{2.24}$$

where

$$g(t, s) = \begin{cases} \dfrac{t(s - 2\pi)}{2\pi} & \text{if } \ 0 \le t \le s \le 2\pi, \\[2ex] \dfrac{(t - 2\pi)s}{2\pi} & \text{if } \ 0 \le s < t \le 2\pi. \end{cases} \tag{2.25}$$

Proof. For any $c_1, c_2 \in \mathbb{R}$, put

$$\sigma(t) = \begin{cases} c_1 + c_2\, t + \displaystyle\int_0^t (t - s)\beta(s)\,\mathrm{d}s & \text{if } \ 0 \le t \le \tau \le 2\pi, \\[2ex] c_1 + c_2\, (t - 2\pi) - \displaystyle\int_t^{2\pi} (t - s)\beta(s)\,\mathrm{d}s & \text{if } \ 0 \le \tau < t \le 2\pi \end{cases} \tag{2.26}$$

and

$$\rho(t) = \begin{cases} c_2 + \displaystyle\int_0^t \beta(s)\,\mathrm{d}s & \text{if } \ 0 \le t \le \tau \le 2\pi, \\[2ex] c_2 - \displaystyle\int_t^{2\pi} \beta(s)\,\mathrm{d}s & \text{if } \ 0 \le \tau < t \le 2\pi. \end{cases} \tag{2.27}$$

Then σ and ρ belong to $\mathbb{BV}[0, 2\pi]$ and they satisfy (2.23), (2.22),

$$\sigma - \Delta^+\sigma(\tau)\chi_{(\tau,2\pi]} \in \mathbb{AC}[0, 2\pi] \quad \text{and} \quad \rho - \Delta^+\rho(\tau)\chi_{(\tau,2\pi]} \in \mathbb{AC}[0, 2\pi].$$

Furthermore, with respect to (2.26), $\Delta^+\sigma(\tau) = 0$ (and consequently σ is absolutely continuous on $[0, 2\pi]$) if and only if

$$c_2 = -\int_0^{2\pi} \frac{\tau - s}{2\pi}\beta(s)\,\mathrm{d}s, \tag{2.28}$$

while $c_1 \in \mathbb{R}$ may be arbitrary. Inserting (2.28) into (2.26) we can check that σ verifies (2.24). Finally, in virtue of (2.27) we have

$$\rho(t) + 2\pi\overline{\beta}\chi_{(\tau,2\pi]}(t) = c_2 + \int_0^t \beta(s)\,\mathrm{d}s,$$

i.e. $\rho^{\mathrm{ac}} = \rho + 2\pi\overline{\beta}\chi_{(\tau,2\pi]}$ and $\rho^{\mathrm{sing}} = -2\pi\overline{\beta}\chi_{(\tau,2\pi]}$. □

2.11. Remark. The Green function g from (2.25) satisfies the estimates

$$-\frac{\pi}{2} \le -\frac{s(2\pi - s)}{2\pi} \le g(t,s) \le 0 \text{ on } [0,2\pi]\times[0,2\pi]. \tag{2.29}$$

2.12. Remark. Notice that if the couple (σ,ρ) is determined by Proposition 2.10, then $\rho \in \mathbb{AC}[0,2\pi]$ whenever $\overline{\beta} = 0$. Furthermore, if $\overline{\beta} = 0$ or $\tau = 0$, then the formula (2.24) reduces to

$$\sigma(t) = c_1 + \int_0^{2\pi} g(t,s)\beta(s)\,\mathrm{d}s \text{ on } [0,2\pi]. \tag{2.30}$$

3 . Lower functions

The following Lemma will be often used in the next two sections.

3.1. Lemma. *Let $p : [0,2\pi] \mapsto \mathbb{R}$ and $p_*,\ p^* \in \mathbb{R}$ be such that $p_* p^* \ge 0$ and $p_* \le p(t) \le p^*$ on $[0,2\pi]$. Then the inequality*

$$\Big|\int_0^{2\pi} p(s)b(s)\,\mathrm{d}s\Big| \le \max\{|p_*|,|p^*|\}\frac{||b||_{\mathbb{L}}}{2}$$

holds for any $b \in \mathbb{L}[0,2\pi]$ such that $\overline{b} = 0$.

Proof. Let $b \in \mathbb{L}[0,2\pi]$ be such that $\overline{b} = 0$. Then $\overline{b^+} = \overline{b^-}$ and $||b||_{\mathbb{L}} = 4\pi\overline{b^+} = 4\pi\overline{b^-}$. Thus, in the case $0 \le p_* \le p^*$, we have

$$-p^*\frac{||b||_{\mathbb{L}}}{2} = -p^* 2\pi\overline{b^-} \le \int_0^{2\pi} p(s)b(s)\,\mathrm{d}s \le p^* 2\pi\overline{b^+} = p^*\frac{||b||_{\mathbb{L}}}{2}.$$

Similarly, we can show that

$$\Big|\int_0^{2\pi} p(s)b(s)\,\mathrm{d}s\Big| \le -p_*\frac{||b||_{\mathbb{L}}}{2}$$

holds if $p_* \le p^* \le 0$. □

3.2. Proposition. *Assume that there are* $a \in \mathbb{R}$, $A \in \mathbb{R}$ *and* $b \in \mathbb{L}[0, 2\pi]$ *such that*

$$a \le 0, \quad \overline{b} = 0 \tag{3.1}$$

and

$$f(t,x) \le a + b(t) \ \text{ for a.e. } t \in [0,2\pi] \ \text{ and all } \ x \in [A,B], \tag{3.2}$$

where

$$B = A + \frac{\pi}{2}\,\|b\|_{\mathbb{L}}. \tag{3.3}$$

Then there exist lower functions (σ, ρ) *of the problem* (1.1) *such that*

$$A \le \sigma(t) \le B \ \text{ on } \ [0, 2\pi]. \tag{3.4}$$

Proof. Let us put $\tau = 0$. By Proposition 2.10, the problem (2.21), (2.22) with $\beta(t) = b(t)$ a.e. on $[0, 2\pi]$ has a unique solution $(\sigma, \rho) \in \mathbb{AC}[0, 2\pi] \times \mathbb{AC}[0, 2\pi]$ for any $c_1 \in \mathbb{R}$. Moreover, σ has the form

$$\sigma(t) = c_1 + \int_0^{2\pi} g(t,s)b(s)\,\mathrm{d}s \ \text{ on } \ [0,2\pi]$$

with g defined by (2.25) (see Remark 2.12). Since Lemma 3.1 and (2.29) give the estimate

$$\Big|\int_0^{2\pi} g(t,s)b(s)\,\mathrm{d}s\Big| \le \frac{\pi}{4}\|b\|_{\mathbb{L}}, \tag{3.5}$$

it follows that

$$c_1 - \frac{\pi}{4}\|b\|_{\mathbb{L}} \le \sigma(t) \le c_1 + \frac{\pi}{4}\|b\|_{\mathbb{L}} \ \text{ on } \ [0,2\pi].$$

Choosing $c_1 = A + \frac{\pi}{4}\,\|b\|_{\mathbb{L}}$ and taking into account (3.3) we verify that (3.4) holds. According to (3.1) and (3.2) this implies that

$$\rho'(t) = b(t) \ge f(t, \sigma(t)) \ \text{ for a.e. } t \in [0,2\pi].$$

Furthermore, with respect to (2.22) we have $\sigma(0) = \sigma(2\pi)$ and $\rho(0+) = \rho(0) = \rho(2\pi) = \rho(2\pi-)$ and hence, by Definition 1.1, the functions (σ, ρ) are lower functions of (1.1). □

The following result is supplementary to Proposition 3.2.

3.3. Proposition. *Assume that there are* $a \in \mathbb{R}$, $A \in \mathbb{R}$, $\tau \in [0, 2\pi)$ *and* $b \in \mathbb{L}[0, 2\pi]$ *such that*

$$a < 0, \quad \overline{b} = 0 \tag{3.6}$$

and

$$f(t,x) \le a + b(t) \quad \textit{for a.e. } t \in [0,2\pi] \quad \textit{and all} \quad x \in [A(t),B(t)], \tag{3.7}$$

where

$$A(t) = A + a\,h(t,\tau),\ \ B(t) = A(t) + \frac{\pi}{2}\|b\|_{\mathbb{L}} \quad \textit{for} \ \ t \in [0,2\pi] \tag{3.8}$$

and

$$h(t,\tau) = \begin{cases} \dfrac{t(2\pi - 2\tau + t)}{2} & \textit{if} \ \ 0 \le t \le \tau \le 2\pi, \\ \dfrac{(2\pi - t)(2\tau - t)}{2} & \textit{if} \ \ 0 \le \tau \le t \le 2\pi. \end{cases} \tag{3.9}$$

Then there exist lower functions (σ,ρ) *of* (1.1) *fulfilling*

$$A(t) \le \sigma(t) \le B(t) \ \ \textit{on} \ \ [0,2\pi]. \tag{3.10}$$

Proof. Let us put $\beta(t) = a + b(t)$ a.e. on $[0,2\pi]$. Then, using (3.9) and Proposition 2.10, we get for the solution (σ,ρ) of the problem (2.21), (2.22)

$$\begin{aligned} \sigma(t) &= c_1 - 2\pi a\, g(t,\tau) + \frac{a}{2}\, t\,(t - 2\pi) + \int_0^{2\pi} g(t,s) b(s)\,\mathrm{d}s \\ &= c_1 + a\,h(t,\tau) + \int_0^{2\pi} g(t,s) b(s)\,\mathrm{d}s \ \text{ on } \ [0,2\pi]. \end{aligned}$$

Thus, if we put again $c_1 = A + \frac{\pi}{4}\|b\|_{\mathbb{L}}$ and take into account (3.5) and (3.8), we obtain (3.10). Furthermore, we have $\rho(0+) \le \rho(0) = \rho(2\pi) = \rho(2\pi-)$ and since by (3.6) $\rho^{\text{sing}} = -2\pi\, a\, \chi_{(\tau,2\pi]}$ is nondecreasing on $[0,2\pi]$, we can complete the proof similarly as that of Proposition 3.2. □

3.4. Remark. Notice that the function h defined in (3.9) fulfils the estimates

$$-\frac{\pi^2}{2} \le -\frac{(\pi - \tau)^2}{2} \le h(t,\tau) \le \frac{\tau(2\pi - \tau)}{2} \le \frac{\pi^2}{2}$$

on $[0,2\pi] \times [0,2\pi]$.

The next assertion provides conditions which ensure the existence of lower functions of (1.1) and which rely upon Corollary 2.7 where α is restricted to the interval $(0,\frac{1}{2}]$ and a need not be nonpositive. (Notice that the proofs of Proposition 3.2 and 3.3 do not admit the case $a > 0$.)

3.5. Proposition. *Assume that there are* $a \in \mathbb{R}$, $A \in \mathbb{R}$, $\tau \in [0,2\pi)$, $\delta \in [0,\infty)$, $\alpha \in (0,\frac{1}{2}]$ *and* $b \in \mathbb{L}[0,2\pi]$ *such that*

$$a \le M, \quad \overline{b} = 0 \tag{3.11}$$

and

$$f(t,x) \le -\alpha^2 x + a + b(t) \quad \text{for a.e. } t \in [0,2\pi] \quad \text{and all} \quad x \in [A(t),B(t)], \tag{3.12}$$

where

$$M = \alpha^2\, A + \frac{\alpha\, \|b\|_{\mathbb{L}}}{4\sin(\alpha\pi)}, \tag{3.13}$$

$$A(t) = A + g(t,\tau)\,\delta, \;\; B(t) = A(t) + \frac{\|b\|_{\mathbb{L}}}{2\alpha\sin(\alpha\pi)} \;\; \text{for} \;\; t \in [0,2\pi] \tag{3.14}$$

and g is given by (2.19).

Then there exist lower functions (σ,ρ) of (1.1) *fulfilling* (3.10).

Proof. Define $\beta(t) = M + b(t)$ for a.e. $t \in [0,2\pi]$. The assumptions (3.11) and (3.12) imply

$$f(t,x) \le -\alpha^2 x + \beta(t) \;\; \text{for a.e. } t \in [0,2\pi] \;\; \text{and all} \;\; x \in [A(t),B(t)].$$

Furthermore, by Corollary 2.7 there is a unique solution (σ,ρ) of (2.18), σ is given by (2.15) and (2.19), $\rho^{\text{sing}} = \delta\chi_{(\tau,2\pi]}$ and $\rho(0+) \le \rho(0) = \rho(2\pi) = \rho(2\pi-)$. Moreover, we have

$$\int_0^{2\pi} g(t,s)\,\mathrm{d}s = \frac{1}{\alpha^2} \;\text{on}\;\; [0,2\pi]$$

and therefore

$$\sigma(t) = g(t,\tau)\,\delta + \frac{M}{\alpha^2} + \int_0^{2\pi} g(t,s)b(s)\,\mathrm{d}s \;\text{on}\;\; [0,2\pi].$$

Now, Lemma 3.1 together with the estimates (2.20) yield

$$\Big|\int_0^{2\pi} g(t,s)b(s)\,\mathrm{d}s\Big| \le \frac{\|b\|_{\mathbb{L}}}{4\alpha\sin(\alpha\pi)} \;\text{on}\;\; [0,2\pi].$$

Consequently, the inequalities

$$g(t,\tau)\,\delta + \frac{M}{\alpha^2} - \frac{\|b\|_{\mathbb{L}}}{4\alpha\sin(\alpha\pi)} \le \sigma(t) \le g(t,\tau)\,\delta + \frac{M}{\alpha^2} + \frac{\|b\|_{\mathbb{L}}}{4\alpha\sin(\alpha\pi)} \tag{3.15}$$

are valid on $[0,2\pi]$. According to (3.13) and (3.14) we can verify that σ satisfies (3.10). This together with (3.11) and (3.12) mean that

$$f(t,\sigma(t)) \le -\alpha^2\sigma(t) + \beta(t) = \rho'(t)$$

holds a.e. on $[0,2\pi]$, i.e. (σ,ρ) are lower functions of (1.1). ◻

The case $\alpha > \frac{1}{2}$ is dealt with by the following immediate corollary of Proposition 3.5.

3.6. Corollary. *Assume that there are* $a \in \mathbb{R}$, $A \in [0, \infty)$, $\tau \in [0, 2\pi)$, $\delta \in [0, \infty)$, $\alpha \in (\frac{1}{2}, \infty)$ *and* $b \in \mathbb{L}[0, 2\pi]$ *such that* (3.11) *and* (3.12) *are satisfied with*

$$M = \frac{1}{8}(2\,A + \|b\|_{\mathbb{L}}), \tag{3.16}$$

$$A(t) = A + g(t, \tau)\,\delta, \quad B(t) = A(t) + \|b\|_{\mathbb{L}} \ \text{on} \ [0, 2\pi] \tag{3.17}$$

and g *given by* (2.19) *with* $\alpha = \frac{1}{2}$.

Then there exist lower functions (σ, ρ) *of* (1.1) *fulfilling* (3.10). □

3.7. Remark. Let us note that if the first inequality in (3.11) falls, i.e. there is $\mu > 0$ such that $a = M + \mu$, then we have to replace $[A(t), B(t)]$ by $[A(t) + \frac{\mu}{\alpha^2}, B(t) + \frac{\mu}{\alpha^2}]$ in (3.12) and in (3.10) to keep the validity of the conclusion of Proposition 3.5. This follows from the estimates

$$g(t, \tau)\,\delta + \frac{a}{\alpha^2} - \frac{\|b\|_{\mathbb{L}}}{4\alpha \sin(\alpha\pi)} \le \sigma(t) \le g(t, \tau)\,\delta + \frac{a}{\alpha^2} + \frac{\|b\|_{\mathbb{L}}}{4\alpha \sin(\alpha\pi)} \ \text{on} \ [0, 2\pi]$$

which can be derived similarly as (3.15) putting $\beta(t) = a + b(t)$ a.e. on $[0, 2\pi]$.

3.8. Remark. Let the assumptions of Proposition 3.5 be satisfied and let $\delta = 0$. Then all the intervals $[A(t), B(t)]$, $t \in [0, 2\pi]$, reduce to $[A, B]$, where

$$B = A + \frac{\|b\|_{\mathbb{L}}}{2\alpha \sin(\alpha\pi)} = \frac{2\,M}{\alpha^2} - A.$$

Further, assume for the simplicity that $A = 0$ and $a = M$. Then (3.12) has the form $f(t, x) \le -\alpha^2\,x + M + b(t)$ for a.e. $t \in [0, 2\pi]$ and all $x \in [0, B]$, which implies

$$f(t, x) \le b(t) \ \text{ for a.e. } t \in [0, 2\pi] \ \text{ and all } \ x \in [\frac{B}{2}, B].$$

Thus, Proposition 3.2 can be also applied to show the existence of lower functions of the problem (1.1) whenever $\frac{B}{2} \ge \frac{\pi}{2}\,\|b\|_{\mathbb{L}}$, i.e. whenever $2\,\alpha\pi\,\sin(\alpha\pi) \le 1$. The function $\varphi(\alpha) = 2\,\alpha\pi\,\sin(\alpha\pi)$ is increasing on $[0, \frac{1}{2}]$, $\varphi(0) = 0$, $\varphi(\frac{1}{2}) = \pi$, which yields that there is exactly one $\alpha_1 \in (0, \frac{1}{2})$ ($\alpha_1 \approx 0.235817$) such that $\varphi(\alpha_1) = 1$. It follows that for $\alpha \in (0, \alpha_1]$ both Proposition 3.2 and Proposition 3.5 guarantee the existence of lower functions (σ, ρ) for (1.1), however Proposition 3.2 states more precise localization of σ ($\sigma(t) \in [\frac{B}{2}, B]$ for all $t \in [0, 2\pi]$). On the other hand, for $\alpha \in (\alpha_1, \frac{1}{2}]$, Proposition 3.2 need not work, in general. Having in mind Corollary 3.6, we can conclude that the conditions (3.11) and (3.12) are proper for getting lower functions provided $\alpha > \alpha_1$, only.

3.9. Proposition. *Assume that there are* $a \in \mathbb{R}$, $A \in \mathbb{R}$, $\tau \in [0, 2\pi)$, $\delta \in [0, \infty)$, $\alpha \in (0, \infty)$ *and* $b \in \mathbb{L}[0, 2\pi]$ *such that*

$$a \le -M, \quad \overline{b} = 0 \tag{3.18}$$

and

$$f(t, x) \le \alpha^2 x + a + b(t) \quad \text{for a.e. } t \in [0, 2\pi] \ \text{ and all } \ x \in [A(t), B(t)], \tag{3.19}$$

where

$$M = \alpha^2\, A + \frac{\alpha \|b\|_{\mathbb{L}} \cosh(\alpha\pi)}{4 \sinh(\alpha\pi)}, \tag{3.20}$$

$$A(t) = A + g(t, \tau)\, \delta, \ \ B(t) = A(t) + \frac{\|b\|_{\mathbb{L}} \cosh(\alpha\pi)}{2\alpha \sinh(\alpha\pi)} \ \text{for} \ \ t \in [0, 2\pi] \tag{3.21}$$

and g *is given by* (2.16).

Then there exist lower functions (σ, ρ) *of* (1.1) *fulfilling* (3.10).

Proof. We can proceed similarly as in the proof of Proposition 3.5. In particular, by Corollary 2.5 the problem (2.14) with $\beta(t) = -M + b(t)$ a.e. on $[0, 2\pi]$ possesses a unique solution (σ, ρ), where σ is given by (2.15) and (2.16), $\rho^{\text{sing}} = \delta \chi_{(\tau, 2\pi]}$ and $\rho(0+) \le \rho(0) = \rho(2\pi) = \rho(2\pi-)$. In view of (2.16) we have

$$\int_0^{2\pi} g(t, s) \mathrm{d}s = -\frac{1}{\alpha^2} \ \text{on} \ \ [0, 2\pi].$$

Further, (2.17) and Lemma 3.1 yield

$$\Big| \int_0^{2\pi} g(t, s) b(s) \mathrm{d}s \Big| \le \|b\|_{\mathbb{L}} \frac{\cosh(\alpha\pi)}{4\alpha \sinh(\alpha\pi)} \ \text{on} \ \ [0, 2\pi].$$

Hence, the inequalities

$$g(t, \tau)\, \delta + \frac{M}{\alpha^2} - \frac{\|b\|_{\mathbb{L}} \cosh(\alpha\pi)}{4\alpha \sinh(\alpha\pi)} \le \sigma(t) \le g(t, \tau)\, \delta + \frac{M}{\alpha^2} + \frac{\|b\|_{\mathbb{L}} \cosh(\alpha\pi)}{4\alpha \sinh(\alpha\pi)} \tag{3.22}$$

are true for all $t \in [0, 2\pi]$. Thus, as in the proof of Proposition 3.5, we can conclude that σ satisfies (3.10) and (σ, ρ) are lower functions of (1.1). □

3.10. Remark. If $a = -M + \mu$ for some $\mu > 0$, then to keep the validity of the conclusion of Proposition 3.9 we have to replace $[A(t), B(t)]$ by $[A(t) - \frac{\mu}{\alpha^2}, B(t) - \frac{\mu}{\alpha^2}]$ in (3.19) and in (3.10). This follows from the estimates

$$g(t, \tau)\, \delta - \frac{a}{\alpha^2} - \frac{\|b\|_{\mathbb{L}} \cosh(\alpha\pi)}{4\alpha \sinh(\alpha\pi)} \le \sigma(t) \le g(t, \tau)\, \delta - \frac{a}{\alpha^2} + \frac{\|b\|_{\mathbb{L}} \cosh(\alpha\pi)}{4\alpha \sinh(\alpha\pi)}$$

which are valid on $[0, 2\pi]$ and can be derived similarly as (3.22) when setting $\beta(t) = a + b(t)$ for a.e. $t \in [0, 2\pi]$.

3.11. Remark. Similarly to Remark 3.8 where we have compared the applicability of Propositions 3.2 and 3.5, we can compare also Propositions 3.2 and 3.9. Indeed, let the assumptions of Proposition 3.9 be satisfied with $\delta = 0$, $A = 0$ and $a = -M$. Denote $B = \frac{2M}{\alpha^2}$. Then the relation (3.19) reduces to

$$f(t,x) \le \alpha^2 x - M + b(t) \quad \text{for a.e. } t \in [0, 2\pi] \quad \text{and all} \quad x \in [0, B].$$

This means that $f(t,x) \le b(t)$ for a.e. $t \in [0, 2\pi]$ and all $x \in [0, \frac{B}{2}]$ and so Proposition 3.2 can be applied to show the existence of lower functions of the problem (1.1) whenever $2\,\alpha\pi\,\tanh(\alpha\pi) \le 1$. The function $\varphi(\alpha) = 2\,\alpha\pi\,\tanh(\alpha\pi)$ is increasing on $[0, \infty)$, $\varphi(0) = 0$, $\lim_{\alpha\to\infty} \varphi(\alpha) = \infty$. Hence, there is exactly one $\alpha_2 \in (0, \infty)$ ($\alpha_2 \approx 0.24564$) such that $\varphi(\alpha_2) = 1$. It follows that for $\alpha \in (0, \alpha_2]$ both Proposition 3.2 and Proposition 3.9 guarantee the existence of lower functions (σ, ρ) for (1.1), however Proposition 3.2 gives a better estimate for σ ($\sigma(t) \in [0, \frac{B}{2}]$ for all $t \in [0, 2\pi]$). On the other hand, for $\alpha \in (\alpha_2, \infty)$, Proposition 3.2 need not work, in general. To summarize, Proposition 3.9 is useful for $\alpha > \alpha_2$, only.

4 . Upper functions

In this section we reformulate the assertions of Section 3 to obtain conditions ensuring the existence of upper functions of (1.1). In the consequence of the duality of the definitions of lower and upper functions (see Definition 1.1) their proofs may be omitted.

4.1. Proposition. *Assume that there are $a \in \mathbb{R}$, $A \in \mathbb{R}$ and $b \in \mathbb{L}[0, 2\pi]$ such that*

$$a \ge 0, \quad \bar{b} = 0 \tag{4.1}$$

and

$$f(t,x) \ge a + b(t) \quad \textit{for a.e. } t \in [0, 2\pi] \quad \textit{and all} \quad x \in [A, B], \tag{4.2}$$

where B is given by (3.3).

Then there exist upper functions (σ, ρ) of the problem (1.1) *with the property* (3.4). □

4.2. Proposition. *Assume that there are $a \in \mathbb{R}$, $A \in \mathbb{R}$, $\tau \in [0, 2\pi)$ and $b \in \mathbb{L}[0, 2\pi]$ such that*

$$a > 0, \quad \bar{b} = 0 \tag{4.3}$$

and

$$f(t,x) \ge a + b(t) \quad \textit{for a.e. } t \in [0, 2\pi] \quad \textit{and all} \quad x \in [A(t), B(t)], \tag{4.4}$$

with the same $A(t)$ and $B(t)$ as in Proposition 3.3.

Then there exist upper functions (σ, ρ) of (1.1) *fulfilling* (3.10). □

4.3. Proposition. *Assume that there are* $a \in \mathbb{R}$, $A \in \mathbb{R}$, $\tau \in [0, 2\pi)$, $\delta \in (-\infty, 0]$, $\alpha \in (0, \frac{1}{2}]$ *and* $b \in \mathbb{L}[0, 2\pi]$ *such that*

$$(4.5) \qquad a \geq M, \quad \overline{b} = 0$$

and

$$(4.6) \qquad f(t, x) \geq -\alpha^2 x + a + b(t) \quad \text{for a.e. } t \in [0, 2\pi] \text{ and all } x \in [A(t), B(t)],$$

with the same M, $A(t)$, *and* $B(t)$ *as in Proposition* 3.5.

Then there exist upper functions (σ, ρ) *of* (1.1) *fulfilling* (3.10).

4.4. Corollary. *Assume that there are* $a \in \mathbb{R}$, $A \in (-\infty, 0]$, $\tau \in [0, 2\pi)$, $\delta \in (-\infty, 0]$, $\alpha \in (\frac{1}{2}, \infty)$ *and* $b \in \mathbb{L}[0, 2\pi]$ *such that* (4.5) *and* (4.6) *are satisfied with the same* M, $A(t)$ *and* $B(t)$ *as in Corollary* 3.6.

Then there exist upper functions (σ, ρ) *of* (1.1) *fulfilling* (3.10). □

4.5. Proposition. *Assume that there are* $a \in \mathbb{R}$, $A \in \mathbb{R}$, $\tau \in [0, 2\pi)$, $\delta \in (-\infty, 0]$, $\alpha \in (0, \infty)$ *and* $b \in \mathbb{L}[0, 2\pi]$ *such that*

$$(4.7) \qquad a \geq -M, \quad \overline{b} = 0$$

and

$$(4.8) \qquad f(t, x) \geq \alpha^2 x + a + b(t) \quad \text{for a.e. } t \in [0, 2\pi] \text{ and all } x \in [A(t), B(t)],$$

with the same M, $A(t)$ *and* $B(t)]$ *as in Proposition* 3.9.

Then there exist upper functions (σ, ρ) *of* (1.1) *fulfilling* (3.10).

4.6. Remark. If $a = M - \mu$ (or $a = -M - \mu$) for some $\mu > 0$, then the conclusion of Proposition 4.3 (or Proposition 4.5) remains valid if we replace the interval $[A(t), B(t)]$ by $[A(t) - \frac{\mu}{\alpha^2}, B(t) - \frac{\mu}{\alpha^2}]$ in (4.6) and (3.10) (or by $[A(t) + \frac{\mu}{\alpha^2}, B(t) + \frac{\mu}{\alpha^2}]$ in (4.8) and (3.10)).

Similarly, putting $A = 0$, $a = M$ (or $a = -M$) and $\delta = 0$ and arguing like in Remark 3.8 (or Remark 3.11) we can deduce that the conditions (4.5), (4.6) (or the conditions (4.7), (4.8)) are profitable for proving the existence of upper functions provided $\alpha > \alpha_1$ (or $\alpha > \alpha_2$), only.

5 . Applications to Lazer-Solimini singular problems

Criteria on the existence of lower and upper functions presented in sections 3 and 4 together with Theorem 1.3 enable us to formulate a number of various theorems on the existence of solutions to the problem (1.1). In this text we shall restrict ourselves just to two examples of such results. In particular, we will consider possibly singular problems of the attractive type

$$u'' + g(u) = e(t), \quad u(0) = u(2\pi), \quad u'(0) = u'(2\pi) \tag{5.1}$$

and of the repulsive type

$$u'' - g(u) = e(t), \quad u(0) = u(2\pi), \quad u'(0) = u'(2\pi), \tag{5.2}$$

where

$$g \in \mathbb{C}(0,\infty) \quad \text{and} \quad e \in \mathbb{L}[0,2\pi] \tag{5.3}$$

and it is allowed that

$$\limsup_{x\to 0+} g(x) = \infty.$$

The problem (5.1) has been studied by Lazer and Solimini in [9] for $e \in \mathbb{C}[0,2\pi]$ and g positive. In [12, Corollary 3.3], their existence result has been extended to $e \in \mathbb{L}[0,2\pi]$ essentially bounded from above. Here, we bring conditions for the existence of solutions to (5.1) without boundedness of e.

5.1. Theorem. *Assume* (5.3) *and let there exist* $A_1, A_2 \in (0,\infty)$ *such that*

$$g(x) \ge \overline{e} \quad \textit{for all} \quad x \in [A_1, B_1], \tag{5.4}$$

$$g(x) \le \overline{e} \quad \textit{for all} \quad x \in [A_2, B_2], \tag{5.5}$$

where

$$B_1 - A_1 = B_2 - A_2 = \frac{\pi}{2}\|e - \overline{e}\|_{\mathbb{L}} \tag{5.6}$$

and $A_2 \ge B_1$.

Then the problem (5.1) *has a solution* u *such that* $A_1 \le u(t) \le B_2$ *on* $[0,2\pi]$.

Proof. Define for a.e. $t \in [0,2\pi]$,

$$f(t,x) = e(t) - \begin{cases} g(A_1) & \text{if } x < A_1, \\ g(x) & \text{if } x \ge A_1. \end{cases}$$

Then $f \in \mathrm{Car}([0,2\pi]\times\mathbb{R})$. Furthermore, by (5.4) and (5.6), f satisfies (3.1)-(3.3) with $a = 0$, $b(t) = e(t) - \overline{e}$ a.e. on $[0,2\pi]$ and $[A,B] = [A_1,B_1]$. Hence, by Proposition 3.2 there exist lower functions $(\sigma_1,\rho_1) \in \mathbb{AC}[0,2\pi]\times\mathbb{AC}[0,2\pi]$ of (1.1) such that $\sigma_1(t) \in [A_1,B_1]$ for all $t \in [0,2\pi]$. Similarly, (5.5), (5.6) and Proposition 4.1 yield the existence of upper functions $(\sigma_2,\rho_2) \in \mathbb{AC}[0,2\pi]\times\mathbb{AC}[0,2\pi]$ of (1.1) such that $\sigma_2(t) \in [A_2,B_2]$ on $[0,2\pi]$. Now, since $A_2 \ge B_1$, we have $\sigma_1(t) \le \sigma_2(t)$ on $[0,2\pi]$ and the assertion (I) of Theorem 1.3 gives the existence of a desired solution u to (1.1) which is also a solution to (5.1), of course. □

Classical Lazer and Solimini's considerations [9] of the repulsive problem (5.2) have been extended by several authors (see e.g. [1], [2], [4], [7], [10] and [17]). Provided $g \in \mathbb{C}(0,\infty)$, e is essentially bounded on $[0,2\pi]$ and

$$\lim_{x\to 0+} g(x) = \infty, \tag{5.7}$$

$$\lim_{x\to 0+} \int_x^1 g(\xi)\,\mathrm{d}\xi = \infty, \tag{5.8}$$

$$\liminf_{x\to\infty} \frac{g(x)}{x} \ge -\frac{1}{4}, \quad \liminf_{x\to\infty} \frac{1}{x^2}\int_1^x g(\xi)\,\mathrm{d}\xi > -\frac{1}{8}, \tag{5.9}$$

$$\text{there is } d>0 \text{ such that } g(x) \le -\overline{e} \text{ for all } x\in[d,\infty), \tag{5.10}$$

Omari and Ye proved in [10, Theorem 1.2] the existence of a solution to (5.2). Here we present a related result, where e need not be essentially bounded.

5.2. Theorem. *Assume* (5.3), (5.8),

$$\liminf_{x\to 0+} g(x) > -\infty, \tag{5.11}$$

and

$$\liminf_{x\to\infty} \frac{g(x)}{x} > -\frac{1}{4}. \tag{5.12}$$

Furthermore, let there exist $A_1, A_2 \in (0,\infty)$ such that

$$g(x) \le -\overline{e} \quad \textit{for all} \quad x\in[A_1,B_1], \tag{5.13}$$

$$g(x) \ge -\overline{e} \quad \textit{for all} \quad x\in[A_2,B_2] \tag{5.14}$$

and (5.6) *are true and $A_1 \ge B_2$.*

Then the problem (5.2) *has a positive solution u such that $u(t_u)\in[A_2,B_1]$ for some $t_u\in[0,2\pi]$.*

5.3. Remark. If e is essentially bounded from below on $[0,2\pi]$, then according to Proposition 4.1, the condition (5.8) implies (5.14) with $A_2 = B_2$.

5.4. Remark. Notice that if $g\in\mathbb{C}(0,\infty)$ satisfies (5.8) then

$$\limsup_{x\to 0+} g(x) = \infty,$$

which implies the existence of a sequence $\{\varepsilon_n\}_{n=1}^{\infty} \subset (0,1)$ such that

$$\lim_n \varepsilon_n = 0, \quad g(\varepsilon_n) > 0 \quad \text{for all} \quad n\in\mathbb{N}. \tag{5.15}$$

For the proof of Theorem 5.2 we will need the following two lemmas, where we deal with the auxiliary family of problems

$$u'' = g_n(u) + e(t), \quad u(0) = u(2\pi), \quad u'(0) = u'(2\pi), \tag{5.16}$$

where $n \in \mathbb{N}$,

$$g_n(x) = \begin{cases} 0 & \text{if } x < 0, \\ g(\varepsilon_n)\frac{x}{\varepsilon_n} & \text{if } x \in [0, \varepsilon_n], \\ g(x) & \text{if } x > \varepsilon_n \end{cases} \tag{5.17}$$

and ε_n are from (5.15).

5.5. Lemma. *Assume that* $g \in \mathbb{C}(0,\infty)$ *satisfies* (5.8), (5.11) *and* (5.12) *and* g_n, $n \in \mathbb{N}$, *are given by* (5.17). *Then there exist* $\eta \in (0, \frac{1}{4})$ *and* $C \geq 0$ *such that*

$$g_n(x)x \geq -(\frac{1}{4} - \eta)x^2 - C|x| \quad \textit{for all } x \in \mathbb{R} \textit{ and all } n \in \mathbb{N}. \tag{5.18}$$

Proof. By (5.12), there are $\eta \in (0, \frac{1}{4})$ and $A \in (1, \infty)$ such that

$$\frac{g(x)}{x} \geq -(\frac{1}{4} - \eta) \quad \text{for all } x \geq A. \tag{5.19}$$

Put

$$p(x) = \begin{cases} 0 & \text{if } x < 0, \\ g(A)\dfrac{x}{A} & \text{if } x \in [0, A], \\ g(x) & \text{if } x > A \end{cases} \tag{5.20}$$

and $q_n(x) = g_n(x) - p(x)$ on $\mathbb{R}$. In virtue of (5.11), there is $C \geq 0$ such that $q_n(x) \geq -C$ for all $x \in \mathbb{R}$ and all $n \in \mathbb{N}$. Thus, since according to (5.19) and (5.20) we also have

$$p(x) \geq -(\frac{1}{4} - \eta)x \quad \text{for all } x \in \mathbb{R},$$

we deduce that (5.18) is true. □

5.6. Lemma. *Assume that* g *and* g_n, $n \in \mathbb{N}$, *are as in Lemma* 5.5. *Then for any* $r > 0$ *and any* $e \in \mathbb{L}[0, 2\pi]$ *there exists* $R > 0$ *such that*

$$u(t) \leq R \textit{ on } [0, 2\pi] \tag{5.21}$$

holds for all $n \in \mathbb{N}$ *and all solutions* u *of* (5.16) *with the property*

$$\min_{t \in [0,2\pi]} u(t) \leq r. \tag{5.22}$$

Proof. Assume that (5.21) does not hold. Then we can choose a subsequence $\{g_k\}_{k=1}^{\infty}$ of the sequence $\{g_n\}_{n=1}^{\infty}$ and sequence of solutions $\{u_k\}_{k=1}^{\infty}$ of the corresponding problems (5.16) satisfying (5.22) and

$$\lim_{k} \max_{t\in[0,2\pi]} u_k(t) = \infty. \tag{5.23}$$

In particular, for any $k \in \mathbb{N}$, there is $t_k \in [0, 2\pi]$ such that

$$u_k(t_k) = r.$$

Furthermore, if we extend the functions u_k, $k \in \mathbb{N}$, and e to functions 2π-periodic on $\mathbb{R}$, we get that

$$u_k''(t) = g_k(u_k(t)) + e(t) \quad \text{for a.e. } t \in \mathbb{R} \tag{5.24}$$

is true for any $k \in \mathbb{N}$.

On the other hand, if we multiply (5.24) by $u_k(t)$, integrate from t_k to $t_k + 2\pi$ and if we take into account Lemma 5.5, we get that there exist $\eta \in (0, \frac{1}{4})$ and $C > 0$ such that for any $k \in \mathbb{N}$

$$\begin{aligned} \|u_k'\|_{\mathbb{L}_2}^2 &= -\int_{t_k}^{t_k+2\pi} g_k(u_k(s))u_k(s)\,\mathrm{d}s - \int_{t_k}^{t_k+2\pi} e(s)u_k(s)\,\mathrm{d}s \\ &\le (\frac{1}{4} - \eta)\|u_k\|_{\mathbb{L}_2}^2 + C\|u_k\|_{\mathbb{L}} + \|e\|_{\mathbb{L}}\|u_k\|_{\mathbb{C}} \end{aligned}$$

holds. Furthermore,

$$\|u_k\|_{\mathbb{C}} \le |u_k(t_k)| + \int_{t_k}^{t_k+2\pi} |u_k'(s)|\,\mathrm{d}s = r + \sqrt{2\pi}\|u_k'\|_{\mathbb{L}_2}. \tag{5.25}$$

Thus,

$$(\|u_k'\|_{\mathbb{L}_2} - \|e\|_{\mathbb{L}}\sqrt{\frac{\pi}{2}})^2 \le (\frac{1}{4} - \eta)\|u_k\|_{\mathbb{L}_2}^2 + C\|u_k\|_{\mathbb{L}} + \|e\|_{\mathbb{L}}r + \frac{\pi}{2}\,\|e\|_{\mathbb{L}}^2. \tag{5.26}$$

Inserting $u_k(t) \equiv v_k(t) + r$ on $\mathbb{R}$ into (5.26), we obtain

$$\frac{(\|v_k'\|_{\mathbb{L}_2} - c)^2}{\|v_k\|_{\mathbb{L}_2}^2} \le \frac{1}{4} - \eta + \frac{a}{\|v_k\|_{\mathbb{L}}} + \frac{b}{\|v_k\|_{\mathbb{L}_2}^2}, \tag{5.27}$$

where $a, b, c \in \mathbb{R}$ do not depend on k. Now, (5.23), (5.25) and (5.26) yield

$$\lim_{k} \|v_k'\|_{\mathbb{L}_2} = \infty \quad \text{and} \quad \lim_{k} \|v_k\|_{\mathbb{L}_2} = \infty. \tag{5.28}$$

Since $v_k(t_k) = v(t_k + 2\pi) = 0$, by Scheeffer's inequality (cf. [8, II.2] or [13, p.207])

$$\|v_k\|^2_{\mathbb{L}_2} \le 4\|v_k'\|^2_{\mathbb{L}_2},$$

we get

$$\frac{(\|v_k'\|_{\mathbb{L}_2} - c)^2}{\|v_k\|^2_{\mathbb{L}_2}} \ge \frac{(\|v_k'\|_{\mathbb{L}_2} - c)^2}{4\|v_k'\|^2_{\mathbb{L}_2}}.$$

Therefore with respect to (5.27) and (5.28) we have

$$\frac{1}{4} = \lim_k \frac{(\|v_k'\|_{\mathbb{L}_2} - c)^2}{4\|v_k'\|^2_{\mathbb{L}_2}} \le \lim_k \Big(\frac{1}{4} - \eta + \frac{a}{\|v_k\|_{\mathbb{L}}} + \frac{b}{\|v_k\|^2_{\mathbb{L}_2}}\Big) = \frac{1}{4} - \eta,$$

a contradiction. □

Proof of Theorem 5.2. Let $R \ge B_1$ be a constant given by Lemma 5.6 for $r = B_1$. In virtue of (5.3) and (5.11) we have $g_* := \inf_{x\in(0,R]} g(x) \in \mathbb{R}$. Put $K = \|e\|_{\mathbb{L}} + |g_*|$ and

$$K^* = K\,\|e\|_{\mathbb{L}} + \int_{A_2}^{R} |g(x)|\mathrm{d}x.$$

It follows from (5.8) and Remark 5.4 that we can choose $\varepsilon \in \{\varepsilon_n\}_{n=1}^{\infty}$ such that $\varepsilon \in (0, A_2)$ and

$$\int_{\varepsilon}^{A_2} g(x)\mathrm{d}x > K^* \quad \text{and} \quad g(\varepsilon) > 0. \tag{5.29}$$

Define

$$\widetilde{g}(x) = \begin{cases} 0 & \text{if } x < 0, \\ g(\varepsilon)\frac{x}{\varepsilon} & \text{if } x \in [0,\varepsilon), \\ g(x) & \text{if } x \in [\varepsilon, R), \\ g(R) & \text{if } x \ge R \end{cases}$$

and

$$f(t,x) = e(t) + \widetilde{g}(x) \quad \text{for a.e. } t \in [0,2\pi] \quad \text{and all} \quad x \in \mathbb{R}.$$

We can see that $f \in \mathrm{Car}([0,2\pi]\times\mathbb{R})$ and (3.1)-(3.3) are satisfied with $a = 0$, $b(t) = e(t) - \overline{e}$ a.e. on $[0,2\pi]$ and $[A,B] = [A_1,B_1]$ wherefrom, due to Proposition 3.2, the existence of lower functions $(\sigma_1,\rho_1) \in \mathbb{AC}[0,2\pi]\times\mathbb{AC}[0,2\pi]$ of (1.1) follows and $\sigma_1(t) \in [A_1,B_1]$ for all $t \in [0,2\pi]$. Similarly, Proposition 4.1 ensures the existence of upper functions $(\sigma_2,\rho_2) \in \mathbb{AC}[0,2\pi]\times\mathbb{AC}[0,2\pi]$ of (1.1) with $\sigma_2(t) \in [A_2,B_2]$ on $[0,2\pi]$. Since $A_1 \ge B_2$, the assertion (II) of Theorem 1.3 (with $m(t) = g_* + e(t)$ a.e.

on $[0,2\pi]$) implies that (1.1) has a solution u such that $u(t_u)\in[A_2,B_1]$ for some $t_u\in[0,2\pi]$ and $\|u'\|_{\mathbb{C}}\le K$. By Lemma 5.6 we have $u(t)\le R$ for all $t\in[0,2\pi]$. It remains to show that $u(t)\ge\varepsilon$ holds on $[0,2\pi]$.

Let t_0 and $t_1\in[0,2\pi]$ be such that

$$u(t_0)=\min_{t\in[0,2\pi]}u(t)\quad\text{and}\quad u(t_1)=\max_{t\in[0,2\pi]}u(t).$$

Clearly, $A_2\le u(t_1)\le R$. With respect to the periodic boundary conditions we have $u'(t_0)=u'(t_1)=0$. Now, multiplying the differential relation $u''(t)=e(t)+\widetilde{g}(u(t))$ by $u'(t)$ and integrating over $[t_0,t_1]$, we get

$$0=\int_{t_0}^{t_1}u''(t)u'(t)\,\mathrm{d}t=\int_{t_0}^{t_1}e(t)u'(t)\,\mathrm{d}t+\int_{t_0}^{t_1}\widetilde{g}(u(t))u'(t)\,\mathrm{d}t,$$

i.e.

$$\int_{u(t_0)}^{u(t_1)}\widetilde{g}(x)\,\mathrm{d}x=-\int_{t_0}^{t_1}e(t)u'(t)\,\mathrm{d}t\le K\,\|e\|_{\mathbb{L}}.$$

Further,

$$\int_{u(t_0)}^{A_2}\widetilde{g}(x)\,\mathrm{d}x\le K\,\|e\|_{\mathbb{L}}+\int_{A_2}^{R}|\widetilde{g}(x)|\,\mathrm{d}x=K^*$$

which, with respect to (5.29), is possible only if $u(t_0)\ge\varepsilon$. Thus, u is a solution to (5.2). □

5.7. Example. Notice that, the function

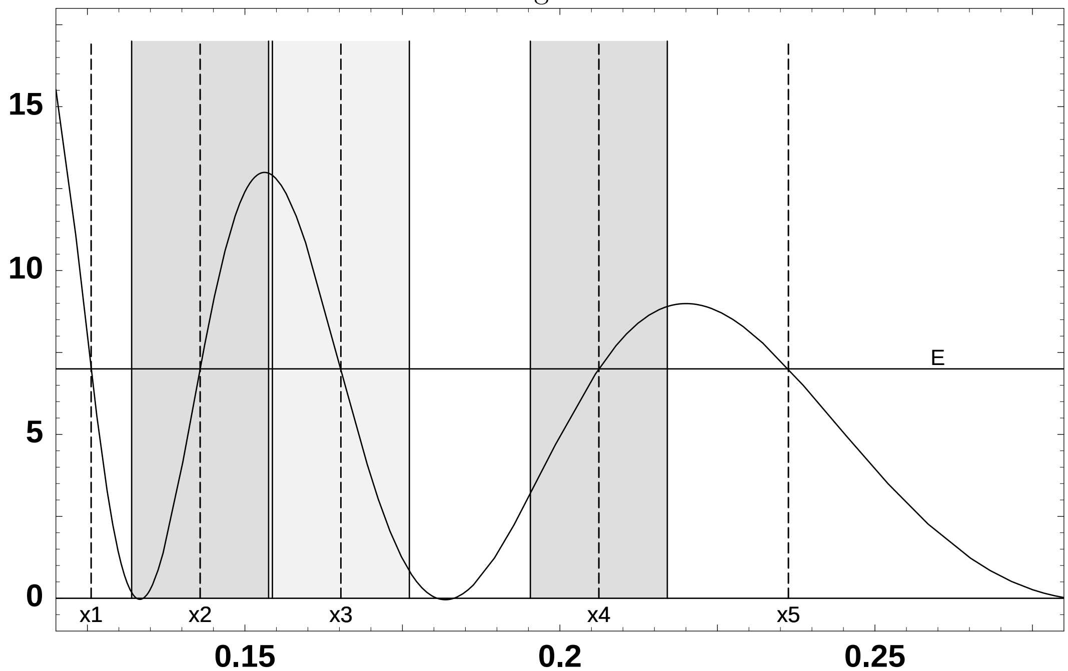
$$g(x)=-0.24x+\frac{1+\sin(\dfrac{\pi}{x})}{x},\quad x\in(0,\infty),$$

verifies the assumptions (5.3), (5.8), (5.11) and (5.12) of Theorem 5.2, while it does not satisfy the condition (5.7) required by Omari and Ye in [10, Theorem 1.2]. Now, let $E=7$ and let us restrict ourselves to $e\in\mathbb{L}[0,2\pi]$ such that $\overline{e}=-E$. It may be shown that the equation $g(x)=E$ has exactly 5 roots x_i, $i=1,2,\dots,5$, in the interval $[0.12,\infty)$ (see Figure 1). In particular, we have $x_1\approx 0.125587$, $x_2\approx 0.142891$, $x_3\approx 0.165230$, $x_4\approx 0.206177$, $x_5\approx 0.236265$, $g(x)>E$ on $(x_2,x_3)\cup(x_4,x_5)$ and $g(x)<E$ on $(x_1,x_2)\cup(x_3,x_4)\cup(x_5,\infty)$. Let

$$d<\frac{x_3-x_2}{2} \tag{5.30}$$

and assume in addition that $\|e-\overline{e}\|_{\mathbb{L}}\le\frac{2}{\pi}\,d$. We have $x_2-x_1>d$ and $x_{i+1}-x_i>2d$ for $i=2,3,4$. We can apply Theorem 5.1 to obtain the existence of solutions u_1 and

Figure 1:



u_2 of the problem (5.2) such that $u_1(t) \in [x_2 - d, x_2 + d]$ and $u_2(t) \in [x_4 - d, x_4 + d]$ on $t \in [0, 2\pi]$, i.e. $u_1(t) < u_2(t)$ on $[0, 2\pi]$. Moreover, by Theorem 5.2 there is a further solution u_3 of (5.2) such that $u_3(t_3) \in [x_3 - d, x_3 + d]$ for some $t_3 \in [0, 2\pi]$. In virtue of (5.30) it means that u_3 can coincide neither with u_1 nor with u_2.

5.8. Remark. In all of the above mentioned results concerning the problem (5.2) the assumption (5.8) is substantial. The existence theorem, which does not need (5.8) has been proved in [12, Corollary 3.7].

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