SUBDIRECTLY IRREDUCIBLE NON-IDEMPOTENT LEFT SYMMETRIC LEFT DISTRIBUTIVE GROUPOIDS

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Abstract. We study groupoids satisfying the identities $x \cdot xy = y$ and $x \cdot yz = xy \cdot xz$. Particularly, we focus our attention at subdirectly irreducible ones, find a description a charecterize small ones.

1. Introduction

A *left symmetric left distributive groupoid* (shortly an *LSLD groupoid*) is a non-empty set equipped with a binary operation (usually denoted multiplicatively) satisfying the equations:

(left symmetry) $\quad x \cdot xy = y$

(left distributivity) $\quad x \cdot yz = xy \cdot xz$

An *LSLDI groupoid* is an idempotent LSLD groupoid, i.e. an LSLD groupoid satisfying the equation $xx = x$. For example, given a group G, the derived operation $x * y = xy^{-1}x$, usually called the *core* of G, is left symmetric, left distributive and idempotent. LSLDI groupoids were introduced in [10] and they (and their applications) were studied by several authors mainly in 1970's and 1980's. A reader is referred to the survey [8] for details. For a long time, it seemed that the non-idempotent case did not play any significant role in self-distributive structures (whether symmetric or not). This was certainly true for the two-sided case, but recently, due to the book [2] of P. Dehornoy, one-sided non-idempotent selfdistributive groupoids enjoyed certain attention. The purpose of the present note is to continue the investigations of non-idempotent LSLD groupoids started in [4] and, in particular, to get a better insight into the structure of subdirectly irreducible ones. Our main results are Theorems 4.2, 4.3 and 5.9.

As far as we know, the only papers concerning non-idempotent LSLD groupoids are [4] and [9]. Subdirectly irreducible idempotent left symmetric medial groupoids were characterized by B. Roszkowska [7] and simple idempotent LSLD groupoids by D. Joyce [3].

Our notation is rather standard and usually follows the book [1]. A reader can look at [5] for various notions concerning groupoids (i.e. sets with a single binary operation).

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Let G be a groupoid. For every $a \in G$, we denote L_a the selfmapping of G defined by $L_a(x) = ax$ for all $x \in G$ and call it the *left translation by a* in G. By an *involution* we mean a permutation of order two.

Lemma 1.1. *Let G be a groupoid. Then*

(1) *G is LSLD, iff every left translation in G is either the identity, or an involutive automorphism of G;*
(2) *if G is LSLD, then $L_{\varphi(a)} = \varphi L_a \varphi^{-1}$ for every $a \in G$ and every automorphism φ of G.*
(3) *if G is LSLD, then the mapping $\lambda : a \mapsto L_a$ is a homomorphism of G into the core of the symmetric group over G.*

Proof. (1) Left symmetry says that every left translation L_a satisfies $L_a^2 = id_G$. Left distributivity says that every L_a is an endomorphism.

(2) Since $\varphi L_a(b) = \varphi(ab) = \varphi(a)\varphi(b) = L_{\varphi(a)}\varphi(b)$ for every $a, b \in G$, we have $\varphi L_a = L_{\varphi(a)}\varphi$ and thus $L_{\varphi(a)} = \varphi L_a \varphi^{-1}$.

(3) It follows from (2) for $\varphi = L_a$ that $L_{ab} = L_a L_b L_a^{-1} = L_a L_b L_a$. □

Example. The following are all (up to an isomorphism) two-element LSLD groupoids (one idempotent, the other not).

S	0	1
0	0	1
1	0	1

T	0	1
0	$\widetilde{0}$	0
$\widetilde{0}$	$\widetilde{0}$	0

Example. The following are all (up to an isomorphism) three-element idempotent LSLD groupoids. $\mathbf{S}_1$ is a right zero groupoid, $\mathbf{S}_2$ is a dual differential groupoid and $\mathbf{S}_3$ is a commutative distributive quasigroup and it forms the smallest Steiner triple system. $\mathbf{S}_3$ is simple and $\mathbf{S}_2$ is subdirectly irreducible.

$\mathbf{S}_1$	0	1	2
0	0	1	2
1	0	1	2
2	0	1	2

$\mathbf{S}_2$	0	1	2
0	0	2	1
1	0	1	2
2	0	1	2

$\mathbf{S}_3$	0	1	2
0	0	2	1
1	2	1	0
2	1	0	2

Example. The following are all (up to an isomorphism) three-element non-idempotent LSLD groupoids. Both are subdirectly irreducible.

$\mathbf{T}_1$	e	0	$\widetilde{0}$
e	e	0	$\widetilde{0}$
$0, \widetilde{0}$	e	$\widetilde{0}$	0

$\mathbf{T}_2$	e	0	$\widetilde{0}$
e	e	$\widetilde{0}$	0
$0, \widetilde{0}$	e	$\widetilde{0}$	0

Example. We define an operation $\circ$ on the Prüfer 2-group $\mathbb{Z}_{2^\infty}(+)$ by $x \circ y = 2x - y + a$, where $a \in \mathbb{Z}_{2^\infty}$ is an element satisfying $a \neq 0 = 2a$. The groupoid $\mathbb{Z}_{2^\infty}(\circ)$ is an infinite subdirectly irreducible idempotent-free LSLD groupoid.

A non-empty subset J of a groupoid G is called a *left ideal* of G, if $ab \in J$ for every $a \in G$ and $b \in J$. Note that the set consisting of all left ideals in a left symmetric groupoid and the empty set is closed under intersection, union and complements. If $\{a\}$ is a left ideal of G, we call the element *a right zero*.

Let G be an LSLD groupoid. We put

$$Id_G = \{x \in G : xx = x\} \quad \text{and} \quad K_G = \{x \in G : xx \neq x\}.$$

Each of Id_G and K_G is either empty or a left ideal of G. Further, we define relations

$$p_G = \{(x, y) \in G \times G : L_x = L_y\}$$
$$q_G = \{(a, b) \in Id_G \times Id_G : L_a|_{K_G} = L_b|_{K_G}\} \cup id_G$$
$$ip_G = \{(x, xx) : x \in G\} \cup id_G$$

and a mapping $o_G : G \to G$ by $o_G(x) = xx$.

Lemma 1.2. *Let G be an LSLD groupoid. Then*

(1) *p_G and q_G are congruences of G and $ip_G \subseteq p_G$;*
(2) *ip_G is a congruence of G, G/ip_G is idempotent and ip_G is the smallest congruence such that the corresponding factor is idempotent; moreover, every non-trivial block of ip_G is isomorphic to $\mathbf{T}$;*
(3) *o_G is either the identity, or an involutive automorphism of G.*

Proof. (1) The relation p_G is the kernel of the homomorphism λ from Lemma 1.1(3), hence it is a congruence.

The relation q_G is an equivalence, so consider $a, b \in Id_G$ such that $L_a|_{K_G} = L_b|_{K_G}$. Then $L_{az}|_{K_G} = L_{bz}|_{K_G}$ for all $z \in G$, since for every $k \in K_G$ we have $az \cdot k = a(z \cdot ak) = a(z \cdot bk) = b(z \cdot bk) = bz \cdot k$ (because $z \cdot bk \in K_G$). And also $L_{za}|_{K_G} = L_{zb}|_{K_G}$ for all $z \in G$, because for every $k \in K_G$ we have $za \cdot k = z(a \cdot zk) = z(b \cdot zk) = zb \cdot k$ (because $zk \in K_G$). Consequently, q_G is a congruence.

Finally, $xy = x(x \cdot xy) = xx \cdot (x \cdot xy) = xx \cdot y$ for every $x, y \in G$ and thus $ip_G \subseteq p_G$.

(2) Since $xx \cdot xx = x \cdot xx = x$ for every $x \in G$, the relation ip_G is symmetric and transitive and every non-trivial block of ip_G consists of two elements and thus is isomorphic to $\mathbf{T}$. Further, $xz = xx{\cdot}z$ for every $z \in G$ due to (1) and $(zx, z{\cdot}xx) \in ip_G$ because $z \cdot xx = zx \cdot zx$; hence ip_G is a congruence. Clearly, G/ip_G is idempotent and ip_G is the smallest congruence with this property.

(3) o_G is an involution (or the identity) according to (2) and $o_G(xy) = xy \cdot xy = x \cdot yy = xx \cdot yy = o_G(x)o_G(y)$ for all $x, y \in G$. □

Corollary 1.3. $\mathbf{T}$ *is the only (up to an isomorphism) simple non-idempotent LSLD groupoid.*

Let G be a groupoid, $e \notin G$ and $\varphi : G \to G$. We denote $G[\varphi]$ the groupoid defined on the set $G \cup \{e\}$ so that G is a subgroupoid of $G[\varphi]$, e is a right zero and $ex = \varphi(x)$ for every $x \in G$.

Lemma 1.4. *Let G be an LSLD groupoid, $e \notin G$ and $\varphi : G \to G$. Then*

(1) *$G[\varphi]$ is an LSLD groupoid, iff $\varphi = id_G$ or φ is an involutive automorphism of G with $L_x = L_{\varphi(x)}$ for all $x \in G$;*
(2) *$G[id_G]$ and $G[o_G]$ are LSLD groupoids and $G[o_G][id_{G[o_G]}]$, $G[id_G][o_{G[id_G]}]$ are isomorphic.*

Proof. This is a straightforward calculation. □

Note that the three-element non-idempotent LSLD groupoids are isomorphic to $\mathbf{T}[id_{\mathbf{T}}]$ and $\mathbf{T}[o_{\mathbf{T}}]$, respectively. One can check that $(\mathbf{T}[id_{\mathbf{T}}])[o_{\mathbf{T}[id_{\mathbf{T}}]}]$ is the only four-element subdirectly irreducible non-idempotent LSLD groupoid.

The following technical lemmas become useful later.

Lemma 1.5. *Let G be an LSLD groupoid and $\varphi \in \{id_G, o_G\}$. Then the set $A_\varphi = \{a \in G : L_a = \varphi\}$ is either empty, or a left ideal of G.*

Proof. Let $a \in A_\varphi$. By Lemma 1.1 $L_{xa} = L_x L_a L_x$ for every $x \in G$. If $L_a = \varphi = id_G$, then $L_{xa} = L_x L_x = id_G = \varphi$. If $L_a = \varphi = o_G$, then $L_{xa}(y) = x o_G(xy) = x(xy \cdot xy) = x(x \cdot yy) = o_G(y)$ for every $y \in G$ and thus $L_{xa} = o_G = L_a$. Hence A_φ is a left ideal. □

Lemma 1.6. *Let G be an LSLD groupoid and J a left ideal of G. Then the relation $\rho_J = ((ip_G)|_J) \cup id_G$ is a congruence of G.*

Proof. The claim follows from Lemma 1.2. □

Lemma 1.7. *Let G be an LSLD groupoid and $a \in G$ a right zero. Then*

(1) *$x \cdot ay = a \cdot xy$ and $xy = ax \cdot y$ for all $x, y \in G$;*
(2) *the relation $\nu_a = \{(x, ax) : x \in G\} \cup id_G$ is a congruence of G; moreover, every non-trivial block of ν_a has two elements.*

Proof. (1) is calculated as follows: $x \cdot ay = xa \cdot xy = a \cdot xy$ and $ax \cdot y = (ax)(a \cdot ay) = a(x \cdot ay) = a(a \cdot xy) = xy$.

(2) Clearly, ν_a is both reflexive and symmetric and it follows from (1) that ν_a is compatible with the multiplication of G. We show that ν_a is transitive. If $(x, y) \in \nu_a$, $(y, z) \in \nu_a$, $x \neq y \neq z$, then $y = ax$ and $z = ay = a \cdot ax = x$ and thus $(x, z) \in \nu_a$. The rest becomes clear now. □

Lemma 1.8. *Let G be an LSLD groupoid and let ρ be a congruence of K_G such that $(u, v) \in \rho$ implies $(au, av) \in \rho$ and $(ua \cdot z, va \cdot z) \in \rho$ for all $a \in Id_G$ and $z \in K_G$. Define a relation σ on Id_G by $(a, b) \in \sigma$ iff $(au, bv) \in \rho$ for every pair $(u, v) \in \rho$. Then $\rho \cup \sigma$ is a congruence of G.*

Proof. This straightforward calculation is omitted. □

2. Basic facts about subdirectly irreducible LSLD groupoids

It is well known that a groupoid G is *subdirectly irreducible* (shortly *SI*), if and only if G possesses a smallest non-trivial congruence (called the *monolith* of G), i.e. a congruence $\mu_G \neq id_G$ such that $\mu_G \subseteq \nu$ for every congruence $\nu \neq id_G$ on G.

Lemma 2.1. *Let G be an SI non-idempotent LSLD groupoid. Then*

(1) *if $J \subseteq K_G$ is a left ideal, then $J = K_G$;*
(2) *ip_G is the monolith of G;*
(3) *$L_a|_{K_G} \neq L_b|_{K_G}$ for every $a, b \in Id_G$ with $a \neq b$; in other words, $q_G = id_G$;*
(4) *$\varphi|_{K_G} \neq \psi|_{K_G}$ for all automorphisms φ, ψ of G with $\varphi \neq \psi$.*

Proof. (1) Let $J \subset K_G$ be a left ideal. Then $J' = K_G \setminus J$ is a left ideal too and $\rho_J, \rho_{J'}$ are non-trivial congruences, since both J and J' contain at least two elements. However, $\rho_J \cap \rho_{J'} = id_G$ yields a contradiction with subdirect irreducibility of G.

(2) We have $\mu_G \subseteq ip_G$. Put $J = \{u \in K_G : (u, uu) \in \mu_G\}$. Then J is a left ideal, because μ_G is a congruence, and thus $J = K_G$ and $\mu_G = ip_G$.

(3) According to Lemma 1.2(1), q_G is a congruence. It is trivial, because $q_G \cap ip_G = id_G$.

(4) Assume that $\varphi|_{K_G} = \psi|_{K_G}$ and we show that $\varphi|_{Id_G} = \psi|_{Id_G}$ too. Observe that $\varphi|_{K_G} = \psi|_{K_G}$ iff $\varphi^{-1}|_{K_G} = \psi^{-1}|_{K_G}$, because every automorphism of G maps

K_G onto itself. Now, given $a \in Id_G$ and $u \in K_G$, we have $\varphi(a)u = \varphi(a)\varphi\varphi^{-1}(u) = \varphi(a\varphi^{-1}(u))$ and, because $a\varphi^{-1}(u) = a\psi^{-1}(u) \in K_G$, we have also $\varphi(a\varphi^{-1}(u)) = \psi(a\psi^{-1}(u)) = \psi(a)u$. Thus $L_{\varphi(a)}|_{K_G} = L_{\psi(a)}|_{K_G}$ and, by (3), $\varphi(a) = \psi(a)$. □

Proposition 2.2. *Let G be a non-idempotent LSLD groupoid and H a subgroupoid of G such that $K_G \subseteq H$. Assume that H is subdirectly irreducible. Then G is subdirectly irreducible, iff $q_G = id_G$.*

Proof. The direct implication was proved in Lemma 2.1(3). So assume $q_G = id_G$ and let ρ be a non-trivial congruence on G. If $\rho|_H \neq id_H$, then $ip_H \subseteq \rho|_H$. But $ip_G = ip_H \cup id_G$ and thus $ip_G \subseteq \rho$. Hence assume that $\rho|_H = id_H$. If $(a, b) \in \rho$ for some $a, b \in Id_G$, $a \neq b$, then $au \neq bu$ for some $u \in K_G$ by Lemma 2.1(3) and we have $(au, bu) \in \rho|_{K_G} = id_{K_G}$, a contradiction. If $(a, u) \in \rho$ for some $a \in Id_G$ and $u \in K_G$, then $(a, uu) = (aa, uu) \in \rho$ and, again, $(u, uu) \in \rho|_{K_G} = id_{K_G}$, a contradiction. Consequently, G is subdirectly irreducible. □

Corollary 2.3. *Let G be a non-idempotent LSLD groupoid such that K_G is subdirectly irreducible. Then G is subdirectly irreducible, iff $q_G = id_G$.*

Lemma 2.4. *Let G be an SI non-idempotent LSLD groupoid and $a, b \in G$ right zeros. Then*

(1) *$L_a \in \{id_G, o_G\}$;*
(2) *$a = b$, iff $L_a = L_b$;*
(3) *G contains at most two right zeros.*

Proof. (1) Let ν_a be the congruence from Lemma 1.7. If $\nu_a = id_G$, then $L_a = id_G$. If $\nu_a \neq id_G$, then $\mu_G = ip_G \subseteq \nu_a$ and thus $L_a|_{K_G} = o_G|_{K_G}$. Hence $L_a = o_G$ according to Lemma 2.1(4).

The statement (2) follows from Lemma 2.1(3) and (3) is an immediate consequence of (1) and (2). □

Lemma 2.5. *Let G be an SI non-idempotent LSLD groupoid and let $a \in G$ be a right zero. Then $H = G \setminus \{a\}$ is an SI non-idempotent LSLD groupoid and it contains no right zero b with $L_b = L_a|_H$.*

Proof. Clearly, H is a left ideal of G and thus a subgroupoid of G. Moreover, if ρ is a non-trivial congruence of H, then $\sigma = \rho \cup \{(a, a)\}$ is a (non-trivial) congruence of G (because $L_a \in \{id_G, o_G\}$) and thus $ip_G = \mu_G \subseteq \sigma$. So $ip_H \subseteq \rho$ and H is subdirectly irreducible. Finally, if b is a right zero in H, then it is also a right zero in G and so $L_b \neq L_a|_H$ by Lemma 2.4. □

Lemma 2.6. *Let G be an SI non-idempotent LSLD groupoid and $\varphi \in \{id_G, o_G\}$. Then $G[\varphi]$ is subdirectly irreducible, iff G contains no right zero a with $L_a = \varphi$.*

Proof. The direct implication follows from Lemma 2.5. On the contrary, if G contains no right zero a with $L_a = \varphi$, then $A_\varphi = \emptyset$ (by Lemmas 1.5 and 2.1(3) $|A_\varphi| \leq 1$, hence any element b with $L_b = \varphi$ is a right zero), so $q_{G[\varphi]} = id$ and Proposition 2.2 applies. □

Corollary 2.7. *Let G be an SI non-idempotent LSLD groupoid with no right zero. Then*

$$G,\ G[id_G],\ G[o_G] \text{ and } G[id_G][o_{G[id_G]}]$$

are pairwise non-isomorphic SI LSLD groupoids.

Corollary 2.8. *Let G be an SI non-idempotent LSLD groupoid and let A be the set of right zeros in G. Then $|A| \le 2$, $H = G \setminus A$ is a left ideal of G, H is an SI non-idempotent LSLD groupoid with no right zero and G is isomorphic to exactly one of*

$$H,\ H[id_H],\ H[o_H] \text{ and } H[id_H][o_{H[id_H]}].$$

3. Groupoids of involutions

Let ε be a binary relation on a non-empty set X. We denote $\mathrm{Inv}(X,\varepsilon)$ the set of all permutations φ of X such that $\varphi^2 = id_X$ and $(x,y) \in \varepsilon$ implies $(\varphi(x),\varphi(y)) \in \varepsilon$. It is easy to see that $\mathrm{Inv}(X,\varepsilon)$ is a subgroupoid of the core of the symmetric group over X and thus it is an idempotent LSLD groupoid.

An equivalence ε is called a *pairing* (a *semipairing*, resp.), if every block of ε consists of (at most, resp.) two elements. Let $\alpha(m) = |\mathrm{Inv}(m,\varepsilon)|$, where ε is a pairing on a cardinal number m ($\alpha(m)$ is defined for even and infinite cardinals only).

Proposition 3.1. *$\alpha(2) = 2$, $\alpha(4) = 6$ and $\alpha(m) = 2\alpha(m-2) + (m-2)\alpha(m-4)$ for every even $6 \le m < \omega$. Further, $\alpha(m) = 2^m$ for every infinite m.*

Proof. Assume that m is finite even and the blocks of ε are the sets $\{2k, 2k+1\}^2$, $k = 0, \dots, \frac{m}{2} - 1$. The claim is trivial for $m \in \{2,4\}$, so assume $m \ge 6$. Let $I_k = \{\varphi \in \mathrm{Inv}(m,\varepsilon) : \varphi(0) = k\}$ for $0 \le k \le m-1$. Then $\mathrm{Inv}(m,\varepsilon) = \bigcup_{k=0}^{m-1} I_k$ and I_k's are pairwise disjoint. If $\varphi \in I_0$, then $\varphi(1) = 1$. If $\varphi \in I_1$, then $\varphi(1) = 0$. Consequently, $|I_0| = |I_1| = \alpha(m-2)$. On the other hand, if $\varphi \in I_k$ for $k \ge 2$, then $\varphi(1) = k'$, where $k' \ne k$ is such that $(k,k') \in \varepsilon$, and thus $\varphi(k) = 0$, $\varphi(k') = 1$. Hence $|I_k| = \alpha(m-4)$ and $|\mathrm{Inv}(m,\varepsilon)| = 2\alpha(m-2) + (m-2)\alpha(m-4)$.

If m is infinite, consider all involutions of the form $(x_1\ y_1)(x_2\ y_2)\dots$, where $\{x_1,y_1\}, \{x_2,y_2\}, \dots$ are pairwise different blocks of ε. They belong to $\mathrm{Inv}(m,\varepsilon)$ and thus $\alpha(m) \ge 2^m$. Hence $\alpha(m) = 2^m$. □

m	2	4	6	8	10	12	14	16	18	20
$\alpha(m)$	2	6	20	76	312	1384	6512	32400	168992	921184

For every semipairing ε on X there is a unique mapping $o_\varepsilon \in \mathrm{Inv}(X,\varepsilon)$ such that $(x, o_\varepsilon(x)) \in \varepsilon$ and $o_\varepsilon(x) = x$ iff $\{x\}$ is a one-element block of ε. It is easy to see that id_X and o_ε are right zeros in $\mathrm{Inv}(X,\varepsilon)$ and that $id_X * \varphi = \varphi$ and $o_\varepsilon * \varphi = \varphi$ for every $\varphi \in \mathrm{Inv}(X,\varepsilon)$. Let $\mathrm{Inv}^-(X,\varepsilon) = \mathrm{Inv}(X,\varepsilon) \setminus \{id_X, o_\varepsilon\}$. Clearly, it is either empty, or a left ideal of $\mathrm{Inv}(X,\varepsilon)$.

Finally, let $\mathrm{Aut}_2(G) = \{\varphi \in \mathrm{Aut}(G) : \varphi^2 = id\}$. If G is an LSLD groupoid, then $\mathrm{Aut}_2(G)$ is a subgroupoid of $\mathrm{Inv}(G, ip_G)$, $L_x \in \mathrm{Aut}_2(G)$ for every $x \in G$ and the mapping $x \mapsto L_x$ is a homomorphism of G into $\mathrm{Aut}_2(G)$. Let $\mathrm{Aut}_2^-(G) = \mathrm{Aut}_2(G) \cap \mathrm{Inv}^-(G, ip_G)$.

Proposition 3.2. *Let G be an SI non-idempotent LSLD groupoid with at least one idempotent element. Then the mapping*

$$\eta : Id_G \to \mathrm{Aut}_2(K_G), \qquad a \mapsto L_a|_{K_G}$$

is an injective homomorphism.

Proof. It follows from Lemmas 1.1 and 2.1(3). □

Corollary 3.3. *Let G be an SI LSLD groupoid with $|K_G| = m \neq 0$. Then*

$$|Id_G| \leq \alpha(m) \quad \text{and} \quad |G| \leq \alpha(m) + m.$$

It will be shown in the next section that the upper bound on $|Id_G|$ is best possible.

4. A description of subdirectly irreducible LSLD groupoids

Lemma 4.1. *Let K be an idempotent-free LSLD groupoid and I a subgroupoid of $\mathrm{Aut}_2(K)$. Put $G = I \cup K$. Then the following conditions are equivalent.*

1. *The operations of I and K can be extended onto G so that G becomes an LSLD groupoid with $\varphi \cdot u = \varphi(u)$ for all $\varphi \in I$, $u \in K$.*
2. *$L_u \varphi L_u \in I$ for all $\varphi \in I$, $u \in K$.*

Moreover, if the conditions are satisfied, the operation of G is uniquely determined and $u \cdot \varphi = L_u \varphi L_u$ for all $\varphi \in I$, $u \in K$.

Proof. Clearly, $u\varphi \in I = Id_G$ for every $u \in K$, $\varphi \in I$. Since $u(\varphi v) = (u\varphi)(uv)$ for every $u, v \in K$, $\varphi \in I$, we have $L_u(\varphi(v)) = (u\varphi)(L_u(v))$ and thus $u\varphi = L_u\varphi(L_u)^{-1} = L_u\varphi L_u$. Indeed, this is possible, iff $L_u\varphi L_u \in I$ for all $\varphi \in I$, $u \in K$. We omit the straightforward calculation showing that the resulting groupoid G is LSLD. □

The groupoid G from Lemma 4.1 will be denoted by $I \sqcup K$. The groupoid $\mathrm{Aut}_2(K) \sqcup K$ will be called the *full extension* of K and denoted $\mathrm{Full}(K)$.

$I \sqcup K$	ψ	v
φ	$\varphi\psi\varphi$	$\varphi(v)$
u	$L_u\psi L_u$	uv

Theorem 4.2. *Let G be an SI non-idempotent LSLD groupoid. Then there exists an injective homomorphism $\eta : G \to \mathrm{Full}(K_G)$ such that*

$$\eta(u) = u \text{ for every } u \in K_G \quad \text{and} \quad \eta(a) = L_a|_{K_G} \text{ for every } a \in Id_G.$$

Thus G is isomorphic (via η) to the subgroupoid $\eta(Id_G) \sqcup K_G$ of $\mathrm{Full}(K_G)$.

Proof. It is straightforward to check that η is a homomorphism and it is injective according to Proposition 3.2. □

Remark. Let K be an idempotent-free LSLD groupoid and assume the set $\mathcal{S}$ of SI subgroupoids G of $\mathrm{Full}(K)$ with $K_G = K$. The set $\mathcal{S}$ is non-empty, iff $\mathrm{Full}(K) \in \mathcal{S}$; in this case, the set $\mathcal{S}$ has minimal elements, say $H_1, \dots, H_k$, and it follows from Proposition 2.2 that $G \in \mathcal{S}$, iff G is a subgroupoid of $\mathrm{Full}(K)$ and $H_i \subseteq G$ for at least one $1 \leq i \leq k$.

Theorem 4.3. *The following conditions are equivalent for an idempotent-free LSLD groupoid K:*

1. *There exists an SI LSLD groupoid G with $K_G = K$.*
2. *The groupoid $\mathrm{Full}(K)$ is SI.*
3. *The groupoid $\mathrm{Full}^-(K)$ is SI.*
4. *If ρ is a non-trivial $\mathrm{Aut}_2(K)$-invariant congruence of K, then $ip_K \subseteq \rho$.*

Proof. The implication (1) $\Rightarrow$ (2) follows from Proposition 2.2, (2) $\Rightarrow$ (3) follows from Lemma 2.5 and (3) $\Rightarrow$ (1) is trivial.

Now, assume that (4) is true and let σ be a non-trivial congruence of $\mathrm{Full}(K)$. If $\sigma|_K \neq id_K$, then $ip_K \subseteq \sigma$ by (4) and thus $\mathrm{Full}(K)$ is SI. So assume that $\rho = \sigma|_K = id_K$. If $(\varphi, \psi) \in \sigma$ for some $\varphi, \psi \in \mathrm{Aut}_2(K)$, $\varphi \neq \psi$, then there is at least one $u \in K$ with $\varphi(u) \neq \psi(u)$ and we have $(\varphi(u), \psi(u)) \in \rho$, a contradiction. Thus $(\varphi, u) \in \sigma$ for some $\varphi \in \mathrm{Aut}_2(K)$, $u \in K$. In this case, $(\varphi, uu) \in \sigma$ and so $(u, uu) \in \rho$, a contradiction again.

Finally, assume (2) and consider a non-trivial $\mathrm{Aut}_2(K)$-invariant congruence ρ of K. Define a relation σ on $\mathrm{Aut}_2(K)$ by $(\varphi, \psi) \in \sigma$ iff $(\varphi(u), \psi(v)) \in \rho$ for every pair $(u, v) \in \rho$. According to Lemma 1.8, $\rho \cup \sigma$ is a congruence of $\mathrm{Full}(K)$ and so $ip_K \subseteq \rho$. □

A groupoid K satisfying the conditions of Theorem 4.3 will be called *pre-SI*.

Example. Let ε be a pairing on a non-empty set K. We equip the set K with an operation such that $L_u = o_\varepsilon$ for every $u \in K$. Clearly, K is an idempotent-free LSLD groupoid and $\mathrm{Aut}_2(K) = \mathrm{Inv}(K, \varepsilon)$. Using Theorem 4.3, we prove that K is pre-SI and thus $G = \mathrm{Full}(K)$ is an SI LSLD groupoid of size $\alpha(|K_G|) + |K_G|$ (cf. Corollary 3.3).

Let ρ be a non-trivial $\mathrm{Aut}_2(K)$-invariant congruence on K. We claim that $ip_K = o_\varepsilon \subseteq \rho$. Indeed, if $(u, o_K(u)) \in \rho$ for some $u \in K$, then for every $v \in K$ the involution $\varphi = (u\ v)(o_K(u)\ o_K(v))$ belongs to $\mathrm{Aut}_2(K)$ and thus $(v, o_K(v)) \in \rho$. Thus $ip_K \subseteq \rho$. On the other hand, if $(u, v) \in \rho$, $u \neq v \neq o_K(u)$, then the involution $\psi = (v\ o_K(v))$ belongs to $\mathrm{Aut}_2(K)$ and thus $(u, o(v)) = (\psi(u), \psi(v)) \in \rho$ and so $(v, o(v)) \in \rho$.

Example. Consider the following four-element groupoid K.

K	0	$\widetilde{0}$	1	$\widetilde{1}$
$0, \widetilde{0}$	$\widetilde{0}$	0	$\widetilde{1}$	1
$1, \widetilde{1}$	0	$\widetilde{0}$	$\widetilde{1}$	1

One can check that K is an LSLD groupoid, $\mathrm{Aut}_2(K) = \{id_K, (0\,\widetilde{0}), (1\,\widetilde{1}), (0\,\widetilde{0})(1\,\widetilde{1})\}$ and the relation $\rho = \{(0, \widetilde{0}), (\widetilde{0}, 0)\} \cup id_K$ is an $\mathrm{Aut}_2(K)$-invariant congruence of K. However, $ip_K \not\subseteq \rho$ and thus K is not pre-SI.

5. Few idempotent elements

In this section, let G be a finite SI non-idempotent LSLD groupoid with $Id_G \neq \emptyset$ and r, s, α, β will denote non-negative integers.

Let $n = |Id_G|$ and $2m = |K_G|$. We put $K_1(a) = \{u \in K_G : au = u\}$, $K_2(a) = \{u \in K_G : au = uu\}$ and $K_3(a) = K_G \smallsetminus (K_1(a) \cup K_2(a))$ for every $a \in Id_G$.

Lemma 5.1. *$|K_1(a)|$, $|K_2(a)|$ are even numbers and $|K_3(a)|$ is divisible by 4.*

Proof. $|K_1(a)|$ is even, because $u \in K_1(a)$, iff $uu \in K_1(a)$ (and analogously for $|K_2(a)|$). Furthermore, the sets $\{v, vv, av, a \cdot vv\}$, $v \in K_3(a)$, are four-element and pairwise disjoint. □

Let $r(a) = \frac{1}{2}|K_1(a)|$ and $s(a) = \frac{1}{2}|K_2(a)|$. Hence $m - r(a) - s(a)$ is a (non-negative) even number.

Lemma 5.2. *$r(xa) = r(a)$ and $s(xa) = s(a)$ for all $a \in Id_G$, $x \in G$.*

Proof. If $v \in K_1(a)$, then $xa \cdot xv = x \cdot av = xv$ and so $xv \in K_1(xa)$. Conversely, if $w \in K_1(xa)$, then $xw = x(xa \cdot w) = (x \cdot xa)(xw) = a \cdot xw$ and so $xw \in K_1(a)$. Thus L_x maps bijectively $K_1(a)$ onto $K_1(xa)$ and, in particular, $r(a) = |K_1(a)| = |K_1(xa)| = r(xa)$. Analogously, $s(a) = s(xa)$. □

Let $I(r,s) = \{a \in Id_G : r(a) = r, s(a) = s\}$. Indeed, if $I(r,s) \neq \emptyset$, then $m-r-s$ is a non-negative even number. It follows from Lemma 5.2 that $I(r,s)$ is either empty, or a left ideal of G.

Lemma 5.3. (1) *If $r \geq m$ and $I(r,s) \neq \emptyset$, then $r = m$, $s = 0$ and $|I(r,s)| = 1$.*
(2) *If $s \geq m$ and $I(r,s) \neq \emptyset$, then $r = 0$, $s = m$ and $|I(r,s)| = 1$.*

Proof. (1) Since $m \geq r+s$, we have $r = m$ and $s = 0$. Consequently, $I(r,s) = I(m,0) = \{a \in Id_G : au = u \text{ for every } u \in K_G\}$, and hence $|I(r,s)| = 1$ by Lemma 2.1(3). (2) is analogous. □

Let $K(r,s,\alpha,\beta)$ be the set of all $u \in K_G$ such that $|\{a \in I(r,s) : u \in K_1(a)\}| = \alpha$ and $|\{a \in I(r,s) : u \in K_2(a)\}| = \beta$.

Lemma 5.4. *Either $K(r,s,\alpha,\beta) = \emptyset$, or $K(r,s,\alpha,\beta) = K_G$.*

Proof. Assume that $J = K(r,s,\alpha,\beta) \neq \emptyset$. We prove that J is a left ideal. Since $a \cdot xu = xu$ iff $xa \cdot u = u$ for every $u \in J$, $x \in G$, $a \in Id_G$, we have $L_x(\{b \in I(r,s) : b \cdot xu = xu\}) = \{c \in I(r,s) : cu = u\}$ (use the fact that $I(r,s)$ is a left ideal) and, in particular, $|\{b \in I(r,s) : xu \in K_1(b)\}| = \alpha$. Similarly, $|\{b \in I(r,s) : xu \in K_2(b)\}| = \beta$ and thus $xu \in J$. Consequently, $J = K_G$ by Lemma 2.1(1). □

Consequently, for every r,s there is a unique pair (α,β) such that $K(r,s,\alpha,\beta) = K_G$ and $K(r,s,\alpha',\beta') = \emptyset$ for all $(\alpha',\beta') \neq (\alpha,\beta)$.

Lemma 5.5. *If $K(r,s,\alpha,\beta) = K_G$, then $\alpha m = rt$ and $\beta m = st$, where $t = |I(r,s)|$.*

Proof. Since $|\{a \in I(r,s) : au = u\}| = \alpha$ and $|\{a \in I(r,s) : au = uu\}| = \beta$ for every $u \in K_G$, we have $|L| = 2\alpha m$, where $L = \{(a,u) \in I(r,s) \times K_G : au = u\}$. On the other hand, $|L| = 2rt$ by the definition of $I(r,s)$. Thus $\alpha m = rt$. Considering the set $\{(a,u) \in I(r,s) \times K_G : au = uu\}$, a similar proof yields $\beta m = st$. □

Lemma 5.6. *If $K(r,s,\alpha,\beta) = K_G$, $I(r,s) \neq \emptyset$ and the numbers m and $t = |I(r,s)|$ are relatively prime, then just one of the following cases takes place:*

(1) $r = s = \alpha = \beta = 0$.
(2) $r = m$, $s = 0$, $\alpha = 1$, $\beta = 0$ *and* $t = 1$.
(3) $r = 0$, $s = m$, $\alpha = 0$, $\beta = 1$ *and* $t = 1$.

Proof. By Lemma 5.5, $\alpha m = rt$ and $\beta m = st$. If $r = s = 0$, then obviously $\alpha = \beta = 0$. If $r \geq 1$, then m divides r and thus $r \geq m$. If $s \geq 1$, then m divides s and thus $s \geq m$. In both cases, Lemma 5.3 applies. □

Proposition 5.7. *If $I(r,s) \neq \emptyset$, $r+s \geq 1$ and the numbers m and $t = |I(r,s)|$ are relatively prime, then G contains a right zero.*

Proof. Choose α,β such that $K(r,s,\alpha,\beta) = K_G$. It follows from Lemma 5.6 that $t = 1$ and thus $I(r,s)$ consists of a right zero. □

Proposition 5.8. *If m is not divisible by any prime number $p \in \{2,\dots,n-2,n\}$, then either G contains a right zero, or $n = 3$, m is even and $u \neq au \neq uu$ for all $a \in Id_G$, $u \in K_G$.*

Proof. If $n = 1$, then $Id_G = \{a\}$ and a is a right zero; so we may assume that $n \geq 2$. Obviously, if $I(r,s) = \emptyset$ for all r, s with $r + s \geq 1$, then $u \neq au \neq uu$ for all $a \in Id_G$, $u \in K_G$, and thus m is divisible by 2 according to Lemma 5.1. Consequently, $2 = n - 1$ and thus $n = 3$.

So assume that there are r, s such that $r + s \geq 1$ and $t = |I(r,s)| \geq 1$. If m and t are relatively prime, then Lemma 5.7 yields the result. If p is a prime dividing both m and t, then $p \leq t \leq n$, and therefore $p = n - 1$, $t = n - 1$ and the only $a \in Id_G \setminus I(r,s)$ is a right zero. □

Theorem 5.9. *Let G be a finite SI non-idempotent LSLD groupoid with $|K_G| = 2m \geq 4$ and let p be the least prime divisor of m. If $|Id_G| < p$, then either Id_G contains precisely three elements which are not right zeros, or every element of Id_G is a right zero and thus $|Id_G| \leq 2$ and K_G is subdirectly irreducible.*

Proof. Let $H = G \setminus A$, where A is the set of all right zeros of G. According to Corollary 2.8, H is an SI LSLD groupoid with no right zeros. However, if $Id_H \neq \emptyset$, then H contains a right zero by Proposition 5.8, a contradiction. The rest follows from Corollary 2.8 too. □

6. Small subdirectly irreducible LSLD groupoids

In this section we apply the theory developed above to search for small SI non-idempotent LSLD groupoids. The procedure for finding all SI LSLD groupoids G with $m > 0$ non-idempotent elements follows.

(1) We find all $\frac{m}{2}$-element LSLDI groupoids.
(2) We find all m-element idempotent-free LSLD groupoids by extending groupoids found in the first step and check which of them are pre-SI (using Theorem 4.3).
(3) For each pre-SI groupoid K found in the second step, we characterize subgroupoids I of $\mathrm{Aut}_2^- K$ with the property 4.1(2) and check which $I \sqcup K$ are subdirectly irreducible.
(4) Each SI LSLD groupoid found in the third step can be extended by id_G, o_G, none or both (see Corollary 2.7).

Two non-idempotents. Let G be an SI LSLD groupoid with $|K_G| = 2$. Then $K_G \simeq \mathbf{T}$ and Id_G is either empty, or isomorphic to a subgroupoid of $\mathrm{Aut}_2(\mathbf{T}) = \mathrm{Inv}(\mathbf{T}, ip_{\mathbf{T}}) = \{id_{\mathbf{T}}, o_{\mathbf{T}}\}$. Hence

$$\mathbf{T},\ \mathbf{T}[id_{\mathbf{T}}],\ \mathbf{T}[o_{\mathbf{T}}] \quad\text{and}\quad \mathbf{T}[id_{\mathbf{T}}][o_{\mathbf{T}[id_{\mathbf{T}}]}]$$

are the only (up to an isomorphism) SI LSLD groupoids with two non-idempotent elements.

Four non-idempotents. Let G be an SI LSLD groupoid with $|K_G| = 4$. Then K_G/ip_{K_G} is isomorphic to $\mathbf{S}$, the only two-element LSLDI groupoid. Clearly, the following groupoids K_1, K_2, K_3 are the only (up to an isomorphism) 4-element idempotent-free LSLD groupoids:

K_1	0	$\widetilde{0}$	1	$\widetilde{1}$
$0, \widetilde{0}$	$\widetilde{0}$	0	$\widetilde{1}$	1
$1, \widetilde{1}$	$\widetilde{0}$	0	$\widetilde{1}$	1

K_2	0	$\widetilde{0}$	1	$\widetilde{1}$
$0, \widetilde{0}$	$\widetilde{0}$	0	1	$\widetilde{1}$
$1, \widetilde{1}$	0	$\widetilde{0}$	$\widetilde{1}$	1

K_3	0	$\widetilde{0}$	1	$\widetilde{1}$
$0, \widetilde{0}$	$\widetilde{0}$	0	$\widetilde{1}$	1
$1, \widetilde{1}$	0	$\widetilde{0}$	$\widetilde{1}$	1

K_1 and K_2 are pre-SI, K_3 is not (see the last example in the fourth section). Hence K_G is isomorphic to one of K_1, K_2. Now, we designate $a = (0\ \widetilde{0})$, $b = (1\ \widetilde{1})$, $c = (0\ 1)(\widetilde{0}\ \widetilde{1})$, $d = (0\ \widetilde{1})(\widetilde{0}\ 1)$ the elements of $I = \mathrm{Aut}_2^-(K_1) = \mathrm{Aut}_2^-(K_2)$. The multiplication table of I is

I	a	b	c	d
a	a	b	d	c
b	a	b	d	c
c	b	a	c	d
d	b	a	c	d

Thus I contains three non-trivial subgroupoids $I_1 = \{a, b\}$, $I_2 = \{c, d\}$ and $I_3 = \{a, b, c, d\}$. Neither K_1 nor K_2 is SI. Since both $I_1 \sqcup K_1$, $I_1 \sqcup K_2$ contain the left ideal $\{0, \widetilde{0}\}$, they are not SI. In $I_2 \sqcup K_1$, the element c is a right zero, because $L_x = o_{K_1}$ for every $x \in K_1$, and thus $L_x c L_x = c$; so $I_2 \sqcup K_1$ is not SI by Corollary 2.8. On the other hand, it is easy to check that $I_2 \sqcup K_2$, $I_3 \sqcup K_1$ and $I_3 \sqcup K_2$ are SI.

Proposition 6.1. *There are 12 (up to an isomorphism) SI LSLD groupoids with four non-idempotent elements:*

$$I_3 \sqcup K_1,\ I_2 \sqcup K_2,\ I_3 \sqcup K_2$$

and their extensions by right zeros.

Six non-idempotents. Let G be an SI LSLD groupoid with $|K_G| = 6$. Then K_G/ip_{K_G} is isomorphic to one of $\mathbf{S}_1$, $\mathbf{S}_2$, $\mathbf{S}_3$ (see the list of three-element LSLDI groupoids in the introduction). $\mathbf{S}_2$ cannot be isomorphic to K_G/ip_{K_G}, because the ip_{K_G}-block corresponding to the element 0 of $\mathbf{S}_2$ is always a proper left ideal inside K_G (every automorphism of G preserves this block), a contradiction with Lemma 2.1(1). Now, one can check that the following groupoids K_4, K_5, K_6, K_7 are the only (up to an isomorphism) 6-element idempotent-free LSLD groupoids such that their factorgroupoid over ip is one of $\mathbf{S}_1$, $\mathbf{S}_3$.

K_4	0	$\widetilde{0}$	1	$\widetilde{1}$	2	$\widetilde{2}$
$0, \widetilde{0}$	$\widetilde{0}$	0	$\widetilde{1}$	1	$\widetilde{2}$	2
$1, \widetilde{1}$	$\widetilde{0}$	0	$\widetilde{1}$	1	$\widetilde{2}$	2
$2, \widetilde{2}$	$\widetilde{0}$	0	$\widetilde{1}$	1	$\widetilde{2}$	2

K_5	0	$\widetilde{0}$	1	$\widetilde{1}$	2	$\widetilde{2}$
$0, \widetilde{0}$	$\widetilde{0}$	0	1	$\widetilde{1}$	2	$\widetilde{2}$
$1, \widetilde{1}$	0	$\widetilde{0}$	$\widetilde{1}$	1	2	$\widetilde{2}$
$2, \widetilde{2}$	0	$\widetilde{0}$	1	$\widetilde{1}$	$\widetilde{2}$	2

K_6	0	$\widetilde{0}$	1	$\widetilde{1}$	2	$\widetilde{2}$
$0, \widetilde{0}$	$\widetilde{0}$	0	$\widetilde{1}$	1	2	$\widetilde{2}$
$1, \widetilde{1}$	0	$\widetilde{0}$	$\widetilde{1}$	1	$\widetilde{2}$	2
$2, \widetilde{2}$	$\widetilde{0}$	0	1	$\widetilde{1}$	$\widetilde{2}$	2

K_7	0	$\widetilde{0}$	1	$\widetilde{1}$	2	$\widetilde{2}$
$0, \widetilde{0}$	$\widetilde{0}$	0	$\widetilde{2}$	2	$\widetilde{1}$	1
$1, \widetilde{1}$	$\widetilde{2}$	2	$\widetilde{1}$	1	$\widetilde{0}$	0
$2, \widetilde{2}$	$\widetilde{1}$	1	$\widetilde{0}$	0	$\widetilde{2}$	2

K_4 and K_5 are pre-SI, K_6 and K_7 aren't. Hence K_G is isomorphic to one of K_4, K_5. One can compute that $I = \mathrm{Inv}^-(K_4, ip_{K_4}) = \mathrm{Aut}_2^-(K_4) = \mathrm{Aut}_2^-(K_5)$ contains the following non-trivial subgroupoids:

$$I_1 = \{(x\ \widetilde{x}) : x = 0,1,2\},$$
$$I_2 = \{(x\ \widetilde{x})(y\ \widetilde{y}) : x,y = 0,1,2,\ x \neq y\},$$
$$I_{3,1} = \{(x\ y)(\widetilde{x}\ \widetilde{y}) : x,y = 0,1,2,\ x \neq y\},$$
$$I_{3,2} = \{(0\ \widetilde{1})(\widetilde{0}\ 1), (0\ \widetilde{2})(\widetilde{0}\ 2), (1\ 2)(\widetilde{1}\ \widetilde{2})\},$$
$$I_{3,3} = \{(0\ \widetilde{1})(\widetilde{0}\ 1), (1\ \widetilde{2})(\widetilde{1}\ 2), (0\ 2)(\widetilde{0}\ \widetilde{2})\},$$
$$I_{3,4} = \{(0\ \widetilde{2})(\widetilde{0}\ 2), (1\ \widetilde{2})(\widetilde{1}\ 2), (0\ 1)(\widetilde{0}\ \widetilde{1})\},$$
$$I_3 = \{(x\ y)(\widetilde{x}\ \widetilde{y}), (x\ \widetilde{y})(\widetilde{x}\ y) : x,y = 0,1,2,\ x \neq y\} = I_{3,1} \cup I_{3,2} \cup I_{3,3} \cup I_{3,4},$$
$$I_{4,1} = \{(x\ \widetilde{y})(\widetilde{x}\ y)(z\ \widetilde{z}) : \{x,y,z\} = \{0,1,2\}\},$$
$$I_{4,2} = \{(0\ 1)(\widetilde{0}\ \widetilde{1})(2\ \widetilde{2}), (0\ 2)(\widetilde{0}\ \widetilde{2})(1\ \widetilde{1}), (1\ \widetilde{2})(\widetilde{1}\ 2)(0\ \widetilde{0})\},$$
$$I_{4,3} = \{(0\ 1)(\widetilde{0}\ \widetilde{1})(2\ \widetilde{2}), (1\ 2)(\widetilde{1}\ \widetilde{2})(0\ \widetilde{0}), (0\ \widetilde{2})(\widetilde{0}\ 2)(1\ \widetilde{1})\},$$
$$I_{4,4} = \{(0\ 2)(\widetilde{0}\ \widetilde{2})(1\ \widetilde{1}), (1\ 2)(\widetilde{1}\ \widetilde{2})(0\ \widetilde{0}), (0\ \widetilde{1})(\widetilde{0}\ 1)(2\ \widetilde{2})\},$$
$$I_4 = \{(x\ \widetilde{y})(\widetilde{x}\ y)(z\ \widetilde{z}), (x\ y)(\widetilde{x}\ \widetilde{y})(z\ \widetilde{z}) : \{x,y,z\} = \{0,1,2\}\} = I_{4,1} \cup \dots \cup I_{4,4},$$
$$I_{3,i} \cup I_{4,i}, \quad i = 1,2,3,4,$$

all unions of I_1, I_2, I_3, I_4.

Clearly, $|I_1| = |I_2| = |I_{3,i}| = |I_{4,i}| = 3$, $i = 1,\dots,4$ and $|I_3| = |I_4| = 6$. Now, none of K_4, K_5 is SI. The following table shows, which of $J \sqcup K_4$, $J \sqcup K_5$ (J a subgroupoid of I) are subdirectly irreducible. (An empty space means it does not satisfy the condition 4.1(2).)

$\sqcup$	I_1	I_2	$I_{3,1}$	$I_{3,2}, I_{3,3}, I_{3,4}$	I_3	$I_{4,1}$	$I_{4,2}, I_{4,3}, I_{4,4}$	I_4
K_4	$-$	$-$	$-$	$-$	$+$	$-$	$-$	$+$
K_5	$-$	$-$			$+$			$+$

$\sqcup$	$I_{3,1} \cup I_{4,1}$	$I_{3,i} \cup I_{4,i}$, $i = 2,3,4$	$I_1 \cup I_2$	$I_i \cup I_j$, $i \neq j$, $\{i,j\} \neq \{1,2\}$	$I_i \cup I_j \cup I_k$, $i \neq j \neq k \neq i$	I
K_4	$-$	$-$	$-$	$+$	$+$	$+$
K_5			$-$	$+$	$+$	$+$

Proposition 6.2. *There are 96 (up to an isomorphism) SI LSLD groupoids with six non-idempotent elements: the 24 without right zeros described in the table above and their extensions by right zeros.*

The following table displays the number of SI LSLD groupoids with 2, 4 and 6 non-idempotent elements and a respective number of idempotent elements.

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	2	1																		
0	0	1	2	3	4	2														
0	0	0	0	0	0	4	8	4	8	16	8	6	12	6	4	8	4	2	4	2

More non-idempotents.

Lemma 6.3. *Let G be an SI LSLD groupoid with $|K_G| = 8$. Then K_G/ip_{K_G} is isomorphic to one of R_1, R_2.*

R_1	0	1	2	3
0	0	1	2	3
1	0	1	2	3
2	0	1	2	3
3	0	1	2	3

R_2	0	1	2	3
0	0	1	3	2
1	0	1	3	2
2	1	0	2	3
3	1	0	2	3

Proof. For every $u \in K_G$, let $t(u)$ be the number of $v \in K_G$ such that $uv \in \{v, vv\}$. We have $t(u) = t(xu)$ for every $x \in G$ (because $xy \cdot z = z$ iff $y \cdot xz = xz$), hence the set $\{u \in K_G : t(u) = t\}$ is a left ideal of G for every t. Consequently, there is t such that $t(u) = t$ for every $u \in K_G$ (see Lemma 2.1(1)) and thus all left translations in $R = K_G/ip_{K_G}$ have the same number $\frac{t}{2}$ of fixed points. Let us denote the elements of R by 0,1,2,3. Clearly, $\frac{t}{2} \geq 1$ is an even number. If $\frac{t}{2} = 4$, then R is the right zero band R_1. Otherwise $\frac{t}{2} = 2$ and we may assume that $0, 1$ are the only fix points of L_0, i.e. $L_0 = (2\ 3)$. Then $1 \cdot 0 = (0 \cdot 1)(0 \cdot 0) = 0(1 \cdot 0)$ (left distributivity) and hence $1 \cdot 0$ is a fix point of L_0. Therefore $1 \cdot 0 = 0$ and so $L_1 = L_0$. Now, $L_{2\cdot 0} = L_2 L_0 L_2 = L_2 L_1 L_2 = L_{2\cdot 1}$. Since $L_2(0), L_2(1) \neq 2$ and $L_0 = L_1 \neq L_3$ (because $L_0(3) \neq L_3(3)$), we have $\{2 \cdot 0, 2 \cdot 1\} = \{0, 1\}$. Hence $L_2 = (0\ 1)$, because it has two fixed points. Analogously also $L_3 = (0\ 1)$. □

Proposition 6.4. *There is no SI idempotent-free LSLD groupoid with 8 elements.*

Proof. Since both R_1, R_2 contain proper left ideals, so does any 8-element SI idempotent-free LSLD groupoid, a contradiction with Lemma 2.1(1). □

Lemma 6.5. *Let G be an SI LSLD groupoid with $|K_G| = 10$. Then K_G/ip_{K_G} is isomorphic to one of R_3, R_4.*

R_3	0	1	2	3	4
0	0	1	2	3	4
1	0	1	2	3	4
2	0	1	2	3	4
3	0	1	2	3	4
4	0	1	2	3	4

R_4	0	1	2	3	4
0	0	2	1	4	3
1	3	1	4	0	2
2	4	3	2	1	0
3	2	4	0	3	1
4	1	0	3	2	4

Proof. Proceed similarly as in the proof of Lemma 6.3. □

Proposition 6.6. *There is no SI idempotent-free LSLD groupoid with 10 elements.*

Proof. Assume that $K = \{0, \widetilde{0}, 1, \widetilde{1}, 2, \widetilde{2}, 3, \widetilde{3}, 4, \widetilde{4}\}$ is an idempotent-free LSLD groupoid, where blocks of ip_K are the sets $\{k, \widetilde{k}\}$ for every $k = 0, \dots, 4$. Then $K/ip_K \simeq R_4$ and without loss of generality we put $0 \cdot 1 = \widetilde{2}$, $0 \cdot 3 = \widetilde{4}$, $1 \cdot 2 = \widetilde{4}$, $1 \cdot 0 = \widetilde{3}$. Then $\widetilde{1} \cdot \widetilde{0} = 3$, $\widetilde{1} \cdot \widetilde{2} = 4$ and thus $2 \cdot 0 = \widetilde{4}$, $2 \cdot 1 = \widetilde{3}$, because L_0 is an automorphism. Also $3 \cdot 0 = \widetilde{2}$, $2 \cdot 1 = \widetilde{4}$, $4 \cdot 0 = \widetilde{1}$, $4 \cdot 2 = \widetilde{3}$, because L_2 is an automorphism, and the operation on K is determined. We see that $\rho = \{0, 1, 2, 3, 4\}^2 \cup \{\widetilde{0}, \widetilde{1}, \widetilde{2}, \widetilde{3}, \widetilde{4}\}^2$ is a congruence on K and $\rho \cap ip_K = id_K$. Hence K is not subdirectly irreducible. □

Proposition 6.7. *The following groupoid is the smallest SI idempotent-free LSLD groupoid with more than two elements.*

K_8	0	$\widetilde{0}$	1	$\widetilde{1}$	2	$\widetilde{2}$	3	$\widetilde{3}$	4	$\widetilde{4}$	5	$\widetilde{5}$
$0,\widetilde{0}$	$\widetilde{0}$	0	1	$\widetilde{1}$	$\widetilde{4}$	4	$\widetilde{5}$	5	$\widetilde{2}$	2	$\widetilde{3}$	3
$1,\widetilde{1}$	0	$\widetilde{0}$	$\widetilde{1}$	1	$\widetilde{5}$	5	$\widetilde{4}$	4	$\widetilde{3}$	3	$\widetilde{2}$	2
$2,\widetilde{2}$	$\widetilde{4}$	4	$\widetilde{5}$	5	$\widetilde{2}$	2	3	$\widetilde{3}$	$\widetilde{0}$	0	$\widetilde{1}$	1
$3,\widetilde{3}$	5	$\widetilde{5}$	4	$\widetilde{4}$	2	$\widetilde{2}$	$\widetilde{3}$	3	1	$\widetilde{1}$	0	$\widetilde{0}$
$4,\widetilde{4}$	$\widetilde{2}$	2	3	$\widetilde{3}$	$\widetilde{0}$	0	1	$\widetilde{1}$	$\widetilde{4}$	4	5	$\widetilde{5}$
$5,\widetilde{5}$	3	$\widetilde{3}$	$\widetilde{2}$	2	$\widetilde{1}$	1	0	$\widetilde{0}$	4	$\widetilde{4}$	$\widetilde{5}$	5

Proof. Subdirect irreducibility of K_8 can be checked easily from the multiplication table and non-existence of a smaller one was proved above. □

7. The group generated by left translations

In the last section, we find another criterion for recognizing that a groupoid is not SI or pre-SI.

Let G be an LSLD groupoid. We denote $\mathrm{L}(G)$ the subgroup of $\mathrm{Aut}(G)$ generated by all left translations in G. For a subset N of $\mathrm{L}(G)$ we define a relation ρ_N by $(x, y) \in \rho_N$, iff there exists $\varphi \in N$ such that $\varphi(x) = y$.

Lemma 7.1. *Let G be an LSLD groupoid and N a normal subgroup of $\mathrm{L}(G)$. Then ρ_N is a congruence of G.*

Proof. Clearly, ρ_N is an equivalence on G. Let $(x, y) \in \rho_N$ and $z \in G$. We have $yz = \varphi(x)z = L_{\varphi(x)}L_x(xz) = \varphi L_x \varphi^{-1} L_x(xz)$, and so $(xz, yz) \in \rho_N$ via the automorphism $\varphi L_x \varphi^{-1} L_x \in N$. Further, $zy = z\varphi(x) = z\varphi(z \cdot zx) = L_z \varphi L_z(zx)$, and so $(zx, zy) \in \rho_N$ via the automorphism $L_z \varphi L_z \in N$. □

Proposition 7.2. *Let G be an SI non-idempotent or a pre-SI idempotent-free LSLD groupoid and let N be a non-trivial normal subgroup of $\mathrm{L}(G)$. Then for every $u \in G$ there exists $\varphi \in N$ such that $\varphi(u) = uu$.*

Proof. If G is SI non-idempotent, then $ip_G \subseteq \rho_N$, because ρ_N is a non-trivial congruence. If G is pre-SI idempotent-free, one must check (in a view of Theorem 4.3) that ρ_N is also $\mathrm{Aut}_2(G)$-invariant. If $(x, y) \in \rho_N$, $\varphi(x) = y$, and $\psi \in \mathrm{Aut}_2(G)$, then $(\psi\varphi\psi^{-1})(\psi(x)) = \psi\varphi(x) = \psi(y)$, and thus $(\psi(x), \psi(y)) \in \rho_N$ via the automorphism $\psi\varphi\psi^{-1} \in N$. □

Example. Recall the groupoid K_3 from the previous section. It is easy to calculate that $\mathrm{L}(K_3) = \{id, (0\ \widetilde{0}), (1\ \widetilde{1}), (0\ \widetilde{0})(1\ \widetilde{1})\}$, and thus $N = \{id, (0\ \widetilde{0})\}$ is a normal subgroup. However, there is no $\varphi \in N$ such that $\varphi(1) = \widetilde{1}$, hence K_3 is not pre-SI by Proposition 7.2.

Remark. Let G be a simple LSLD groupoid. Then the subgroup of $\mathrm{L}(G)$ generated by all $L_x L_y$, $x, y \in G$, is a smallest non-trivial normal subgroup of $\mathrm{L}(G)$ and thus $\mathrm{L}(G)$ is subdirectly irreducible. This is a result of H. Nagao [6] and it can be proved similarly. However, due to Corollary 1.3, it is interesting in the idempotent case only.

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