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SOME THEOREMS ON THE LATTICE OF LOCAL INTERPRETABILITY TYPES

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§ 0. Introduction

In [4] J. MYCIELSKI introduced a very general notion of multidimensional local interpretability of first order theories. If we define the relation $\leqq$ between theories T, S by $T \leqq S$ iff T is multidimensionaly locally interpretable in S, then $\leqq$ is a preordering. The induced partial ordering is a lattice ordering, it is called *the lattice of local interpretability types.*

We consider theories formalized in the first order logic without equality; equality is considered as a congruence and hence need not be interpreted absolutely. The languages of the theories are considered without function symbols, i.e. an n-ary function has to be included as an $(n + 1)$-ary relation. We regard two theories as equal iff they have the same theorems.

A theory T is *locally interpretable* in a theory S, in symbols $T \leqq S$, iff each theorem of T is interpretable in S. Equivalently: $T \leqq S$ iff each finite part of T is interpretable in S. The interpretation may have parameters, variables may be translated as k-tuples of variables (then we speak about *k-dimensional interpretations*).

By definition different theorems of T may have different interpretations of atomic relations in S. If there is an interpretation of atomic relations common to all theorems of T, we speak about *global interpretation* of T in S; in symbols $T \leqq_g S$. If each theorem of T is interpretable k-dimensionally in S, we write $T \leqq_k S$.

The class of all theories T such that $T \leqq S$ and $S \leqq T$ forms the *local interpretability type of* S, denoted by $|S|$. So $T \in |S|$ iff $T \leqq S$ and $S \leqq T$.

If we replace in the previous paragraph $\leqq$ by $\leqq_g$ (resp. by $\leqq_k$), we obtain the definition of *global* (resp. *k-dimensional*) *interpretability type of* S.

The relation $\leqq$ determines a partial ordering of the local interpretability types which is a distributive and complete lattice. The largest type is the type of the inconsistent theories, the lowest type is the type of the theories whose each theorem has a 1-point model. Among the types which do not contain inconsistent theory, there is the largest type which is the type of theory $\mathrm{Th}(\omega, +, \cdot)$ (as it is well known that for every consistent sentence we can define a model in arithmetic). There also exists the lowest type among the types which are not the lowest in the whole lattice. It is the type of theories having locally (i.e., each theorem has) finite models but which do not have 1-point models. Hence it is, for example, the type of the theory based on the axiom $(\exists x, y)\, (P(x)\ \&\ \neg P(y))$.

The sublattice of the lattice of the types which we obtain by tearing off the largest and the lowest types is therefore also a distributive and complete lattice. Let us (according to [1]) denote it by $\mathscr{M}$. More details can be found in [4], [5] and in the joint manuscript [1] (a revised version of it is being prepared for publication), where one can find also a number of first results and problems about the lattice $\mathscr{M}$. For completeness of the text some of these results are recapitulated in § 1.

In [1] J. MYCIELSKI conjectured that all mathematically interesting theories are prime, i.e., their types are joint-irreducible, and he proved that the theory of linear order without maximal element is prime. In [5] P. PUDLÁK essentially confirmed MYCIELSKI's conjecture by proving that each sequential theory (see [5]) is prime; in particular PA, ZF, GB, $\mathrm{Th}(\omega, +, \cdot)$ are prime. In the same paper he stated the problem whether $\mathrm{Th}(\omega, <)$ is prime. § 5 of the present paper is motivated by this problem; we study there mutual interpretability of various theories of order.

In [1] A. EHRENFEUCHT studies the dual notion of coprime theory, i.e. a theory with a meet-irreducible type, and he proves that many "strong" (i.e. high in the lattice $\mathcal{M}$) theories are not coprime. Namely he proves this theorem: Each consistent recursively enumerable extension of PA is not coprime. In the same paper the problem is posed whether something similar holds for "weak" theories. In § 2 we will prove the following characterization: A theory T is coprime iff T has the same type as some complete theory. It follows that there exist many "strong" and many "weak" theories which are coprime.

In [4] J. MYCIELSKI asked whether in each type there is a theory with a finite language. We solve this problem affirmatively in § 3. This result and the above mentioned characterization of coprime theories were independently proved by A. STERN in his thesis (Berkeley, 1984). In § 4 we prove a few technical lemmas which are needed for § 5. In § 6 we state a number of open problems.

This is also a suitable place to thank P. PUDLÁK without whose help this paper would never have come into existence. Especially in § 3 P. PUDLÁK suggested to us how to extend our earlier result to the final theorem. We thank also J. MYCIELSKI for his remarks which we used in the final preparation of the paper. In his new terminology a local interpretability type should be called a *chapter*.

§ 1. Preliminaries

In this part we repeat a few results from [1], [4]. The proofs can be found in those papers.

1.1. Let α^I, α^J be two interpretations of a sentence α in the language of a theory S and let $S \vdash \alpha^I \vee \alpha^J$ hold. Then $\alpha \leqq S$.

1.2. For a model $M \models S$ we denote by $\mathrm{Th}(M)$ the theory of M in the language of S (i.e. without the absolute equality of M). Then $T \leqq_k S$ iff $T \leqq_k \mathrm{Th}(M)$ for each model $M \models S$.

1.3. The type $|T| \wedge |S|$ is the type of the theory $\{\alpha \vee \beta \mid \alpha \in T, \beta \in S\}$.

1.4. $\mathcal{M}$ is a complete distributive lattice.

1.5. The compact types are exactly the types of finitely axiomatizable theories.

1.6. In each type there is a countable theory.

1.7. The lattice $\mathcal{M}$ is algebraic of countable character, i.e., every type is a join of countably many compact types.

§ 2. Coprime theories

2.1. A theory T is *coprime* iff its type is meet-irreducible. So T is coprime iff $(\forall a, b \in \mathcal{M})$ $(a, b \geqq |T| \;\&\; a \wedge b = |T| \rightarrow a = |T| \vee b = |T|)$, or equivalently (because $\mathcal{M}$ is a distributive lattice) $(\forall a, b \in \mathcal{M})$ $(a \wedge b \leqq |T| \rightarrow a \leqq |T| \vee b \leqq |T|)$.

2.2. Theorem. *A type $t \in \mathcal{M}$ is meet-irreducible iff it contains a complete theory (i.e. a theory T is coprime iff it has the same type as some complete theory).*

Proof. (a) Let $S \in t$ and S be complete. Let further be $t \geqq |S_1| \wedge |S_2|$. By 1.3, $|S_1| \wedge |S_2| = |S_3|$, where $S_3 = \{\alpha \vee \beta \mid \alpha \in S_1, \beta \in S_2\}$. If $|S_1| \nleqq t$, then there exists α_0 such that $S_1 \vdash \alpha_0$ and $\alpha_0 \nleqq S$. Then, for each interpretation I of α_0 in the language of S, $S \nvdash \alpha_0^I$. By completeness of S it is surely $S \vdash \neg(\alpha_0^I)$. Since by assumption $|S_3| \leqq |S|$, in particular, $S_2 \vdash \beta$ implies $\alpha_0 \vee \beta \leqq S$ for each β. Therefore for $S_2 \vdash \beta$ there exists an interpretation I_β of the language of S_3 in the language of S such that $S \vdash (\alpha_0 \vee \beta)^{I_\beta}$ and so also $S \vdash \alpha_0^{I_\beta} \vee \beta^{I_\beta}$. But we know that $S \vdash \neg(\alpha_0^{I_\beta})$, hence $S \vdash \beta^{I_\beta}$ must hold. Therefore $|S_2| \leqq |S|$.

(b) Suppose that the type $t \in \mathscr{M}$ is meet-irreducible. Let us take a countable theory $S \in t$ (by 1.6) and let $\sigma_0, \sigma_1, \ldots$ be a list of all sentences in the language of S. We will construct theories T_i, $i = 0, 1, \ldots$, such that $|T_i| = |S|$. Define $T_0 = S$ (so $|T_0| = |S|$). Suppose $|T_i| = |S|$ and construct T_{i+1} as follows:

$$T_{i+1} = \begin{cases} T_i + \sigma_i & \text{if } T_i + \sigma_i \leqq T_i, \\ T_i + \neg\sigma_i & \text{if } T_i + \neg\sigma_i \leqq T_i. \end{cases}$$

If both cases are possible, choose arbitrarily one. Always at least one is possible since

$$\begin{aligned} |T_i + \sigma_i| \wedge |T_i + \neg\sigma_i| = |\{\alpha \vee \sigma_i \mid \alpha \in T_i\} \cup \{\beta \vee \neg\sigma_i \mid \beta \in T_i\} \cup \\ \cup \{\alpha \vee \beta \mid \alpha, \beta \in T_i\} \cup \{\sigma_i \vee \neg\sigma_i\}| = |T_i|. \end{aligned}$$

By induction hypothesis $|T_i| = |S|$ $(= t)$, hence $|T_i|$ is meet-irreducible. Therefore $T_i + \sigma_i \leqq T_i$ or $T_i + \neg\sigma_i \leqq T_i$ holds. Take $T = \bigcup_i T_i$. Then $|T| = |S|$ and T is complete.

2.3. Remark. If in the preceeding proof the theory S is finitely axiomatizable, then we can search for an interpretation of $T_i + \sigma_i$ (resp. $T_i + \neg\sigma_i$) in some systematic way. Hence the resulting theory T will be decidable.

By an analogical construction for S such that $|S| \leqq t$ and t is meet-irreducible, we can obtain a complete extension T of S such that $|T| \leqq t$. If S is finitely axiomatizable and in the type t there is some recursively axiomatizable theory, then we can construct such an extension in decidable way.

2.4. Corollary. *If S is a finitely axiomatizable and essentially undecidable theory and R is its recursively axiomatizable extension, then R is not coprime.*

2.5. Remark. The theorem of A. Ehrenfeucht in [1] (see § 0) follows from 2.4.

Proof. By [6] the theory Q, Robinson's arithmetic, is essentially undecidable and also $Q \subseteqq PA$.

2.6. Theorem. *Let S be a countable theory and α a sentence (not necessarily in the language of S). Let also $\alpha \nleqq S$. Then there exists a complete extension T of S such that $\alpha \nleqq T$ still holds.*

Proof. It is enough to construct extensions T_i, $i = 0, 1, \ldots$, of S such that $\alpha \nleqq T_i$ and $\bigcup T_i$ is a consistent and complete theory. This can be done analogically as in the proof of 2.2.

2.7. Corollary. *Let $t, s \in \mathscr{M}$ be types such that $s \nleqq t$. Then there exists a meet-irreducible type $t' \geqq t$ such that still $s \nleqq t'$ holds.*

Proof. Let T be a countable theory such that $|T| = t$. Since $s \nleqq t$, there exists a sentence γ of some theory S, $|S| = s$, such that $\gamma \nleqq T$. By 2.6 then there exists a complete extension T' of T such that still $\gamma \nleqq T'$. So $t' = |T'|$ is the required type.

Immediately from 2.7 it follows:

2.8. Corollary. *Above each type* $t < |\mathrm{Th}(\omega, +, \cdot)|$ $(= 1_{\mathcal{M}})$ *there exists a meet-irreducible type* t' *such that still* $t' < |\mathrm{Th}(\omega, +, \cdot)|$.

2.9. Remark. It follows from 2.7 that there exist many "weak" theories and also (explicitly from 2.8) many "strong" theories which are coprime.

§ 3. Theories with finite languages are in each type

In [4] J. Mycielski posed the problem (see [4], problem 5) whether in each type there exists a theory with a finite language. According to [4] it is only known (by results of R. L. Vaught) that each recursively enumerable theory is globally interpretable in finitely axiomatizable one. In this part we will solve this problem affirmatively.

For any sentence φ we define the sentence $\varphi_=$ by

$\varphi_= \equiv \varphi$ & "axioms of equality with a new binary predicate symbol with respect to the predicates of φ".

3.1. Lemma. $|\varphi| = |\varphi_=|$.

Proof. Define the relation of indiscernibility.

3.2. Let $\mathcal{L} = \{D, T, L, R\}$ a finite language, where D, T are unary and L, R are binary predicates.

3.3. Lemma. *For each sentence* ψ *there exists a sentence* ψ^I *in the language* $\mathcal{L}$ *of the same type.*

Proof. By 3.1 we can suppose that ψ is a sentence with equality. Let R_n^k $(k \leqq k_n,\ n \leqq n_0)$ be all predicates in ψ of arity n, where n_0 is the greatest arity. Define an interpretation I of the predicates R_k^n in the language $\mathcal{L}$ in the following way:

(i) choose parameters c_n^k $(k \leqq k_n,\ n \leqq n_0)$;

(ii) $$\begin{aligned} R_n^k(x_1, \ldots, x_n)^I \equiv (\exists y_1, \ldots, y_n)\,(L(x_1, y_1) \mathbin{\&} R(x_2, y_1) \\ \mathbin{\&} L(y_1, y_2) \mathbin{\&} R(x_3, y_2) \mathbin{\&} \ldots \\ \mathbin{\&} L(y_{n-2}, y_{n-1}) \mathbin{\&} R(x_n, y_{n-1}) \\ \mathbin{\&} L(y_{n-1}, y_n) \mathbin{\&} R(c_n^k, y_n) \mathbin{\&} T(y_n)); \end{aligned}$$

(iii) relativize the quantifiers of ψ to the domain D.

Let ψ^I be the formula obtained from ψ in this way. So obviously $\psi \leqq \psi^I$ and it remains to show that $\psi^I \leqq \psi$ holds too. For this purpose we will define an interpretation J. We will need enough mutually different parameters. But using multidimensional interpretation we can guarantee this even for sentences with a two-point model. Thus from now till the end of the proof we will assume that this is already done. (Because our aim is to prove that in the type of ψ there is a sentence with a finite language and this is known for sentences with the lowest type in $\mathcal{M}$, we can alternatively assume that all models of ψ are infinite.) Let us define an $(n_0 + 2)$-dimensional an interpretation J of the language $\mathcal{L}$ in the language of ψ as follows:

(i) choose mutually different(!) parameters $b_1, \ldots, b_{n_0+1}$, c_n^k $(k \leqq k_n,\ n \leqq n_0)$;

(ii) for an $(n_0 + 2)$-dimensional vector x of variables "$x(i) = a$" stands for "a is the ith coordinate of x"; define:

$$D(x) \equiv x(n_0 + 2) = b_1,$$

$$L(x, y) \equiv \bigvee_{1 \leq i \leq n_0} [x(n_0 + 2) = b_i \;\&\; y(n_0 + 2) = b_{i+1} \;\&\; \bigwedge_{j \leq i} x(j) = y(j)],$$

$$R(x, y) \equiv \bigvee_{2 \leq i \leq n_0 + 1} [x(n_0 + 2) = b_1 \;\&\; y(n_0 + 2) = b_i \;\&\; x(1) = y(i)],$$

$$T(y) \equiv \bigvee_{\substack{k \leq k_n \\ n \leq n_0}} [y(n_0 + 2) = b_{n+1} \;\&\; y(n + 1) = c_n^k \;\&\; R_n^k(y(1), \ldots, y(n))].$$

Now, since we choose all parameters mutually different (and, since the above assumption that ψ has only infinite models, it is provable from ψ that there exist sufficiently many different elements), for all $k \leq k_n$, $n \leq n_0$, and for all vectors from domain D the following holds:

$$R_n^k(x_1(1), \ldots, x_n(1)) \equiv [[R_n^k]^I]^J (x_1, \ldots, x_n, c_1^1, \ldots, c_{n_0}^{k_{n_0}}, b_1, \ldots, b_{n_0+1}).$$

Hence $\psi^I \leq \psi$, and we are done.

3.4. Theorem. *In each type there is a theory in the language $\mathscr{L}$.*

Proof. For the type $t \in \mathscr{M}$ choose any countable theory $S \in t$. Let $\eta_0, \eta_1, \ldots$ be the axioms of S. Define $\sigma_i \equiv \eta_0 \;\&\; \ldots \;\&\; \eta_i$ and $\psi_i \equiv (\sigma_i)_=$ (see lemma 3.1). For the equality in the ψ_i's we take the same symbol for each $i \in \omega$. The theory $T = \{\psi_i^I \mid i \in \omega\}$ is in the language $\mathscr{L}$ and evidently $S \leq T$. On the other hand, clearly $\psi_{i+1}^I \to \psi_i^I$ holds for all $i \in \omega$. Hence for all $r \in \omega$, $\{\psi_0^I, \ldots, \psi_r^I\} \leq \{\psi_r^I\}$. But also $\psi_r^I \leq \psi_r \leq \sigma_r \leq S$ and so $T \leq S$ holds, and we are done.

3.5. Remark. Using an easy coding we can replace the language $\mathscr{L}$ by a language with exactly one binary predicate.

§ 4. Some technical lemmas

In this part we will introduce some notions and we will prove some lemmas about them which will turn out to be useful in § 5.

4.1. Let S, T be theories, S with equality, and let I be a global interpretation of S in T. Let n be the dimension of I, δ_I the domain of I and let $M \models T$ be any model of T. Consider I with fixed parameters from M. We shall denote by M_S^I the structure with universe $(M^n \cap \delta_I)/{=_I}$, where $=_I$ translates the equality. The relations of M_S^I are the translations of the predicates from S. So $M_S^I \models S$ holds.

4.2. Let S, T, M, I denote the same as in 4.1 and let $\mathrm{Aut}^I(M)$ be the group of all automorphisms of the model M which preserve the parameters of I. For $f \in \mathrm{Aut}^I(M)$ define $f^I\colon M_S^I \to M_S^I$ as follows:

$$f^I(\langle x_1, \ldots, x_n \rangle) = \langle f(x_1), \ldots, f(x_n) \rangle.$$

Since f preserves the parameters of I, f^I preserves all the I-relations (in particular the domain δ_I) and hence it is an automorphism of structure M_S^I.

4.3. Lemma. *Let $(N, <)$ be a model of linear order and $x \in N$. Then for $f \in \mathrm{Aut}(N)$, $f(x) \neq x$ implies $f^{(k)} \neq x$ for each $k > 0$.*

Proof. Obvious.

4.4. Corollary. *Let M be any model and I an interpretation of linear order in M. Let $f \in \mathrm{Aut}^I(M)$, $x \in \delta_I$ be such that for some $k \geqq 1$, $x = f^{(k)}(x)$. Then $x =_I f(x)$ holds.*

Proof. Immediate from 4.3.

4.5. For a theory T and M a model of T, let us denote by $D^n(M)$ the system of all subsets of M^n definable with parameters (from M^n). $D^n(M)$ with ordering by inclusion is a lattice. Let again S be a theory and I its global n-dimensional interpretation in T with absolute equality (if equality is in S).

4.6. Observation. *The lattice $D(M^I_S)$ is embeddable into the lattice $D^n(M)$.*

Proof. Obvious.

4.7. Corollary. *If S is finitely axiomatizable and $S \leqq \mathrm{Th}(\omega, <)$, then there exists a countable model $M \models S$ such that the lattice $D(M)$ is embeddable into the lattice $D^n(\omega)$, for suitable n.*

Proof. Since S is finitely axiomatizable, it is also globally interpretable in $\mathrm{Th}(\omega, <)$. Since ω^n is lexicographically well-ordered, the equality may be translated absolutely. By 4.6, ω^I_S is the required model.

4.8. Let us consider a structure M and let $2^{<\omega}$ be the set of all finite words on the alphabet $\{0, 1\}$. For $\xi, \eta \in 2^{<\omega}$ we define:

$$\xi \leqq \eta \text{ iff } (\exists \varrho)\, \xi^\frown \varrho = \eta \quad (^\frown \text{ denotes concatenation}).$$

A set $A \in D^n(M)$ is called *breakable* iff there exists a function $f\colon 2^{<\omega} \to D^n(M)$ such that for every ξ, ϱ:

(i) $f(\Lambda) = A$ (Λ denotes the empty word), (ii) $f(\xi)$ is infinite,

(iii) $f(\xi^\frown 0) \cap f(\xi^\frown 1) = \emptyset$, (iv) $f(\xi^\frown \varrho) \subseteqq f(\xi)$.

4.9. Observation. *If S, T, M, I are as in 4.5 and $D(M^I_S)$ contains a breakable set, then $D^n(M)$ too.*

§ 5. Mutual interpretability of theories of order

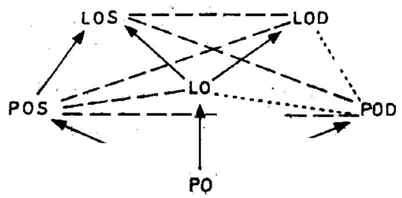
This part deals with mutual interpretability of some theories of partial and linear order. All theories from 5.1 are in the language $\{=, <\}$ ("$=$" the equality).

5.1. In the following theories we assume implicitly the axioms of equality and "$x \leqq y$" stands for "$x = y \vee x < y$".

$$\begin{aligned}
\mathrm{PO} &= \{(\forall x)\, \neg x < x,\ (\forall x, y, z)\,(x < y \,\&\, y < z \to x < z),\ (\forall x)\,(\exists y)\, x < y\};\\
\mathrm{POS} &= \mathrm{PO} + \{(\exists! x)\,(\forall y)\, x \leqq y,\\
&\qquad (\forall x)\,(\forall y > x)\,(\exists z > x)\,(\forall t > x)\,(y \geqq z \,\&\, (y \geqq t \to t \geqq z)),\\
&\qquad (\forall x)\,(\forall y < x)\,(\exists z < x)\,(\forall t < x)\,(y \leqq z \,\&\, (y \leqq t \to t \leqq z)\};\\
\mathrm{POD} &= \mathrm{PO} + \{(\forall x)\,(\exists y)\, x > y,\ (\forall x)\,(\forall y > x)\,(\exists t)\,(x < t \,\&\, t < y)\};\\
\mathrm{LO} &= \mathrm{PO} + \{(\forall x, y)\,(x \leqq y \vee y \leqq x\};\\
\mathrm{LOS} &= \mathrm{LO} + \mathrm{POS}; \qquad \mathrm{LOD} = \mathrm{LO} + \mathrm{POD}.
\end{aligned}$$

It is well known (see [2], for example) that $\mathrm{LOS} = \mathrm{Th}(\omega, <)$ and $\mathrm{LOD} = \mathrm{Th}(\mathbf{Q}, <)$, where $\mathbf{Q}$ are the rational numbers.

5.2. The main program of the remaining paragraphs is to prove the validity of the following diagram ($T \to S$ denotes $T < S$, T— — — —S denotes the incomparability of T and S, and T··········S denotes that we do not know the relation):



5.3. Lemma. *Let* $M = (Q, <)$ *be a countable model of* LOD (*i.e. up to an isomorphism the rationals*). *Then for arbitrary natural numbers* n, k *and for arbitrary* $a_1, \ldots, a_k \in Q$ *the structure* $(M; a_1, \ldots, a_k)$ *realizes only finitely many* n*-types of the language* $\{=, <\} \cup \{c_1, \ldots, c_k\}$ (*the* c_i*'s are new constants*).

Proof. For $\vec{x}, \vec{y} \in Q^n$ we denote by x_i (resp. y_i) the ith coordinate of $\vec{x}$ (resp. $\vec{y}$) and define

$$\vec{x} \parallel \vec{y} \text{ iff (i) } M \models x_i \leqq x_j \equiv y_i \leqq y_j \ (i, j \leqq n) \text{ and}$$
$$\text{(ii) } M \models x_i \leqq a_j \equiv y_i \leqq a_j \ (i \leqq n, j \leqq k).$$

Now let $\vec{x}, \vec{y}$ be such that $\vec{x} \parallel \vec{y}$. Clearly there exists an automorphism f of the structure $(M; a_1, \ldots, a_k)$ such that $f(x_i) = y_i$ for all $i \leqq n$. It follows immediately that $\vec{x}, \vec{y}$ realize the same n-type in $(M; a_1, \ldots, a_k)$. Now it is sufficient to notice that the relation $\parallel$ is an equivalence with finitely many blocks.

5.4. Theorem. POS $\nleq$ LOD.

Proof. Suppose there is an n-dimensional interpretation of POS in the model $(Q, <) \models$ LOD with parameters $a_1, \ldots, a_k$. Take some $\vec{b} \in Q^n$ from the domain of this interpretation. For an arbitrary x_0 from any model $M \models$ POS it is possible to construct arbitrarily many different 1-types which are realizable in the structure $(M; x_0)$. For example we can define (for $r \in \omega$):

$$\psi_r(y, x_0) \equiv \text{“}y > x_0 \text{ and there are exactly } r \text{ elements between } x_0 \text{ and } y\text{”}.$$

Each such formula ψ_r is in $(M; x_0)$ realized by an other element. So in the structure $(Q; a_1, \ldots, a_k, b_1, \ldots, b_n)$ arbitrarily many $(n + k)$-types should be realizable. This is a contradiction with 5.3.

5.5. Lemma. *Let* M *be any model of* POD. *Then* $D(M)$ *contains a breakable set.*

Proof. Each set of the form $\{x \mid a < x < b\} \in D(M)$, for $a < b$, is infinite. Therefore it is sufficient to “halve” the interval (a, b).

5.6. Lemma. *Let* $(\omega, <)$ *be the standard model of* LOS *and* $\psi_1, \ldots, \psi_r$ *be formulae with* n *free variables* $x_1, \ldots, x_n$. *Then there exists* $m \in \omega$ *such that the following holds:*

$$\omega \models (\forall x_1, \ldots, x_n \geqq m) \bigwedge_{i \leqq r} \psi_i(x_1, \ldots, x_n) \equiv \psi_i(x_1 + 1, \ldots, x_n + 1),$$

where $x + 1$ *denotes the successor of* x.

Proof. Take a model $M \models$ LOS, $M \cong \omega + (\omega^* + \omega)$, where ω^* denotes the inverse ordering of ω (i.e., M is composed from the natural numbers which are succeded by the integers). Let us denote by f the automorphism of M which preserves the standard elements of M (it must) and shifts nonstandard ones to their successors. Let $x_1, \ldots, x_n$ be nonstandard elements of M. The automorphism f maps $(x_1, \ldots, x_n)$ on $(x_1 + 1, \ldots, x_n + 1)$ and so these n-tuples realize the same n-type in M. Choose $m \in M$ nonstandard, so if $x \geqq m$, then x is nonstandard. But from this, in particular, for the formulae $\psi_1, \ldots, \psi_r$ it follows

$$M \models (\forall x_1, \ldots, x_n \geqq m) \bigwedge_{i \leqq r} \psi_i(x_1, \ldots, x_n) \equiv \psi_i(x_1 + 1, \ldots, x_n + 1).$$

Hence also $M \models (\exists m)(\forall x_1, \ldots, x_n \geqq m) \bigwedge_{i \leqq r} \psi_i(x_1, \ldots, x_n) \equiv \psi_i(x_1 + 1, \ldots, x_n + 1)$. Since $M \equiv \omega$ we are done.

5.7. Corollary. *Let for a formula $\psi(x)$ with one free variable x, $\omega \models (\forall x)(\exists y > x)\,\psi(y)$. Then also $\omega \models (\exists y)(\forall x \geqq y)\,\psi(x)$.*

5.8. Corollary. POD $\nleqq_1$ LOS.

Proof. Not only in this proof but in the whole part we use the fact 1.2, i.e. for proving $T \nleqq_k S$ it is enough to find a model $M \models S$ such that $T \nleqq_k \mathrm{Th}(M)$. It follows from 5.7 immediately that in ω there are not two definable disjoint infinite sets (in definitions in ω we need not consider parameters because each element of ω is in ω definable without parameters). In particular $D(\omega)$ does not contain a breakable set. By 5.5 and 4.9 we are done.

5.9. Becausee w have already used brief "arithmetical" notation (as $x + 1$ for the successor of x) we continue with this in the following selfexplanatory definition: For vectors $\vec{x}, \vec{y} \in \omega^s$ we write $\vec{x} \sim \vec{y}$ iff "$x_i - y_i = \text{const}$ for $i = 1, \ldots, s$" (i.e., $x \sim x + 1$, $(2, 3) \sim (4, 5)$ but not $(2, 3) \sim (4, 6), \ldots$). This is no definition inside ω^s. We will use this notion in the proof of lemma 5.11.

Define yet for $\vec{x}, \vec{y} \in \omega^s$: $\vec{x} \leqq \vec{y}$ iff "$x_i \leqq y_i$ for $i = 1, \ldots, s$".

Lemma. (a) *Let the set $A \subseteqq \{a\} \times \omega^n$ be definable in ω^{n+1} for $a \in \omega$.* (b) *Let the set $B = \{\vec{y} \in \omega^{n+1} \mid \vec{b} \leqq \vec{y} \;\&\; \vec{b} \sim \vec{y}\}$ be definable in ω^{n+1} for $\vec{b} \in \omega^{n+1}$. Then both sets A, B are definable isomorphic in ω to definable sets $A', B' \subseteqq \omega^n$, namely*

$$A' = \{\vec{x} \in \omega^n \mid (a, \vec{x}) \in A\}, \qquad B' = \{\vec{x} \in \omega^n \mid (\exists y)\,(y, \vec{x}) \in B\}.$$

Proof. Obvious.

5.10. Remark. It follows from 5.7 and 5.9 that it is not possible in $\omega \times n$ (with the usual ordering) to define $n + 1$ disjoint infinite sets. If we define $\mathrm{LOS}_n = \mathrm{Th}(\omega \times n, <)$, it follows that $\mathrm{LOS}_n <_1 \mathrm{LOS}_{n+1}$. But in the lattice $\mathscr{M}$ evidently all these theories are in the type |LOS| (actually $\mathrm{LOS}_n \leqq_2 \mathrm{LOS}_1$).

5.11. Lemma. *For any n, $D^n(\omega)$ does not contain a breakable set.*

Proof. We will prove this lemma by induction on n. For $n = 1$ we are (by the proof of 5.8) done. Suppose that for n the lemma holds, and we prove it for $n + 1$:

1. Suppose the lemma does not hold for $n + 1$. Let $A \in D^{n+1}(\omega)$ be breakable and f be the function $2^{<\omega} \to D^{n+1}(\omega)$ from definition 4.8, i.e. $f(\Lambda) = A, \ldots$ and so on.

2. Let ψ_ξ be the formula which defines $f(\xi)$. We define m_k to be the minimal m from lemma 5.6 for the formulae ψ_ξ, $|\xi| \leqq k$ ($|\xi|$ denotes the length of ξ).

3. We define the sets $K(m)$ and $R_i(m)$, $i \leqq n + 1$, by

$$K(m) = \{(x_1, \ldots, x_{n+1}) \mid x_j \geqq m \text{ for all } j \leqq n + 1\},$$
$$R_i(m) = \{\vec{x} \in K(m) \mid x_i = m\}.$$

So $K(m)$ is a quadrant and $\bigcup_i R_i(m)$ is its border.

4. For $\xi \in 2^{<\omega}$ and $i \leqq n + 1$ we define a function $\langle \xi, i \rangle : 2^{<\omega} \times \{1, \ldots, n+1\} \to \to 2^{<\omega}$ as follows:

(i) Consider the function $g: 2^{<\omega} \to D^n(\omega)$ which satisfies the following definition:

$$(y_1, \ldots, y_n) \in g(\eta) \text{ iff } (\exists \vec{x} \in \omega^{n+1}) (\vec{x} \sim (y_1, \ldots, y_{i-1}, m_{|\xi|}, y_i, \ldots, y_n) \\ \& \vec{x} \in R_i(m_{|\xi^\frown\eta|}) \;\&\; \vec{x} \in f(\xi^\frown \eta)).$$

Observe that, since the m_k's are fixed, the right-hand formula is, in fact, a definition inside ω. So $g(\eta) \in D^n(\omega)$.

(ii) By induction hypothesis no set from $D^n(\omega)$ is breakable, so g does not satisfy the conditions from 4.8 for breaking $g(\Lambda)$. But g evidently satisfies (iii), (iv) of 4.8 (because f satisfies them). Hence condition (ii) of 4.8 must necessarily fail, i.e., there exists η such that $g(\eta)$ is finite.

(iii) Define $\langle \xi, i \rangle = \xi^\frown \eta$, where η is minimal with respect to the lexicographical ordering of $2^{<\omega}$ such that $g(\eta)$ is finite.

5. We define ξ_j by $\xi_0 = \Lambda$, $\quad \xi_{i+1} = \langle \xi_i, i+1 \rangle \quad (i \leqq n_0)$.

By the definition of the function $\langle \xi, i \rangle$ we have $f(\xi_i) \supseteqq f(\xi_{i+1})$, and $f(\xi_i) \cap R_i(m_{|\xi_i|})$ is finite.

6. By the conclusion of 5 the set $B = f(\xi_{n+1})$ satisfies the condition:

$$(\forall i)\; B \cap R_i(m_{|\xi_{n+1}|}) \text{ is finite.}$$

7. By definition 2 of m_k now the following must hold:

$$(\forall x_1, \ldots, x_{n+1} \geqq m_{|\xi_{n+1}|})\, (x_1, \ldots, x_{n+1}) \in B \equiv (x_1 + 1, \ldots, x_{n+1} + 1) \in B.$$

So the set $B \cap K(m_{|\xi_{n+1}|})$ is the union of finitely many (see 6) sets of the form

(a) $\qquad \{\vec{y} \in K(m_{|\xi_{n+1}|}) \mid \vec{y} \sim \vec{x}\}$, for some $\vec{x} \in \bigcup_i R_i(m_{|\xi_{n+1}|})$.

8. The set $B \setminus K(m_{|\xi_{n+1}|})$ is a part of the union of finitely many sets of the form

(b) $\qquad \{\vec{y} \in \omega^{n+1} \mid y_i = r\}$, for some $i \leqq n + 1$ and $r < m_{|\xi_{n+1}|}$.

9. By lemma 5.9 each definable set of the form (a) or (b) is definable isomorphic to a definable subset of ω^n and hence it is not (by induction hypothesis) breakable.

10. It is not difficult to see that a finite disjoint union of unbreakable sets is unbreakable, and also neither of its parts is breakable.

11. From this it follows that $B = f(\xi_{n+1})$ is unbreakable. This is a contradiction with assumption 1.

5.12. Theorem. POD $\not\leqq$ LOS.

Proof. Suppose that I is an interpretation of POD in LOS and so, in particular, in ω. Because ω is well-ordered, it is sufficient to consider I with absolute equality. Further, because POD is finite, I is global. By 5.5, $D(\omega^I_{\mathrm{POD}})$ contains a breakable set and hence, by 4.9, $D^n(\omega)$ too. This is a contradiction to lemma 5.11.

5.13. Lemma. *If T is a theory, then* LO $\leqq T$ *implies* LO $\leqq_1 T$.

Altough this lemma is not a consequence of a "similar" Mycielski's lemma (see [1], 2.vi), but the idea of the proof is in fact identical. Therefore we omit the proof.

5.14. Theorem. LO $\not\leq$ POS.

Proof. The proof of this theorem is rather long and therefore we will divide it into claims:

1. Let N be a the structure $(2^{<\omega}; {}^\frown, \leqq, \wedge, \sim)$, where the reduct $(2^{<\omega}; {}^\frown, \leqq)$ is isomorphic to the structure from 4.8, $\xi \wedge \eta$ denotes the $\leqq$-meet of ξ, η and $N \models \xi \sim \eta$ iff $|\xi| = |\eta|$.

2. We observe that $N \models (\forall \xi)(\forall \eta \leqq \xi)(\exists \varrho)(\xi \sim \varrho \;\&\; \xi \wedge \varrho = \eta)$.

3. We fix any proper elementary extension M of N, i.e. $N \prec M$.

4. Let $\xi \in M$ be a nonstandard element of M. Then there exists an automorphism $f \in \mathrm{Aut}(M)$ such that (i) $f(\xi) = \xi^\frown 1$, (ii) $f(\eta) = \eta$ for all η such that $\xi \wedge \eta \in N$.

Proof. (a) Suppose that ξ is a sequence of 1's (i.e. $(\forall \eta < \xi)\, \eta^\frown 1 \leqq \xi$). Each element $\varrho \in M$ can be written uniquely as $(\xi \wedge \varrho)^\frown \bar{\varrho}$. Now we define the function f_1 as follows

$$f_1(\varrho) = \begin{cases} \varrho & \text{if } \xi \wedge \varrho \in N, \\ (\xi \wedge \varrho)^\frown 1^\frown \bar{\varrho} & \text{otherwise.} \end{cases}$$

Evidently $f_1 \in \mathrm{Aut}(M)$.

(b) Clearly, for ξ general, there exists an automorphism $g \in \mathrm{Aut}(M)$ such that $g(\xi^\frown 1)$ is a sequence of 1's. Now, using f_1 from (a) (constructed for $g(\xi^\frown 1)$), we define $f = g^{-1} f_1 g$. Again $f \in \mathrm{Aut}(M)$ and f satisfies the required conditions. Hence we are done.

5. Let I be a 1-dimensional interpretation of LO in POS and let us fix some parameters of I in N. Since $N \prec M$, these parameters are also the parameters of I in M. Let $=_I$ interpret $=$, δ_I be the domain and $<_I$ interpret $<$.

6. We take all words $\beta_0, \ldots, \beta_{2^n-1}$ of length n, where $n - 1$ is the maximal length of the parameters of I. On at least one of the domains $\{\xi \mid \xi \geqq \beta_i\}$ the preorder $<_I$ is unbounded. So we can further suppose that $\delta_I \subseteq \{\xi \mid \xi \geqq \beta_0\}$ and no parameter of I is in δ_I.

7. Evidently for $\xi, \eta \in N$ we have: if $\xi, \eta \geqq \beta_0$ and $\xi \sim \eta$, then there exists an automorphism $f \in \mathrm{Aut}^I(N)$ such that $f(\xi) = \eta$ and $f^{(2)}(\xi) = \xi$.

8. By 4.4 for such an f necessarily $f(\xi) =_I \xi$, and from this (by 7) follows

$$N \models (\forall \xi, \eta \geqq \beta_0)(\xi \sim \eta \rightarrow (\xi \in \delta_I \equiv \eta \in \delta_I \;\&\; (\xi, \eta \in \delta_I \rightarrow \xi =_I \eta))).$$

9. Since the domain δ_I is necessarily infinite (and $N \prec M$), for some nonstandard $\xi_0 \in M$, $M \models \xi_0 \in \delta_I$. Further let ξ_0 be such a fixed nonstandard element of M.

10. We consider the formula $\psi(\eta)$ defined by $(\exists \varrho \in \delta_I)(\xi_0 \wedge \varrho = \eta \;\&\; \xi_0 =_I \varrho)$. So, in particular, $M \models \psi(\eta) \rightarrow \eta \leqq \xi_0$. Since in N each definable (and surely nondefinable too) subset has a $\leqq$-minimum so it does in M. Thus let η_0 be a $\leqq$-minimum of $\{\eta \mid \psi) \eta)\}$ in M.

11. We prove that $\eta_0 = \beta_0$. Otherwise, since clearly $\eta_0 \geqq \beta_0$, there exists a $\leqq$-predecessor δ_0 of η_0 such that $\delta_0 \geqq \beta_0$. By 2, 8 and from $N \prec M$ there exists a $\varrho_0 \in M$ such that $M \models \xi_0 \wedge \varrho_0 = \delta_0 \;\&\; \xi_0 =_I \varrho_0$, and hence $M \models \psi(\delta_0)$, too. But this is a contradiction with the choice of η_0.

12. Now let ϱ_0 be such that

$$M \models \xi_0 \wedge \varrho_0 = \beta_0 \;\&\; \xi_0 =_I \varrho_0.$$

For $\delta \geqq \varrho_0$ also $\xi_0 \wedge \delta = \beta_0$. By 4 then there exists an automorphism $f \in \mathrm{Aut}^I(M)$ such that $f(\xi_0) = \xi_0$ and $f(\delta) = \delta^\frown 1$. Hence it holds

$$M \models (\forall \delta \geqq \varrho_0)(\xi_0 =_I \delta \equiv \xi_0 =_I \delta^\frown 1) \;\&\; (\delta \in \delta_I \equiv \delta^\frown 1 \in \delta_I)),$$

and therefore $M \models (\exists \xi, \varrho \in \delta_I)(\forall \delta \geqq \varrho)((\xi =_I \delta \equiv \xi =_I \delta^\frown 1) \;\&\; (\delta \in \delta_I \equiv \delta^\frown 1 \in \delta_I))$.

13. Again from $N \prec M$, the last formula from 12 holds in N. Because 8 it means for N that there exists an n such that for all $\xi, \eta \in N$: if $|\xi|, |\eta| \geqq n$, then $N \models (\xi, \eta \in \delta_I \to \xi =_I \eta)$.

14. We are done, because this is a contradiction with the infiniteness of N^I_{LO}, i.e. $\delta_I / =_I$.

5.15. Theorem. *The diagram of* 5.2 *is correct.*

The proof follows easily from 5.4, 5.12 and 5.14.

5.16. In addition to 5.15 a little more holds for LO:

Theorem. $|\mathrm{LO}| = |\mathrm{LOS}| \wedge |\mathrm{LOD}|$.

Proof. From 5.15 it is clear that $|\mathrm{LO}| \leqq |\mathrm{LOS}| \wedge |\mathrm{LOD}|$. Hence we must prove the inverse relation. To prove this, by 1.2 and 1.3, it is sufficient to interpret in each model $M \models \mathrm{LO}$ either LOS or LOD. Let $M \models \mathrm{LO}$. We call a closed interval $\langle u, v\rangle$ *isolated* iff each its interior points is isolated in $\langle u, v\rangle$ (in particular each interval $\langle a, a\rangle$ is isolated). A closed isolated interval which is maximal (with respect to inclusion) with this property will be called an *MCI-interval* (maximal closed isolated interval). There are two possibilities:

1. $M \models$ "each closed isolated interval is contained in an MCI-interval".

The property of x, y: "$\langle x, y\rangle$ is an MCI-interval" can be easily expressed in the language of linear order and clearly each MCI-interval can be represented by its left endpoint. Now we can define a 1-dimensional interpretation I of LOD in $\mathrm{Th}(M)$ by

$$\delta_I(x) \equiv \text{"}x \text{ is the left endpoint of an MCI-interval"}, \quad x <_I y \equiv x < y.$$

It is easy to see that the assumption of this paragraph implies $M \models \mathrm{LOD}^I$.

2. $M \models$ "there exists a closed isolated interval $\langle a, b\rangle$ which is not contained in any MCI-interval".

Let, for example, there do not exist the right endpoint for such an MCI-interval. Then we define a 1-dimensional interpretation J of LOS in $\mathrm{Th}(M)$ as follows:

$$\delta_J(x) \equiv \text{"}b \leqq x \text{ \& the interval } \langle b, x\rangle \text{ is isolated"}, \quad x <_J y \equiv x < y.$$

Again it is clear that $M \models \mathrm{LOS}^J$. Thus we are done.

5.17. In this last section of this paragraph we will prove a lemma which will allow us to use the preceding theorem for a small contribution to the problem whether LOS is prime.

Lemma. *Let the types* $a, b \in \mathscr{M}$ *be incomparable and the type* $c = a \wedge b$ *be join-irreducible. If* $t, s < a$ *are types such that* $s \vee t = a$, *then* (i) *if* $s \geqq c$ (*resp.* $t \geqq c$), *then* s (*resp.* t) *is not meet-irreducible and* (ii) $s \geqq c$ *or* $t \geqq c$.

Proof. (i) Let $s \geqq c$ and s be meet-irreducible. Then $s \geqq a \wedge b$ and so $s \geqq a$ or $s \geqq b$. Because $s < a$ we have $s \geqq b$. But from this follows $a > b$, which is a contradiction to the incomparability of a, b. (ii) By $s \vee t = a$ we have $a \wedge b = (s \vee t) \wedge b = (s \wedge b) \vee (t \wedge b) = c$. Since c is join-irreducible we have, for example, $s \wedge b = c$. But then immediately $s \geqq c$. We are done.

Now we can apply this lemma to $a = |\mathrm{LOS}|$, $b = |\mathrm{LOD}|$ and $c = |\mathrm{LO}|$ (by [1] LO is prime, see also § 0).

§ 6. Problems

In this last paragraph we want to call attention to a few problems.

6.1. Results 1.5 from [4] about compact types, 2.2 about coprime types and paper [5] about prime types promise that more lattice-theoretically defined properties of types have also their "logical" characterizations. In particular: which types are cocompact?

6.2. Examples of compact prime types ($|\mathrm{LO}|$, $|\mathrm{I}\Delta_0|$, $|\mathrm{GB}|$) or cocompact coprime types (if they exist?) prove that in $\mathcal{M}$ there are intervals of length 1. But no explicit example is known!

6.3. With regard to 6.2 we ask: are some of the intervals in the diagram 5.2 of length 1? How do the intervals of the types between $|\mathrm{LO}|$ and $|\mathrm{LOS}|$, $|\mathrm{PO}|$ and $|\mathrm{LO}|, \ldots$ look like?

6.4. To the "perfection" of the diagram 5.2 it remains to prove that POD < LOD and that LO, POD are incomparable. Is it true?

6.5. Diagram 5.2 suggests that the position of LOS and LOD in this part of the lattice $\mathcal{M}$ should be in a sense "symmetrical". So it is reasonable to ask: is LOD prime? (In fact, if one considers only 1-dimensional interpretations, then it is so. See [5].)

6.6. Many questions about the structure of the lower part of $\mathcal{M}$ "near" theories of order are open. For example: how wide is the lattice $\mathcal{M}$ here? (In [4] J. Mycielski proved, using R. Montague's work [3], that in $\mathcal{M}$ there exists an antichain of length $2^{\aleph_0}$. But it follows from [3] that this antichain is at least above the type of $\mathrm{I}\Delta_0$.)

6.7. Which are the differences between $\mathcal{M}$ and the lattice of 1-dimensional interpretation "near" theories of order? Remark: 5.10 proves that there are some.

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