Linear Integral Equations in the Space of Regulated Functions

Milan Tvrdý *

Mathematical Institute
Academy of Sciences of the Czech Republic
115 67 PRAHA 1, Žitná 25, Czech Republic
(e-mail: tvrdy@math.cas.cz)

Abstract. In this paper we investigate systems of linear integral equations in the space $\mathbf{G}_L^n$ of n-vector valued functions which are regulated on the closed interval $[0,1]$ (i.e. such that can have only discontinuities of the first kind in $[0,1]$) and left-continuous in the corresponding open interval $(0,1)$. In particular, we are interested in systems of the form

$$x(t) - A(t)x(0) - \int_0^1 B(t,s)\mathrm{d}[x(s)] = f(t),$$

where $f \in \mathbf{G}_L^n$, the columns of the $n \times n$-matrix valued function A belong to $\mathbf{G}_L^n$, the entries of $B(t,.)$ have a bounded variation on $[0,1]$ for any $t \in [0,1]$ and the mapping $t \in [0,1] \mapsto B(t,.)$ is regulated on $[0,1]$ and left-continuous on $(0,1)$ as the mapping with values in the space of $n \times n$-matrix valued functions of bounded variation on $[0,1]$. The integral stands for the Perron-Stieltjes one treated as the special case of the Kurzweil-Henstock integral.

In particular, we prove basic existence and uniqueness results for the given equation and obtain the explicit form of its adjoint equation. A special attention is paid to the Volterra (causal) type case. It is shown that in that case the given equation possesses a unique solution for any right-hand side from $\mathbf{G}_L^n$, and its representation by means of resolvent operators is given.

The results presented cover e.g. the results known for systems of linear generalized differential equations

$$x(t) - x(0) - \int_0^t \mathrm{d}[A(s)]x(s) = f(t) - f(0)$$

as well as systems of Stieltjes integral equations

$$x(t) - \int_0^1 [\mathrm{d}_s K(t,s)]x(s) = g(t) \quad \text{or} \quad x(t) - \int_0^t [\mathrm{d}_s K(t,s)]x(s) = g(t).$$

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0 . Introduction

The paper is devoted to linear operator equations of the form

$$x - \mathscr{L}x = f, \tag{0.1}$$

where $\mathscr{L}$ is a linear compact operator on the space $\mathbf{G}_L^n$ of column n-vector valued functions $x : [0,1] \mapsto \mathbb{R}^n$ which are regulated on $[0,1]$ and left-continuous on $(0,1)$, and $f \in \mathbf{G}_L^n$ is given. Due to Schwabik (cf. [15, Theorem]) it is known that $\mathscr{L}$ is a linear compact operator on $\mathbf{G}_L^n$ if and only if there are $n \times n$-matrix valued functions $A(t)$ and $B(t,s)$ respectively defined on $[0,1]$ and $[0,1] \times [0,1]$ and such that

$$(\mathscr{L}x)(t) = A(t)x(0) + \int_0^1 B(t,s)\mathrm{d}[x(s)] \ \text{ for } \ x \in \mathbf{G}_L^n \ \text{ and } \ t \in [0,1], \tag{0.2}$$

while the columns of A belong to $\mathbf{G}_L^n$ ($A \in \mathbf{G}_{\mathrm{L}}^{n\times n}$), the entries of $B(t,.)$ have a bounded variation on $[0,1]$ for any $t \in [0,1]$ ($B(t,.) \in \mathbf{BV}^{n\times n}$) and the mapping

$$\mathscr{M}_B : t \in [0,1] \mapsto \mathscr{M}_B(t) = B(t,.) \in \mathbf{BV}^{n\times n}$$

is regulated on $[0,1]$ and left-continuous on $(0,1)$ (i.e. $B \in \mathscr{K}_L^{n\times n}$, see Definitions 2.2 and 2.3). The integral on the right-hand side of (0.2) stands for the Perron-Stieltjes one treated as the special case of the Kurzweil-Henstock integral.

In Sections 3 and 4 we prove basic existence and uniqueness results for the equation (0.1) and obtain the explicit form of its adjoint equation. An important tool for the proofs of our main results is in particular the theorem on the interchange of the integration order for Stieltjes type integrals (i.e. the Bray Theorem). Its proof for the Perron-Stieltjes integral is given in Sec.2 (cf. Theorem 2.13).

Special attention (cf. Sec. 5) is paid to the causal case, i.e. to the Volterra-Stieltjes integral equations of the form

$$x(t) - A(t)x(0) - \int_0^t B(t,s)\mathrm{d}[x(s)] = f(t), \quad t \in [0,1],$$

where $A(0) = 0$.

Similar problems in the space of regulated functions were treated e.g. by Ch. S. Hönig [6], [7], L. Fichmann [3] and L. Barbanti [2], where the interior (Dushnik) integral was used.

1 . Preliminaries

1.1. Notation. Throughout the paper by $\mathbb{R}^{p\times q}$ we denote the space of real $p\times q$-matrices, $\mathbb{R}^n=\mathbb{R}^{n\times 1}$ stands for the space of real column n-vectors, $\mathbb{R}^{1\times 1}=\mathbb{R}^1=\mathbb{R}$. Given a $p\times q$-matrix M, its elements are denoted by $m_{i,j}$, i.e.

$$M=(m_{i,j})_{\substack{i=1,2,\dots,p,\\ j=1,2,\dots,q}}\,,\;\; |M|=\max_{j=1,2,\dots,q}\sum_{i=1}^{p}|m_{i,j}| \text{ and } \;M^T=(m_{j,i})_{\substack{j=1,2,\dots,q\\ i=1,2,\dots,p}}.$$

In particular,

$$|x|=\sum_{i=1}^{n}|x_i|,\;\; x^T=(x_1,x_2,\dots,x_n) \text{ and } \;|x^T|=\max_{j=1,\dots,n}|x_j| \text{ for } \;x\in\mathbb{R}^n.$$

Furthermore, for a given matrix $M\in\mathbb{R}^{p\times q}$, its columns are denoted by $m^{[j]}$ and we write $M=(m^{[j]})_{j=1,2,\dots,q}$. Obviously, we have

$$|M|=\max_{j=1,2,\dots,q}|m^{[j]}| \quad \text{for all} \quad M\in\mathbb{R}^{p\times q}.$$

The symbols I and 0 stand respectively for the identity and the zero matrix of the proper type. Given an $n\times n$-matrix M, $\det(M)$ denotes its determinant.

If $-\infty<a<b<\infty$, then $[a,b]$ and (a,b) denote the corresponding closed and open intervals, respectively. Furthermore, $[a,b)$ and $(a,b]$ are the corresponding half-open intervals. The sets $d=\{t_0,t_1,\dots,t_m\}$ of points in the closed interval $[a,b]$ such that $a=t_0<t_1<\dots<t_m=b$ are called *divisions* of $[a,b]$. The set of all divisions of the interval $[a,b]$ is denoted by $\mathcal{D}(a,b)$.

Given $M\subset\mathbb{R}$, χ_M denotes its characteristic function.

1.2. Regulated functions. Any function $f:[a,b]\mapsto\mathbb{R}$ which possesses finite limits

$$f(t+)=\lim_{\tau\to t+}f(\tau) \quad \text{and} \quad f(s-)=\lim_{\tau\to s-}f(\tau)$$

for all $t\in[a,b)$ and $s\in(a,b]$ is said to be *regulated* on $[a,b]$. A $p\times q$-matrix valued function $F:[a,b]\mapsto\mathbb{R}^{p\times q}$ is said to be regulated on $[a,b]$ if all its components $f_{i,j}$ ($i=1,2,\dots,p,\ j=1,2,\dots,q$) are regulated on $[a,b]$. The linear space of $p\times q$-matrix valued functions regulated on $[a,b]$ is denoted by $\mathbf{G}^{p\times q}(a,b)$, $\mathbf{G}_L^{p\times q}(a,b)$ denotes the space of all functions from $\mathbf{G}^{p\times q}(a,b)$ which are left-continuous on (a,b). It is easy to see that any function regulated on $[a,b]$ is bounded on $[a,b]$. For $F\in\mathbf{G}^{p\times q}(a,b)$ we put

$$\|F\|=\sup_{t\in[a,b]}|F(t)|.$$

It is well known that both $\mathbf{G}^{p\times q}(a,b)$ and $\mathbf{G}_L^{p\times q}(a,b)$ are Banach spaces with respect to this norm (cf. [6, Theorem I.3.6]). Given $F\in\mathbf{G}^{p\times q}(a,b)$, $t\in[a,b)$ and $s\in(a,b]$, we put

$$\Delta^+F(t)=F(t+)-F(t) \quad \text{and} \quad \Delta^-F(s)=F(s)-F(s-).$$

A function $F\in\mathbf{G}^{p\times q}$ is said to be a *finite step function* on $[0,1]$, if there exist a division $d=\{t_0,t_1,\dots,t_m\}$ of the interval $[0,1]$ and real numbers $c_{i,j}^{[r]}$ and $d_{i,j}^{[r]}$, $r=1,2\dots,m$, $i=1,2,\dots,p$, $j=1,2,\dots,q$, such that

$$f_{i,j}(t)=\sum_{r=0}^{m}c_{i,j}^{[r]}\chi_{[t_r,1]}(t)+\sum_{r=0}^{m}d_{i,j}^{[r]}\chi_{(t_r,1]}(t) \quad \text{on} \quad [0,1]$$

for any component $f_{i,j}$, $i=1,2,\dots,p$, $j=1,2,\dots,q$, of the function F. It is well-known (cf. in [6, Theorem I.3.1]) that $F\in\mathbf{G}^{p\times q}$ if and only if there is a sequence $\{F_k\}_{k=1}^{\infty}$ of finite step functions on $[0,1]$ such that

$$\lim_{k\to\infty}\|F_k-F\|=0.$$

1.3. Functions of bounded variation. For a given function $F:[a,b]\mapsto\mathbb{R}^{p\times q}$ and a given division $d=\{t_0,t_1,\dots,t_m\}$ of $[a,b]$ $(d\in\mathcal{D}(a,b))$ we define

$$S(F,d)=\sum_{j=1}^{m}|F(t_j)-F(t_{j-1})|.$$

If

$$\operatorname{var}_a^bF=\sup_{d\in\mathcal{D}(a,b)}S(F,d)<\infty,$$

we say that the function F has a *bounded variation* var_a^bF on the interval $[a,b]$. $\mathbf{BV}^{p\times q}(a,b)$ denotes the Banach space of $p\times q$-matrix valued functions of bounded variation on $[a,b]$ equipped with the norm

$$F\in\mathbf{BV}^{p\times q}(a,b)\mapsto\|F\|_{\mathbf{BV}}=|F(a)|+\operatorname{var}_a^bF.$$

For a given $F\in\mathbf{BV}^{p\times q}(a,b)$, we define

$$v_F(t)=\operatorname{var}_a^tF \quad \text{for} \quad t\in[a,b].$$

It is well known (cf. sections II.4.7, II.6.1 and the introduction to Section II.7 in [5]) that the relations

$$\Delta^+v_F(t)=\Delta^+F(t) \quad \text{for all} \quad t\in[a,b) \tag{1.1}$$

and

$$\Delta^- v_F(s) = \Delta^- F(s) \quad \text{for all} \quad s \in (a, b] \tag{1.2}$$

are true.

For more details concerning regulated functions or functions of bounded variation see [[1]], [6], [4] or [5], respectively.

1.4. Notation. In the case $[a, b] = [0, 1]$ we write simply $\mathcal{D}$, $\mathbf{G}^{p\times q}$, $\mathbf{G}_L^{p\times q}$ and $\mathbf{BV}^{p\times q}$ instead of $\mathcal{D}(0, 1)$, $\mathbf{G}^{p\times q}(0, 1)$, $\mathbf{G}_L^{p\times q}(0, 1)$ and $\mathbf{BV}^{p\times q}(0, 1)$, respectively. Furthermore, $\mathbf{G}^{n\times 1} = \mathbf{G}^n$, $\mathbf{G}_L^{n\times 1} = \mathbf{G}_L^n$ and $\mathbf{BV}^{n\times 1} = \mathbf{BV}^n$.

1.5. Functions of two real variables. If a $p \times q$-matrix valued function K is defined on $[0, 1] \times [0, 1]$ and $t, s \in [a, b]$ are given, then the symbols $K(t, .)$ and $K(., s)$ denote the functions

$$K(t, .) : \tau \in [0, 1] \mapsto K(t, \tau) \in \mathbb{R}^{p\times q}$$

and

$$K(., s) : \tau \in [0, 1] \mapsto K(\tau, s) \in \mathbb{R}^{p\times q},$$

respectively. Furthermore, if $s \in [0, 1]$ and $K(., s) \in \mathbf{G}^{p\times q}$, then we put

$$\Delta_1^- K(\tau, s) = K(\tau, s) - K(\tau-, s) \quad \text{for} \quad \tau \in (0, 1]$$

and

$$\Delta_1^+ K(\tau, s) = K(\tau+, s) - K(\tau, s) \quad \text{for} \quad \tau \in [0, 1).$$

Similarly, if $t \in [0, 1]$ and $K(t, .) \in \mathbf{G}^{p\times q}$, then we put

$$\Delta_2^- K(t, \sigma) = K(t, \sigma) - K(t, \sigma-) \quad \text{for} \quad \sigma \in (0, 1]$$

and

$$\Delta_2^+ K(t, \sigma) = K(t, \sigma+) - K(t, \sigma) \quad \text{for} \quad \sigma \in [0, 1).$$

1.6. Notation. For given linear spaces $\mathbb{X}$ and $\mathbb{Y}$, the symbols $\mathcal{L}(\mathbb{X}, \mathbb{Y})$ and $\mathcal{K}(\mathbb{X}, \mathbb{Y})$ denote the linear space of all linear bounded mappings of $\mathbb{X}$ into $\mathbb{Y}$ and the linear space of linear compact mappings of $\mathbb{X}$ into $\mathbb{Y}$, respectively. If $\mathbb{X} = \mathbb{Y}$ we write $\mathcal{L}(\mathbb{X})$ and $\mathcal{K}(\mathbb{X})$. If $\mathscr{A} \in \mathcal{L}(\mathbb{X}, \mathbb{Y})$, then $\mathcal{R}(\mathscr{A})$, $\mathcal{N}(\mathscr{A})$ and $\mathscr{A}^*$ denote its range, null space and adjoint operator, respectively.

1.7. Integrals. The integrals which occur in this paper are the Perron-Stieltjes ones. For the original definition, see [19] or [10]. We use the equivalent summation definition due to Kurzweil (cf. [8], [9], [16]).

Let the functions f, g be regulated on $[a,b]$. If the integral $\int_a^b f(s)\,\mathrm{d}[g(s)]$ has a finite value, then by Theorem 1.3.4 from [8] the function

$$h : t \in [a,b] \mapsto \int_a^t f(s)\,\mathrm{d}[g(s)] \in \mathbb{R}$$

is regulated on $[a,b]$. Let us note that if both the functions f, g are regulated on $[a,b]$ and at least one of them has a bounded variation on $[a,b]$, then the integral

$$\int_a^b f(s)\,\mathrm{d}[g(s)]$$

has a finite value (cf. [18, Theorem2.8]). In this case the above mentioned Theorem 1.3.4 from [8] implies that

$$h(t+) = h(t) + f(t)\Delta^+ g(t) \quad \text{and} \quad h(s-) = h(s) - f(s)\Delta^- g(s)$$

holds for all $t \in [a,b)$ and $s \in (a,b]$. Moreover, if $g \in \mathbf{BV}$ then $h \in \mathbf{BV}$, as well.

Further basic properties of the Perron-Stieltjes integral with respect to scalar regulated functions were described in [18].

Given a $p \times q$-matrix valued function F and a $q \times n$-matrix valued function G defined on $[a,b]$ and such that all the integrals

$$\int_a^b f_{i,k}(t)\,\mathrm{d}[g_{k,j}(t)] \quad (i = 1,2,\dots,p;\, k = 1,2,\dots,q;\, j = 1,2,\dots,n)$$

exist (i.e. they have finite values), then

$$\int_a^b F(t)\,\mathrm{d}[G(t)] = \Big(\sum_{k=1}^q \int_a^b f_{i,k}(t)\,\mathrm{d}[g_{k,j}(t)] \Big)_{\substack{i=1,2,\dots,p \\ j=1,2,\dots,n}}.$$

The integrals

$$\int_a^b \mathrm{d}[F(t)]G(t) \quad \text{and} \quad \int_a^b F(t)\,\mathrm{d}[G(t)]H(t)$$

for matrix valued functions F, G, H of proper types are defined analogously. The extension of the results obtained in [18] for scalar functions to vector valued or matrix valued functions is obvious and hence for the basic facts concerning integrals with respect to regulated functions we will refer to the corresponding assertions from [18].

In particular, the following lemma follows easily from [18, Theorem 3.8].

1.8. Lemma. *Φ is a linear bounded mapping of $\mathbf{G}_L^n$ into $\mathbb{R}^m$ if and only if there exists an $m \times n$-matrix M and an $m \times n$-matrix valued function $K(t)$ of bounded variation on $[0,1]$ such that*

$$\Phi x = M x(0) + \int_0^1 K(t) \mathrm{d}[x(t)] \quad \text{for all} \quad x \in \mathbf{G}_L^n.$$

Furthermore, for a given $m \times n$-matrix M and an $m \times n$-matrix valued function $K(t)$ of bounded variation on $[0,1]$, the relation

$$M x(0) + \int_0^1 K(t) \mathrm{d}[x(t)] = 0 \quad \text{for all} \quad x \in \mathbf{G}_L^n$$

holds if and only if

$$M = 0 \quad \text{and} \quad K(t) \equiv 0 \quad \text{on} \quad [0,1].$$

By a slight modification of Corollary 2 from [15], we can obtain a result analogous to Lemma 1.8 also for linear bounded mappings of $\mathbf{G}_L^n$ into $\mathbf{G}^n$.

1.9. Lemma. *$\mathscr{L}$ is a linear bounded mapping of $\mathbf{G}_L^n$ into $\mathbf{G}^n$ if and only if there exist $n \times n$-matrix valued functions $A \in \mathbf{G}^{n\times n}$ and $B : [0,1] \times [0,1] \mapsto \mathbb{R}^{n\times n}$ such that*

(1.3) $\quad B(.,s) \in \mathbf{G}^{n\times n}$ *for all* $s \in [0,1]$,

(1.4) $\quad B(t,.) \in \mathbf{BV}^{n\times n}$ *for all* $t \in [0,1]$,

(1.5) *there is a* $\beta < \infty$ *such that* $\mathrm{var}_0^1 B(t,.) \le \beta$ *for all* $t \in [0,1]$

and $\mathscr{L}$ is given by (0.2). Furthermore, for given $n \times n$-matrix valued functions $A \in \mathbf{G}^{n\times n}$ and $B(t,s)$ fulfilling (1.3)-(1.5) the relation

$$A(t) x(0) + \int_0^1 B(t,s) \mathrm{d}[x(s)] \equiv 0 \quad \text{on} \quad [0,1]$$

holds for all $x \in \mathbf{G}^n$ if and only if

$$A(t) \equiv 0 \quad \text{on} \quad [0,1] \quad \text{and} \quad B(t,s) \equiv 0 \quad \text{on} \quad [0,1] \times [0,1].$$

2 . Functions of the class $\mathscr{K}^{n\times n}$ and the Bray Theorem

In this section we shall study the properties of the class $\mathscr{K}^{n\times n}$ of $n \times n$-matrix valued functions which will play a crucial role in our investigations of equations of the form (0.1.

2.1. Notation. For a given function $K : [0,1] \times [0,1] \mapsto \mathbb{R}^{n\times n}$ such that $K(t,.) \in \mathbf{BV}^{n\times n}$ for any $t \in [0,1]$, we denote by $\mathscr{M}_K$ the mapping of $[0,1]$ into $\mathbf{BV}^{n\times n}$ defined by

$$\mathscr{M}_K : t \in [0,1] \mapsto \mathscr{M}_K(t) = K(t,.) \in \mathbf{BV}^{n\times n}. \tag{2.1}$$

2.2. Definition. We say that a matrix-valued function $K : [0,1] \times [0,1] \mapsto \mathbb{R}^n \times n$ belongs to the class $\mathscr{K}^{n\times n}$ if it satisfies the following hypothesis:

(H_1) $K(t,.) \in \mathbf{BV}^n \times n$ for any $t \in [0,1]$;

(H_2)(i) for any $t \in [0,1)$ there exists a function $K_t^+ = \mathscr{M}_K(t+) \in \mathbf{BV}^{n\times n}$ such that

$$\lim_{\tau\to t+} \|\mathscr{M}_K(\tau) - K_t^+\|_{\mathbf{BV}} = 0,$$

(H_2)(ii) for any $t \in (0,1]$ there exists a function $K_t^- = \mathscr{M}_K(t-) \in \mathbf{BV}^{n\times n}$ such that

$$\lim_{\tau\to t-} \|\mathscr{M}_K(\tau) - K_t^-\|_{\mathbf{BV}} = 0.$$

2.3. Definition. We say that a matrix-valued function $K : [0,1] \times [0,1] \mapsto \mathbb{R}^n \times n$ belongs to the class $\mathscr{K}_L^{n\times n}$ if $K \in \mathscr{K}^{n\times n}$ and the mapping $\mathscr{M}_K : [0,1] \mapsto \mathbf{BV}^{n\times n}$ given by (2.1) is left-continuous on $(0,1)$, i.e.

$$\lim_{\tau\to t-} \|K(\tau,.) - K(t,.)\|_{\mathbf{BV}} = 0$$

holds for any $t \in (0,1)$.

2.4. Remark. Let a matrix-valued function $K : [0,1] \times [0,1] \mapsto \mathbb{R}^{n\times n}$ be such that $K(t,.) \in \mathbf{BV}^{n\times n}$ for any $t \in [0,1]$ and let the mapping $\mathscr{M}_K : [0,1] \mapsto \mathbf{BV}^{n\times n}$ be defined by (2.1). We say that $\mathscr{M}_K$ is regulated on $[0,1]$ if the condition (H_2) from Definition 2.2 is satisfied. Obviously, (H_2) is true if and only if the following assertions are true:

(H_2)(i′) for any $t \in [0,1)$ and any $\varepsilon > 0$ there exists a $\delta > 0$ such that $t + \delta < 1$ and $\|K(\tau_2,.) - K(\tau_1,.)\|_{\mathbf{BV}} < \varepsilon$ for all $\tau_1, \tau_2 \in (t, t+\delta)$

(H_2)(ii′) for any $t \in (0,1]$ and any $\varepsilon > 0$ there exists a $\delta > 0$ such that $t - \delta > 0$ and $\|K(\tau_2,.) - K(\tau_1,.)\|_{\mathbf{BV}} < \varepsilon$ for all $\tau_1, \tau_2 \in (t-\delta, t)$.

The following assertion due to Schwabik (cf. [15, Theorem 4]) has been already mentioned in the introduction.

2.5. Theorem. *$\mathscr{L}$ is a linear compact mapping of $\mathbf{G}_L^n$ into $\mathbf{G}^n$ if and only if there exist $n\times n$-matrix valued functions $A \in \mathbf{G}^{n\times n}$ and $B : [0,1]\times[0,1] \mapsto \mathbb{R}^{n\times n}$ such that $B \in \mathscr{K}^{n\times n}$ and $\mathscr{L}$ is given by (0.2). Furthermore, $\mathscr{L}$ is a linear compact mapping of $\mathbf{G}_L^n$ into $\mathbf{G}_L^n$ if and only if there exist $n \times n$-matrix valued functions $A \in \mathbf{G}_{\mathrm{L}}^{n\times n}$ and $B : [0,1] \times [0,1] \mapsto \mathbb{R}^{n\times n}$ such that $B \in \mathscr{K}_L^{n\times n}$ and $\mathscr{L}$ is given by (0.2).*

Let us summarize some of the further properties of functions of the class $\mathscr{K}^{n\times n}$.

2.6. Lemma. *If $K \in \mathscr{K}^{n\times n}$, then $K(.,s) \in \mathbf{G}^{n\times n}$ for any $s \in [0,1]$.*

Proof. Let $t \in [0,1)$ and $\varepsilon > 0$ be given. By (H$_2$)(i′) (cf. Remark 2.4) there exists $\delta > 0$ such that $t + \delta < 1$ and

$$\|K(\tau_2,.) - K(\tau_1,.)\|_{\mathbf{BV}} < \varepsilon \quad \text{for all} \quad \tau_1,\tau_2 \in (t,t+\delta).$$

Consequently, if $s \in [0,1]$ and $\tau_1,\tau_2 \in (t,t+\delta)$, then

$$\begin{aligned}&|K(\tau_2,s) - K(\tau_1,s)|\\ &\quad\le |K(\tau_2,0) - K(\tau_1,0)| + |K(\tau_2,s) - K(\tau_1,s) - K(\tau_2,0) + |K(\tau_1,0)|\\ &\quad\le \|K(\tau_2,.) - K(\tau_1,.)\|_{\mathbf{BV}} < \varepsilon.\end{aligned}$$

This implies that $K(.,s)$ possesses a limit $\lim_{\tau\to t+} K(\tau,s) = K(t+,s) \in \mathbb{R}^n$ for any $t \in [0,1)$ and any $s \in [0,1]$. Analogously, $K(.,s)$ possesses a limit $\lim_{\tau\to t-} K(\tau,s) = K(t-,s) \in \mathbb{R}^n$ for any $t \in (0,1]$ and any $s \in [0,1]$. □

2.7. Lemma. *If $K \in \mathscr{K}^{n\times n}$, then*

$$\varkappa := \sup_{t\in[0,1]} \|K(t,.)\|_{\mathbf{BV}} < \infty.$$

Proof follows directly from Definition 2.2 by means of the Vitali Covering Theorem (cf. also Remark 2.4). □

2.8. Lemma. *If $K \in \mathscr{K}^{n\times n}$ and $\mathscr{M}_K$ is given by (2.1), then*

$$\mathscr{M}_K(t+) = K(t+,.) \in \mathbf{BV}^{n\times n} \quad \textit{for all} \quad t \in [0,1) \tag{2.2}$$

and

$$\mathscr{M}_K(t-) = K(t-,.) \in \mathbf{BV}^{n\times n} \quad \textit{for all} \quad t \in (0,1]. \tag{2.3}$$

Proof. Let $t \in [0,1)$ be given. By (H$_2$)(ii) there exists $H \in \mathbf{BV}^{n\times n}$ such that

$$\lim_{\tau\to t+} \|K(\tau,.) - H\|_{\mathbf{BV}} = 0,$$

i.e. $H = \mathscr{M}_K(t+)$. In particular, in virtue of Lemma 2.6 we have

$$K(t+,s) = \lim_{\tau\to t+} K(\tau,s) = H(s) \quad \text{for all} \quad s \in [0,1]$$

wherefrom the relation (2.2) immediately follows. Analogously we can prove that the relation (2.3) is true, as well. □

As a direct consequence of Lemma 2.8 we have the following

2.9. Corollary. *If $K \in \mathscr{K}^{n\times n}$, then the relations*

$$\lim_{\tau\to t+} \|K(\tau,.) - K(t+,.)\|_{\mathbf{BV}} = 0 \quad \textit{for all} \quad t \in [0,1)$$

and

$$\lim_{\tau\to t-} \|K(\tau,.) - K(t-,.)\|_{\mathbf{BV}} = 0 \quad \textit{for all} \quad t \in (0,1]$$

are true.

2.10. Lemma. *Let $K \in \mathscr{K}^{n\times n}$, then for any $x \in \mathbf{G}^n$ the integrals*

$$\int_0^1 K(t,s)\,\mathrm{d}[x(s)], \qquad t \in [0,1], \tag{2.4}$$

$$\int_0^1 K(t+,s)\,\mathrm{d}[x(s)], \quad t \in [0,1) \tag{2.5}$$

and

$$\int_0^1 K(t-,s)\,\mathrm{d}[x(s)], \quad t \in (0,1] \tag{2.6}$$

have sense and the relations

$$\lim_{\tau\to t+} \int_0^1 K(\tau,s)\,\mathrm{d}[x(s)] = \int_0^1 K(t+,s)\,\mathrm{d}[x(s)] \quad \textit{for} \quad t \in [0,1) \tag{2.7}$$

and

$$\lim_{\tau\to t-} \int_0^1 K(\tau,s)\,\mathrm{d}[x(s)] = \int_0^1 K(t-,s)\,\mathrm{d}[x(s)] \quad \textit{for} \quad t \in (0,1] \tag{2.8}$$

are true.

Proof. All the integrals (2.4) -(2.6) have sense according to [18, Theorem 2.8]. The relations (2.7) and (2.8) follow then immediately by [18, Theorem 2.7] and by Corollary 2.9. □

2.11. Corollary. *If $K \in \mathscr{K}^{n\times n}$, then the integral*

$$\int_0^1 K(t,s)\,\mathrm{d}[x(s)]$$

is defined for any $x \in \mathbf{G}^n$ and any $t \in [0,1]$ and the function $h : [0,1] \mapsto \mathbb{R}^n$ defined by

$$h(t) = \int_0^1 K(t,s)\,\mathrm{d}[x(s)]$$

is regulated on $[0,1]$ ($h \in \mathbf{G}^n$).

Moreover, if $K \in \mathscr{K}_L^{n\times n}$, then $h \in \mathbf{G}_L^n$.

2.12. Lemma. *If $K \in \mathscr{K}^{n\times n}$, then the integrals*

$$\int_0^1 y^T(s)\,\mathrm{d}_s[K(s,t)], \quad t \in [0,1] \tag{2.9}$$

are defined for any $y \in \mathbf{BV}^n$ and the function $h : [0,1] \mapsto \mathbb{R}^n$ defined by

$$h^T(t) = \int_0^1 y^T(s)\,\mathrm{d}_s[K(s,t)]$$

has a bounded variation on $[0,1]$ for any $y \in \mathbf{BV}^n$.

Proof. a) The existence of the integrals (2.9) follows by [18, Theorem 2.8].

b) To prove that $h \in \mathbf{BV}^n$, let us first assume that $n = 1$, $k \in \mathscr{K}^{n\times n}$ and $d = \{t_0, t_1, \dots, t_m\} \in \mathcal{D}$. Then for all $x_i \in \mathbb{R}, i = 1, 2, \dots, m$ such that $|x_i| \leq 1$ we have by [18, Theorem 2.8] and Lemma 2.7

$$\begin{aligned}
\Big|\sum_{i=1}^m [h(t_i) - h(t_{i-1})]x_i\Big| &= \Big|\int_0^1 y(s)\,\mathrm{d}_s\Big[\Big(\sum_{i=1}^m (k(s,t_i) - k(s,t_{i-1}))\Big)\Big]x_i\Big| \\
&\leq 2\|y\|_{\mathbf{BV}}\Big(\sup_{\substack{s\in[0,1]\\ |x_i|\leq 1}}\Big|\sum_{i=1}^m (k(s,t_i) - k(s,t_{i-1}))x_i\Big|\Big) \\
&\leq 2\|y\|_{\mathbf{BV}}\left(\sup_{\substack{s\in[0,1]\\ |x_i|\leq 1}}\Big(\sum_{i=1}^m |k(s,t_i) - k(s,t_{i-1})||x_i|\Big)\right) \\
&\leq 2\|y\|_{\mathbf{BV}}\sup_{s\in[0,1]}\big(\mathrm{var}_0^1 k(s,.)\big) = 2\|y\|_{\mathbf{BV}}\,\varkappa < \infty.
\end{aligned}$$

In particular, if we put

$$x_i = \mathrm{sign}[h(t_i) - h(t_{i-1})]$$

for $i = 1, 2, \dots, m$ we obtain that the inequality

$$S(h,d) = \sum_{i=1}^m |h(t_i) - h(t_{i-1})| \leq 2\varkappa\|y\|_{\mathbf{BV}}$$

holds for any division $d = \{t_0, t_1, \dots, t_m\} \in \mathcal{D}$ of the interval $[0,1]$ and any $y \in \mathbf{BV}$, i.e.

$$\mathrm{var}_0^1 h \leq 2\varkappa\|y\|_{\mathbf{BV}} < \infty \quad \text{for any} \quad y \in \mathbf{BV}.$$

c) In the general case of $n \in \mathbb{N}$, $n \geq 1$, we have for any $j = 1, 2, \ldots, n$, any $y \in \mathbf{BV}^n$ and any $t \in [0,1]$

$$h_j(t) = \sum_{i=1}^{n} \int_0^1 y_i(s) \mathrm{d}_s[k_{i,j}(s,t)].$$

Consequently, by the second part of the proof of this lemma the inequalities

$$\mathrm{var}_0^1 h_j \leq 2\Big(\sum_{i=1}^{n} \|y_i\|_{\mathbf{BV}}\Big)\varkappa = 2\|y\|_{\mathbf{BV}}\varkappa$$

are true. It follows easily that $h \in \mathbf{BV}^n$ for any $y \in \mathbf{BV}^n$. □

2.13. Theorem. (Bray Theorem) *If $K \in \mathscr{K}^{n\times n}$, then for any $x \in \mathbf{G}^n$ and any $y \in \mathbf{BV}^n$ the relation*

$$\int_0^1 y^T(t) \mathrm{d}_t\Big[\int_0^1 K(t,s)\mathrm{d}[x(s)]\Big] = \int_0^1 \Big(\int_0^1 y^T(t)\mathrm{d}_t[K(t,s)]\Big)\mathrm{d}[x(s)] \tag{2.10}$$

is true.

Proof. a) Both iterated integrals occurring in (2.10) have sense by Corollary 2.11, Lemma 2.12 and by [18, Theorem 2.8].

b) Let us first assume $n = 1$, $k \in \mathscr{K}^{n\times n}$ and $y \in \mathbf{BV}$. Let $f \in \mathbf{G}$ be a finite step function, i.e. there is a division $\{t_0, t_1, \ldots, t_m\}$ of the interval $[0,1]$ such that f is on $[0,1]$ a linear combination of the functions

$$\big\{\chi_{[t_r,1]},\ r = 0, 1, \ldots, m,\ \chi_{(t_j,1]},\ j = 0, 1, \ldots, m-1\big\}.$$

To show that the relation

$$\int_0^1 y(t) \mathrm{d}_t\Big[\int_0^1 k(t,s)\mathrm{d}[f(s)]\Big] = \int_0^1 \Big(\int_0^1 y(t)\mathrm{d}_t[k(t,s)]\Big)\mathrm{d}[f(s)] \tag{2.11}$$

is true for any finite step function f on $[0,1]$, it is sufficient to show that (2.11) is true for any function from the set

$$\Big\{\chi_{[\tau,1]},\ \tau \in [0,1]\Big\} \quad \cup \quad \Big\{\chi_{(\sigma,1]},\ \sigma \in [0,1)\Big\}.$$

If $f = \chi_{[0,1]}$, i.e. $f(t) \equiv 1$ on $[0,1]$, then obviously both sides of (2.11) equal 0. Furthermore, let $\tau \in (0,1]$ and $f = \chi_{[\tau,1]}$. Then by [18, Proposition 2.3],

$$\int_0^1 k(t,s)\mathrm{d}[f(s)] = k(t,\tau),$$

i.e.

$$\int_0^1 y(t)\,\mathrm{d}_t\Big[\int_0^1 k(t,s)\,\mathrm{d}[f(s)]\Big] = \int_0^1 y(t)\,\mathrm{d}_t[k(t,\tau)].$$

On the other hand, we have by [18, Proposition 2.3],

$$\int_0^1 \Big(\int_0^1 y(t)\,\mathrm{d}_t[k(t,s)]\Big)\,\mathrm{d}[f(s)] = \int_0^1 y(t)\,\mathrm{d}_t[k(t,\tau)],$$

as well.

Analogously we would prove that (2.11) holds also for $f=\chi_{(\sigma,1]}$, $\sigma\in[0,1)$. Now, if $x\in\mathbf{G}$, let $\{x_r\}_{r=1}^{\infty}$ be a sequence of finite step functions on $[0,1]$ such that x_r tends to x uniformly on $[0,1]$ as $r\to\infty$. By the previous part of the proof, we have

$$\int_0^1 y(t)\,\mathrm{d}_t\Big[\int_0^1 k(t,s)\,\mathrm{d}[x_r(s)]\Big] = \int_0^1 \Big(\int_0^1 y(t)\,\mathrm{d}_t[k(t,s)]\Big)\,\mathrm{d}[x_r(s)]$$

for any $r\in\mathbb{N}$. According to [18, Corollary 2.9] it follows that

$$\lim_{r\to\infty}\bigg(\int_0^1 \Big(\int_0^1 y(t)\,\mathrm{d}_t[k(t,s)]\Big)\,\mathrm{d}[x_r(s)]\bigg) = \int_0^1 \Big(\int_0^1 y(t)\,\mathrm{d}_t[k(t,s)]\Big)\,\mathrm{d}[x(s)].$$

On the other hand, by Lemma 2.7 and by [18, Theorem 2.8] we have for any $r\in\mathbb{N}$ and any $t\in[0,1]$

$$\begin{aligned}\Big|\int_0^1 k(t,s)\,\mathrm{d}[x_r(s)] - \int_0^1 k(t,s)\,\mathrm{d}[x(s)]\Big| &= \Big|\int_0^1 k(t,s)\,\mathrm{d}[x_r(s)-x(s)]\Big|\\ &\le 2\|k(t,.)\|_{\mathbf{BV}}\|x_r-x\| \le 2\varkappa\|x_r-x\|\end{aligned}$$

and consequently

$$\lim_{r\to\infty}\Big(\int_0^1 k(t,s)\,\mathrm{d}[x_r(s)]\Big) = \int_0^1 k(t,s)\,\mathrm{d}[x(s)]$$

uniformly with respect to $t\in[0,1]$. Thus, making use of [18, Corollary 2.9] once more, we obtain that the relation

$$\lim_{r\to\infty}\int_0^1 y(t)\,\mathrm{d}_t\Big[\int_0^1 k(t,s)\,\mathrm{d}[x_r(s)]\Big] = \int_0^1 y(t)\,\mathrm{d}_t\Big[\int_0^1 k(t,s)\,\mathrm{d}[x(s)]\Big]$$

is true. It follows immediately that the relation (2.11) is true for any $y\in\mathbf{BV}$ and any $f\in\mathbf{G}$.

c) The proof can be extended to the general case $n\in\mathbb{N}$, $n\ge 1$, similarly as it was done at the end of the proof of Lemma 2.12. □

2.14. Remark. For the proof of the Bray Theorem in the case of the interior Dushnik integral see [6, Theorem II.1.1].

In the following text we shall make use of the following assertion, as well.

2.15. Lemma. *Let $K \in \mathscr{K}^{n\times n}$ and let*

$$H(t,s) = \begin{cases} K(t,s+) & \text{for} \quad t\in[0,1] \quad \text{and} \quad s\in[0,1), \\ K(t,1-) & \text{for} \quad t\in[0,1] \quad \text{and} \quad s=1. \end{cases}$$

Then $H \in \mathscr{K}^{n\times n}$. Moreover, if $K \in \mathscr{K}_L^{n\times n}$, then $H \in \mathscr{K}_L^{n\times n}$, as well.

Proof. Analogously as in the proofs of Lemma 2.12 and of Theorem 2.13 it is sufficient to show that the assertion of the lemma is true in the scalar case $n = 1$.

Let $n = 1$, $k \in \mathscr{K}^{n\times n}$ and

$$h(t,s) = \begin{cases} k(t,s+) & \text{for} \quad t\in[0,1] \quad \text{and} \quad s\in[0,1), \\ k(t,1-) & \text{for} \quad t\in[0,1] \quad \text{and} \quad s=1. \end{cases}$$

a) Let $d = \{s_0, s_1, \dots, s_m\}$ be an arbitrary division of the interval $[0,1]$ ($d \in \mathcal{D}$). Then

$$\begin{aligned} S(h,d) &= \sum_{j=1}^{m} |h(t,s_j) - h(t,s_{j-1})| \\ &= \sum_{j=1}^{m-1} |k(t,s_j+) - k(t,s_{j-1}+)| + |k(t,1-) - k(t,s_{m-1}+)|. \end{aligned}$$

Let $\delta > 0$ be such that

$$s_{m-1} + \delta < 1 - \delta$$

and let us denote

$$\sigma_0 = 0,\ \sigma_j = s_{j-1} + \delta \quad \text{for} \quad j = 1,2,\dots,m,\ \sigma_{m+1} = 1-\delta,\ \sigma_{m+2} = 1. \tag{2.12}$$

Then

$$d_\delta = \{\sigma_0, \sigma_1, \dots, \sigma_{m+2}\} \in \mathcal{D} \tag{2.13}$$

and according to (H_2), for any $\delta > 0$ sufficiently small we have

$$\begin{aligned} S(k,d_\delta) &= |k(t,\delta) - k(t,0)| + \sum_{j=1}^{m-1} |k(t,s_j+\delta) - k(t,s_{j-1}+\delta)| \\ &= |k(t,1-\delta) - k(t,s_{m-1}+\delta)| + |k(t,1) - k(t,1-\delta)| \\ &\le \operatorname{var}_0^1 k(t,.) < \infty. \end{aligned}$$

Thus

$$\infty > \lim_{\delta\to 0+} S(k,d_\delta) = S(h,d) + |\Delta_2^+ k(t,0)| + |\Delta_2^- k(t,1)|$$

and consequently the inequality

$$S(h,d) \le \operatorname{var}_0^1 k(t,.) - |\Delta_2^+ k(t,0)| - |\Delta_2^- k(t,1)|$$

holds for any division $d \in \mathcal{D}$. Hence

$$\begin{aligned}\|h(t,.)\|_{\mathbf{BV}} &= |k(t,0+)| + \operatorname{var}_0^1 h(t,.)\\ &\le |k(t,0)| + |\Delta_2^+ k(t,0)| + \operatorname{var}_0^1 k(t,.) - |\Delta_2^+ k(t,0)| - |\Delta_2^- k(t,1)|\\ &\le \|k(t,.)\|_{\mathbf{BV}},\end{aligned}$$

i.e. h fulfils (H_1).

b) Let $t \in [0,1)$ and $\varepsilon > 0$ be given. According to (H_2)(i') there is a $\delta_0 > 0$ such that $t+\delta_0 < 1$ and

$$\|k(\tau_2,.) - k(\tau_1,.)\|_{\mathbf{BV}} < \varepsilon$$

holds for any couple $\tau_1, \tau_2 \in (t, t+\delta_0)$. In particular,

$$S(k(\tau_2,.) - k(\tau_1,.), \Delta) < \varepsilon \tag{2.14}$$

for any division $\Delta \in \mathcal{D}$ and any couple $\tau_1, \tau_2 \in (t, t+\delta_0)$. Now, let an arbitrary division $d = \{s_0, s_1, \dots, s_m\} \in \mathcal{D}$ be given and let $\delta > 0$ be such that $\delta < \delta_0$ and $s_{m-1} + \delta < 1 - \delta$. Let us define the division $d_\delta = \{\sigma_0, \sigma_1, \dots, \sigma_m\} \in \mathcal{D}$ as in (2.12) and (2.13). Making use of (2.14) we obtain

$$\begin{aligned}S(h(\tau_2,.)&-h(\tau_1,.),d)\\ &= |k(\tau_2,s_1+) - k(\tau_1,s_1+) - k(\tau_2,0+) + k(\tau_1,0+)|\\ &\quad + \sum_{j=2}^{m-1} |k(\tau_2,s_j+) - k(\tau_1,s_j+) - k(\tau_2,s_{j-1}+) + k(\tau_1,s_{j-1}+)|\\ &\quad + |k(\tau_2,1-) - k(\tau_1,1-) - k(\tau_2,s_{m-1}+) + k(\tau_1,s_{m-1}+)|\\ &= \lim_{\delta\to 0+} \Big(\sum_{j=1}^{m} |k(\tau_2,\sigma_{j+1}) - k(\tau_1,\sigma_{j+1}) - k(\tau_2,\sigma_j) + k(\tau_1,\sigma_j)| \Big)\\ &= \lim_{\delta\to 0+} \big(S(k(\tau_2,.) - k(\tau_1,.), d_\delta) \big)\\ &\quad - |\Delta_2^+(k(\tau_2,0) - k(\tau_1,0))| - |\Delta_2^-(k(\tau_2,1) - k(\tau_1,1))| < \varepsilon.\end{aligned}$$

This means that for any couple $\tau_1, \tau_2 \in (t, t+\delta)$ we have

$$\|h(\tau_2,.) - h(\tau_1,.)\|_{\mathbf{BV}} < \mathrm{e},$$

i.e. h fulfils (H_2)(i'). Similarly we could show that h fulfils also (H_2)(ii). Thus $h \in \mathcal{K}^{1\times 1}$.

c) Let $\mathscr{M}_k : t \in [0,1] \mapsto k(t,.) \in \mathbf{BV}$ be left-continuous on $(0,1)$ and let $\varepsilon > 0$ be given. Then there is a $\delta_0 > 0$ such that $t - \delta_0 > 0$ and

$$S(k(t,.) - k(\tau,.), \Delta) < \varepsilon \tag{2.15}$$

holds for any $\tau \in (t - \delta_0, t)$ and any $\Delta \in \mathcal{D}$. Let an arbitrary division $d = \{s_0, s_1, \dots, s_m\} \in \mathcal{D}$ be given and let $d_\delta = \{\sigma_0, \sigma_1, \dots, \sigma_{m+2}\} \in \mathcal{D}$ be given for $\delta \in (0, \min\{\delta_0, \frac{1-s_{m-1}}{2}\})$ by (2.12) and (2.13). Then making use of (2.15) we obtain similarly as in part b) of this proof

$$\begin{aligned}
S(h(t,.)-&h(\tau,.), d)\\
&= \lim_{\delta\to 0+} \Big(\sum_{j=1}^{m} |k(t,\sigma_{j+1}) - k(\tau,\sigma_{j+1}) - k(t,\sigma_j) + k(\tau,\sigma_j)| \Big)\\
&= \lim_{\delta\to 0+} \big(S(k(t,.) - k(\tau,.), d_\delta) \big)\\
&\quad - |\Delta_2^+(k(t,0) - k(\tau,0))| - |\Delta_2^-(k(t,1) - k(\tau,1))| < \varepsilon,
\end{aligned}$$

wherefrom the desired relation

$$\lim_{\tau\to t-} \|h(t,.) - h(\tau,.)\|_{\mathbf{BV}} = 0$$

easily follows. □

2.16. Remark. Analogously we could show that if $K \in \mathscr{K}^{n\times n}$ and if

$$H(t,s) = \begin{cases} K(t,0+) & \text{for } t \in [0,1] \text{ and } s = 0,\\ K(t,s-) & \text{for } t \in [0,1] \text{ and } s \in (0,1], \end{cases}$$

then $H \in \mathscr{K}^{n\times n}$. Moreover, if $K \in \mathscr{K}_L^{n\times n}$, then $H \in \mathscr{K}_L^{n\times n}$, as well.

2.17. Lemma. *Let $K \in \mathscr{K}^{n\times n}$ and let*

$$H(t,s) = \begin{cases} K(t+,s) & \text{for } t \in [0,1) \text{ and } s \in [0,1],\\ K(1-,s) & \text{for } t = 1 \text{ and } s \in [0,1] \end{cases}$$

and

$$G(t,s) = \begin{cases} K(0+,s) & \text{for } t = 0 \text{ and } s \in [0,1],\\ K(t-,s) & \text{for } t \in (0,1] \text{ and } s \in [0,1]. \end{cases}$$

Then $H \in \mathscr{K}^{n\times n}$ and $G \in \mathscr{K}_L^{n\times n}$.

Proof. We shall prove that under the assumptions of the lemma $H \in \mathscr{K}^{n\times n}$. The proof of the latter relation would be quite similar.

Let $t < 1$ and let $d \in \mathcal{D}$ be an arbitrary division of $[0,1]$. Then for any $\delta \in (0, 1-t)$ we have by Lemma 2.7

$$S(K(t+\delta,.), d) \leq \operatorname{var}_0^1 K(t+\delta,.) \leq \varkappa < \infty.$$

Letting $\delta \to 0+$ we immediately obtain that the inequality

$$S(H(t,.), d) \leq \varkappa < \infty$$

is true for any $d \in \mathcal{D}$. It means that

$$\operatorname{var}_0^1 H(t,.) \leq \varkappa < \infty.$$

Now, let an arbitrary $\varepsilon > 0$ be given. By $(\mathrm{H}_2)(\mathrm{i}')$ there is a $\delta > 0$ such that

$$\|K(\tau_2,.) - K(\tau_1,.)\|_{\mathbf{BV}} < \tfrac{\varepsilon}{2}$$

holds whenever $t < \tau_1 < \tau_2 < t+\delta$. It means that for all $t_1, t_2 \in (t, t+\frac{\delta}{2})$ and any $\tau \in (0, \frac{\delta}{2})$ we have

$$\|K(t_2+\tau,.) - K(t_1+\tau,.)\|_{\mathbf{BV}} < \tfrac{\varepsilon}{2}.$$

In particular, we have for any division $d \in \mathcal{D}$

$$|K(t_2+\tau,0) - K(t_1+\tau,0)| \quad < \tfrac{\varepsilon}{2}$$

and

$$S(K(t_2+\tau,.) - K(t_1+\tau,.), d) < \tfrac{\varepsilon}{2}$$

wherefrom we obtain easily that the relation

$$\|H(t_2,.) - H(t_1,.)\|_{\mathbf{BV}} < \varepsilon$$

is true whenever $t < t_1 < t_2 < t + \frac{\delta}{2}$.

Analogously we would prove that if $t > 0$, then for any $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\|H(t_2,.) - H(t_1,.)\|_{\mathbf{BV}} < \varepsilon$$

is true whenever $t - \frac{\delta}{2} < t_1 < t_2 < t$. □

2.18. Lemma. *Let* $K \in \mathscr{K}^{n\times n}$, $t_1, s_1 \in [0,1)$, *and* $t_2, s_2 \in (0,1]$. *Then all the limits*

$$K(t_1+, s_1+) = \lim_{\substack{(\tau,\sigma)\to(t_1,s_1)\\ \tau>t_1,\sigma>s_1}} K(\tau,\sigma), \quad K(t_1+, s_2-) = \lim_{\substack{(\tau,\sigma)\to(t_1,s_2)\\ \tau>t_1,\sigma<s_2}} K(\tau,\sigma),$$
$$K(t_2-, s_1+) = \lim_{\substack{(\tau,\sigma)\to(t_2,s_1)\\ \tau<t_2,\sigma>s_1}} K(\tau,\sigma), \quad K(t_2-, s_2-) = \lim_{\substack{(\tau,\sigma)\to(t_2,s_2)\\ \tau<t_2,\sigma<s_2}} K(\tau,\sigma)$$

are defined in $\mathbb{R}^{n\times n}$.

Proof. We will restrict ourselves to proving the existence of the limits

$$K(t_1+, s_1+) \in \mathbb{R}^{n\times n} \quad \text{for} \quad t_1, s_1 \in [0,1).$$

The modifications of the proofs in the remaining cases are obvious.

Let $t_1 \in [0,1)$ and $s_1 \in [0,1)$ be given. By Lemma 2.15 there exists $M \in \mathbb{R}^{n\times n}$ such that

$$\lim_{\sigma\to s+} K(t_1+, \sigma) = \lim_{\sigma\to s+} \Big(\lim_{\tau\to t_1+} K(\tau,\sigma) \Big) = M.$$

Furthermore, since in virtue of Corollary 2.9

$$\lim_{\tau\to t_1+} \|K(\tau,.) - K(t_1+,.)\| = 0,$$

i.e.

$$\lim_{\tau\to t_1+} K(\tau,\sigma) = K(t_1+,\sigma) \quad \text{uniformly with respect to } \ \sigma \in [0,1],$$

it follows that

$$\lim_{\substack{(\tau,\sigma)\to(t_1,s_1)\\ \tau>t_1,\sigma>s_1}} K(\tau,\sigma) = M.$$

□

2.19. Lemma. *Let* $K \in \mathscr{K}^{n\times n}$, $s \in (0,1]$ *and* $t \in [0,1)$. *Then*

$$\lim_{\tau\to t+} K(\tau,\tau-) = \lim_{\tau\to t+} K(\tau,\tau+) = K(t+,t+),$$
$$\lim_{\tau\to t+} K(\tau-,\tau) = \lim_{\tau\to t+} K(\tau+,\tau) = K(t+,t+),$$
$$\lim_{\tau\to s-} K(\tau,\tau-) = \lim_{\tau\to s-} K(\tau,\tau+) = K(s-,s-)$$

and

$$\lim_{\tau\to s-} K(\tau-,\tau) = \lim_{\tau\to s-} K(\tau+,\tau) = K(s-,s-).$$

Proof. We will restrict ourselves to the proof of the relations

$$\lim_{\tau\to t+} K(\tau,\tau-) = K(t+,t+), \quad t\in[0,1).$$

The proofs of the remaining assertions of the lemma would be quite analogous. By Lemma 2.18 there exists a $\delta\in(0,1-t)$ such that

$$|K(\tau,\sigma) - K(t+,t+)| < \tfrac{\varepsilon}{2}$$

holds whenever $t<\tau<t+\delta$ and $t<\sigma<t+\delta$. Furthermore, for any $\tau\in(t,t+\delta)$ we may choose a $\sigma_\tau\in(t,\tau)$ such that

$$|K(\tau,\tau-) - K(\tau,\sigma_\tau)| < \tfrac{\varepsilon}{2}$$

is true. Thus for any $\tau\in(t,t+\delta)$ we have

$$\begin{aligned}|K(\tau,\tau-) - K(t+,t+)| &\le |K(\tau,\tau-) - K(\tau,\sigma_\tau)| + |K(\tau,\sigma_\tau) - K(t+,t+)|\\ &< \varepsilon.\end{aligned}$$

□

2.20. Remark. A matrix valued function $K:[0,1]\times[0,1]\mapsto\mathbb{R}^{n\times n}$ is said to be of bounded *Vitali variation* on $[0,1]\times[0,1]$ if

$$\begin{aligned}&v_{[0,1]\times[0,1]}(K)\\ &\quad= \sup_D \sum_{i,j=1}^m \Big|K(t_i,s_j) - K(t_{i-1},s_j) - K(t_i,s_{j-1}) + K(t_{i-1},s_{j-1})\Big| < \infty,\end{aligned}$$

where the supremum is taken over all net subdivisions

$$D = \Big\{0=t_0<t_1<\dots<t_m=1; 0=s_0<s_1<\dots<s_m=1\Big\}$$

of the interval $[0,1]\times[0,1]$. A matrix valued function $K:[0,1]\times[0,1]\mapsto\mathbb{R}^{n\times n}$ is said to be of *strongly bounded variation* on $[0,1]\times[0,1]$ if

$$v_{[0,1]\times[0,1]}(K) + \operatorname{var}_0^1 K(0,.) + \operatorname{var}_0^1 K(0,.) < \infty.$$

Let us denote the set of $n\times n$-matrix valued functions of strongly bounded variation on $[0,1]\times[0,1]$ by $\mathcal{SBV}^{n\times n}$. It follows by [16, Corollaries I.6.15 and I.6.16] that $\mathcal{SBV}^{n\times n}\subset\mathscr{K}^{n\times n}$.

On the other hand, the set $\mathcal{G}.\mathcal{BV}^{n\times n}$ of $n\times n$-matrix valued functions K of the form

$$K(t,s) = F(t)G(s), \quad (t,s)\in[0,1]\times[0,1],$$

where $F\in\mathbf{G}^{n\times n}$ and $G\in\mathbf{BV}^{n\times n}$, provides the simplest example of the class of kernels which satisfy the assumptions of this paper, but do not belong in general to the class $\mathcal{SBV}^{n\times n}$. In fact, it is easy to verify that $\mathcal{G}.\mathcal{BV}^{n\times n}\subset\mathscr{K}^{n\times n}$ holds.

2.21. Lemma. *Let $K \in \mathscr{K}^{n\times n}$ and $t \in [0,1)$. Then*

(2.16) *for any* $\varepsilon > 0$ *there exists a* $\delta \in (0, 1-t)$ *such that* $\operatorname{var}_{t_1}^{t_2} K(t_2,.) < \varepsilon$ *holds whenever* $0 < t < t_1 < t_2 < t + \delta \leq 1$.

Proof (due to I. Vrkoč). Let $t \in [0,1)$ be given and let us assume that there is a $\gamma > 0$ and sequences $\{t_k^1\}$ and $\{t_k^2\}$ of points in $(t,1]$ such that

$$t < t_{k+1}^1 < t_{k+1}^2 < t_k^1 < t_k^2 < 1 \quad \text{holds for any} \quad k \in \mathbb{N},$$
$$\lim_{k\to\infty} t_k^1 = \lim_{k\to\infty} t_k^2 = t \quad \text{and} \quad \operatorname{var}_{t_k^1}^{t_k^2} K(t_2^k,.) > 2\gamma.$$

On the other hand, by (H_2)(ii) there is a natural number k_0 such that

$$\operatorname{var}_0^1 (K(t_2^k,.) - K(t_2^{k_0},.)) < \gamma.$$

This means that in the case that (2.16) does not hold we obtain

$$\begin{aligned}\operatorname{var}_0^1 K(t_2^{k_0},.) &\geq \sum_{k\geq k_0} \operatorname{var}_{t_1^k}^{t_2^k} K(t_2^{k_0},.)\\ &\geq \sum_{k\geq k_0} \Big[\operatorname{var}_{t_1^k}^{t_2^k} K(t_2^k,.) - \operatorname{var}_{t_1^k}^{t_2^k}(K(t_2^k,.) - K(t_2^{k_0},.))\Big]\\ &\geq \sum_{k\geq k_0} \gamma = \infty.\end{aligned}$$

This being impossible in virtue of the assumption (H_1), it follows that the assertion (2.16) is true and this completes the proof of the lemma. □

Analogously we could prove the following assertion, as well.

2.22. Lemma. *Let $K \in \mathscr{K}^{n\times n}$ and $t \in (0,1]$. Then*

(2.17) *for any* $\varepsilon > 0$ *there exists a* $\delta \in (0,t)$ *such that* $\operatorname{var}_{t_1}^{t_2} K(t_2,.) < \varepsilon$ *holds whenever* $0 \leq t - \delta < t_1 < t_2 < t$.

3. Fredholm-Stieltjes integral equations in the space $\mathbf{G}_L^n$

In this section we will consider linear integral equations of the form

$$x(t) - A(t)x(0) - \int_0^1 B(t,s)\,\mathrm{d}[x(s)] = f(t), \quad t \in [0,1], \tag{3.1}$$

where

$$A \in \mathbf{G}_{\mathrm{L}}^{n\times n} \quad \text{and} \quad B \in \mathscr{K}_L^{n\times n}.$$

3.1. Remark. Let us recall that the operator $\mathscr{L}$ given by (0.2), i.e.

$$(\mathscr{L}x)(t) = A(t)x(0) + \int_0^1 B(t,s)\mathrm{d}[x(s)], \quad x \in \mathbf{G}_L^n, t \in [0,1] \tag{3.2}$$

is the general form of a linear compact operator on the space $\mathbf{G}_L^n$ (cf. Theorem 2.5). The equation (3.1) may be written as the operator equation

$$x - \mathscr{L}x = f, \tag{3.3}$$

as well.

3.2. Remark. It is also known (cf. [18, Theorem 3.8]) that the dual space $(\mathbf{G}_L^n)^*$ to $\mathbf{G}_L^n$ is isomorphic to the space $\mathbf{BV}^n \times \mathbb{R}^n$, while for a given couple $(y,\gamma) \in \mathbf{BV}^n \times \mathbb{R}^n$ the corresponding linear bounded functional on $\mathbf{G}_L^n$ is given by

$$x \in \mathbf{G}_L^n \mapsto \langle x, (y,\gamma)\rangle := \gamma^T x(0) + \int_0^1 y^T(s)\mathrm{d}[x(s)] \in \mathbb{R}. \tag{3.4}$$

The compactness of the operator $\mathscr{L}$ immediately implies that the following Fredholm alternative type assertions 3.3-3.5 are true.

3.3. Proposition. *Let $A \in \mathbf{G}_{\mathrm{L}}^{n\times n}$ and $B \in \mathscr{K}_L^{n\times n}$. Then the given equation* (3.1) *possesses a unique solution $x \in \mathbf{G}_L^n$ for any $f \in \mathbf{G}_L^n$ if and only if the corresponding homogeneous equation $x - \mathscr{L}x = 0$, i.e.*

$$x(t) - A(t)x(0) - \int_0^1 B(t,s)\mathrm{d}[x(s)] = 0, \quad t \in [0,1],$$

possesses only the trivial solution.

3.4. Proposition. *Let $A \in \mathbf{G}_{\mathrm{L}}^{n\times n}$, $B \in \mathscr{K}_L^{n\times n}$ and $f \in \mathbf{G}_L^n$. Then the equation* (3.1) *possesses a solution in $\mathbf{G}_L^n$ if and only if*

$$\gamma^T f(0) + \int_0^1 y^T(s)\mathrm{d}[f(s)] = 0 \tag{3.5}$$

holds for any solution $(y,\gamma) \in \mathbf{BV}^n \times \mathbb{R}^n$ of the operator equation

$$(y,\gamma) - \mathscr{L}^*(y,\gamma) = 0 \in \mathbf{BV}^n \times \mathbb{R}^n \tag{3.6}$$

adjoint to (3.1).

3.5. Proposition. *Let $A \in \mathbf{G}_{\mathrm{L}}^{n\times n}$ and $B \in \mathscr{K}_L^{n\times n}$. Then the relations*

$$\dim \mathcal{N}(\mathrm{I} - \mathscr{L}) = \dim \mathcal{N}(\mathrm{I} - \mathscr{L}^*) < \infty$$

hold for the dimensions of the null spaces $\mathcal{N}(\mathrm{I} - \mathscr{L})$ and $\mathcal{N}(\mathrm{I} - \mathscr{L}^*)$ corresponding to the operator $\mathscr{L}$ and its adjoint $\mathscr{L}^*$, respectively.

3.6. Corollary. *Let $A \in \mathbf{G}_{\mathrm{L}}^{n\times n}$ and $B \in \mathscr{K}_L^{n\times n}$. Then the given equation (3.1) possesses a unique solution $x \in \mathbf{G}_L^n$ for any $f \in \mathbf{G}_L^n$ if and only if the corresponding homogeneous equation*

$$x - \mathscr{L}x = 0$$

possesses only the trivial solution.

Making use of the above mentioned explicit representation (3.4) of the dual space to $\mathbf{G}_L^n$ and of the Bray Theorem we can derive the explicit form of the adjoint operator $\mathscr{L}^*$ to $\mathscr{L}$.

3.7. Theorem. *Let $A \in \mathbf{G}_{\mathrm{L}}^{n\times n}$ and $B \in \mathscr{K}_L^{n\times n}$. Then the adjoint operator $\mathscr{L}^*$ to the operator $\mathscr{L}$ is given by*

$$\mathscr{L}^* : (y,\gamma) \in \mathbf{BV}^n \times \mathbb{R}^n \mapsto \big(\mathscr{L}_1^*(y,\gamma), \mathscr{L}_2^*(y,\gamma)\big) \in \mathbf{BV}^n \times \mathbb{R}^n,$$

where

$$\big(\mathscr{L}_1^*(y,\gamma)\big)(t) = B^T(0,t)\gamma + \int_0^1 \mathrm{d}_s\big[B^T(s,t)\big]y(s), \quad t \in [0,1]$$

and

$$\mathscr{L}_2^*(y,\gamma) = A^T(0)\gamma + \int_0^1 \mathrm{d}\big[A^T(s)\big]y(s).$$

Proof. Given $x \in \mathbf{G}_L^n$, $y \in \mathbf{BV}^n$ and $\gamma \in \mathbb{R}^n$, we have by (3.4) and by Theorem 2.13

$$\begin{aligned}
\langle \mathscr{L}x, (y,\gamma)\rangle &= \gamma^T\Big(A(0)x(0) + \int_0^1 B(0,t)\mathrm{d}[x(t)]\Big) \\
&\quad + \int_0^1 y^T(t)\mathrm{d}_s\Big[A(t)x(0) + \int_0^1 B(t,s)\mathrm{d}[x(s)]\Big] \\
&= \Big(\gamma^T A(0) + \int_0^1 y^T(s)\mathrm{d}[A(s)]\Big)x(0) \\
&\quad + \int_0^1 \Big(\gamma^T B(0,t) + \int_0^1 y^T(s)\mathrm{d}_s[B(s,t)]\Big)\mathrm{d}[x(t)] \\
&= (\mathscr{L}_2^*(y,\gamma))^T x(0) + \int_0^1 (\mathscr{L}_1^*(y,\gamma))^T(t)\mathrm{d}[x(t)] \\
&= \langle x, (\mathscr{L}_1^*(y,\gamma), \mathscr{L}_2^*(y,\gamma))\rangle
\end{aligned}$$

wherefrom the proof of the theorem immediately follows. □

Proposition 3.4 and Theorem 3.7 immediately yield the following assertion.

3.8. Theorem. *Let* $A \in \mathbf{G}_{\mathrm{L}}^{n\times n}$, $B \in \mathscr{K}_L^{n\times n}$ *and* $f \in \mathbf{G}_L^n$. *Then the equation* (3.1) *possesses a solution* $x \in \mathbf{G}_L^n$ *if and only if* (3.5) *holds for any solution* $(y,\gamma) \in \mathbf{BV}^n \times \mathbb{R}^n$ *of the system*

$$\begin{aligned} y(t) - B^T(0,t)\gamma - \int_0^1 \mathrm{d}_s\big[B^T(s,t)\big]y(s) &= 0, \quad t \in [0,1],\\ \gamma - A^T(0)\gamma \quad - \int_0^1 \mathrm{d}\big[A^T(s)\big]y(s) \quad &= 0. \end{aligned}$$

3.9. Remark. Let us notice that in virtue of Corollary 2.9, for any solution $x \in \mathbf{G}^n$ of (3.1) on $[0,1]$ we have

$$\begin{aligned} x(t+) &= A(t+)x(0) + \int_0^1 B(t+,s)\mathrm{d}[x(s)] + f(t+) \quad \text{for all} \quad t \in [0,1),\\ x(t-) &= A(t-)x(0) + \int_0^1 B(t-,s)\mathrm{d}[x(s)] + f(t-) \quad \text{for all} \quad t \in (0,1]. \end{aligned}$$

In particular, if $A \in \mathbf{G}_{\mathrm{L}}^{n\times n}$, $B \in \mathscr{K}_L^{n\times n}$ and $f \in \mathbf{G}_L^n$, then any solution x of (3.1 on $[0,1]$ is left-continuous on $(0,1)$.

3.10. Example. Let us consider a linear Stieltjes integral equation

$$x(t) - \int_0^1 \mathrm{d}_s[P(t,s)]x(s) = f(t), \quad t \in [0,1] \tag{3.7}$$

with $P \in \mathscr{K}_L^{n\times n}$ and $f \in \mathbf{G}_L^n$. Such equations with kernels P of strongly bounded variation on $[0,1]\times[0,1]$ (cf. Remark 2.20) were treated in [16].

Let $t \in [0,1]$ and $x \in \mathbf{G}_L^n$ be given. Let us put

$$Q(t,s) = \begin{cases} P(t,s+) & \text{for} \quad s < 1,\\ P(t,1-) & \text{for} \quad s = 1 \end{cases}$$

and

$$Z(t,s) = P(t,s) - Q(t,s) \quad \text{for} \quad (t,s) \in [0,1]\times[0,1].$$

Then

$$Z(t,s) = \begin{cases} -\Delta_2^+ P(t,s) & \text{for} \quad s < 1,\\ \Delta_2^- P(t,1) & \text{for} \quad s = 1. \end{cases}$$

Since obviously $Q(t,.)$ and $Z(t,.) \in \mathbf{BV}^{n\times n}$, $\lim_{\sigma\to s+} P(t,\sigma+) = P(t,s+)$ if $s \in [0,1)$ and $\lim_{\sigma\to s-} P(t,\sigma+) = P(t,s-)$ if $s \in (0,1]$, it is easy to verify that

$$Z(t,s-) = 0 \quad \text{for all} \quad s \in [0,1) \quad \text{and} \quad Z(t,s+) = 0 \quad \text{for all} \quad s \in (0,1].$$

Since $Z(t,.) \in \mathbf{BV}^{n\times n}$, this implies that there is an at most countable set $W \subset [0,1]$ of points in $[0,1]$ such that $Z(t,s) = 0$ holds for any $s \in [0,1] \setminus W$. Making use of [17, Proposition 2.13] we obtain that

$$\int_0^1 \mathrm{d}_s[Z(t,s)]x(s) = Z(t,1)x(1) - Z(t,0)x(0).$$

This implies that the relation

$$\int_0^1 \mathrm{d}_s[P(t,s)]x(s) = \int_0^1 \mathrm{d}_s[Q(t,s)]x(s) + \Delta_2^+ P(t,0)x(0) + \Delta_2^- P(t,1)x(1)$$

is true. Furthermore, according to the integration-by-parts formula (cf. [18, Theorem 2.15]) we have

$$\begin{aligned}
&\int_0^1 \mathrm{d}_s[P(t,s)]x(s) \\
&\qquad = Q(t,1)x(1) - Q(t,0)x(0) - \int_0^1 Q(t,s)\,\mathrm{d}[x(s)] \\
&\qquad\quad + \big[P(t,0+) - P(t,0)\big]x(0) + \big[P(t,1) - P(t,1-)\big]x(1) \\
&\qquad = P(t,1)x(1) - P(t,0)x(0) - \int_0^1 Q(t,s)\,\mathrm{d}[x(s)] \\
&\qquad = \big[P(t,1) - P(t,0)\big]x(0) + \int_0^1 \big(P(t,1) - Q(t,s)\big)\,\mathrm{d}[x(s)] \\
&\qquad = \big[P(t,1) - P(t,0)\big]x(0) \\
&\qquad\qquad + \int_0^1 \left\{ \begin{array}{ll} P(t,1) - P(t,s+), & s < 1 \\ P(t,1) - P(t,1-), & s = 1 \end{array} \right\} \mathrm{d}[x(s)]
\end{aligned}$$

and hence

$$\int_0^1 \mathrm{d}_s[P(t,s)]x(s) = C(t)x(0) + \int_0^1 D(t,s)\,\mathrm{d}[x(s)],$$

where

$$C(t) = \mathrm{I} + P(t,1) - P(t,0)$$

and

$$D(t,s) = \begin{cases} P(t,1) - P(t,s+) & \text{for} \quad s \in [0,1), \\ P(t,1) - P(t,1-) & \text{for} \quad s = 1. \end{cases}$$

Obviously, under our assumptions we have $C \in \mathbf{G}_{\mathrm{L}}^{n\times n}$ and $D \in \mathscr{K}_L^{n\times n}$ (cf. Lemma 2.15). Thus, if $P \in \mathscr{K}_L^{n\times n}$ and $f \in \mathbf{G}_L^n$, then the given equation (3.7) may be transformed to an equation of the form (3.1) with coefficients A, B and f fulfilling the assumptions of Theorem 3.8.

4 . The resolvent couple for the Fredholm - Stieltjes integral equation

In this section we will consider the special case when the equation (3.1) possesses a unique solution $x \in \mathbf{G}_L^n$ for any $f \in \mathbf{G}_L^n$. In particular, in addition to $A \in \mathbf{G}_{\mathrm{L}}^{n\times n}$, $B \in \mathscr{K}_L^{n\times n}$ we will assume that

$$\dim \mathscr{N}(\mathrm{I} - \mathscr{L}) = 0 \tag{4.1}$$

(cf. Corollary 3.6).

Under these assumptions the Bounded Inverse Theorem [11, Section III.4.1] implies that the linear bounded operator $\mathrm{I} - \mathscr{L} : \mathbf{G}_L^n \to \mathbf{G}_L^n$ possesses a bounded inverse operator $(\mathrm{I} - \mathscr{L})^{-1} : \mathbf{G}_L^n \to \mathbf{G}_L^n$. Furthermore, as

$$(\mathrm{I} - \mathscr{L})^{-1} = \mathrm{I} + (\mathrm{I} - \mathscr{L})^{-1}\mathscr{L},$$

it follows immediately that the inverse operator $(\mathrm{I} - \mathscr{L})^{-1}$ may be expressed in the form

$$(\mathrm{I} - \mathscr{L})^{-1} = \mathrm{I} + \mathit{\Gamma}, \tag{4.2}$$

where $\mathit{\Gamma}$ is a linear compact operator ($\mathit{\Gamma} \in \mathcal{K}(\mathbf{G}_L^n, \mathbf{G}_L^n)$). By Theorem 2.5 there exist functions $U \in \mathbf{G}_{\mathrm{L}}^{n\times n}$, $V \in \mathscr{K}_L^{n\times n}$ such that $\mathit{\Gamma}$ is given by

$$\mathit{\Gamma} : f \in \mathbf{G}_L^n \to U(t)f(0) + \int_0^1 V(t,s)\,\mathrm{d}[f(s)]. \tag{4.3}$$

The following assertion now follows from Lemma 1.9 and Theorem 2.5.

4.1. Theorem. *Let us assume that* $A \in \mathbf{G}_{\mathrm{L}}^{n\times n}$ *and* $B \in \mathscr{K}_L^{n\times n}$ *are such that* (4.1) *holds. Then there exists a uniquely defined couple of functions* $U \in \mathbf{G}_{\mathrm{L}}^{n\times n}$, $V \in \mathscr{K}_L^{n\times n}$ *such that for any* $f \in \mathbf{G}_L^n$ *the corresponding solution* $x \in \mathbf{G}_L^n$ *to* (3.1) *is given by*

$$x(t) = f(t) + U(t)f(0) + \int_0^1 V(t,s)\,\mathrm{d}[f(s)], \quad t \in [0,1]. \tag{4.4}$$

4.2. Theorem. *Let us assume that $A \in \mathbf{G}_{\mathrm{L}}^{n\times n}$ and $B \in \mathscr{K}_L^{n\times n}$ are such that* (4.1) *holds. Then the functions U, V given by Theorem 4.1 satisfy the matrix equations*

$$U(t) - A(t)U(0) \quad - \int_0^1 B(t,\tau)\,\mathrm{d}[U(\tau)] \quad = A(t) \tag{4.5}$$

and

$$V(t,s) - A(t)V(0,s) - \int_0^1 B(t,\tau)\,\mathrm{d}_\tau[V(\tau,s)] = B(t,s) \tag{4.6}$$

for all $t, s \in [0,1]$.

Proof. Let Γ be a linear compact operator defined by (4.2). Inserting (4.2) into (3.1) we obtain that under our assumptions Γ has to satisfy the relation

$$\Gamma f - \mathscr{L}(\Gamma f) = \mathscr{L} f \quad \text{for all} \quad f \in \mathbf{G}_L^n. \tag{4.7}$$

Inserting (4.3) into (4.7) and making use of the Bray Theorem (cf. Theorem 2.13) we obtain furthermore that

$$\begin{aligned}
&\Big(U(t) - A(t)U(0) - \int_0^1 B(t,\tau)\,\mathrm{d}[U(\tau)]\Big) f(0) \\
&\quad + \int_0^1 \Big(V(t,s) - A(t)V(0,s) - \int_0^1 B(t,\tau)\,\mathrm{d}_\tau[V(\tau,s)]\Big)\,\mathrm{d}[f(s)] \\
&= A(t)f(0) + \int_0^1 B(t,s)\,\mathrm{d}[f(s)]
\end{aligned}$$

has to be true for any $f \in \mathbf{G}_L^n$, wherefrom by Lemma 1.9 the assertion of the theorem immediately follows. □

4.3. Definition. We say that a couple of functions $U \in \mathbf{G}_{\mathrm{L}}^{n\times n}$, $V \in \mathscr{K}_L^{n\times n}$ is the *resolvent couple* for the equation (3.1) if for any $f \in \mathbf{G}_L^n$ the unique solution $x \in \mathbf{G}_L^n$ is given by (4.3).

5 . Volterra-Stieltjes integral equations in $\mathbf{G}_L^n$

It is natural to expect that the linear operator equation (3.3) could possess for any $f \in \mathbf{G}_L^n$ a unique solution if the operator $\mathscr{L}$ is causal.

5.1. Definition. An operator $\mathscr{L} \in \mathcal{L}(\mathbf{G}_L^n)$ is said to be *causal* if

(5.1) $$(\mathscr{L}x)(0) = 0 \quad \text{for any} \quad x \in \mathbf{G}_L^n,$$

and for a given $t \in (0,1)$

(5.2) $$(\mathscr{L}x)(t) = 0 \quad \text{whenever} \quad x \in \mathbf{G}_L^n \ \text{ and } \ x(\tau) = 0 \ \text{ on } \ [0,t].$$

5.2. Lemma. *If $A \in \mathbf{G}_{\mathrm{L}}^{n\times n}$ and $B \in \mathscr{K}_{\mathrm{L}}^{n\times n}$, then the linear operator $\mathscr{L} \in \mathcal{L}(\mathbf{G}_L^n)$ given by* (3.2) *is causal if and only if*

(5.3) $$A(0) = 0 \quad \textit{and} \quad B(t,s) = 0 \qquad \textit{for all} \quad t \in [0,1) \quad \textit{and} \quad s \in [t,1].$$

Proof. a) If (5.3) is satisfied, then obviously the relation

$$\int_0^1 B(t,s)\,\mathrm{d}[x(s)] = \int_0^t B(t,s)\,\mathrm{d}[x(s)]$$

is true for any $x \in \mathbf{G}_L^n$ and any $t \in [0,1]$ whence the causality of $\mathscr{L}$ immediately follows.

b) On the other hand, let us assume that $\mathscr{L}$ is causal. Then by (5.1) the relation

$$A(0)x(0) + \int_0^1 B(0,s)\,\mathrm{d}[x(s)] = 0$$

has to be satisfied for any $x \in \mathbf{G}_L^n$. By Lemma 1.8 this means that the relations

$$A(0) = 0 \quad \text{and} \quad B(0,s) = 0 \quad \text{for all} \quad s \in [0,1]$$

have to be satisfied, as well. Furthermore, if $t \in (0,1)$, then (5.2) is true if and only if

$$\int_t^1 B(t,s)\,\mathrm{d}[x(s)] = 0 \quad \text{for all} \quad x \in \mathbf{G}_L^n.$$

An obvious modification of Lemma 1.8 implies that this may hold only if

$$B(t,s) = 0 \quad \text{for all} \quad s \in [t,1],$$

wherefrom the assertion of the lemma immediately follows. □

5.3. Remark. Let us notice that the condition (5.3) does not necessarily imply that $B(1,1)=0$. On the other hand, it is easy to verify that the operator $\mathscr{L}\in\mathcal{L}(\mathbf{G}_L^n)$ given by (3.2) fulfils a somewhat stronger causality properties (5.1) and

$$(5.4)\quad (\mathscr{L}x)(t)=0 \ \text{ for all } \ t\in(0,1] \ \text{ and } \ x\in\mathbf{G}_L^n \ \text{ such that } \ x(\tau)=0 \ \text{ on } \ [0,t)$$

if and only if

$$A(0)=0 \quad\text{and}\quad B(t,s)=0 \quad\text{whenever}\quad 0\le t\le s\le 1.$$

In fact, if $x(\tau)=0$ on $[0,1)$, then

$$(\mathscr{L}x)(1)=B(1,1)x(1)=0$$

holds for any $x(1)\in\mathbb{R}^n$ if and only if $B(1,1)=0$.

5.4. Remark. As noticed in the proof of Lemma 5.2, if the assumptions of Lemma 5.2 and the conditions (5.3) are satisfied, then the Fredholm-Stieltjes equation (3.1) reduces to the Volterra-Stieltjes equation

$$(5.5)\qquad x(t)-A(t)x(0)-\int_0^t B(t,s)\,\mathrm{d}[x(s)]=f(t),\quad t\in[0,1].$$

To show that the equation (5.5) possesses a unique solution $x\in\mathbf{G}_L^n$ for each $f\in\mathbf{G}_L^n$, it is by Proposition 3.4 sufficient to show that the corresponding homogeneous equation

$$(5.6)\qquad x(t)=A(t)x(0)+\int_0^t B(t,s)\,\mathrm{d}[x(s)],\quad t\in[0,1]$$

possesses only the trivial solution $x\equiv 0$.

Let $x\in\mathbf{G}_L^n$ be an arbitrary solution of (5.6) on $[0,1]$. Then obviously $x(0)=0$. Furthermore, since by (5.3) $B(0+,s)=0$ whenever $s>0$, we have by Lemma 2.10

$$\begin{aligned} x(0+)&=\lim_{t\to0+}\int_0^t B(t,s)\,\mathrm{d}[x(s)]=\lim_{t\to0+}\int_0^1 B(t,s)\,\mathrm{d}[x(s)]\\ &=\int_0^1 B(0+,s)\,\mathrm{d}[x(s)]\quad =B(0+,0)\Delta^+x(0)=B(0+,0)x(0+), \end{aligned}$$

i.e.

$$\Big[\mathrm{I}-B(0+,0)\Big]x(0+)=0.$$

Thus we have $x(0+)=0$ whenever

$$\det\big[\mathrm{I}-B(0+,0)\big]\ne 0.$$

Analogously, if we assume that $x(\tau) \equiv 0$ on $[0,t]$ holds for a given $t \in (0,1)$, then

$$x(t+) = \int_t^1 B(t+,s)\,\mathrm{d}[x(s)] = B(t+,t)x(t+),$$

and thus necessarily $x(t+) = 0$ whenever $\det\big(\mathrm{I} - B(t+,t)\big) \neq 0$. Finally, if we assume that $x(\tau) \equiv 0$ on $[0,1)$, then the equation (5.6) reduces to

$$\Big[\mathrm{I} - B(1,1)\Big]x(1) = x(1).$$

This indicates that it is possible to expect that the equation (5.6) will possess only the trivial solution $x \equiv 0$ on $[0,1]$ if the relations

$$\text{(5.7)} \qquad \det\big[\mathrm{I} - B(1,1)\big] \neq 0 \quad \text{and} \quad \det\big[\mathrm{I} - B(t+,t)\big] \neq 0 \quad \text{for all} \quad t \in [0,1)$$

will be satisfied.

5.5. Theorem. *Let $A \in \mathbf{G}_{\mathrm{L}}^{n\times n}$, $B \in \mathscr{K}_L^{n\times n}$ and let the condition* (5.3) *be satisfied. Then the Volterra-Stieltjes equation* (5.5) *possesses a unique solution $x \in \mathbf{G}_L^n$ for any $f \in \mathbf{G}_L^n$ if and only if the relations* (5.7) *are satisfied.*

Proof. First, let us assume that the relations (5.7) are satisfied. We shall show that then the equation (5.6) possesses only the trivial solution. Indeed, let $x \in \mathbf{G}_L^n$ be a solution of (5.6). Then $x(0+) = x(0) = 0$ and as in Remark 5.4 we have

$$\int_0^t B(0+,s)\,\mathrm{d}[x(s)] = B(0+,0)\Delta^+ x(0) = 0 \quad \text{for all} \quad t \in [0,1].$$

Consequently, the equation (5.6) can be rewritten as

$$x(t) = \int_0^t (B(t,s) - B(0+,s))\,\mathrm{d}[x(s)].$$

In virtue of [18, Theorem 2.8], this yields that the inequality

$$|x(t)| \leq 2\|B(t,.) - B(0+,.)\|_{\mathbf{BV}} \Big(\sup_{s\in[0,t]} |x(s)|\Big)$$

is true for any $t \in [0,1]$. Furthermore, by Corollary 2.9 there is a $\delta > 0$ such that

$$\|B(t,.) - B(0+,.)\|_{\mathbf{BV}} < \tfrac{1}{4} \quad \text{whenever} \ \ t \in (0,\delta]$$

and hence also

$$\sup_{t\in[0,\delta]} |x(s)| < \tfrac{1}{2} \sup_{t\in[0,\delta]} |x(s)|,$$

wherefrom the relation

$$x \equiv 0 \quad \text{on} \quad [0,\delta]$$

follows. Now, let us put

$$t^* = \sup\Big\{\delta \in [0,1] : x(t) = 0 \ \text{ on } \ [0,\delta]\Big\}.$$

We know that $t^* \in (0,1]$ and $x(t) = 0$ on $[0,t^*)$. Since x is left-continuous on $(0,1)$ (cf. Remark 3.9), it follows that if $t^* < 1$, then $x(t^*) = x(t^*-) = 0$, as well. We close the first part of the proof by showing that $t^* = 1$ and $x(1) = 0$.

Indeed, if $t^* < 1$, taking into account the hypothesis (5.3) and Lemma 2.10 we would obtain

$$\begin{aligned} x(t^*+) &= \lim_{t \to t^*+} \int_0^t B(t,s)\,\mathrm{d}[x(s)] = \int_0^1 B(t^*+,s)\,\mathrm{d}[x(s)] \\ &= B(t^*+,t^*)x(t^*+) \end{aligned}$$

and consequently

$$\Big[\mathrm{I} - B(t^*+,t^*)\Big]x(t^*+) = 0.$$

Hence according to (5.7) we would have $x(t^*+) = 0$. By an argument analogous to that used above for 0 in the place of t^*, we can get that there exists $\delta > 0$ such that $x(t) = 0$ on $[0, t^* + \delta]$, which contradicts the definition of t^*. Finally, as we have obviously $x(t) \equiv 0$ on $[0,1)$ and hence also $x(1-) = 0$, the relation (5.6) reduces to $x(1) = B(1,1)x(1)$ or

$$\Big[\mathrm{I} - B(1,1)\Big]x(1) = 0,$$

wherefrom the desired relation $x(1) = 0$ immediately follows taking into account our assumption (5.7).

To show the necessity of the conditions (5.7) for the unique solvability of (5.5) for any $f \in \mathbf{G}_L^n$, let us assume that the set

$$\mathcal{S}_B := \Big\{t \in [0,1) : \det\big[\mathrm{I} - B(t+,t)\big] = 0\Big\}$$

is nonempty. Let us denote

$$t^\star = \inf \mathcal{S}_B.$$

Then t^* is not a point of accumulation of $\mathcal{S}_B$. In fact, if this were not true, then there would exist a sequence $\{t_k\}_{k=1}^\infty$ of points in $\mathcal{S}_B$ such that $t_k > t^*$ for any $k \in \mathbb{N}$ and $\lim_{k\to\infty} t_k = t^*$. Since in virtue of (5.3) we have for any $\sigma > t^*$

$$\lim_{\tau \to t^*+} B(\tau,\sigma) = 0,$$

it follows by Lemma 2.18 that

$$B(t^*+,t^*+) = \lim_{\substack{(\tau,\sigma)\to(t^*,t^*)\\ \tau>t^*,\sigma>t^*}} B(\tau,\sigma) = \lim_{\sigma\to t^*+}\big(\lim_{\tau\to t^*+} B(\tau,\sigma)\big) = 0$$

and consequently

$$0 = \lim_{k\to\infty} \det\big(\mathrm{I} - B(t_k+,t_k)\big) = \det\big(\mathrm{I} - B(t^*+,t^*+)\big) = \det(\mathrm{I}) = 1.$$

In particular, $t^* \in \mathcal{S}_B$ and

$$\det\big(\mathrm{I} - B(t^*+,t^*)\big) = 0.$$

Hence there is a $d \in \mathbb{R}^n$ such that there is no $c \in \mathbb{R}^n$ such that

$$\big[\mathrm{I} - B(t^*+,t^*)\big]c = d.$$

Now, let us put

$$f(t) = \begin{cases} 0 & \text{for} \quad t \le t^*, \\ d & \text{for} \quad t > t^*. \end{cases}$$

By the first part of the proof, for any possible solution $x \in \mathbf{G}^n_L$ of the equation (5.5 on $[0,1]$ we have $x(t) = 0$ on $[0,t^*)$ and thus

$$x(t^*) = \lim_{t\to t^*-} x(t) = 0.$$

By an argument analogous to that used above we can further deduce that the limit $x(t^*+)$ of any possible solution x of (5.5) has to verify the relation

$$\Big[\mathrm{I} - B(t^*+,t^*+)\Big]x(t^*+) = f(t^*+) = d.$$

However, by the definition of d this is not possible and consequently the set $\mathcal{S}_B$ is empty. This completes the proof of the theorem. □

5.6. Corollary. *Let $A \in \mathbf{G}^{n\times n}_{\mathrm{L}}$, $B \in \mathscr{K}^{n\times n}_L$ and let the condition* (5.3) *be satisfied. Then the homogeneous equation* (5.6) *possesses only the trivial solution $x \equiv 0$ if and only if the relations* (5.7) *are satisfied.*

The *proof* follows immediately from Proposition 3.3 and Theorem 5.5. □

Similarly, the proof of the following assertion is an easy consequence of Theorems 4.1 and 4.2 and Corollary 5.6.

5.7. Corollary. *Let $A \in \mathbf{G}_{\mathrm{L}}^{n\times n}$, $B \in \mathscr{K}_L^{n\times n}$ and let the conditions* (5.3) *and* (5.7) *be satisfied. Then there exists a resolvent couple $U \in \mathbf{G}_{\mathrm{L}}^{n\times n}$, $V \in \mathscr{K}_L^{n\times n}$ for the equation* (5.5)*. The functions U and V satisfy in addition the relations*

$$U(0) = 0 \qquad \text{and} \qquad V(t,s) = 0 \qquad \text{for all} \quad s \in [0,1), t \in [0,s], \tag{5.8}$$

$$U(t) - \int_0^t B(t,\tau)\,\mathrm{d}_\tau[U(\tau)] = A(t) \qquad \text{for all} \quad t \in [0,1], \tag{5.9}$$

and

$$V(t,s) - \int_0^t B(t,\tau)\,\mathrm{d}_\tau[V(\tau,s)] = B(t,s) \quad \text{for all} \quad t,s \in [0,1]. \tag{5.10}$$

Proof. Let $A \in \mathbf{G}_{\mathrm{L}}^{n\times n}$, $B \in \mathscr{K}_L^{n\times n}$ and let the conditions (5.3) and (5.7) be satisfied. Then by Theorems 4.1 and 4.2 and Corollary 5.6 there exists a resolvent couple $U \in \mathbf{G}_{\mathrm{L}}^{n\times n}$, $V \in \mathscr{K}_L^{n\times n}$ for the equation (5.5) and the functions U, V satisfy the matrix equations (4.5) and (4.6. Furthermore, as in virtue of (5.3) we have $A(0) = 0$, it follows easily from (4.5) that $U(0) = 0$ holds. Consequently, the relation (4.5) reduces to (5.9).

Furthermore, let an arbitrary $s \in (0,1)$ be given. Since by (5.3) we have $B(t,s) = 0$ whenever $t \le s$, it follows easily that the function $V(.,s)$ fulfils the relation

$$V(t,s) = A(t)V(0,s) + \int_0^1 B(t,\tau)\,\mathrm{d}_\tau[V(\tau,s)] \quad \text{for all} \quad t \in [0,s].$$

By an argument analogous to that used in the proof of Corollary 5.6 we can deduce now that $V(t,s) = 0$ has to be true for any $t \in [0,s]$. Finally, as by the assumption (5.3) we have $B(0,s) = 0$ for any $s \in [0,1]$, it follows immediately from (4.6) that $V(0,0) = 0$ on $[0,1]$, as well. Thus the relations (5.8) are true and consequently the relation (4.6) reduces to (5.10). □

.

5.8. Remark. It is easy to verify that under the assumption of Corollary 5.7 the resolvent couple (U,V) of (5.5) satisfies in addition to the relations (5.8)-(5.10) the following relations, as well.

$$V(t,1) \equiv 0 \text{ on } [0,1) \quad \text{and} \quad V(1,1) = \Big[\mathrm{I} - B(1,1)\Big]^{-1} B(1,1).$$

To show that the results of this section cover also the Volterra analogue of the equation mentioned in Example 3.10 the following lemma is essential.

5.9. Lemma. *Let $K \in \mathscr{K}^{n\times n}$ and let $K^\triangle$ be given by*

$$K^\triangle(t,s) = \begin{cases} K(t,s) & \text{for} \quad t \in [0,1] \quad \text{and} \quad s \in [0,t], \\ K(t,t) & \text{for} \quad t \in [0,1] \quad \text{and} \quad s \in [t,1]. \end{cases} \tag{5.11}$$

Then $K^{\triangle} \in \mathscr{K}^{n\times n}$. *Moreover, if* $K \in \mathscr{K}^{n\times n}_L$ *and*

$$K(t,t-) = K(t,t) \quad \text{for all} \quad t \in (0,1), \tag{5.12}$$

then $K^{\triangle} \in \mathscr{K}^{n\times n}_L$, *as well.*

Proof. Let $t \in (0,1]$ and $\varepsilon > 0$ be given. Then by assumption and by Lemma 2.21 there exists a $\delta \in (0,t)$ such that

$$\|K(t_2,.) - K(t_1,.)\|_{\mathbf{BV}} < \tfrac{\varepsilon}{2} \quad \text{and} \quad \operatorname{var}_{t_1}^{t_2} K(t_2,.) < \tfrac{\varepsilon}{2}$$

whenever $0 \le t - \delta < t_1 \le t_2 < t$. Now, let an arbitrary couple $t_1, t_2 \in [0,1]$ such that $t - \delta < t_1 \le t_2 < t$ be given. Then by (5.11) we have

$$K^{\triangle}(t_2,s) - K^{\triangle}(t_1,s) = \begin{cases} K(t_2,s) - K(t_1,s) & \text{for} \quad 0 \le s \le t_1, \\ K(t_2,s) - K(t_1,t_1) & \text{for} \quad t_1 \le s \le t_2, \\ K(t_2,t_2) - K(t_1,t_1) & \text{for} \quad t_2 \le s \end{cases}$$

and it is easy to see that this implies that

$$\begin{aligned} &\|K^{\triangle}(t_2,.) - K^{\triangle}(t_1,.)\|_{\mathbf{BV}} \\ &\quad \le |K(t_2,0) - K(t_1,0)| + \operatorname{var}_0^{t_1}(K(t_2,.) - K(t_1,.)) \\ &\qquad + \operatorname{var}_{t_1}^{t_2}(K(t_2,.) - K(t_1,t_1)) \\ &\quad \le \|K(t_2,.) - K(t_1,.)\|_{\mathbf{BV}} + \operatorname{var}_{t_1}^{t_2} K(t_2,.) < \varepsilon \end{aligned}$$

holds for any couple $t_1, t_2 \in [0,1]$ such that $t - \delta < t_1 \le t_2 < t$. Analogously we would show that for any $\varepsilon > 0$ there exists a $\delta \in (0,t)$ such that

$$\|K^{\triangle}(t_2,.) - K^{\triangle}(t_1,.)\|_{\mathbf{BV}} < \varepsilon$$

holds for any couple $t_1, t_2 \in [0,1]$ such that $t < t_1 \le t_2 < t + \delta$, wherefrom the relation $K^{\triangle} \in \mathscr{K}^{n\times n}$ immediately follows.

Furthermore, if $K^{\triangle} \in \mathscr{K}^{n\times n}_L$ and (5.12) holds, then we obviously have

$$\begin{aligned} &\lim_{\tau \to t-} \|K^{\triangle}(t,.) - K^{\triangle}(\tau,.)\|_{\mathbf{BV}} \\ &\quad \le \lim_{\tau \to t-} \|K(t,.) - K(\tau,.)\|_{\mathbf{BV}} + \lim_{\tau \to t-} \operatorname{var}_{\tau}^{t} K(t,.) = 0 \end{aligned}$$

for any $t \in [0,1]$. □

5.10. Remark. It follows easily from Lemmas 2.18 and 2.19 that if $K \in \mathscr{K}^{n\times n}_L$, then for any $x \in \mathbf{G}^n_L$ the function $z(t)$ given by

$$z(t) = \int_0^t \mathrm{d}_s[K(t,s)]x(s) \quad \text{for} \quad t \in [0,1]$$

is left-continuous on $(0,1)$ if and only if (5.12) holds.

5.11. Example. Let us consider the linear Volterra-Stieltjes integral equation

$$x(t)-\int_0^t \mathrm{d}_s[K(t,s)]x(s)=f(t), \quad t\in[0,1] \tag{5.13}$$

with $K\in\mathscr{K}_L^{n\times n}$ fulfilling the relation (5.12) and $f\in\mathbf{G}_L^n$. (Such equations with kernels K of strongly bounded variation on $[0,1]\times[0,1]$ (cf. Remark 2.20) were treated in [16].)

Let us define the function $K^\triangle:[0,1]\times[0,1]\to\mathbb{R}^{n\times n}$ again by (5.11). Then by Lemma 5.9 we have $K^\triangle\in\mathscr{K}_L^{n\times n}$. Obviously,

$$\int_0^t \mathrm{d}_s[K(t,s)]x(s)=\int_0^1 \mathrm{d}_s[K^\triangle(t,s)]x(s) \tag{5.14}$$

holds for any $x\in\mathbf{G}^n$. Let $t\in[0,1]$ and $x\in\mathbf{G}_L^n$ be given. Analogously as in Example 3.10 we could show that then

$$\int_0^1 \mathrm{d}_s[K^\triangle(t,s)]x(s)=A(t)x(0)+\int_0^1 B(t,s)\mathrm{d}[x(s)], \tag{5.15}$$

where

$$A(t)=\mathrm{I}+K^\triangle(t,1)-K^\triangle(t,0) \quad \text{for} \quad t\in[0,1]$$

and

$$B(t,s)=\begin{cases} K^\triangle(t,1)-K^\triangle(t,s+) & \text{for} \quad t\in[0,1] \quad \text{and} \quad s\in[0,1),\\ K^\triangle(t,1)-K^\triangle(t,1-) & \text{for} \quad t\in[0,1] \quad \text{and} \quad s=1.\end{cases}$$

It is easy to verify that $A\in\mathbf{G}_{\mathrm{L}}^{n\times n}$ and $B\in\mathscr{K}_L^{n\times n}$ (cf. Lemma 2.15 and Lemma 5.9) and

$$A(t)=\mathrm{I}+K(t,t)-K(t,0) \quad \text{for} \quad t\in[0,1]$$

and

$$B(t,s)=\begin{cases} K(t,t)-K(t,s+) & \text{if} \ \ 0\le s<t\le 1,\\ K(t,t)-K(t,t) & \text{if} \ \ 0\le t\le s<1,\\ K(t,t)-K(t,t) & \text{if} \ \ 0\le t<s=1,\\ K(1,1)-K(1,1-) & \text{if} \ \ t=s=1.\end{cases}$$

In particular, we have

$$A(0)=0 \quad \text{and} \quad B(t,s)=0 \quad \text{whenever} \quad 0\le t\le s\le 1 \quad \text{and} \quad t<1.$$

Furthermore, for an arbitrary $t\in[0,1)$ we have

$$B(t+,t)=\lim_{\tau\to t+}(K(\tau,\tau)-K(\tau,t+))=K(t+,t+)-K(t+,t+)=0$$

(cf. Lemma 2.18). It means that under the above assumptions the Volterra-Stieltjes integral equation (5.13) may be converted to the causal integral equation of the form (5.5) whose coefficients A and B satisfy the assumptions of Corollary 5.7 if in addition we would assume that the relation

$$\det\big(\mathrm{I} - (K(1,1) - K(1,1-))\big) \neq 0$$

is satisfied.

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