LINEAR BOUNDED FUNCTIONALS ON THE SPACE OF REGULAR REGULATED FUNCTIONS

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Abstract. A representation of a general linear bounded functional on the space of regular regulated functions on a compact interval is given by means of the Perron-Stieltjes integral.

0. Introduction

Making use of the Perron-Stieltjes integral a representation of the dual space to the space of functions regulated on a compact interval $[a,b]$ and left-continuous on its interior (a,b) was obtained in [Tv1]. This result together with the properties of the Perron-Stieltjes integral derived in [Tv1] by means of the equivalent definition due to Kurzweil [Ku1] were utilized in [Tv2], where boundary value problems for generalized linear differential equations on $[a,b]$ with left-continuous on (a,b) regulated solutions were dealt with. Since then it turned out that functions regulated on $[a,b]$ and regular on (a,b) (i.e. functions $f:[a,b]\to\mathbb{R}$ such that for any $t\in(a,b)$ the value $f(t)$ is an arithmetical mean of the corresponding one-sided limits $\lim_{s\to t-} f(s)$ and $\lim_{s\to t+} f(s)$) are important for the investigation of differential equations with distributional coefficients (cf. [Pe-Tv] and [Tv3]). The aim of this note is to derive a representation of the dual space to the space of such functions.

1. Preliminaries

Throughout the paper $\mathbb{R}$ denotes the space of real numbers and $a,b\in\mathbb{R}$ are given such that $a<b$. For given numbers $c,d\in\mathbb{R}$ such that $c<d$, $[c,d]$ stands for the closed interval $c\le t\le d$, (c,d) is the open interval $c<t<d$, while $[c,d)$ and $(c,d]$ are the corresponding half-open intervals and $[c]$ stands for the one point set $\{c\}$.

The sets $D=\{t_0,t_1,\dots,t_N\}$ of points in $[a,b]$ such that

$$a=t_0<t_1<\cdots<t_N=b$$

are called *divisions* of $[a,b]$.

Given a subset M of $\mathbb{R}$, χ_M denotes its *characteristic function* ($\chi_M(t)=1$ for $t\in M$, $\chi_M(t)=0$ for $t\in\mathbb{R}\setminus M$.) In particular, for $\tau\in\mathbb{R}$ we have $\chi_{[\tau]}(\tau)=1$ and $\chi_{[\tau]}(t)=0$ for $t\ne\tau$.

Any function $f:[a,b]\to\mathbb{R}$ which possesses finite limits

$$f(t+)=\lim_{\tau\to t+} f(\tau) \quad \text{and} \quad f(s-)=\lim_{\tau\to s-} f(\tau)$$

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for all $t \in [a,b)$ and $s \in (a,b]$ is said to be *regulated* on $[a,b]$. The space of functions regulated on $[a,b]$ is denoted by G. It is known (cf. [Hö, Corollary 3.2a]) that for any $f \in$ G the set of its discontinuities on the interval $[a,b]$ is at most countable. Obviously any function regulated on $[a,b]$ is bounded on $[a,b]$. For a given $f \in$ G we define

$$\|f\| = \sup_{t\in[a,b]} |f(t)|.$$

By [Hö,Theorem 3.6], when endowed with this norm, G becomes a Banach space.

For given $f \in \mathrm{G}, t \in [a,b), s \in (a,b]$ and $r \in (a,b)$, we put $\Delta^+ f(t) = f(t+) - f(t)$, $\Delta^- f(s) = f(s) - f(s-)$ and $\Delta f(r) = f(r+) - f(r-)$.

The subset of G consisting of all functions regulated on $[a,b]$ and such that

$$f(t) = \tfrac{1}{2}[f(t-) + f(t+)] \quad \text{for all} \quad t \in (a,b),$$

will be denoted by G_{reg}. The functions belonging to G_{reg} are usually called *regular* on (a,b). It is easy to see that G_{reg} is closed in G.

A function $f : [a,b] \to \mathbb{R}$ is said to be a *finite step function* on $[a,b]$ if there exists a division $D = \{t_0, t_1, \dots, t_N\}$ of $[a,b]$ such that f is constant on every open interval $(t_{j-1}, t_j), j = 1, 2, \dots, N$. The set of all finite step functions on $[a,b]$ is denoted by S. It is known (cf. [Hö, Theorem 3.1]) that S is dense in G ($c\ell(\mathrm{S}) = \mathrm{G}$). It means that any function regulated on $[a,b]$ is a uniform limit of a sequence of finite step functions on $[a,b]$. The set of all finite step functions which are regular on (a,b) is denoted by S_{reg}.

$\mathrm{B}V$ denotes the space of functions of bounded variation on $[a,b]$ equipped with the norm

$$f \in \mathrm{B}V \longrightarrow |f(a)| + \operatorname{var}_a^b f,$$

where $\operatorname{var}_a^b f$ stands for the variation of f over the interval $[a,b]$.

The integrals which occur in this paper are the Perron-Stieltjes ones. In particular we make use of their equivalent definition due to J. Kurzweil (cf.[Ku1] or [Ku2]). The basic properties of the integrals of the form

$$\int_c^d f(t)\mathrm{d}[g(t)],$$

where $-\infty < a < c < d < b < \infty$, the functions f and g are functions regulated on $[a,b]$, while at least one of them has a bounded variation on $[a,b]$, were summarized in [Tv1]. Some more details concerning the Perron-Stieltjes integral with respect to functions of bounded variation see e.g. in [Sch1] or [STV].

Given a Banach space X, X^* stands for its dual (the space of all linear bounded functionals on X).

2. Linear bounded functionals on the space of regular regulated functions

In this section we shall show that linear bounded functionals on G_{reg} may be represented in the form

$$x \in \mathrm{G}_{reg} \to q\, x(a) + \int_a^b p(s)\mathrm{d}[x(s)],$$

where $p \in \mathrm{B}V$ and $q \in \mathbb{R}$. To this aim the following lemmas will be helpful.

2.1. Lemma. *A function $f : [a,b] \to \mathbb{R}$ is a finite step function on $[a,b]$ which is regular on (a,b) ($f \in \mathrm{S}_{reg}$) if and only if there are real numbers $\alpha_1, \alpha_2, \dots, \alpha_N$ and a division $D = \{t_0, t_1, \dots, t_N\}$ of $[a,b]$ such that*

$$f(t) = \sum_{j=0}^{N} \alpha_j h_j(t) \quad \text{on} \quad [a,b],$$

where $h_0 = 1$, $h_1 = \chi_{(a,b]}$, $h_j = \frac{1}{2}\chi_{[t_j]} + \chi_{(t_j,b]}$ for $j = 2,3,\dots,N-1$ and $h_N = \chi_{[b]}$.

Proof. Obviously a function $f : [a,b] \to \mathbb{R}$ belongs to S_{reg} if and only if there are real numbers $c_0, c_1, \dots, c_{N+1}$ and a division $D = \{t_0, t_1, \dots, t_N\}$ of $[a,b]$ such that

$$\begin{aligned} f(a) &= c_0, \\ f(t) &= c_j, \quad t \in (t_{j-1}, t_j), \quad j = 1,2,\dots,N, \\ f(t_j) &= \tfrac{c_j + c_{j+1}}{2}, \quad j = 1,2,\dots,N-1, \\ f(b) &= c_{N+1} \end{aligned}$$

i.e.

$$f(t) = c_0\chi_{[a]}(t) + \sum_{j=1}^{N} c_j \chi_{(t_{j-1},t_j)}(t) + \frac{1}{2}\left(\sum_{j=1}^{N-1}(c_j + c_{j+1})\chi_{[t_j]}(t)\right) + c_{N+1}\chi_{[b]}(t) \quad \text{for} \quad t \in [a,b]. \tag{2.1}$$

It is easy to verify that the right-hand side of (2.1) may be rearranged as follows

$$\begin{aligned} f = \;& c_0\chi_{[a,b]} - c_0\chi_{(a,b]} + \sum_{j=1}^{N} c_j\chi_{(t_{j-1},b]} - \sum_{j=1}^{N-1} c_j\chi_{(t_j,b]} \\ & - \frac{1}{2}\sum_{j=1}^{N-1} c_j\chi_{[t_j]} - c_N\chi_{[b]} + \frac{1}{2}\sum_{j=1}^{N-1} c_{j+1}\chi_{[t_j]} + c_{N+1}\chi_{[b]} \\ = \;& c_0\chi_{[a,b]} - c_0\chi_{(a,b]} + \sum_{j=0}^{N-1} c_{j+1}\chi_{(t_j,b]} - \sum_{j=1}^{N-1} c_j\chi_{(t_j,b]} \\ & - \frac{1}{2}\sum_{j=1}^{N-1} c_j\chi_{[t_j]} + \frac{1}{2}\sum_{j=1}^{N-1} c_{j+1}\chi_{[t_j]} + c_{N+1}\chi_{[b]} - c_N\chi_{[b]} \\ = \;& c_0\chi_{[a,b]} + (c_1 - c_0)\chi_{(a,b]} + \sum_{j=1}^{N-1}(c_{j+1} - c_j)\Big(\chi_{(t_j,b]} + \frac{1}{2}\chi_{[t_j]}\Big) \\ & + (c_{N+1} - c_N)\chi_{[b]}, \end{aligned}$$

wherefrom the assertion of the lemma immediately follows. ■

2.2. Lemma. *The set* S_{reg} *is dense in* G_{reg}.

Proof. Let $x \in \mathrm{G}_{reg}$ and $\varepsilon > 0$ be given. Since $c\ell(\mathrm{S}) = \mathrm{G}$, there exist a $\xi \in \mathrm{S}$ such that $|x(t) - \xi(t)| < \varepsilon$ holds for any $t \in [a,b]$. Consequently, we have

$$(2.2) \qquad |x(t-) - \xi(t-)| < \varepsilon, \quad |x(s+) - \xi(s+)| < \varepsilon \quad \text{for all } \ t \in [a,b), s \in (a,b].$$

Let us put $\xi^*(a) = \xi(a)$, $\xi^*(t) = \frac{1}{2}\Big(\xi(t+) + \xi(t-)\Big)$ for $t \in (a,b)$ and $\xi^*(b) = \xi(b)$. Obviously $\xi^*(t-) = \xi(t-)$ and $\xi^*(s+) = \xi(s+)$ for all $t \in (a,b]$ and $s \in [a,b)$, respectively. In particular, $\xi^*(t) = \xi(t)$ for any point t of continuity of ξ. It follows that $\xi^* \in \mathrm{S}_{reg}$. Furthermore, in virtue of (2.2) we have for any $t \in (a,b)$

$$|x(t) - \xi^*(t)| = \tfrac{1}{2}\big|[x(t-) - \xi(t-)] + [x(t+) - \xi(t+)]\big| < \varepsilon,$$

wherefrom the assertion of the lemma immediately follows. ■

2.3. Lemma. *Let F be an arbitrary linear bounded functional on* G_{reg}. *Let us define*

$$(2.3) \qquad p(t) = \begin{cases} F(\chi_{(a,b]}), & \textit{for } \ t = a, \\ F(\frac{1}{2}\chi_{[t]} + \chi_{(t.b]}), & \textit{for } \ t \in (a,b), \\ F(\chi_{[b]}), & \textit{for } \ t = b. \end{cases}$$

Then

$$\mathrm{var}_a^b\, p \le \|F\| = \sup_{x \in \mathrm{G}_{reg}, \|x\| \le 1} |F(x)|$$

(i.e. $p \in \mathrm{BV}$).

Proof. Let $D = \{t_0, t_1, \dots, t_N\}$ be an arbitrary division of $[a,b]$ and let $\alpha_j \in \mathbb{R}$, $j = 1, 2, \dots, N$, be such that $|\alpha_j| \le 1$ for all $j = 1, 2, \dots, N$. Then

$$(2.4) \qquad \begin{aligned} \sum_{j=1}^{N} \alpha_j [p(t_j) - p(t_{j-1})] &= \alpha_1 \Big[F(\tfrac{1}{2}\chi_{[t_1]} + \chi_{(t_1,b]}) - F(\chi_{(a,b]})\Big] \\ &+ \sum_{j=2}^{N-1} \alpha_j \Big[F(\tfrac{1}{2}\chi_{[t_j]} + \chi_{(t_j,b]}) - F(\tfrac{1}{2}\chi_{[t_{j-1}]} + \chi_{(t_{j-1},b]})\Big] \\ &+ \alpha_N \Big[F(\chi_{[b]}) - F(\tfrac{1}{2}\chi_{[t_{N-1}]} + \chi_{(t_{N-1},b]})\Big] = F(h), \end{aligned}$$

where

$$\begin{aligned} h = \ & \alpha_1 \Big[\tfrac{1}{2}\chi_{[t_1]} + \chi_{(t_1,b]} - \chi_{(a,b]}\Big] + \alpha_N \Big[\chi_{[b]} - \tfrac{1}{2}\chi_{[t_{N-1}]} - \chi_{(t_{N-1},b]}\Big] \\ & + \sum_{j=2}^{N-1} \alpha_j \Big[\tfrac{1}{2}\chi_{[t_j]} + \chi_{(t_j,b]} - \tfrac{1}{2}\chi_{[t_{j-1}]} - \chi_{(t_{j-1},b]}\Big] \\ = \ & \alpha_1 \Big[\tfrac{1}{2}\chi_{[t_1]} - \chi_{(a,t_1]}\Big] - \alpha_N \Big[\tfrac{1}{2}\chi_{[t_{N-1}]} + \chi_{(t_{N-1},b)}\Big] \\ & + \sum_{j=2}^{N-1} \alpha_j \Big[\tfrac{1}{2}\chi_{[t_j]} - \chi_{(t_{j-1},t_j]} - \tfrac{1}{2}\chi_{[t_{j-1}]}\Big] \end{aligned}$$

$$\begin{aligned} &= -\alpha_1\Big[\tfrac{1}{2}\chi_{[t_1]}+\chi_{(a,t_1)}\Big]-\alpha_N\Big[\tfrac{1}{2}\chi_{[t_{N-1}]}+\chi_{(t_{N-1},b)}\Big]\\ &\qquad -\tfrac{1}{2}\sum_{j=2}^{N-1}\alpha_j\chi_{[t_j]}-\tfrac{1}{2}\sum_{j=2}^{N-1}\alpha_j\chi_{[t_{j-1}]}-\sum_{j=2}^{N-1}\alpha_j\chi_{(t_{j-1},t_j)}\\ &= -\tfrac{1}{2}\sum_{j=1}^{N-1}\alpha_j\chi_{[t_j]}-\tfrac{1}{2}\sum_{j=2}^{N}\alpha_j\chi_{[t_{j-1}]}-\sum_{j=1}^{N}\alpha_j\chi_{(t_{j-1},t_j)}\\ &= -\sum_{j=1}^{N-1}\tfrac{\alpha_j+\alpha_{j+1}}{2}\chi_{[t_j]}-\sum_{j=1}^{N}\alpha_j\chi_{(t_{j-1},t_j)}. \end{aligned}$$

It is easy to see that $h \in \mathrm{S}_{reg}$ and $|h(t)| \le 1$ for all $t \in [a,b]$. Consequently, by (2.4) , we have that

$$\sup_{|\alpha_j|\le 1, j=1,2,\dots,N}\Big|\sum_{j=1}^{N}\alpha_j[p(t_j)-p(t_{j-1})]\Big| \le \sup_{x\in \mathrm{G}_{reg},\|x\|\le 1}\|F(x)\|$$

holds for any division $D=\{t_0,t_1,\dots,t_N\}$ of $[a,b]$. In particular, choosing $\alpha_j=\mathrm{sign}[p(t_j)-p(t_{j-1})]$, for $j=1,2,\dots,N$, we get

$$\sum_{j=1}^{N}|p(t_j)-p(t_{j-1})| \le \sup_{x\in \mathrm{G}_{reg},\|x\|\le 1}\|F(x)\| < \infty$$

and hence $\mathrm{var}_a^b\, p \le \|F\| < \infty$. ■

2.4. Lemma. *Let F be an arbitrary linear bounded functional on G_{reg}. Let $\eta=(p,q)$, where $p\in \mathrm{BV}$ is given by (2.3) and $q=F(\chi_{[a,b]})$. Let us define*

$$F_\eta(x) = q\,x(a) + \int_a^b p(s)\mathrm{d}[x(s)] \quad \text{for} \ \ x\in \mathrm{G}. \tag{2.5}$$

Then F_η is a linear bounded functional on G,

$$F_\eta(x) = F(x) \tag{2.6}$$

holds for any $x\in \mathrm{G}_{reg}$ and

$$\sup_{x\in \mathrm{G},\|x\|\le 1}|F_\eta(x)| \le \Big(|q| + 2\Big(|p(a)| + \mathrm{var}_a^b\, p\Big)\Big). \tag{2.7}$$

Proof. By [Tv1, Theorem 2.8], $F_\eta(x)$ is defined and

$$|F_\eta(x)| \le (|q|+|p(a)|+|p(b)|+\mathrm{var}_a^b\, p)\|x\| \quad \text{for all} \ \ x\in \mathrm{G} \tag{2.8}$$

It means that F_η is a linear bounded functional on G and the inequality (2.7) is true. It is easy to verify that the relation (2.6) holds for any function h from the set

$$\left\{\chi_{[a,b]},\chi_{(a,b]},\tfrac{1}{2}\chi_{[\tau]}+\chi_{(\tau,b]},\chi_{[b]};\tau\in(a,b)\right\}.$$

According to Lemmas 2.1 and 2.2 this implies that (2.6) holds for all $x\in \mathrm{G}_{reg}$. ■

2.5. Lemma. *Let $\eta = (p,q) \in \mathrm{BV} \times \mathbb{R}$. Then $F_\eta(x) = 0$ for all $x \in \mathrm{S}_{reg}$ only if $q = 0$ and $p(t) \equiv 0$ on $[a,b]$.*

Proof. Let $\eta = (p,q) \in \mathrm{BV} \times \mathbb{R}$ and let $F_\eta(x) = 0$ for all $x \in \mathrm{S}_{reg}$. Then $F(\chi_{[a,b]}) = q = 0$. Furthermore, by [Tv1, Proposition 2.3] we have

$$F_\eta(\chi_{(a,b]}) = p(a) = 0,$$
$$F_\eta(\tfrac{1}{2}\chi_{[\tau]} + \chi_{(\tau,b]}) = p(\tau) = 0 \quad \text{for} \quad \tau \in (a,b)$$

and

$$F_\eta(\chi_{[b]}) = p(b) = 0.$$

By Lemma 2.1 this completes the proof. ■

2.6. Remark. Let us notice that if $x \in \mathrm{G}_{reg}$, then $F_\eta(x) = 0$ for all $\eta = (p,q) \in \mathrm{BV} \times \mathbb{R}$ if and only if $x(t) \equiv 0$ on $[a,b]$. In fact, let $x \in \mathrm{G}$ and let $F_\eta(x) = 0$ for all $\eta = (p,q) \in \mathrm{BV} \times \mathbb{R}$. Then by [Tv1, Corollary 3.4], we have

$$x(a) = x(a+) = x(t-) = x(t+) = x(b-) = x(b) = 0 \quad \text{for all} \quad t \in (a,b).$$

In particular, if $x \in \mathrm{G}_{reg}$, then $x(t) = 0$ for any $t \in [a,b]$.

2.7. Theorem. *A mapping $F : \mathrm{G}_{reg} \to \mathbb{R}$ is a linear bounded functional on G_{reg} ($F \in \mathrm{G}^*_{reg}$) if and only if there is an $\eta = (p,q) \in \mathrm{BV} \times \mathbb{R}$ such that $F = F_\eta$, where F_η is given by (2.5). The mapping $\Phi : \eta \in \mathrm{BV} \times \mathbb{R} \to \mathrm{G}^*_{reg}$ generates an isomorphism between $\mathrm{BV} \times \mathbb{R}$ and G^*_{reg}.*

Proof. By Lemmas 2.4 and 2.5 and by the inequality (2.7) the mapping Φ is a bounded linear one-to-one mapping of $\mathrm{BV} \times \mathbb{R}$ onto G^*_{reg}. Consequently, by the Bounded Inverse Theorem Φ^{-1} is bounded, as well. ■

2.8. Remark. Similarly as in [Sch3] it is possible to modify the proof of Theorem 2.7 to obtain a representation of general linear bounded operators from G_{reg} into G.

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