Charles University in Prague
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DOCTORAL THESIS

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Shape optimization for Navier–Stokes equations with viscosity

Department of Numerical Mathematics

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Study program: Mathematics
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Univerzita Karlova v Praze
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DISERTAČNÍ PRÁCE

Jan Stebel

Tvarová optimalizace pro Navierovy–Stokesovy rovnice s viskozitou

Katedra numerické matematiky

Školitel: Prof. RNDr. Jaroslav Haslinger, DrSc.
Studijní program: Matematika
Studijní obor: M6 Vědecko-technické výpočty

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I declare that this thesis was written solely by myself and exclusively with help of the cited resources. I agree to borrowing of the thesis.

Prague, 22^{nd} June 2007 Jan Stebel

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Abstrakt: V práci se řeší problém optimalizace tvaru vstupní komory, která je součástí strojňa výrobu papíru a která přivádí směs "voda+dřevní hmota" do výrobního procesu. Cílem je navrhnout takový tvar, který zajišťuje a priori daný průběh rychlosti směsi na výtokové části. Z matematického hlediska se jedná o úlohu optimálního řízení, kdy řídící proměnnou je tvar oblasti, která představuje vstupní komoru, stavovou úlohou je zobecněný Navier-Stokesův systém s netriviálními okrajovými podmínkami. Cílem je teoretické studium této úlohy (důkaz existence řešení), její diskretizace a numerická realizace.
Klíčová slova: Tvarová optimalizace, vstupní komora papírenského stroje, nestlačitelná nenewtonovská tekutina, algebraický model turbulence

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Abstract: We study the shape optimization problem for the paper machine headbox which distributes a mixture of water and wood fibers in the paper making process. The aim is to find a shape which a priori ensures the given velocity profile on the outlet part. The mathematical formulation leads to the optimal control problem in which the control variable is the shape of the domain representing the header, the state problem is represented by the generalised Navier-Stokes system with nontrivial boundary conditions. The objective is to analyze theoretically this problem (proof of the existence of a solution), its discretization and the numerical realization.
Keywords: Optimal shape design, paper machine headbox, incompressible non-Newtonian fluid, algebraic turbulence model

List of Figures

Notation

Domains and sets

$\mathbb{N}$	set of all positive integers
$\mathbb{R}$	set of all real numbers
$\mathbb{R}^n$	Euclidian space of dimension n
$\mathbb{R}^{m\times n}$	space of real matrices with m rows and n columns
Ω	bounded domain in $\mathbb{R}^n$
$\overline{\Omega}$	closure of Ω
$\partial\Omega$	boundary of Ω
$\mathcal{C}^{0,1}$	system of bounded domains with Lipschitz continuous boundaries

Spaces of functions

$\mathcal{C}^k(A,B)$	functions defined in A, taking values in B, continuously differentiable in the Fréchet sense up to order $k\in\mathbb{N}\cup\{0,\infty\}$ ($\mathcal{C}^k(A):=\mathcal{C}^k(A,\mathbb{R})$)
$\mathcal{C}^k(\overline{\Omega})$	functions whose derivatives up to order k are continuous in $\overline{\Omega}$, $k\in\mathbb{N}\cup\{0,\infty\}$ ($\mathcal{C}(\overline{\Omega}):=\mathcal{C}^0(\overline{\Omega})$)
$\mathcal{C}^k_0(\overline{\Omega})$	functions from $\mathcal{C}^k(\overline{\Omega})$ vanishing in the vicinity of $\partial\Omega$
$\mathcal{C}^{0,1}(\overline{\Omega})$	Lipschitz continuous functions in $\overline{\Omega}$
$L^p(\Omega)$	Lebesgue integrable functions in Ω, $p\in[1,\infty]$
$W^{k,p}(\Omega)$	functions whose derivatives (in the sense of distributions) up to order $k\in\{0\}\cup\mathbb{N}$ are in $L^p(\Omega)$
$W^{k,p}_0(\Omega)$	functions from $W^{k,p}(\Omega)$ whose derivatives in the sense of traces up to order $(k-1)$ are equal to zero on $\partial\Omega$

Convergences

$\to$ in X	convergence in the norm of a normed space X (strong convergence)
$\rightharpoonup$ in X	weak convergence in a normed linear space X
$\rightrightarrows$ in X	uniform convergence of a sequence of continuous functions in X

Linear algebra

$\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{v}$	column vectors in $\mathbb{R}^n$
$\boldsymbol{x}^{\mathrm{T}}$	transpose of $\boldsymbol{x}$
$\mathbb{A}, \mathbb{B}$	matrices in $\mathbb{R}^{n\times n}$
$\mathbb{A}^{\mathrm{T}}$	transpose of $\mathbb{A}$
$\mathbb{A}^{-1}$	inverse of $\mathbb{A}$
$\mathbb{I}$	identity matrix

Mappings

$A : X \to Y$	A maps space X to space Y
A^{-1}	inverse of A
$\mathcal{R}(A)$	range of $A : X \to Y$, i.e. $\{A(x);\ x \in X\}$
$f \circ g$ (also $f(g)$)	composite function
$\mathrm{Tr}\, f$	trace of f

Differential calculus

$\frac{\partial f(\boldsymbol{x})}{\partial \boldsymbol{x}_i}$, $\frac{\partial^2 f(\boldsymbol{x})}{\partial \boldsymbol{x}_i \partial \boldsymbol{x}_j}$	first and second order generalized derivatives, respectively, of $f : \mathbb{R}^n \to \mathbb{R}$
$\frac{\partial f}{\partial \boldsymbol{\nu}}$, $\frac{\partial f}{\partial \boldsymbol{\tau}}$	normal and tangential derivatives, respectively, of f on $\partial\Omega$
∇f	gradient of f
$\mathrm{div}\, \boldsymbol{f}$	divergence of $\boldsymbol{f}$
$\mathrm{curl}\, f$	rotation of $f : \mathbb{R}^2 \to \mathbb{R}$, i.e. $\left(\frac{\partial f}{\partial \boldsymbol{x}_2}, -\frac{\partial f}{\partial \boldsymbol{x}_1}\right)^{\mathrm{T}}$

Norms and scalar products

$\|\cdot\|_X$	norm in a normed space X
$\lvert\cdot\rvert_X$	seminorm in a normed space X
$\boldsymbol{f}\cdot\boldsymbol{g}$	scalar product of two vectors or vector-valued functions, i.e. $\sum_{i=1}^n \boldsymbol{f}_i\boldsymbol{g}_i$
$\boldsymbol{f}\otimes\boldsymbol{g}$	tensor product of two vectors, i.e. $(\boldsymbol{f}\otimes\boldsymbol{g})_{ij} := f_i g_j$
$\boldsymbol{f}\otimes_s\boldsymbol{g}$	symmetrized tensor product, i.e. $\frac{1}{2}(\boldsymbol{f}\otimes\boldsymbol{g}+\boldsymbol{g}\otimes\boldsymbol{f})$
$\mathbb{A}:\mathbb{B}$	scalar product of two matrices or matrix-valued functions, i.e. $\sum_{i,j=1}^n \mathbb{A}_{ij}\mathbb{B}_{ij}$
$\lvert\mathbb{A}\rvert$	matrix norm of $\mathbb{A}$, i.e. $\sqrt{\mathbb{A}:\mathbb{A}}$
$(f,g)_\Omega$	scalar product in $L^2(\Omega)$ or, more generally, $\int_\Omega fg\,dx$ if $fg\in L^1(\Omega)$
$\|\cdot\|_{p,\Omega}$	norm in $L^p(\Omega)$
$\|\cdot\|_{k,p,\Omega}$	norm in $W^{k,p}(\Omega)$

Miscellaneous

δ_{ij}	Kronecker symbol
$\boldsymbol{\nu}$	unit outward normal vector to $\partial\Omega$
$\boldsymbol{\tau}$	unit tangent vector to $\partial\Omega$

Introduction and derivation of the model

For many years paper belongs to the most used everyday tools. About 19 centuries ago ancient Chinese developed the paper manufacturing technique using the bark and hemp. Since that time many improvements have been made in order to reduce the costs and enhance the quality, production speed and environmental compatibility. Today paper production presents a complex process.

Recently the paper machine technology has been achieved mostly through the experimental work in pilot plants. With increasing speeds and sophisticated machines this approach has become too expensive and time-consuming so that more effective methods must be used to bring further development. One of such methods is mathematical modelling in the framework of continuum mechanics resulting in the numerical simulations for a proposed model. The experimental research is still needed to verify the simulated results.

The first component in the paper making process is the headbox which is located at the wet end of a paper machine. The headbox shape and the fluid flow phenomena taking place there largely determine the quality of the produced paper. The first flow passage in the headbox is a dividing manifold, called the header. It is designed to distribute the fibre suspension on the wire so that the produced paper has an optimal basis weight and fibre orientation across the whole width of a paper machine. The aim of this work is to find an optimal shape for the back wall of the header so that the outlet flow rate distribution from the headbox results in an optimal paper quality.

The paper making pulp (also called the fibre suspension, furnish or stock) is a mixture of wood fibres, water, filler clays and various chemicals at concentration of 1% solids to 99 % water by weight. In the large-scale simulation it seems reasonable to model this complex mixture as a single continuum, with the fluid being an incompressible liquid described by the Navier–Stokes equations

$$\rho \boldsymbol{u}_{,t} + \rho\,\mathrm{div}(\boldsymbol{u} \otimes \boldsymbol{u}) = -\nabla q + \mathrm{div}\,(\mu_0 \mathbb{D}(\boldsymbol{u})) + \rho \boldsymbol{f}, \quad \mathrm{div}\ \boldsymbol{u} = 0, \tag{1}$$

where $\boldsymbol{u}, q, \rho, \mu_0$, $\boldsymbol{f}$ are the velocity, the pressure, the density, the viscosity and the body force (e.g. gravity). The symbol

$$\mathbb{D}(\boldsymbol{u}) = \frac{1}{2}\left(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^{\mathrm{T}}\right)$$

means the symmetric part of the gradient of $\boldsymbol{u}$ and $|\mathbb{D}(\boldsymbol{u})|$ is its norm.

The turbulence character of the flow in the header is a desirable phenomenon in the paper making process. Typically, the input Reynolds number defined as $Re = \frac{\ell V}{\mu_0}$, where V denotes the magnitude of the input velocity and ℓ is the diameter of the input channel, is about 10^6. In the modelling of turbulence, the velocity field $\boldsymbol{u}$ is usually decomposed into the sum of the average velocity $\boldsymbol{v}$ and its fluctuation $\boldsymbol{v}'$. Averaging of (1) then leads to the system

$$\rho\boldsymbol{v}_{,t} + \rho\,\mathrm{div}(\boldsymbol{v}\otimes\boldsymbol{v}) = -\nabla p + \mathrm{div}\left(\mu_0\mathbb{D}(\boldsymbol{v}) + \overline{\mathbb{R}}\right) + \rho\boldsymbol{f}, \quad \mathrm{div}\;\boldsymbol{v} = 0, \qquad (2)$$

where $\overline{\mathbb{R}}$ denotes the so-called Reynolds tensor given as the average of $-\boldsymbol{v}'\otimes\boldsymbol{v}'$. Since the flow in the header is steady and it is expected that the geometry of the domain changes only in the part of the boundary, we use a classical algebraic model, where

$$\overline{\mathbb{R}} = \rho l_{m,\alpha}^2|\mathbb{D}(\boldsymbol{v})|\mathbb{D}(\boldsymbol{v}) \qquad (3)$$

with experimentally determined mixing length $l_{m,\alpha}^2$, specified later. Note that inserting (3) into (2) yields a closed system for unknowns $\boldsymbol{v}$ and p.

Setting $\mu_1 = \rho l_{m,\alpha}^2$ and

$$\mathbb{T}\left(p, \mathbb{D}(\boldsymbol{v})\right) = -p\mathbb{I} + \left(\mu_0 + \mu_1|\mathbb{D}(\boldsymbol{v})|\right)\mathbb{D}(\boldsymbol{v}) \qquad (4)$$

we obtain the model appearing also in non-Newtonian fluid mechanics. The models where the Cauchy stress $\mathbb{T}(p, \mathbb{D}(\boldsymbol{v}))$ takes the form

$$\mathbb{T}(p, \mathbb{D}(\boldsymbol{v})) = -p\mathbb{I} + \nu\left(|\mathbb{D}(\boldsymbol{v})|\right)\mathbb{D}(\boldsymbol{v}) \qquad (5)$$

represent a class of non-Newtonian fluids called the fluids with shear-dependent viscosity. Since in the case (4) the viscosity increases with increasing shear rate (in a simple shear flow), (4) is a model for fluids that have the ability to shear thicken, see [20, 17, 16] for more details on non-Newtonian fluids and their mathematical analysis.

On Figure 1 the geometry of the header is shown. The inlet is on the left and the so-called recirculation on the right hand side. Typically about 10 % of the fluid flows out through the recirculation. The main outlet is performed by a number (usually several hundreds or thousands) of small

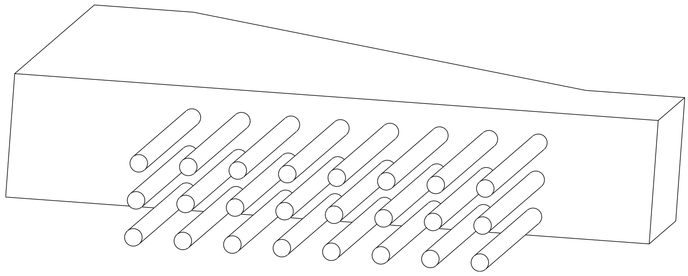

Figure 1: The header.

tubes. This fact presents a difficulty in the numerical simulation and thus the complicated geometry of the tube bank is replaced by an effective medium using the homogenization technique. It introduces a nonstandard boundary condition of the form

$$\mathbb{T}\boldsymbol{\nu} \cdot \boldsymbol{\nu} = \sigma |v_\nu| v_\nu, \tag{6}$$

where $\mathbb{T}$, $\boldsymbol{\nu}$, v_ν, σ are the stress tensor, the unit outer normal vector, the normal component of the velocity and the coefficient of suction, respectively.

This work was motivated by some previous papers: The fluid flow model which is used here has been derived and studied numerically in [10]. The shape optimization problem has also been solved numerically and the results are presented in [11], see also [12]. Both fluid flow model and shape optimization problem have been studied there formally without establishing existence results. Therefore our primary goal is to give the theoretical analysis of the flow equations and of the whole optimization problem.

The text is organized as follows. In Chapter 1 we present the fluid flow model and analyze the existence of a solution. The existence proof is based on appropriate energy estimates and the Galerkin method. A shape optimization problem is then formulated in Chapter 2 and the existence of an optimal shape is established. The continuous dependence of solutions to state problems with respect to shape variations is the most important result of this part. An approximation of the fluid flow model and of the shape optimization problem is studied in Chapter 3 and 4, respectively. Finally, Chapter 5 describes an implementation and example numerical computations.

Part I

Existence analysis of the continuous problem

Chapter 1

Steady flow of a non-Newtonian fluid

In the introduction we have shown that the fluid flow model used for the modelling of the pulp is very similar to the one of non-Newtonian fluids with shear-dependent viscosity. Hence the methods used in this chapter for the existence analysis come from the mathematical theory of non-Newtonian fluids. However the turbulence model makes the situation more involved, requiring special function spaces to be introduced.

At the beginning we make some simplifications:

(i) the fluid motion in the header will be assumed to be stationary;

(ii) the flow is almost negligible in the vertical direction.

Thus we will restrict ourselves to a plane geometry given by the horizontal cut of the real 3D header. Since the gravity is perpendicular to the plane, the body force vanishes. First we specify the geometry of the problem.

1.1 Description of admissible domains

Let $L_1, L_2, L_3 > 0$, $\alpha_{max} \geq H_1 \geq H_2 \geq \alpha_{min} > 0$, $\gamma > 0$ be given and suppose that $\alpha \in \mathcal{U}_{ad}$, where

$$\mathcal{U}_{ad} = \Big\{\alpha \in \mathcal{C}^{0,1}([0,L]);\ \alpha_{min} \leq \alpha \leq \alpha_{max},\\ \alpha_{|[0,L_1]} = H_1,\ \alpha_{|[L_1+L_2,L]} = H_2, |\alpha'| \leq \gamma \text{ a.e. in } [0,L]\Big\}. \quad (1.1)$$

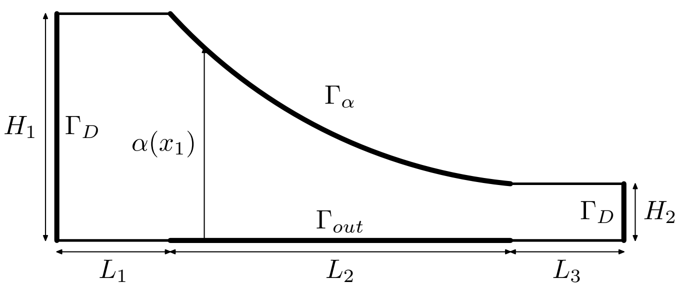

Figure 1.1: Geometry of $\Omega(\alpha)$ and parts of the boundary $\partial\Omega(\alpha)$.

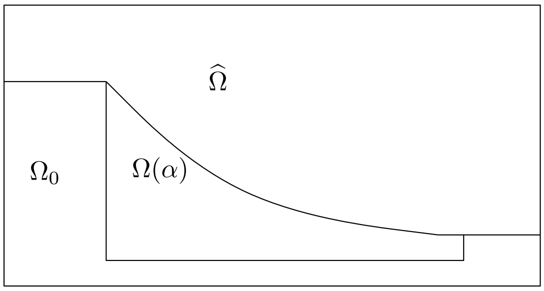

Figure 1.2: Domains: Ω_0, $\Omega(\alpha)$ and $\widehat{\Omega}$.

Here $L = L_1 + L_2 + L_3$. With any function $\alpha \in \mathcal{U}_{ad}$ we associate the domain $\Omega(\alpha)$, see Fig. 1.1:

$$\Omega(\alpha) = \Big\{\boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2; 0 < x_1 < L, 0 < x_2 < \alpha(x_1)\Big\} \tag{1.2}$$

and introduce the system of admissible domains

$$\mathcal{O} = \big\{\Omega;\ \exists\ \alpha \in \mathcal{U}_{ad} :\ \Omega = \Omega(\alpha)\big\}.$$

Further we will need the domains $\widehat{\Omega} = (0, L) \times (0, \alpha_{max})$ and $\Omega_0 = \big((0, L_1) \times (0, H_1)\big) \cup \big((0, L) \times (0, \alpha_{min})\big) \cup \big((L_1 + L_2, L) \times (0, H_2)\big)$, see Fig. 1.2. Notice that $\Omega_0 \subset \Omega \subset \widehat{\Omega}$ for all $\Omega \in \mathcal{O}$.

Clearly $\Omega(\alpha) \in \mathcal{C}^{0,1}$ for all $\alpha \in \mathcal{U}_{ad}$. We will denote the parts of the boundary $\partial\Omega(\alpha)$ as follows (see Fig. 1.1):

$$\begin{aligned}
\Gamma_D &= \Big\{\boldsymbol{x} \in \partial\Omega(\alpha); x_1 = 0 \text{ or } x_1 = L\Big\}\\
\Gamma_{out} &= \Big\{\boldsymbol{x} \in \partial\Omega(\alpha); L_1 \le x_1 \le L_1 + L_2, x_2 = 0\Big\}\\
\Gamma_\alpha &= \Big\{\boldsymbol{x} \in \partial\Omega(\alpha); L_1 \le x_1 \le L_1 + L_2, x_2 = \alpha(x_1)\Big\}\\
\Gamma_f &= \partial\Omega(\alpha) \setminus \big(\Gamma_D \cup \Gamma_{out} \cup \Gamma_\alpha\big).
\end{aligned}$$

The components Γ_D, Γ_{out} and Γ_f are fixed for every $\alpha \in \mathcal{U}_{ad}$.

1.2 Classical formulation of the state problem

The fluid motion in $\Omega(\alpha)$ is described by the generalised Navier–Stokes system

$$\left.\begin{array}{rcl} -\operatorname{div}\ \mathbb{T}(p,\mathbb{D}(\boldsymbol{v})) + \rho\operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{v}) & = & \mathbf{0} \\ \operatorname{div}\ \boldsymbol{v} & = & 0 \end{array}\right\} \text{in } \Omega(\alpha). \tag{1.3}$$

Here $\boldsymbol{v}$ means the velocity, p the pressure, ρ is the density of the fluid and the stress tensor $\mathbb{T}$ is defined by the following formulae:

$$\mathbb{T}(p,\mathbb{D}(\boldsymbol{v})) = -p\mathbb{I} + 2\mu(|\mathbb{D}(\boldsymbol{v})|)\mathbb{D}(\boldsymbol{v}),$$
$$\mu(|\mathbb{D}(\boldsymbol{v})|) := \mu_0 + \mu_t(|\mathbb{D}(\boldsymbol{v})|) = \mu_0 + \rho l_{m,\alpha}^2|\mathbb{D}(\boldsymbol{v})|,$$

where $\mu_0 > 0$ is a constant laminar viscosity and $\mu_t(|\mathbb{D}(\boldsymbol{v})|)$ stands for a turbulent viscosity. The function $l_{m,\alpha}$ represents a mixing length in the algebraic model of turbulence and it has the following form (see [11] for more details):

$$l_{m,\alpha}(\boldsymbol{x}) = \frac{1}{2}\alpha(x_1)\left(0.14 - 0.08\left(1 - \frac{2d_\alpha(\boldsymbol{x})}{\alpha(x_1)}\right)^2 - 0.06\left(1 - \frac{2d_\alpha(\boldsymbol{x})}{\alpha(x_1)}\right)^4\right),$$

where $d_\alpha(\boldsymbol{x}) = \min\{x_2, \alpha(x_1) - x_2\}, \boldsymbol{x} \in \Omega(\alpha)$. In Figure 1.3, $l_{m,\alpha}$ is depicted (for a particular choice of $\alpha \in \mathcal{U}_{ad}$).

The equations are completed by the following boundary conditions:

$$\begin{array}{rcll} \boldsymbol{v} & = & \mathbf{0} & \text{on } \Gamma_f \cup \Gamma_\alpha, \\ \boldsymbol{v} & = & \boldsymbol{v}_D & \text{on } \Gamma_D, \\ \boldsymbol{v}\cdot\boldsymbol{\tau} = v_1 & = & 0 & \text{on } \Gamma_{out}, \\ T_{22} := \mathbb{T}\boldsymbol{\nu}\cdot\boldsymbol{\nu} & = & -\sigma|v_2|v_2 & \text{on } \Gamma_{out}, \end{array} \tag{1.4}$$

where $\boldsymbol{\nu}, \boldsymbol{\tau}$ stands for the unit normal, tangential vector to Γ_{out}, respectively and $\sigma > 0$ is a given suction coefficient. The condition $(1.4)_4$ originates in the homogenization of a complex geometry that is placed on Γ_{out} (for more details we refer to [10]).

By a classical solution we mean any velocity field $\boldsymbol{v} \in (\mathcal{C}^2(\Omega(\alpha)))^2 \cap (\mathcal{C}^1(\overline{\Omega}(\alpha)))^2$ and a pressure $p \in \mathcal{C}^1(\Omega(\alpha)) \cap \mathcal{C}(\overline{\Omega}(\alpha))$ satisfying (1.3) and (1.4).

1.3 Weak formulation of the state problem

Throughout the paper we assume that there exists a function $\boldsymbol{v}_0 \in (W^{1,3}(\Omega_0))^2$, which satisfies the Dirichlet boundary conditions in the sense of traces, i.e.

$$\boldsymbol{v}_0|_{\Gamma_D} = \boldsymbol{v}_D, \quad \boldsymbol{v}_0|_{\partial\Omega_0\setminus(\Gamma_D\cup\Gamma_{\text{out}})} = \mathbf{0}, \quad \boldsymbol{v}_0\cdot\boldsymbol{\tau}|_{\Gamma_{\text{out}}} = 0$$

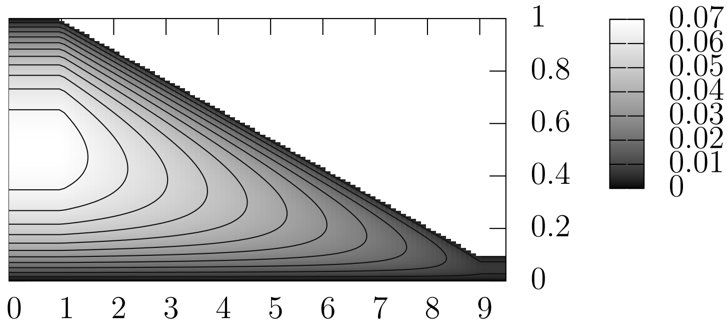

Figure 1.3: Contours of the mixing length $l_{m,\alpha}$ for α linear in $[1,9]$.

and, in addition, $\operatorname{div} \boldsymbol{v}_0 = 0$ in Ω_0. We extend $\boldsymbol{v}_0$ by zero on $\widehat{\Omega} \setminus \overline{\Omega}_0$ so that $\boldsymbol{v}_0 \in (W^{1,3}(\widehat{\Omega}))^2$ and $\operatorname{div} \boldsymbol{v}_0 = 0$ in $\widehat{\Omega}$ (the extended function $\boldsymbol{v}_0$ will be denoted by the same symbol). Note that such $\boldsymbol{v}_0$ is independent of $\alpha \in \mathcal{U}_{ad}$.

1.3.1 Function spaces

For any $\alpha \in \mathcal{U}_{ad}$ we denote

$$\mathcal{V}_0(\alpha) = \Big\{ \boldsymbol{\varphi} = (\varphi_1, \varphi_2) \in \mathcal{C}_0^\infty(\Omega(\alpha)) \times \mathcal{C}^\infty(\overline{\Omega}(\alpha)); \\ \operatorname{dist}(\operatorname{supp}(\varphi_2), \partial\Omega(\alpha) \setminus \Gamma_{\text{out}}) > 0 \Big\}$$

and define the spaces for the velocity

$$W(\alpha) = \overline{(\mathcal{C}^\infty(\overline{\Omega}(\alpha)))^2}^{\|\cdot\|_\alpha}, \tag{1.5}$$

$$\tag{1.6}$$

$$W_0(\alpha) = \overline{\mathcal{V}_0(\alpha)}^{\|\cdot\|_\alpha}, \tag{1.7}$$

where the closure is taken in the norm

$$\begin{aligned} \|\boldsymbol{v}\|_\alpha &:= \|\boldsymbol{v}\|_{1,2,\Omega(\alpha)} + \|M_\alpha \mathbb{D}(\boldsymbol{v})\|_{3,\Omega(\alpha)} + \|\operatorname{div} \boldsymbol{v}\|_{3,\Omega(\alpha)} \\ &= \|\boldsymbol{v}\|_{1,2,\Omega(\alpha)} + \big(\textstyle\sum_{i,j=1}^2 \|M_\alpha \mathbb{D}(\boldsymbol{v})_{ij}\|_{3,\Omega(\alpha)}^3 \big)^{1/3} + \|\operatorname{div} \boldsymbol{v}\|_{3,\Omega(\alpha)}, \\ M_\alpha(\boldsymbol{x}) &:= \big(l_{m,\alpha}(\boldsymbol{x}) \big)^{2/3}, \ \boldsymbol{x} \in \overline{\Omega}(\alpha). \end{aligned}$$

Finally, let

$$W_{\boldsymbol{v}_0}(\alpha) = \big\{\boldsymbol{v} \in W(\alpha);\ \boldsymbol{v} - \boldsymbol{v}_0 \in W_0(\alpha)\big\}.$$

We say that $\boldsymbol{v} \in W(\alpha)$ satisfies the stable boundary conditions $(1.4)_{1-2}$ in the weak sense iff $\boldsymbol{v} \in W_{\boldsymbol{v}_0}(\alpha)$.

Remark 1.1. *It is very easy to verify that the norms* $\|\cdot\|_\alpha$ *and* $\|\cdot\|_{1,2,\Omega(\alpha)} + \|M_\alpha|\mathbb{D}(\cdot)|\|_{3,\Omega(\alpha)} + \|\operatorname{div}\cdot\|_{3,\Omega(\alpha)}$ *are equivalent in* $W(\alpha)$*.*

Remark 1.2. *Due to the Friedrichs inequality, the seminorm*

$$|\boldsymbol{v}|_\alpha := \|\nabla \boldsymbol{v}\|_{2,\Omega(\alpha)} + \|M_\alpha \mathbb{D}(\boldsymbol{v})\|_{3,\Omega(\alpha)} + \|\operatorname{div}\boldsymbol{v}\|_{3,\Omega(\alpha)}$$

is a norm in $W_0(\alpha)$*, which is equivalent with* $\|\boldsymbol{v}\|_\alpha$*.*

Further properties of the spaces $W(\alpha)$ and $W_0(\alpha)$ such as reflexivity and separability are studied in Appendix B.3.

Remark 1.3. *Since* $M_\alpha = 0$ *on* $\partial\Omega(\alpha) \setminus \Gamma_D$*, it can be extended by zero on* $\widehat{\Omega} \setminus \overline{\Omega}(\alpha)$*. The resulting function, which is continuous in* $\overline{\widehat{\Omega}}$ *and which will be used in the next analysis, will be denoted by* $\widetilde{M}_\alpha$*. The same convention of notation will hold for the function* $l_{m,\alpha}$*.*

The following lemma is needed in order to prove a useful relation between the functions $\alpha \in \mathcal{U}_{ad}$ and $l_{m,\alpha}$.

Lemma 1.1. *Let* (X_1, ρ_1), (X_2, ρ_2), (X_3, ρ_3) *be metric spaces and consider functions* $f_n : X_1 \to X_2$, $n \in \mathbb{N}$, $f : X_1 \to X_2$, $g : X_2 \to X_3$ *such that* g *is uniformly continuous in* X_2*, i.e.*

$$\forall \delta > 0\ \exists \eta > 0 : \ \rho_2(y_1, y_2) < \eta \Rightarrow \rho_3(g(y_1), g(y_2)) < \delta,$$

and

$$f_n \rightrightarrows f \ \text{ in } X_1.$$

Then

$$g \circ f_n \rightrightarrows g \circ f \ \text{ in } X_1.$$

Proof. Choose $\delta > 0$. Then there exists $\eta > 0$ such that for every $y_1, y_2 \in X_2$, $\rho_2(y_1, y_2) < \eta$,

$$\rho_3(g(y_1), g(y_2)) < \delta.$$

Further there exists $n_0 \in \mathbb{N}$ such that for every $x \in X_1$ and $n \geq n_0$ it holds:

$$\rho_2(f_n(x), f(x)) < \eta,$$

from which the lemma follows. □

Now we present some important properties of the weight function M_α, which will be used in the further analysis.

Lemma 1.2. *(Some properties of M_α, $\alpha \in \mathcal{U}_{ad}$)*

(i) *M_α is continuous in $\overline{\Omega}(\alpha)$, positive in $\Omega(\alpha)$;*

(ii) *If $\alpha_n \rightrightarrows \alpha$ in $[0, L]$ then $\tilde{M}_{\alpha_n} \rightrightarrows \tilde{M}_\alpha$ in $\overline{\hat{\Omega}}$;*

(iii) *$M_\alpha \approx r_\alpha^{\frac{2}{3}}$, i.e. there exist positive numbers β_1, β_2 such that*

$$\beta_1 r_\alpha^{\frac{2}{3}} \leq M_\alpha \leq \beta_2 r_\alpha^{\frac{2}{3}} \text{ in } \Omega(\alpha),$$

where $r_\alpha(\boldsymbol{x}) := \operatorname{dist}(\boldsymbol{x}, \partial\Omega(\alpha) \setminus \Gamma_D)$. In addition, β_1 and β_2 do not depend on $\alpha \in \mathcal{U}_{ad}$.

Proof. We drop (i) as an easy exercise. Note that $M_\alpha = 0$ on $\partial\Omega \setminus \Gamma_D$.

(ii) Let us define the function l_m by the formulae

$$l_m(\boldsymbol{x}) := \frac{1}{2} x_1 \left(0.14 - 0.08 d^2(\boldsymbol{x}) - 0.06 d^4(\boldsymbol{x})\right),$$

$$d(\boldsymbol{x}) := \left(1 - \frac{2\min\{x_2, x_1 - x_2\}}{x_1}\right),$$

$\boldsymbol{x} \in [\alpha_{min}, \alpha_{max}] \times [0, \alpha_{max}]$. Then, since l_m is continuous in $[\alpha_{min}, \alpha_{max}] \times [0, \alpha_{max}]$, it is uniformly continuous as well. Moreover

$$l_{m,\alpha}(\boldsymbol{x}) = l_m(\alpha(x_1), x_2) \quad \forall \alpha \in \mathcal{U}_{ad}.$$

From this and Lemma 1.1 it follows that

$$\tilde{l}_{m,\alpha_n} \rightrightarrows \tilde{l}_{m,\alpha} \text{ in } \overline{\hat{\Omega}}.$$

(iii) Using that $0 \leq \left(\frac{2d_\alpha(\boldsymbol{x})}{\alpha(x_1)}\right) \leq 1$, $\boldsymbol{x} \in \Omega(\alpha)$ and the inequality

$$0.14t \leq \left(0.14 - 0.08\,(1-t)^2 - 0.06\,(1-t)^4\right) \leq 0.4t \;\; \forall t \in [0,1]$$

we obtain the following estimate:

$$0.14 d_\alpha(\boldsymbol{x}) \leq l_{m,\alpha}(\boldsymbol{x}) \leq 0.4 d_\alpha(\boldsymbol{x}), \;\; \boldsymbol{x} \in \Omega(\alpha). \tag{1.8}$$

From the definition of $\Omega(\alpha)$, $\alpha \in \mathcal{U}_{ad}$ it follows that

$$r_\alpha(\boldsymbol{x}) \leq d_\alpha(\boldsymbol{x}) \leq \sqrt{1+\gamma^2}\, r_\alpha(\boldsymbol{x}), \;\; \boldsymbol{x} \in \Omega(\alpha),$$

which together with (1.8) yields (iii), where the constants are $\beta_1 := 0.14^{\frac{2}{3}}$, $\beta_2 := \left(0.4\sqrt{1+\gamma^2}\right)^{\frac{2}{3}}$. □

Definition 1.1. *Define the operator $A_\alpha : W(\alpha) \to \big(W(\alpha)\big)^*$ by the formula*

$$\langle A_\alpha(\boldsymbol{v}), \boldsymbol{w}\rangle_\alpha := 2\rho \int_{\Omega(\alpha)} M_\alpha^3 |\mathbb{D}(\boldsymbol{v})| \mathbb{D}(\boldsymbol{v}) : \mathbb{D}(\boldsymbol{w}); \ \ \boldsymbol{v}, \boldsymbol{w} \in W(\alpha).$$

Here $\langle \cdot, \cdot \rangle_\alpha$ denotes the duality pairing between $\big(W(\alpha)\big)^*$ and $W(\alpha)$.

The fact that $A_\alpha(\boldsymbol{v}) \in \big(W(\alpha)\big)^*$, $\boldsymbol{v} \in W(\alpha)$, follows from Hölder's inequality (see Appendix B).

Convention. *In what follows we will use the Einstein summation convention, i.e. $a_i b_i := \sum_{i=1}^n a_i b_i$.*

Lemma 1.3. *(Some properties of A_α, $\alpha \in \mathcal{U}_{ad}$)*

(i) *A_α is monotone in $W(\alpha)$ in the following sense:*

$$\langle A_\alpha(\boldsymbol{v}) - A_\alpha(\boldsymbol{w}), \boldsymbol{v} - \boldsymbol{w}\rangle_\alpha \geq C \| M_\alpha \mathbb{D}(\boldsymbol{v} - \boldsymbol{w})\|_{3,\Omega}^3 \ \forall \boldsymbol{v}, \boldsymbol{w} \in W(\alpha),$$

where $C > 0$ is independent of α.

(ii) *A_α is continuous in $W(\alpha)$.*

Proof. (i) We use Lemma A.VI from Appendix A to show that the matrix function $S : \mathbb{A} \mapsto |\mathbb{A}|\mathbb{A}$ is strongly monotone in $\mathbb{R}^{2\times 2}$, i.e.

$$\exists C > 0 : \ \ (S(\mathbb{A}) - S(\mathbb{B})) : (\mathbb{A} - \mathbb{B}) \geq C|\mathbb{A} - \mathbb{B}|^3, \ \forall \mathbb{A}, \mathbb{B} \in \mathbb{R}^{2\times 2}.$$

Indeed, the assumptions of Lemma A.VI are satisfied:

∘ $S(\mathbb{O}) = \mathbb{O}$;

∘ $\frac{\partial S_{ij}(\mathbb{A})}{\partial A_{kl}} = \left\{ \begin{array}{ll} 0 & ; \ \mathbb{A} = \mathbb{O} \\ \frac{A_{ij}^2}{|\mathbb{A}|} + |\mathbb{A}| & ; \ (i,j) = (k,l) \\ \frac{A_{ij} A_{kl}}{|\mathbb{A}|} & ; \ \text{otherwise} \end{array} \right\}$, thus $S \in \mathcal{C}^1(\mathbb{R}^{2\times 2}, \mathbb{R}^{2\times 2})$;

∘ $\frac{\partial S(\mathbb{A})}{\partial \mathbb{A}} : (\mathbb{B} \otimes \mathbb{B}) = \left\{ \begin{array}{ll} 0 & ; \ \mathbb{A} = \mathbb{O} \\ \frac{(\mathbb{A}:\mathbb{B})^2}{|\mathbb{A}|} + |\mathbb{A}||\mathbb{B}|^2 & ; \ \text{otherwise} \end{array} \right\} \geq |\mathbb{A}||\mathbb{B}|^2$, i.e. $r = 3$.

Then we have:

$$\begin{aligned} &\langle A_\alpha(\boldsymbol{v}) - A_\alpha(\boldsymbol{w}), \boldsymbol{v} - \boldsymbol{w}\rangle_\alpha \\ &\quad = 2\rho \int_{\Omega(\alpha)} (S(M_\alpha \mathbb{D}(\boldsymbol{v})) - S(M_\alpha \mathbb{D}(\boldsymbol{w}))) : (M_\alpha \mathbb{D}(\boldsymbol{v} - \boldsymbol{w})) \\ &\qquad \geq C \| M_\alpha \mathbb{D}(\boldsymbol{v} - \boldsymbol{w})\|_{3,\Omega}^3. \end{aligned}$$

(ii) Let $\boldsymbol{v}_n \to \boldsymbol{v}$ in $W(\alpha)$. Then also $M_\alpha |\mathbb{D}(\boldsymbol{v}_n)| \to M_\alpha |\mathbb{D}(\boldsymbol{v})|$ in $L^3(\Omega(\alpha))$. We want to show that $A_\alpha(\boldsymbol{v}_n) \to A_\alpha(\boldsymbol{v})$ in $(W(\alpha))^*$. Indeed:

$$\begin{aligned} &|\langle A_\alpha(\boldsymbol{v}_n) - A_\alpha(\boldsymbol{v}), \boldsymbol{w}\rangle_\alpha| \\ &\quad \leq \left| \int_{\Omega(\alpha)} M_\alpha^3 \left(|\mathbb{D}(\boldsymbol{v}_n)| \mathbb{D}(\boldsymbol{v}_n - \boldsymbol{v}) + (|\mathbb{D}(\boldsymbol{v}_n)| - |\mathbb{D}(\boldsymbol{v})|) \mathbb{D}(\boldsymbol{v}) \right) : \mathbb{D}(\boldsymbol{w}) \right| \\ &\leq \|\boldsymbol{w}\|_\alpha \left(\|\boldsymbol{v}_n\|_\alpha \|\boldsymbol{v}_n - \boldsymbol{v}\|_\alpha + \|\boldsymbol{v}\|_\alpha \|M_\alpha (|\mathbb{D}(\boldsymbol{v}_n)| - |\mathbb{D}(\boldsymbol{v})|)\|_{3,\Omega(\alpha)} \right) \to 0 \end{aligned}$$

holds for every $\boldsymbol{w} \in W(\alpha)$. Therefore

$$\|A_\alpha(\boldsymbol{v}_n) - A_\alpha(\boldsymbol{v})\|_{(W(\alpha))^*} = \sup_{\boldsymbol{w} \in W(\alpha) \setminus \{\boldsymbol{0}\}} \frac{|\langle A_\alpha(\boldsymbol{v}_n) - A_\alpha(\boldsymbol{v}), \boldsymbol{w}\rangle_\alpha|}{\|\boldsymbol{w}\|_\alpha} \to 0.$$

1.3.2 Definition of a weak solution

Now we are ready to give a weak formulation of the state problem. It can be formally derived by multiplying the equations (1.3) by a smooth test function $\boldsymbol{\varphi} \in \mathcal{V}_0(\alpha)$ and integrating over $\Omega(\alpha)$ with the use of the Green theorem. The scalar product in $L^2(\Omega(\alpha))$ will be denoted by $(\cdot,\cdot)_\alpha$ in what follows.

Definition 1.2. *A pair $(\boldsymbol{v}, p) \in W(\alpha) \times L^{\frac{3}{2}}(\Omega(\alpha))$ is said to be a weak solution of the state problem $(\mathcal{P}(\alpha))$ iff*

(i) $\boldsymbol{v} \in W_{\boldsymbol{v}_0}(\alpha)$,

(ii) for every $\boldsymbol{\varphi} \in W_0(\alpha)$ it holds:

$$\begin{aligned} 2\mu_0(\mathbb{D}(\boldsymbol{v}), \mathbb{D}(\boldsymbol{\varphi}))_\alpha + \rho(v_j \frac{\partial v_i}{\partial x_j}, \varphi_i)_\alpha + \langle A_\alpha(\boldsymbol{v}), \boldsymbol{\varphi}\rangle_\alpha \\ + \sigma \int_{\Gamma_{out}} |v_2| v_2 \varphi_2 - (p, \operatorname{div} \boldsymbol{\varphi})_\alpha = 0, \end{aligned}$$

(iii) for every $\psi \in L^{\frac{3}{2}}(\Omega(\alpha))$ it holds: $(\psi, \operatorname{div} \boldsymbol{v})_\alpha = 0$.

Convention. *In the following sections the existence of a weak solution to $(\mathcal{P}(\alpha))$ on a fixed domain $\Omega(\alpha)$, $\alpha \in \mathcal{U}_{ad}$ will be analyzed. Thus for simplicity of notation the letter α in the argument will be usually omitted (we will write $\Omega := \Omega(\alpha)$, $W := W(\alpha)$, $A := A_\alpha$, $(\cdot,\cdot) := (\cdot,\cdot)_\alpha$ etc. in what follows).*

1.4 Existence of a weak solution

Recall that the function $\boldsymbol{v}_0$ is now defined in the whole $\widehat{\Omega}$ and it does not depend on $\alpha \in \mathcal{U}_{ad}$. This fact will be used further in order to establish estimates which are independent of $\alpha \in \mathcal{U}_{ad}$.

1.4.1 The main result

At the beginning of Section 1.4 we state the main existence theorem which will be proved stepwise in the sequel.

Theorem 1.4. *Let*

$$\sigma > \frac{\rho}{2}. \tag{1.9}$$

Then

(i) for every $\alpha \in \mathcal{U}_{ad}$ there exists at least one weak solution of $(\mathcal{P}(\alpha))$;

(ii) there exists a constant $C_E := C_E(\mu_0, \rho, \sigma, \|\nabla \boldsymbol{v}_0\|_{3,\widehat{\Omega}}) > 0$ such that for any weak solution $(\boldsymbol{v}, p)$ of $(\mathcal{P}(\alpha))$, $\alpha \in \mathcal{U}_{ad}$, the following estimate holds:

$$\|\nabla \boldsymbol{v}\|_{2,\Omega}^2 + \|M|\mathbb{D}(\boldsymbol{v})|\|_{3,\Omega}^3 + \|v_2\|_{3,\Gamma_{out}}^3 + \|p\|_{\frac{3}{2},\Omega}^{\frac{3}{2}} \leq C_E. \tag{1.10}$$

In addition the constant C_E does not depend on $\alpha \in \mathcal{U}_{ad}$;

(iii) if $(\boldsymbol{v}, p^1)$ and $(\boldsymbol{v}, p^2)$ are two weak solutions of $(\mathcal{P}(\alpha))$, $\alpha \in \mathcal{U}_{ad}$, then $p^1 = p^2$. Moreover, for $\|\nabla \boldsymbol{v}_0\|_{3,\widehat{\Omega}}$ small enough (independently of $\alpha \in \mathcal{U}_{ad}$) there exists a unique weak solution.

Proof will be done as follows: In Section 1.4.2 we will solve a regularized problem $(\mathcal{P}(\alpha)^\varepsilon)$ using the Galerkin method and apriori estimates. In Section 1.4.3 we will show that solutions of $(\mathcal{P}(\alpha)^\varepsilon)$ converge to a solution of the original problem $(\mathcal{P}(\alpha))$ and in Section 1.4.4 we will discuss uniqueness of the solution.

1.4.2 Regularized problem $(\mathcal{P}(\alpha)^\varepsilon)$

Let us note that $(\mathcal{P}(\alpha))$ has the saddle-point structure. In particular, p is the Lagrange multiplier of the incompressibility constraint $\operatorname{div} \boldsymbol{v} = 0$ in Ω. This structure does not allow us to obtain apriori estimates of $\boldsymbol{v}$ and p simultaneously, using them as test functions.

In general, there are at least two ways of solving saddle-point problems:

1. *Decoupling the system* – one proves first the existence of $\boldsymbol{v}$ and then "reconstructs" p.
2. *Regularization (or penalization)* – the incompressibility constraint is perturbed by adding a small regularizing term such that the resulting system can be solved and such that the additional term vanishes in the limit passage.

Both approaches are more or less equivalent. However, the second one could be sometimes useful for numerical realization.

We will use the second approach, using $\varepsilon|p^\varepsilon|^{-\frac{1}{2}}p^\varepsilon$ as the additional regularizing term, where $(\boldsymbol{v}^\varepsilon, p^\varepsilon)$ denotes a solution of the regularized system and $\varepsilon > 0$ is given. For some reasons, which will be explained in the proof of apriori estimates, we have to add a term containing $\operatorname{div} \boldsymbol{v}^\varepsilon$ to the momentum equation as well. With this choice we obtain apriori estimate of p^ε in the norm of $L^{\frac{3}{2}}(\Omega)$. Moreover, we will show that p^ε can be expressed by means of $\boldsymbol{v}^\varepsilon$ (see (1.44)) and thus eliminated from the system. As a consequence we obtain the formulation with a penalty term for the constraint $\operatorname{div} \boldsymbol{v} = 0$ in Ω, see (1.45).

Let $\varepsilon > 0$ be given. We consider a regularized problem $(\mathcal{P}(\alpha)^\varepsilon)$:

Find $(\boldsymbol{v}^\varepsilon, p^\varepsilon) \in W \times L^{\frac{3}{2}}(\Omega)$ *such that*

(i) $\boldsymbol{v}^\varepsilon \in W_{\boldsymbol{v}_0}$,

(ii) for every $\boldsymbol{\varphi} \in W_0$ it holds:

$$2\mu_0(\mathbb{D}(\boldsymbol{v}^\varepsilon), \mathbb{D}(\boldsymbol{\varphi})) + \rho(v_j^\varepsilon \frac{\partial v_i^\varepsilon}{\partial x_j}, \varphi_i) + \frac{\rho}{2}((\operatorname{div} \boldsymbol{v}^\varepsilon)\,(\boldsymbol{v}^\varepsilon - \boldsymbol{v}_0)\,, \boldsymbol{\varphi})$$
$$+ \langle A(\boldsymbol{v}^\varepsilon), \boldsymbol{\varphi}\rangle + \sigma \int_{\Gamma_{\mathrm{out}}} |v_2^\varepsilon| v_2^\varepsilon \varphi_2 - (p^\varepsilon, \operatorname{div} \boldsymbol{\varphi}) = 0, \quad (1.11)$$

(iii) for every $\psi \in L^{\frac{3}{2}}(\Omega)$ it holds:

$$\varepsilon(\psi, |p^\varepsilon|^{-\frac{1}{2}}p^\varepsilon) + (\psi, \operatorname{div} \boldsymbol{v}^\varepsilon) = 0. \quad (1.12)$$

Remark 1.4. $(\mathcal{P}(\alpha)^\varepsilon)$ *represents a weak formulation of the problem*

$$\left.\begin{aligned} -\operatorname{div} \mathbb{T}(p^\varepsilon, \mathbb{D}(\boldsymbol{v}^\varepsilon)) + \rho(\boldsymbol{v}^\varepsilon \cdot \nabla)\boldsymbol{v}^\varepsilon + \tfrac{\rho}{2}(\operatorname{div} \boldsymbol{v}^\varepsilon)(\boldsymbol{v}^\varepsilon - \boldsymbol{v}_0) &= \mathbf{0} \\ \varepsilon|p^\varepsilon|^{-\frac{1}{2}}p^\varepsilon + \operatorname{div} \boldsymbol{v}^\varepsilon &= 0 \end{aligned}\right\} \text{ in } \Omega$$

with the same boundary conditions as in (1.4).

Existence of $(\boldsymbol{v}^\varepsilon, p^\varepsilon)$ solving $(\mathcal{P}(\alpha)^\varepsilon)$ will be proved using the Galerkin method.

Apriori estimates

An apriori estimate is the keystone of the existence proof. It enables us to prove solvability of the Galerkin system and then passing to the limit, also of the original problem.

In the proof we will need the following useful inequalities:

$$\|\boldsymbol{w}\|_{q,\Omega} \leq C_I(q)\|\boldsymbol{w}\|_{1,2,\Omega},\ q \in [1,\infty), \tag{1.13}$$
$$\|\boldsymbol{w}\|_{1,2,\Omega} \leq C_F\|\nabla \boldsymbol{w}\|_{2,\Omega}, \tag{1.14}$$
$$C_K\|\nabla \boldsymbol{w}\|_{2,\Omega} \leq \|\mathbb{D}(\boldsymbol{w})\|_{2,\Omega}, \tag{1.15}$$

which hold for every $\boldsymbol{w} \in W_{\boldsymbol{v}_0}$. In addition, $C_I(q)$, C_F, $C_K > 0$ are positive constants independent of $\alpha \in \mathcal{U}_{ad}$. Indeed, let us denote for any $q \in [1,\infty)$ the space

$$\tilde{W}^{1,q}(\widehat{\Omega}) := \left\{\boldsymbol{u} \in (W^{1,q}(\widehat{\Omega}))^2,\ \operatorname{Tr}\boldsymbol{u}|_{(0,L)\times\{\alpha_{max}\}} = \boldsymbol{0}\right\}. \tag{1.16}$$

Extending $\boldsymbol{w}$ by zero on $\widehat{\Omega} \setminus \overline{\Omega}$ we see that the resulting function $\tilde{\boldsymbol{w}}$ belongs to $\tilde{W}^{1,2}(\widehat{\Omega})$, in which the imbedding, the Friedrichs and the Korn inequality, respectively, hold with the corresponding constants.

The following lemma helps us to estimate the convective term and will be used in the proof of apriori estimate.

Lemma 1.5. *There exists a constant* $C_c := C_c(\mu_0, \rho, \|\nabla \boldsymbol{v}_0\|_{3,\widehat{\Omega}}) > 0$*, such that for any* $\boldsymbol{w} \in W_{\boldsymbol{v}_0}$ *it holds:*

$$\left|(w_j\frac{\partial v_{0i}}{\partial x_j}, w_i)\right| \leq C_c + \frac{\mu_0 C_K^2}{2\rho}\|\nabla \boldsymbol{w}\|_{2,\Omega}^2 + \frac{1}{3}\|M|\mathbb{D}(\boldsymbol{w})|\|_{3,\Omega}^3, \tag{1.17}$$

where C_K *is from* (1.15)*. Moreover, if* $\|\nabla \boldsymbol{v}_0\|_{3,\widehat{\Omega}}$ *is small enough then* C_c *is also small.*

Proof. First let the domain $\Omega_\delta := \Omega_\delta(\alpha)$, $\delta > 0$ be defined by

$$\Omega_\delta := \{\boldsymbol{x} \in \Omega;\ \operatorname{dist}(\boldsymbol{x}, \partial\Omega \setminus \Gamma_D) > \delta\}$$

(see Figure 1.4). Notice that for δ small enough Ω_δ has the Lipschitz continuous boundary and that

$$|\Omega \setminus \Omega_\delta| \leq C_1\delta, \tag{1.18}$$

where $C_1 > 0$ is independent of δ and $\alpha \in \mathcal{U}_{ad}$, as follows from the definition of $\mathcal{U}_{ad}$.

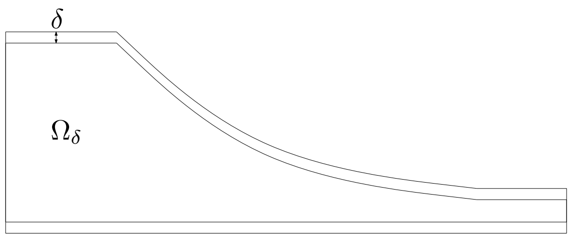

Figure 1.4: Geometry of $\Omega_\delta \subset \Omega$.

We write:

$$(w_j \frac{\partial v_{0i}}{\partial x_j}, w_i) = \int_{\Omega\setminus\Omega_\delta} w_j \frac{\partial v_{0i}}{\partial x_j} w_i + \int_{\Omega_\delta} w_j \frac{\partial v_{0i}}{\partial x_j} w_i.$$

The first term on the right can be estimated as follows:

$$\begin{aligned}\left|\int_{\Omega\setminus\Omega_\delta} w_j \frac{\partial v_{0i}}{\partial x_j} w_i\right| &\leq |\Omega \setminus \Omega_\delta|^{1/3} \|\nabla \boldsymbol{v}_0\|_{3,\widehat{\Omega}} \|\boldsymbol{w}\|^2_{3,\Omega} \\ &\leq (C_1\delta)^{1/3} \|\nabla \boldsymbol{v}_0\|_{3,\widehat{\Omega}} C_I^2(3) C_F^2 \|\nabla \boldsymbol{w}\|^2_{2,\Omega}, \quad (1.19)\end{aligned}$$

using the Hölder inequality, (1.13)-(1.15) and (1.18). Let us fix $\delta > 0$ such that

$$(C_1\delta)^{1/3} \|\nabla \boldsymbol{v}_0\|_{3,\widehat{\Omega}} C_I^2(3) C_F^2 < \frac{\mu_0 C_K^2}{4\rho}. \quad (1.20)$$

Then (1.19) becomes:

$$\left|\int_{\Omega\setminus\Omega_\delta} w_j \frac{\partial v_{0i}}{\partial x_j} w_i\right| \leq \frac{\mu_0 C_K^2}{4\rho} \|\nabla \boldsymbol{w}\|^2_{2,\Omega}. \quad (1.21)$$

Let $\Gamma_\delta := \Gamma_D \cap \partial\Omega_\delta$. We claim that for every $\eta > 0$ there exists a constant $C_2 := C_2(\eta) > 0$ independent of $\alpha \in \mathcal{U}_{ad}$ such that the inequality

$$\|\boldsymbol{w}\|_{3,\Omega_\delta} \leq C_2 \left(\|\boldsymbol{w}\|_{3,\Gamma_\delta} + \|\mathbb{D}(\boldsymbol{w})\|_{3,\Omega_\delta}\right) + \eta \|\nabla \boldsymbol{w}\|_{2,\Omega} \quad (1.22)$$

holds for every $\boldsymbol{w} \in W_{\boldsymbol{v}_0}$. The estimate (1.22) with a suitable choice of η will be used to handle the second integral:

$$\begin{aligned}\left|\int_{\Omega_\delta} w_j \frac{\partial v_{0i}}{\partial x_j} w_i\right| &\leq \|\nabla \boldsymbol{v}_0\|_{3,\widehat{\Omega}} \|\boldsymbol{w}\|^2_{3,\Omega_\delta} \\ &\leq \|\nabla \boldsymbol{v}_0\|_{3,\widehat{\Omega}} \left(C_2\left(\|\boldsymbol{w}\|_{3,\Gamma_\delta} + \|\mathbb{D}(\boldsymbol{w})\|_{3,\Omega_\delta}\right) + \eta\|\nabla \boldsymbol{w}\|_{2,\Omega}\right)^2 \\ &\qquad (\boldsymbol{w}_{|\Gamma_\delta} = \boldsymbol{v}_{0|\Gamma_\delta} \text{ and Young's ineq. }) \\ &\leq \|\nabla \boldsymbol{v}_0\|_{3,\widehat{\Omega}} 3C_2^2 \left(\|\boldsymbol{v}_0\|^2_{3,\Gamma_\delta} + \|\mathbb{D}(\boldsymbol{w})\|^2_{3,\Omega_\delta}\right) + 3\eta^2 \|\nabla \boldsymbol{v}_0\|_{3,\widehat{\Omega}} \|\nabla \boldsymbol{w}\|^2_{2,\Omega} \\ &\qquad \text{(trace theorem and Hölder's ineq.)} \\ &\leq C_3 \|\nabla \boldsymbol{v}_0\|^3_{3,\widehat{\Omega}} + 3C_2^2 \|\nabla \boldsymbol{v}_0\|_{3,\widehat{\Omega}} \|M^{-2}\|_{\infty,\Omega_\delta} \|M\mathbb{D}(\boldsymbol{w})\|^2_{3,\Omega_\delta} + 3\eta^2 \|\nabla \boldsymbol{v}_0\|_{3,\widehat{\Omega}} \|\nabla \boldsymbol{w}\|^2_{2,\Omega}.\end{aligned}$$

From (iii) of Lemma 1.2 we know that there is a constant $\beta_1 > 0$ independent of $\alpha \in \mathcal{U}_{ad}$ such that $M_{|\Omega_\delta} \geq \beta_1 \operatorname{dist}(\cdot, \partial\Omega \setminus \Gamma_D)^{\frac{2}{3}}_{|\Omega_\delta} \geq \beta_1 \delta^{\frac{2}{3}}$, thus

$$\|M^{-2}\|_{\infty,\Omega_\delta} \leq \beta_1^{-2}\delta^{-\frac{4}{3}}.$$

Therefore

$$\begin{aligned}\left|\int_{\Omega_\delta} w_j \frac{\partial v_{0i}}{\partial x_j} w_i\right| &\leq C_3\|\nabla \boldsymbol{v}_0\|^3_{3,\widehat{\Omega}} + 3C_2^2\|\nabla \boldsymbol{v}_0\|_{3,\widehat{\Omega}}\beta_1^{-2}\delta^{-\frac{4}{3}}2^{\frac{2}{3}}\frac{\|M\mathbb{D}(\boldsymbol{w})\|^2_{3,\Omega}}{2^{\frac{2}{3}}} \\ &\quad + 3\eta^2\|\nabla \boldsymbol{v}_0\|_{3,\widehat{\Omega}}\|\nabla \boldsymbol{w}\|^2_{2,\Omega} \\ &\qquad \text{(Young's ineq.)} \\ &\leq (C_3 + 36C_2^6\beta_1^{-6}\delta^{-4})\|\nabla \boldsymbol{v}_0\|^3_{3,\widehat{\Omega}} + \frac{1}{3}\|M\mathbb{D}(\boldsymbol{w})\|^3_{3,\Omega} \\ &\quad + 3\eta^2\|\nabla \boldsymbol{v}_0\|_{3,\widehat{\Omega}}\|\nabla \boldsymbol{w}\|^2_{2,\Omega}. \end{aligned} \tag{1.23}$$

Denoting $C_c := (C_3 + 36C_2^6\beta_1^{-6}\delta^{-4})\|\nabla \boldsymbol{v}_0\|^3_{3,\widehat{\Omega}}$, choosing $\eta > 0$ such that

$$3\eta^2\|\nabla \boldsymbol{v}_0\|_{3,\widehat{\Omega}} \leq \frac{\mu_0 C_K^2}{4\rho}$$

and using (1.21) we arrive at (1.17). Since δ depends on μ_0 and ρ, as seen from (1.20), so does C_c.

Finally, let us assume that $\|\nabla \boldsymbol{v}_0\|_{3,\widehat{\Omega}}$ is small enough so that (1.20) holds for $\delta := 1$. Then (1.23) reads:

$$\begin{aligned}\left|\int_{\Omega_\delta} w_j \frac{\partial v_{0i}}{\partial x_j} w_i\right| &\leq (C_3 + 36C_2^6\beta_1^{-6})\|\nabla \boldsymbol{v}_0\|^3_{3,\widehat{\Omega}} + \frac{1}{3}\|M\mathbb{D}(\boldsymbol{w})\|^3_{3,\Omega} \\ &\quad + 3\eta^2\|\nabla \boldsymbol{v}_0\|_{3,\widehat{\Omega}}\|\nabla \boldsymbol{w}\|^2_{2,\Omega}, \end{aligned} \tag{1.24}$$

from which we see that $C_c = (C_3 + 36C_2^6\beta_1^{-6})\|\nabla \boldsymbol{v}_0\|^3_{3,\widehat{\Omega}}$ can be arbitrarily small, provided that $\|\nabla \boldsymbol{v}_0\|_{3,\widehat{\Omega}}$ is small enough.

To complete the proof, (1.22) has to be proven. Assume for contradiction that $\exists \eta > 0\ \forall n \in \mathbb{N}\ \exists \alpha_n \in \mathcal{U}_{ad}\ \exists \boldsymbol{w}_n \in W_{\boldsymbol{v}_0}(\alpha_n)$:

$$\|\boldsymbol{w}_n\|_{3,\Omega_\delta(\alpha_n)} > n(\|\mathbb{D}(\boldsymbol{w}_n)\|_{3,\Omega_\delta(\alpha_n)} + \|\boldsymbol{w}_n\|_{3,\Gamma_\delta}) + \eta\|\nabla \boldsymbol{w}_n\|_{2,\Omega(\alpha_n)} \tag{1.25}$$

and, additionally, $\|\boldsymbol{w}_n\|_{3,\Omega_\delta(\alpha_n)} = 1$. Due to compactness of $\mathcal{U}_{ad}$ (see Lemma 2.1), we may assume without loss of generality that $\alpha_n \rightrightarrows \alpha \in \mathcal{U}_{ad}$. Then, for n sufficiently large, we have $|\alpha_n - \alpha| < \frac{\delta}{2}$ from which the inclusions

$$\Omega_\delta(\alpha_n) \subset \Omega_{\frac{\delta}{2}}(\alpha) \subset \Omega(\alpha_n)$$

follow. Let Ω' be an open set such that $\Omega' \subset \Omega_\delta(\alpha_n)$ for all n.

From the above inclusions and (1.25) we easily obtain the estimates:

$$\|\boldsymbol{w}_n\|_{3,\Omega'} \leq \|\boldsymbol{w}_n\|_{3,\Omega_\delta(\alpha_n)} = 1,$$

$$\|\mathbb{D}(\boldsymbol{w}_n)\|_{2,\Omega_{\frac{\delta}{2}}(\alpha)} \leq \|\mathbb{D}(\boldsymbol{w}_n)\|_{2,\Omega(\alpha_n)} \leq \|\nabla \boldsymbol{w}_n\|_{2,\Omega(\alpha_n)} < \frac{1}{\eta}.$$

Now we use Korn's inequality (B.3) (see Appendix B) and obtain:

$$\|\boldsymbol{w}_n\|_{1,2,\Omega_{\frac{\delta}{2}}(\alpha)} \leq C_4(\|\mathbb{D}(\boldsymbol{w}_n)\|_{2,\Omega_{\frac{\delta}{2}}(\alpha)} + \|\boldsymbol{w}_n\|_{3,\Omega'}) \leq C_4(1+\frac{1}{\eta}), \tag{1.26}$$

with a fixed positive constant C_4. Thus there exists $\boldsymbol{w} \in (W^{1,2}(\Omega_{\frac{\delta}{2}}(\alpha)))^2$ and a subsequence of $\{\boldsymbol{w}_n\}$ (denoted by the same index n) such that

$$\boldsymbol{w}_n \rightharpoonup \boldsymbol{w} \text{ weakly in } W^{1,2}(\Omega_{\frac{\delta}{2}}(\alpha)), \tag{1.27}$$

$$\boldsymbol{w}_n \to \boldsymbol{w} \text{ strongly in } L^3(\Omega_{\frac{\delta}{2}}(\alpha)), \tag{1.28}$$

as follows from the imbedding theorem. Since $\Omega_\delta(\alpha) \subset \Omega_{\frac{\delta}{2}}(\alpha)$, it also holds that $\|\boldsymbol{w}_n\|_{3,\Omega_\delta(\alpha)} \to \|\boldsymbol{w}\|_{3,\Omega_\delta(\alpha)}$, meaning that

$$\|\boldsymbol{w}\|_{3,\Omega_\delta(\alpha)} = 1. \tag{1.29}$$

On the other hand, the inequality (1.25) yields that $\mathbb{D}(\boldsymbol{w}_n) \to 0$ strongly in $L^3(K)$ for every $K \subset \overline{K} \subset \Omega_\delta(\alpha)$, consequently $\mathbb{D}(\boldsymbol{w}_n) \to 0$ a.e. in $\Omega_\delta(\alpha)$. This and the estimate (1.26) enable us to use the Vitali theorem A.V from Appendix A to prove that $\mathbb{D}(\boldsymbol{w}_n) \to 0$ in $L^{2-\lambda}(\Omega_\delta(\alpha))$ for any $\lambda \in (0,1)$. Therefore, using also (1.27), we have that $\mathbb{D}(\boldsymbol{w}) = 0$ a.e. in $\Omega_\delta(\alpha)$. Similarly we can show that $\mathrm{Tr}\, \boldsymbol{w}_{|\Gamma_\delta} = \mathbf{0}$, consequently $\boldsymbol{w} = \mathbf{0}$ a.e. in $\Omega_\delta(\alpha)$, which contradicts to (1.29). □

Let us emphasize that the constant C_c in (1.17) is independent of $\alpha \in \mathcal{U}_{ad}$. This fact will be used for establishing apriori estimates which are independent of $\alpha \in \mathcal{U}_{ad}$.

Theorem 1.6. *Let*

$$\sigma > \frac{\rho}{2}. \tag{1.30}$$

Then there exists a constant $C_E := C_E(\mu_0, \rho, \sigma, \|\nabla \boldsymbol{v}_0\|_{3,\widehat{\Omega}}) > 0$ *independent of* $\varepsilon > 0$ *and* $\alpha \in \mathcal{U}_{ad}$ *such that for any solution* $(\boldsymbol{v}^\varepsilon, p^\varepsilon)$ *of* $\left(\mathcal{P}(\alpha)^\varepsilon\right)$ *the following apriori estimate holds:*

$$\|\nabla \boldsymbol{v}^\varepsilon\|_{2,\Omega}^2 + \|M|\mathbb{D}(\boldsymbol{v}^\varepsilon)|\|_{3,\Omega}^3 + \|v_2^\varepsilon\|_{3,\Gamma_{out}}^3 + \varepsilon\|p^\varepsilon\|_{\frac{3}{2},\Omega}^{\frac{3}{2}} \leq C_E. \tag{1.31}$$

Proof. Let $(\boldsymbol{v}^\varepsilon, p^\varepsilon)$ be a solution of $(\mathcal{P}(\alpha)^\varepsilon)$. Choose $\boldsymbol{\varphi} := \boldsymbol{v}^\varepsilon - \boldsymbol{v}_0$ and $\psi := p^\varepsilon$. Then (1.12) and (1.11) give:

$$\varepsilon \|p^\varepsilon\|_{\frac{3}{2},\Omega}^{\frac{3}{2}} = -(p^\varepsilon, \operatorname{div} \boldsymbol{v}^\varepsilon), \tag{1.32}$$

$$\begin{aligned} 2\mu_0(\mathbb{D}(\boldsymbol{v}^\varepsilon), \mathbb{D}(\boldsymbol{v}^\varepsilon - \boldsymbol{v}_0)) + \rho(v_j^\varepsilon \frac{\partial v_i^\varepsilon}{\partial x_j}, v_i^\varepsilon - v_{0i}) + \frac{\rho}{2}((\operatorname{div} \boldsymbol{v}^\varepsilon)\,(\boldsymbol{v}^\varepsilon - \boldsymbol{v}_0)\,, \boldsymbol{v}^\varepsilon - \boldsymbol{v}_0) \\ + \langle A(\boldsymbol{v}^\varepsilon), \boldsymbol{v}^\varepsilon - \boldsymbol{v}_0\rangle + \sigma \int_{\Gamma_{\text{out}}} |v_2^\varepsilon| v_2^\varepsilon\,(v_2^\varepsilon - v_{02}) + \varepsilon \|p^\varepsilon\|_{\frac{3}{2},\Omega}^{\frac{3}{2}} = 0, \end{aligned} \tag{1.33}$$

making use (1.32) and $\operatorname{div} \boldsymbol{v}_0 = 0$ in $\widehat{\Omega}$. Now we rearrange the terms in the previous identity:

$$\begin{aligned} &2\mu_0 \|\mathbb{D}(\boldsymbol{v}^\varepsilon)\|_{2,\Omega}^2 + 2\rho \|M|\mathbb{D}(\boldsymbol{v}^\varepsilon)|\|_{3,\Omega}^3 + \sigma \|v_2^\varepsilon\|_{3,\Gamma_{\text{out}}}^3 + \varepsilon \|p^\varepsilon\|_{\frac{3}{2},\Omega}^{\frac{3}{2}} \\ &= 2\mu_0(\mathbb{D}(\boldsymbol{v}^\varepsilon), \mathbb{D}(\boldsymbol{v}_0)) - \rho(v_j^\varepsilon \frac{\partial v_i^\varepsilon}{\partial x_j}, v_i^\varepsilon - v_{0i}) - \frac{\rho}{2}((\operatorname{div} \boldsymbol{v}^\varepsilon)\,(\boldsymbol{v}^\varepsilon - \boldsymbol{v}_0)\,, \boldsymbol{v}^\varepsilon - \boldsymbol{v}_0) \\ &\qquad + \langle A(\boldsymbol{v}^\varepsilon), \boldsymbol{v}_0\rangle + \sigma \int_{\Gamma_{\text{out}}} |v_2^\varepsilon| v_2^\varepsilon v_{02} =: E_1 + \ldots + E_5. \end{aligned} \tag{1.34}$$

In the rest of the proof we will estimate the terms $E_1, \ldots, E_5$. Hölder's and Young's inequality give:

$$|E_1| \le \mu_0 \|\mathbb{D}(\boldsymbol{v}^\varepsilon)\|_{2,\Omega}^2 + \mu_0 \|\mathbb{D}(\boldsymbol{v}_0)\|_{2,\Omega}^2, \tag{1.35}$$

$$|E_4| \le \frac{4}{3}\rho \|M|\mathbb{D}(\boldsymbol{v}^\varepsilon)|\|_{3,\Omega}^3 + \frac{2}{3}\rho \|M|\mathbb{D}(\boldsymbol{v}_0)|\|_{3,\Omega}^3, \tag{1.36}$$

$$|E_5| \le \delta_5 \|v_2^\varepsilon\|_{3,\Gamma_{\text{out}}}^3 + C_{\delta_5} \|v_{02}\|_{3,\Gamma_{\text{out}}}^3, \tag{1.37}$$

where $\delta_5 > 0$ is arbitrary and $C_{\delta_5} > 0$ depends only on δ_5 and σ. Next we rewrite E_2:

$$\begin{aligned} E_2 = -\rho(v_j^\varepsilon \frac{\partial (v_i^\varepsilon - v_{0i})}{\partial x_j}, v_i^\varepsilon - v_{0i}) - \rho(v_j^\varepsilon \frac{\partial v_{0i}}{\partial x_j}, v_i^\varepsilon) + \rho(v_j^\varepsilon \frac{\partial v_{0i}}{\partial x_j}, v_{0i}) \\ =: E_{21} + E_{22} + E_{23}. \end{aligned} \tag{1.38}$$

Applying Green's theorem to the first term we obtain:

$$E_{21} = -\frac{\rho}{2}\int_\Omega v_j^\varepsilon \frac{\partial}{\partial x_j}\left(|\boldsymbol{v}^\varepsilon - \boldsymbol{v}_0|^2\right) =$$

(Green's thm.)

$$= -\frac{\rho}{2}\int_{\partial\Omega}(\boldsymbol{v}^\varepsilon\cdot\boldsymbol{\nu})|\boldsymbol{v}^\varepsilon - \boldsymbol{v}_0|^2 + \frac{\rho}{2}(\operatorname{div}\boldsymbol{v}^\varepsilon(\boldsymbol{v}^\varepsilon - \boldsymbol{v}_0), \boldsymbol{v}^\varepsilon - \boldsymbol{v}_0)$$

$$= \frac{\rho}{2}\int_{\Gamma_{\mathrm{out}}} v_2^\varepsilon|v_2^\varepsilon - v_{02}|^2 - E_3 = \frac{\rho}{2}\int_{\Gamma_{\mathrm{out}}} v_2^\varepsilon\left(|v_2^\varepsilon|^2 - 2v_2^\varepsilon v_{02} + |v_{02}|^2\right) - E_3 \leq$$

(Young's ineq.)

$$\leq \frac{\rho}{2}(1+\delta_1)\|v_2^\varepsilon\|_{3,\Gamma_{\mathrm{out}}}^3 + C_{\delta_1}\|v_{02}\|_{3,\Gamma_{\mathrm{out}}}^3 - E_3, \quad (1.39)$$

where $\delta_1 > 0$ is arbitrary and $C_{\delta_1} > 0$ depends only on δ_1 and ρ. Due to Lemma 1.5 we have:

$$|E_{22}| \leq \rho C_c + \frac{\mu_0 C_K^2}{2}\|\nabla\boldsymbol{v}^\varepsilon\|_{2,\Omega}^2 + \frac{\rho}{3}\|M|\mathbb{D}(\boldsymbol{v}^\varepsilon)|\|_{3,\Omega}^3. \quad (1.40)$$

Finally we make use of imbedding and Friedrichs' inequality in $W^{1,3}(\widehat{\Omega})$ to obtain:

$$|E_{23}| \;\leq\; \rho\|\boldsymbol{v}_0\|_{3,\widehat{\Omega}}\|\nabla\boldsymbol{v}_0\|_{3,\widehat{\Omega}}\|\boldsymbol{v}^\varepsilon\|_{3,\Omega} \;\leq\; \delta_3\|\nabla\boldsymbol{v}^\varepsilon\|_{2,\Omega}^2 + C_{\delta_3}\|\nabla\boldsymbol{v}_0\|_{3,\widehat{\Omega}}^4 \quad (1.41)$$

where $\delta_3 > 0$ and $C_{\delta_3} > 0$ depends only on δ_3 and ρ.

Altogether, (1.38), (1.39), (1.40) and (1.41) yield:

$$|E_2 + E_3| \leq \frac{\rho}{2}(1+\delta_1)\|v_2^\varepsilon\|_{3,\Gamma_{\mathrm{out}}}^3 + \left(\delta_3 + \frac{\mu_0 C_K^2}{2}\right)\|\nabla\boldsymbol{v}^\varepsilon\|_{2,\Omega}^2$$
$$+ \frac{\rho}{3}\|M|\mathbb{D}(\boldsymbol{v}^\varepsilon)|\|_{3,\Omega}^3 + \rho C_c + C_{\delta_1}\|v_{02}\|_{3,\Gamma_{\mathrm{out}}}^3 + C_{\delta_3}\|\nabla\boldsymbol{v}_0\|_{3,\widehat{\Omega}}^4. \quad (1.42)$$

Using the estimate of $E_1, \ldots, E_5$ and putting all the terms containing $\boldsymbol{v}^\varepsilon$ from the right hand side of (1.34) to its left we obtain:

$$\left(\frac{\mu_0 C_K^2}{2} - \delta_3\right)\|\nabla\boldsymbol{v}^\varepsilon\|_{2,\Omega}^2 + \rho\|M|\mathbb{D}(\boldsymbol{v}^\varepsilon)|\|_{3,\Omega}^3$$
$$+ \left(\sigma - \frac{\rho}{2}(1+\delta_1) - \delta_5\right)\|v_2^\varepsilon\|_{3,\Gamma_{\mathrm{out}}}^3 + \varepsilon\|p^\varepsilon\|_{\frac{3}{2},\Omega}^{\frac{3}{2}}$$
$$\leq \text{ terms containing solely } \boldsymbol{v}_0 \;\leq C_E. \quad (1.43)$$

Here C_K stands for the constant of Korn's inequality from (1.15). We also use the fact that all terms on the right hand side can be collectively estimated by an expression $C_E := C_E(\mu_0, \rho, \sigma, \|\nabla \boldsymbol{v}_0\|_{3,\widehat{\Omega}})$. We choose $\delta_3 < \frac{\mu_0 C_K^2}{2}$, δ_1 and δ_5 in such a way that

$$\delta_5 + \frac{\rho}{2}(1+\delta_1) < \sigma$$

and fix them (here we used the assumption $\sigma > \frac{\rho}{2}$). Hence we arrive at (1.31). □

Remark 1.5. *Let us comment on the assumption and the statement of Theorem 1.6.*

(i) The condition $\sigma > \frac{\rho}{2}$ can be possibly satisfied by adjusting the outflow properties of the headbox.

(ii) From the proof it can be easily seen that the right hand side of (1.43) *is given by the sum*

$$\rho C_c + \mu_0 \|\mathbb{D}(\boldsymbol{v}_0)\|^2_{2,\widehat{\Omega}} + \frac{2}{3}\rho \| M|\mathbb{D}(\boldsymbol{v}_0)| \|^3_{3,\widehat{\Omega}} + (C_{\delta_1} + C_{\delta_5})\|v_{02}\|^3_{3,\Gamma_{out}} + C_{\delta_3}\|\nabla \boldsymbol{v}_0\|^4_{3,\widehat{\Omega}}.$$

Using the fact that all the norms can be estimated by $\|\nabla \boldsymbol{v}_0\|_{3,\widehat{\Omega}}$, the constant C_E can be written in the form

$$C_E(\mu_0, \rho, \sigma, \|\nabla \boldsymbol{v}_0\|_{3,\widehat{\Omega}}) = C_1(\mu_0)\|\nabla \boldsymbol{v}_0\|^2_{3,\widehat{\Omega}} + C_2(\mu_0,\rho,\sigma)\|\nabla \boldsymbol{v}_0\|^3_{3,\widehat{\Omega}} + C_3(\rho)\|\nabla \boldsymbol{v}_0\|^4_{3,\widehat{\Omega}},$$

where $C_1, C_2, C_3 > 0$ depend only on the indicated parameters, implying that

$$\lim_{t\to 0+} C_E(\mu_0, \rho, \sigma, t) = 0.$$

This property will be used in the proof of uniqueness of the solution to $(\mathcal{P}(\alpha))$.

(iii) Let us recall once more that C_E is independent of $\varepsilon > 0$ and $\alpha \in \mathcal{U}_{ad}$. This is important in order to be able to pass to the limit with respect to these parameters.

Solvability of the Galerkin approximation of the regularized problem

Let us observe that from (1.12) one can express the approximate pressure:

$$p^\varepsilon = -\frac{1}{\varepsilon^2}|\operatorname{div} \boldsymbol{v}^\varepsilon| \operatorname{div} \boldsymbol{v}^\varepsilon. \tag{1.44}$$

Therefore p^ε can be eliminated from $(\mathcal{P}(\alpha)^\varepsilon)$ and the regularized velocity $\boldsymbol{v}^\varepsilon$ is given by the equation

$$2\mu_0(\mathbb{D}(\boldsymbol{v}^\varepsilon), \mathbb{D}(\boldsymbol{\varphi})) + \rho(v_j^\varepsilon \frac{\partial v_i^\varepsilon}{\partial x_j}, \varphi_i) + \frac{\rho}{2}((\operatorname{div} \boldsymbol{v}^\varepsilon)(\boldsymbol{v}^\varepsilon - \boldsymbol{v}_0), \boldsymbol{\varphi}) \\ + \langle A(\boldsymbol{v}^\varepsilon), \boldsymbol{\varphi}\rangle + \sigma \int_{\Gamma_{\text{out}}} |v_2^\varepsilon| v_2^\varepsilon \varphi_2 + \frac{1}{\varepsilon^2}(|\operatorname{div} \boldsymbol{v}^\varepsilon| \operatorname{div} \boldsymbol{v}^\varepsilon, \operatorname{div} \boldsymbol{\varphi}) = 0, \ \forall \boldsymbol{\varphi} \in W_0. \tag{1.45}$$

Convention. *In this subsection the parameter ε will be fixed. Thus, for simplicity of notation we will write $\boldsymbol{v}$ instead of $\boldsymbol{v}^\varepsilon$, p instead of p^ε etc. For the same reason we also assume that $2\mu_0 = \rho = \sigma = 1$ in what follows.*

Let $\{\boldsymbol{\omega}^s\}_{s=1}^{\infty}$ be a dense set in W_0 of linearly independent functions and denote its finite-dimensional subspace

$$K_N := \operatorname{span}\{\boldsymbol{\omega}^1, \dots, \boldsymbol{\omega}^N\}.$$

For every $N = 1, 2, \dots$ we solve the Galerkin problem:
Find $\boldsymbol{v}^N \in W$ such that

- $\boldsymbol{v}^N - \boldsymbol{v}_0 \in K_N$,
- $\forall \boldsymbol{\varphi} \in K_N$

$$(\mathbb{D}(\boldsymbol{v}^N), \mathbb{D}(\boldsymbol{\varphi})) + (v_j^N \frac{\partial v_i^N}{\partial x_j}, \varphi_i) + \frac{1}{2}\big((\operatorname{div} \boldsymbol{v}^N)\big(\boldsymbol{v}^N - \boldsymbol{v}_0\big), \boldsymbol{\varphi}) \\ + \langle A(\boldsymbol{v}^N), \boldsymbol{\varphi}\rangle + \int_{\Gamma_{out}} |v_2^N| v_2^N \varphi_2 + \frac{1}{\varepsilon^2}(|\operatorname{div} \boldsymbol{v}^N| \operatorname{div} \boldsymbol{v}^N, \operatorname{div} \boldsymbol{\varphi}) = 0. \tag{1.46}$$

Define a mapping $\boldsymbol{P}_N : \mathbb{R}^N \to \mathbb{R}^N$ as follows: For $s = 1, \dots, N$, the s-th component $\boldsymbol{P}_N(\boldsymbol{d}^N)_s$ equals the left hand side of (1.46) with $\boldsymbol{\varphi} := \boldsymbol{\omega}^s$ and

$$\boldsymbol{v}^N(\boldsymbol{x}) := \boldsymbol{v}_0(\boldsymbol{x}) + \sum_{r=1}^{N} d_r^N \boldsymbol{\omega}^r(\boldsymbol{x}), \ \boldsymbol{d}^N = (d_1^N, \dots, d_N^N).$$

Then the Galerkin problem is equivalent to:

$$\textit{Find } \overline{\boldsymbol{d}}^N \in \mathbb{R}^N \textit{ such that } \boldsymbol{P}_N(\overline{\boldsymbol{d}}^N) = \boldsymbol{0}. \tag{1.47}$$

Next we show that this nonlinear algebraic system has a solution by using Brouwer's theorem (see Appendix A, Corollary A.III).

Theorem 1.7. *Under the assumptions of Theorem 1.6, system* (1.47) *has a solution.*

Proof. Clearly, the mapping $\boldsymbol{P}_N$ is continuous. To prove the existence of a solution to (1.47) we need to verify that there exists $R > 0$ such that

$$\forall \boldsymbol{d}^N \in \mathbb{R}^N, |\boldsymbol{d}^N| = R : \ \boldsymbol{P}_N(\boldsymbol{d}^N) \cdot \boldsymbol{d}^N > 0.$$

Using the same technique as in the proof of the apriori estimate (1.31) we obtain:

$$\boldsymbol{P}_N(\boldsymbol{d}^N)\cdot\boldsymbol{d}^N \geq C\left(\|\nabla\boldsymbol{v}^N\|_{2,\Omega}^2 + \|M|\mathbb{D}(\boldsymbol{v}^N)|\|_{3,\Omega}^3 + \int_{\Gamma_{\text{out}}} |v_2^N|^3 + \frac{1}{\varepsilon^2}\|\operatorname{div}\boldsymbol{v}^N\|_{3,\Omega}^3\right) \\ - C_E(\|\nabla\boldsymbol{v}_0\|_{3,\widehat{\Omega}}) \geq C\|\nabla\boldsymbol{v}^N\|_{2,\Omega}^2 - C_E(\|\nabla\boldsymbol{v}_0\|_{3,\widehat{\Omega}})$$

where $C > 0$ is independent of ε. For $|\boldsymbol{d}^N|$ large enough the last term is positive. Indeed:

$$\|\nabla\boldsymbol{v}^N\|_{2,\Omega}^2 = \\ = \|\nabla\boldsymbol{v}_0\|_{2,\Omega}^2 + 2\sum_{r=1}^N d_r^N(\nabla\boldsymbol{v}_0, \nabla\boldsymbol{\omega}^r) + \sum_{r,s=1}^N d_r^N d_s^N(\nabla\boldsymbol{\omega}^r, \nabla\boldsymbol{\omega}^s) \\ \geq \|\nabla\boldsymbol{v}_0\|_{2,\Omega}^2 + 2\sum_{r=1}^N d_r^N(\nabla\boldsymbol{v}_0, \nabla\boldsymbol{\omega}^r) + \beta|\boldsymbol{d}^N|^2 \to \infty \text{ as } |\boldsymbol{d}^N| \to \infty.$$

Here we used the fact that the Gramm matrix of the linearly independent system $\{\boldsymbol{\omega}^s\}_{s=1}^N$ is positive definite with a constant $\beta > 0$.

From Appendix A, Corollary A.III the existence of $\overline{\boldsymbol{d}}^N \in \mathbb{R}^N$ solving (1.47) follows. □

Limit passage $N \to \infty$

Theorem 1.8. *Let the assumptions of Theorem 1.6 hold. Then for every* $\alpha \in \mathcal{U}_{ad}$ *and* $\varepsilon > 0$ *problem* $(\mathcal{P}(\alpha)^\varepsilon)$ *has a solution.*

Proof. Following the proof of Theorem 1.6, one can show that every solution $\boldsymbol{v}^N$ of the Galerkin system satisfies the estimate

$$\|\nabla\boldsymbol{v}^N\|_{2,\Omega}^2 + \|M|\mathbb{D}(\boldsymbol{v}^N)|\|_{3,\Omega}^3 + \|\boldsymbol{v}^N\|_{3,\Gamma_{out}}^3 + \frac{1}{\varepsilon^2}\|\operatorname{div}\boldsymbol{v}^N\|_{3,\Omega}^3 \leq C_E \tag{1.48}$$

where C_E is independent of $\varepsilon > 0$, $\alpha \in \mathcal{U}_{ad}$ and N. For any $\boldsymbol{w} \in W$ denote $d(\boldsymbol{v}^N) := \frac{1}{\varepsilon^2}|\operatorname{div}\boldsymbol{v}^N|\operatorname{div}\boldsymbol{v}^N$. Using this notation, there exist weak limits $\boldsymbol{v} := \boldsymbol{v}^\varepsilon \in (W^{1,2}(\Omega))^2$, $\overline{A} \in W^*$ and $\overline{d} \in L^{\frac{3}{2}}(\Omega)$ such that

$$\boldsymbol{v}^N \rightharpoonup \boldsymbol{v} \text{ in } (W^{1,2}(\Omega))^2, \tag{1.49}$$

$$A(\boldsymbol{v}^N) \rightharpoonup \overline{A} \text{ in } W^*, \tag{1.50}$$

$$d(\boldsymbol{v}^N) \rightharpoonup \overline{d} \text{ in } L^{\frac{3}{2}}(\Omega) \tag{1.51}$$

as $N \to \infty$ (here and later we consider a subsequence of $\{\boldsymbol{v}^N\}$, denoted by the same symbol). Next we show that $\boldsymbol{v} \in W_{\boldsymbol{v}_0}$. By virtue of the compact imbedding of $W^{1,2}(\Omega)$ into $L^q(\Omega)$ and $L^q(\partial\Omega)$, $q \in [1,\infty)$ we have:

$$\boldsymbol{v}^N \to \boldsymbol{v} \text{ in } (L^q(\Omega))^2 \text{ and in } (L^q(\partial\Omega))^2,\ N \to \infty. \tag{1.52}$$

We now show that

$$M\mathbb{D}(\boldsymbol{v}^N) \rightharpoonup M\mathbb{D}(\boldsymbol{v}) \text{ in } (L^3(\Omega))^{2\times 2},\ N \to \infty. \tag{1.53}$$

Indeed, from (1.48) it follows that $M\mathbb{D}(\boldsymbol{v}^N)$ converges weakly to some $\mathbb{A}$ in $(L^3(\Omega))^{2\times 2}$. It means that for any $\mathbb{B} \in (L^{\frac{3}{2}}(\Omega))^{2\times 2}$

$$\int_\Omega M\mathbb{D}(\boldsymbol{v}^N) : \mathbb{B} \to \int_\Omega \mathbb{A} : \mathbb{B}.$$

However for $\mathbb{B} \in (L^2(\Omega))^{2\times 2}$ we have:

$$\int_\Omega M\mathbb{D}(\boldsymbol{v}^N) : \mathbb{B} \to \int_\Omega M\mathbb{D}(\boldsymbol{v}) : \mathbb{B},$$

as follows from (1.49) and the fact that $M \in L^\infty(\Omega)$. Since $(L^2(\Omega))^{2\times 2}$ is dense in $(L^{\frac{3}{2}}(\Omega))^{2\times 2}$, it follows that $\mathbb{A} = M\mathbb{D}(\boldsymbol{v}) \in (L^3(\Omega))^{2\times 2}$. Therefore $\boldsymbol{v} \in W_{\boldsymbol{v}_0}$.

Let $\boldsymbol{\varphi} \in K_J$ where $J \in \mathbb{N}$ is fixed. Then, using (1.49)-(1.52), one can pass to the limit in the Galerkin system so that

$$(\mathbb{D}(\boldsymbol{v}), \mathbb{D}(\boldsymbol{\varphi})) + (v_j \frac{\partial v_i}{\partial x_j}, \varphi_i) + \frac{1}{2}((\operatorname{div}\boldsymbol{v})(\boldsymbol{v} - \boldsymbol{v}_0), \boldsymbol{\varphi})$$
$$+ \langle \overline{A}, \boldsymbol{\varphi}\rangle + \int_{\Gamma_{\text{out}}} |v_2|v_2\varphi_2 + (\overline{d}, \operatorname{div}\boldsymbol{\varphi}) = 0. \tag{1.54}$$

Consequently, (1.54) holds for any $\boldsymbol{\varphi} \in W_0$. It remains to prove that

$$\langle \overline{A}, \boldsymbol{\varphi}\rangle + (\overline{d}, \operatorname{div}\boldsymbol{\varphi}) = \langle A(\boldsymbol{v}), \boldsymbol{\varphi}\rangle + (d(\boldsymbol{v}), \operatorname{div}\boldsymbol{\varphi}).$$

We use the monotonicity of the mappings A and d introduced in Definition 1.1 and (1.51). Let $\boldsymbol{\varphi} \in W$. Then

$$
\begin{aligned}
0 &\le \langle A(\boldsymbol{v}^N) - A(\boldsymbol{\varphi}), \boldsymbol{v}^N - \boldsymbol{\varphi}\rangle + (d(\boldsymbol{v}^N) - d(\boldsymbol{\varphi}), \operatorname{div}(\boldsymbol{v}^N - \boldsymbol{\varphi})) \\
&= -(\mathbb{D}(\boldsymbol{v}^N), \mathbb{D}(\boldsymbol{v}^N - \boldsymbol{v}_0)) - (v_j^N \frac{\partial v_i^N}{\partial x_j}, v_i^N - v_{0i}) - \frac{1}{2}(\left(\operatorname{div} \boldsymbol{v}^N\right)\left(\boldsymbol{v}^N - \boldsymbol{v}_0\right), \boldsymbol{v}^N - \boldsymbol{v}_0) \\
&\quad - \int_{\Gamma_{\text{out}}} |v_2^N| v_2^N (v_2^N - v_{02}) + \langle A(\boldsymbol{v}^N), \boldsymbol{v}_0 - \boldsymbol{\varphi}\rangle + (d(\boldsymbol{v}^N), \operatorname{div}(\boldsymbol{v}_0 - \boldsymbol{\varphi})) \\
&\quad - \langle A(\boldsymbol{\varphi}), \boldsymbol{v}^N - \boldsymbol{\varphi}\rangle - (d(\boldsymbol{\varphi}), \operatorname{div}(\boldsymbol{v}^N - \boldsymbol{\varphi})), \qquad (1.55)
\end{aligned}
$$

making use of (1.46). Letting $N \to \infty$ and using lower semicontinuity of $\|\mathbb{D}(\boldsymbol{v}^N)\|_{2,\Omega}$ and continuity of the remaining terms we obtain:

$$
\begin{aligned}
0 &\le -(\mathbb{D}(\boldsymbol{v}), \mathbb{D}(\boldsymbol{v} - \boldsymbol{v}_0)) - (v_j \frac{\partial v_i}{\partial x_j}, v_i - v_{0i}) - \frac{1}{2}((\operatorname{div} \boldsymbol{v})(\boldsymbol{v} - \boldsymbol{v}_0), \boldsymbol{v} - \boldsymbol{v}_0) \\
&\quad - \int_{\Gamma_{\text{out}}} |v_2| v_2 (v_2 - v_{02}) + \langle \overline{A}, \boldsymbol{v}_0 - \boldsymbol{\varphi}\rangle + (\overline{d}, \operatorname{div}(\boldsymbol{v}_0 - \boldsymbol{\varphi})) \\
&\quad - \langle A(\boldsymbol{\varphi}), \boldsymbol{v} - \boldsymbol{\varphi}\rangle - (d(\boldsymbol{\varphi}), \operatorname{div}(\boldsymbol{v} - \boldsymbol{\varphi})). \qquad (1.56)
\end{aligned}
$$

From (1.54) and (1.56) we arrive at the inequality

$$0 \le \langle \overline{A} - A(\boldsymbol{\varphi}), \boldsymbol{v} - \boldsymbol{\varphi}\rangle + (\overline{d} - d(\boldsymbol{\varphi}), \operatorname{div}(\boldsymbol{v} - \boldsymbol{\varphi})), \qquad (1.57)$$

which holds for any $\boldsymbol{\varphi} \in W$. We now use the so-called Minty trick. Instead of $\boldsymbol{\varphi}$ we insert into (1.57) a function $\boldsymbol{v} \pm \lambda\boldsymbol{\psi}$, where $\lambda > 0$, $\boldsymbol{\psi} \in W$:

$$0 \le \langle \overline{A} - A(\boldsymbol{v} \pm \lambda\boldsymbol{\psi}), \mp\lambda\boldsymbol{\psi}\rangle + (\overline{d} - d(\boldsymbol{v} \pm \lambda\boldsymbol{\psi}), \operatorname{div}(\mp\lambda\boldsymbol{\psi})). \qquad (1.58)$$

Dividing this inequality by λ we obtain for $\lambda \to 0+$:

$$0 \le \pm\left(\langle \overline{A} - A(\boldsymbol{v}), \boldsymbol{\psi}\rangle + (\overline{d} - d(\boldsymbol{v}), \operatorname{div}(\boldsymbol{\psi}))\right), \qquad (1.59)$$

making use of continuity of A and d. Thus

$$\langle \overline{A}, \boldsymbol{\varphi}\rangle + (\overline{d}, \operatorname{div} \boldsymbol{\varphi}) = \langle A(\boldsymbol{v}), \boldsymbol{\varphi}\rangle + (d(\boldsymbol{v}), \operatorname{div} \boldsymbol{\varphi}) \ \forall \boldsymbol{\varphi} \in W_0.$$

Inserting this into (1.54) we arrive at (1.45). Finally we define $p := -d(\boldsymbol{v})$ and conclude that $(\boldsymbol{v}^\varepsilon, p^\varepsilon) := (\boldsymbol{v}, p)$ is a solution of $(\mathcal{P}(\alpha)^\varepsilon)$. □

Uniform estimate of the pressure

Until now we know that the regularized pressure p^ε admits the estimate (see (1.31))

$$\varepsilon \|p^\varepsilon\|^{\frac{3}{2}}_{\frac{3}{2},\Omega} \le C_E.$$

The aim of this subsection is to obtain a similar estimate of p^ε, but independent of ε, i.e. to prove that there exists a constant $C_p > 0$ independent of ε and $\alpha \in \mathcal{U}_{ad}$ such that

$$\|p^\varepsilon\|_{\frac{3}{2},\Omega} \le C_p.$$

If Dirichlet b.c. were prescribed on the whole $\partial\Omega$, one could construct a special test function $\boldsymbol{\varphi}^\varepsilon$ which satisfies:

(i) $\boldsymbol{\varphi}^\varepsilon \in (W^{1,3}_0(\Omega))^2$,

(ii) $\operatorname{div} \boldsymbol{\varphi}^\varepsilon = |p^\varepsilon|^{-\frac{1}{2}} p^\varepsilon$,

(iii) $\|\boldsymbol{\varphi}^\varepsilon\|_{1,3,\Omega} \le C_{\text{div}} \|p^\varepsilon\|^{\frac{1}{2}}_{\frac{3}{2}}$ where $C_{\text{div}} > 0$ is independent of ε,

(see Appendix B, Theorem B.VI). If such test function existed then inserting $\boldsymbol{\varphi}^\varepsilon$ into (1.11) we would obtain:

$$\|p^\varepsilon\|^{\frac{3}{2}}_{\frac{3}{2},\Omega} \le C \|\boldsymbol{\varphi}^\varepsilon\|_{1,3,\Omega} \le C C_{\text{div}} \|p^\varepsilon\|^{\frac{1}{2}}_{\frac{3}{2},\Omega}.$$

This approach however automatically gives:

$$\int_\Omega |p^\varepsilon|^{-\frac{1}{2}} p^\varepsilon = \int_\Omega \operatorname{div} \boldsymbol{\varphi}^\varepsilon = \int_{\partial\Omega} \boldsymbol{\varphi}^\varepsilon \cdot \boldsymbol{\nu} = 0.$$

In our case there is no reason to expect that the mean-value of $|p^\varepsilon|^{-\frac{1}{2}} p^\varepsilon$ over Ω is zero. The situation is here more involved due to the outflow boundary condition. In what follows we will use a modification of this idea.

Let us refer to Section B.2 in Appendix B for the fundamental results concerning the divergence equation (ii).

Theorem 1.9. *There exists a constant $C_{\text{div}} > 0$ such that for any $\varepsilon > 0$ there exists $\boldsymbol{\varphi}^\varepsilon \in (W^{1,3}(\Omega))^2$ which satisfies*

(i) $\operatorname{div} \boldsymbol{\varphi}^\varepsilon = |p^\varepsilon|^{-\frac{1}{2}} p^\varepsilon$ *a.e. in* Ω,

(ii) $\operatorname{Tr} \boldsymbol{\varphi}^\varepsilon = \mathbf{0}$ *on* $\partial\Omega \setminus \Gamma_{out}$,

(iii) $\operatorname{Tr} \varphi^\varepsilon_1 = 0$ *on* Γ_{out},

(iv) $\|\boldsymbol{\varphi}^\varepsilon\|_{1,3,\Omega} \le C_{\text{div}} \|p^\varepsilon\|^{\frac{1}{2}}_{\frac{3}{2},\Omega}$.

Proof. We will use Corollary B.3 from Appendix B. All we need is to find a function $\boldsymbol{\mathcal{A}}^\varepsilon \in (W^{1,3}(\Omega))^2$ such that $\operatorname{Tr} \boldsymbol{\mathcal{A}}^\varepsilon = \mathbf{0}$ on $\partial\Omega \setminus \Gamma_{\text{out}}$, $\operatorname{Tr} \mathcal{A}_1^\varepsilon = 0$ on Γ_{out}, $\|\boldsymbol{\mathcal{A}}^\varepsilon\|_{1,3,\Omega} \le C \|p^\varepsilon\|^{\frac{1}{2}}_{\frac{3}{2},\Omega}$ and

$$\int_{\Gamma_{\text{out}}} \mathcal{A}_2^\varepsilon = \int_\Omega |p^\varepsilon|^{-\frac{1}{2}} p^\varepsilon. \tag{1.60}$$

Having such $\boldsymbol{\mathcal{A}}^\varepsilon$ at our disposal, we can immediately use Corollary B.3 to obtain $\boldsymbol{\varphi}^\varepsilon$. It is easy to construct such $\boldsymbol{\mathcal{A}}^\varepsilon$. Indeed: We choose $\boldsymbol{\xi} \in (\mathcal{C}^\infty(\overline{\Omega}))^2$ such that

(i) $\xi_1 \equiv 0$ in $\overline{\Omega}$,

(ii) $\operatorname{supp} \xi_2 \cap \partial\Omega \subset \Gamma_{\text{out}}$,

(iii) $\int_{\Gamma_{\text{out}}} \xi_2 = 1$.

Finally we define $\boldsymbol{\mathcal{A}}^\varepsilon := \left(\int_\Omega |p^\varepsilon|^{-\frac{1}{2}} p^\varepsilon \right) \boldsymbol{\xi}$. □

We are ready to prove the uniform apriori estimate for pressure.

Theorem 1.10. *Let the assumptions of Theorem 1.6 be satisfied. Then there exists a constant* $C_p := C_p(\mu_0, \rho, \sigma, \|\nabla \boldsymbol{v}_0\|_{3,\widehat{\Omega}}) > 0$ *independent of* $\varepsilon > 0$ *such that for any solution* $(\boldsymbol{v}^\varepsilon, p^\varepsilon)$ *of* $(\mathcal{P}(\alpha)^\varepsilon)$ *it holds:*

$$\|p^\varepsilon\|_{\frac{3}{2},\Omega} \le C_p. \tag{1.61}$$

Proof. We use $\boldsymbol{\varphi}^\varepsilon$ from Theorem 1.9 as a test function in (1.11) (note that $\boldsymbol{\varphi}^\varepsilon \in W_0$). Using apriori estimate (1.31) we obtain

$$\|p^\varepsilon\|^{\frac{3}{2}}_{\frac{3}{2},\Omega} \le C(\|\nabla \boldsymbol{v}_0\|_{3,\widehat{\Omega}}) C_{\text{div}} \|p^\varepsilon\|^{\frac{1}{2}}_{\frac{3}{2},\Omega}. \tag{1.62}$$

Dividing by $\|p^\varepsilon\|^{\frac{1}{2}}_{\frac{3}{2},\Omega}$ we arrive at (1.61). □

Remark 1.6. *Since the constant* C_{div} *depends on* Ω*, in general, the same holds for* C_p *in* (1.61)*. In Chapter 2 we will prove however that both* C_{div} *and* C_p *can be found independently of* $\alpha \in \mathcal{U}_{ad}$.

1.4.3 Limit passage $\varepsilon \to 0+$

The apriori estimate (1.31) and the uniform estimate of pressure (1.61) together with (1.44) yield:

$$\|\nabla \boldsymbol{v}^\varepsilon\|_{2,\Omega}^2 + \|M|\mathbb{D}(\boldsymbol{v}^\varepsilon)|\|_{3,\Omega}^3 + \|v_2^\varepsilon\|_{3,\Gamma_{\text{out}}}^3 + \varepsilon\|p^\varepsilon\|_{\frac{3}{2},\Omega}^{\frac{3}{2}} \le C_E, \tag{1.63}$$

$$\|p^\varepsilon\|_{\frac{3}{2},\Omega} = \frac{1}{\varepsilon^3}\|\operatorname{div}\boldsymbol{v}^\varepsilon\|_{3,\Omega}^3 \le C_p, \tag{1.64}$$

where $0 < C_E := C_E(\mu_0, \rho, \sigma, \|\nabla \boldsymbol{v}_0\|_{3,\widehat{\Omega}})$ and $0 < C_p := C_p(\mu_0, \rho, \sigma, \|\nabla \boldsymbol{v}_0\|_{3,\widehat{\Omega}}, \alpha)$ are independent of ε. Hence one can pass to the limit with $\varepsilon \to 0+$, i.e. there exists a triple $(\boldsymbol{v}, p, \overline{A}) \in (W^{1,2}(\Omega))^2 \times L^{\frac{3}{2}}(\Omega) \times W^*$ such that

$$\boldsymbol{v}^\varepsilon \rightharpoonup \boldsymbol{v} \text{ in } (W^{1,2}(\Omega))^2, \tag{1.65}$$

$$A(\boldsymbol{v}^\varepsilon) \rightharpoonup \overline{A} \text{ in } W^*, \tag{1.66}$$

$$p^\varepsilon \rightharpoonup p \text{ in } L^{\frac{3}{2}}(\Omega),\ \varepsilon \to 0+. \tag{1.67}$$

To prove that $\boldsymbol{v} \in W_{\boldsymbol{v}_0}$ we will again use the compact imbedding of $W^{1,2}(\Omega)$ into $L^q(\Omega)$ and $L^q(\partial\Omega)$, $q \in [1, \infty)$ so that

$$\boldsymbol{v}^\varepsilon \to \boldsymbol{v} \text{ in } (L^q(\Omega))^2 \text{ and in } (L^q(\partial\Omega))^2,\ \varepsilon \to 0+. \tag{1.68}$$

Using the same arguments as in the proof of Theorem 1.8 we obtain:

$$M\mathbb{D}(\boldsymbol{v}^\varepsilon) \rightharpoonup M\mathbb{D}(\boldsymbol{v}) \text{ in } (L^3(\Omega))^{2\times 2},\ \varepsilon \to 0+. \tag{1.69}$$

Let us observe that (1.64) implies:

$$\operatorname{div}\boldsymbol{v}^\varepsilon \to 0 \text{ in } L^3(\Omega),\ \varepsilon \to 0+. \tag{1.70}$$

From this and (1.65) it follows that

$$\operatorname{div}\boldsymbol{v} = 0 \text{ in } \Omega. \tag{1.71}$$

Finally, (1.69) and (1.71) imply that $\boldsymbol{v} \in W_{\boldsymbol{v}_0}$. Using (1.65), (1.68) and (1.71) we obtain:

$$\frac{1}{2}((\operatorname{div}\boldsymbol{v}^\varepsilon)(\boldsymbol{v}^\varepsilon - \boldsymbol{v}_0), \boldsymbol{\varphi}) \to \frac{1}{2}((\operatorname{div}\boldsymbol{v})(\boldsymbol{v} - \boldsymbol{v}_0), \boldsymbol{\varphi}) = 0\ \forall \boldsymbol{\varphi} \in W_0. \tag{1.72}$$

Letting $\varepsilon \to 0+$ in (1.11), in view of (1.65)-(1.68) and (1.72), we obtain:

$$(\mathbb{D}(\boldsymbol{v}), \mathbb{D}(\boldsymbol{\varphi})) + (v_j \frac{\partial v_i}{\partial x_j}, \varphi_i) + \langle \overline{A}, \boldsymbol{\varphi}\rangle + \int_{\Gamma_{\text{out}}} |v_2| v_2 \varphi_2 - (p, \operatorname{div}\boldsymbol{\varphi}) = 0 \tag{1.73}$$

for all $\boldsymbol{\varphi} \in W_0$. We will use strong monotonicity of A to prove that $\boldsymbol{v}^\varepsilon \to \boldsymbol{v}$ in W, which then yields that $A(\boldsymbol{v}^\varepsilon) \to A(\boldsymbol{v}) = \overline{A}$ in W^*:

$$\begin{aligned}
0 \le C &\left(\|\mathbb{D}(\boldsymbol{v}^\varepsilon - \boldsymbol{v})\|_{2,\Omega}^2 + \|M|\mathbb{D}(\boldsymbol{v}^\varepsilon - \boldsymbol{v})|\|_{3,\Omega}^3\right) \\
&\le \|\mathbb{D}(\boldsymbol{v}^\varepsilon - \boldsymbol{v})\|_{2,\Omega}^2 + \langle A(\boldsymbol{v}^\varepsilon) - A(\boldsymbol{v}), \boldsymbol{v}^\varepsilon - \boldsymbol{v}\rangle \\
= -(\mathbb{D}(\boldsymbol{v}), \mathbb{D}(\boldsymbol{v}^\varepsilon - \boldsymbol{v})) &- (v_j^\varepsilon \frac{\partial v_i^\varepsilon}{\partial x_j}, (v_i^\varepsilon - v_i)) - \frac{1}{2}((\operatorname{div} \boldsymbol{v}^\varepsilon)(\boldsymbol{v}^\varepsilon - \boldsymbol{v}_0), \boldsymbol{v}^\varepsilon - \boldsymbol{v}) \\
&- \sigma \int_{\Gamma_{\text{out}}} |v_2^\varepsilon| v_2^\varepsilon (v_2^\varepsilon - v_2) + (p^\varepsilon, \operatorname{div}(\boldsymbol{v}^\varepsilon - \boldsymbol{v})) - \langle A(\boldsymbol{v}), \boldsymbol{v}^\varepsilon - \boldsymbol{v}\rangle, \quad (1.74)
\end{aligned}$$

making use of (1.11). From this we see that

$$\lim_{\varepsilon \to 0+} \left(\|\mathbb{D}(\boldsymbol{v}^\varepsilon - \boldsymbol{v})\|_{2,\Omega}^2 + \|M|\mathbb{D}(\boldsymbol{v}^\varepsilon - \boldsymbol{v})|\|_{3,\Omega}^3\right) = 0. \quad (1.75)$$

Indeed,

$$(p^\varepsilon, \operatorname{div}(\boldsymbol{v}^\varepsilon - \boldsymbol{v})) = -\varepsilon \|p^\varepsilon\|_{\frac{3}{2},\Omega}^{\frac{3}{2}} \to 0, \quad (1.76)$$

using (1.12), (1.64) and (1.71). The remaining terms on the right hand side of (1.74) tend to zero due to (1.65), (1.68), (1.69) and (1.72). This completes the proof of (i) and (ii) of Theorem 1.4.

1.4.4 Uniqueness results

Proposition 1.11 (Uniqueness of the pressure). *The pressure is uniquely determined by the velocity.*

Proof. Let $(\boldsymbol{v}, p^1)$ and $(\boldsymbol{v}, p^2)$ be two weak solutions of $(\mathcal{P}(\alpha))$. Then the definition of $(\mathcal{P}(\alpha))$ yields

$$(p^1 - p^2, \operatorname{div} \boldsymbol{\varphi}) = 0 \ \forall \boldsymbol{\varphi} \in W_0.$$

From this it follows that $p^1 = p^2$ under the condition that the set $\operatorname{div}(W_0)$ is dense in $L^3(\Omega)$. We show that even $\operatorname{div}(W_0) = L^3(\Omega)$.

Let $f \in L^3(\Omega)$. Following the proof of Theorem 1.9, we choose $\boldsymbol{\xi} \in (\mathcal{C}^\infty(\overline{\Omega}))^2$ such that

(i) $\xi_1 \equiv 0$ in $\overline{\Omega}$,

(ii) $\operatorname{supp} \xi_2 \cap \partial\Omega \subset \Gamma_{\text{out}}$,

(iii) $\int_{\Gamma_{\text{out}}} \xi_2 = 1$.

Then we define $\boldsymbol{\mathcal{A}} := \left(\int_\Omega f\,dx\right)\boldsymbol{\xi}$. Finally we apply Corollary B.3 from Appendix B to obtain $\boldsymbol{\varphi} \in W_0$ such that $\operatorname{div}\boldsymbol{\varphi} = f$ in Ω and $\operatorname{Tr}\boldsymbol{\varphi} = \operatorname{Tr}\boldsymbol{\mathcal{A}}$ on $\partial\Omega$. □

We finish this chapter by the uniqueness analysis of the velocity.

Lemma 1.12. *If the constant C_E from Theorem 1.4 satisfies*

$$C_E < \frac{\mu_0}{\rho}\left(\frac{C_K}{C_F C_I(4)}\right)^2, \tag{1.77}$$

where C_K, C_F, and $C_I(q)$, $q \in [1,\infty)$ are specified by (1.13)-(1.15), *then Problem* $(\mathcal{P}(\alpha))$ *has a unique solution.*

Proof. Let $(\boldsymbol{v}^1, p^1)$ and $(\boldsymbol{v}^2, p^2)$ be two solutions of $(\mathcal{P}(\alpha))$. We subtract the weak formulations for $\boldsymbol{v}^1$ and $\boldsymbol{v}^2$ with $\boldsymbol{\varphi} = \boldsymbol{v}^1 - \boldsymbol{v}^2 \in W_0$ as a test function. We obtain:

$$2\mu_0\|\mathbb{D}(\boldsymbol{v}^1-\boldsymbol{v}^2)\|^2_{2,\Omega} + \underbrace{\langle A(\boldsymbol{v}^1) - A(\boldsymbol{v}^2), \boldsymbol{v}^1 - \boldsymbol{v}^2\rangle}_{\geq\, 0} + \sigma\underbrace{\int_{\Gamma_{\text{out}}}(|v_2^1|v_2^1 - |v_2^2|v_2^2)(v_2^1 - v_2^2)}_{\geq\, 0}$$
$$= \rho((v^2 - v^1)_j\frac{\partial v_i^2}{\partial x_j}, (v_i^2 - v_i^1)) + \rho(v_j^1\frac{\partial(v_i^2 - v_i^1)}{\partial x_j}, (v_i^2 - v_i^1)).$$

We estimate the terms on the right hand side, making use of the Hölder inequality, the imbedding of $W^{1,2}(\widehat{\Omega})$ into $L^4(\widehat{\Omega})$, the Friedrichs inequality in $W^{1,2}(\widehat{\Omega})$ and the apriori estimates:

$$((v_j^2-v_j^1)\frac{\partial v_i^2}{\partial x_j}, (v_i^2-v_i^1)) \leq \|\nabla\boldsymbol{v}^2\|_{2,\Omega}\|\boldsymbol{v}^1-\boldsymbol{v}^2\|^2_{4,\Omega} \leq C_E(C_F C_I(4))^2\|\nabla(\boldsymbol{v}^1-\boldsymbol{v}^2)\|^2_{2,\Omega},$$

$$(v_j^1\frac{\partial(v_i^2 - \boldsymbol{v}_i^1)}{\partial x_j}, (v_i^2 - v_i^1)) \leq \|\boldsymbol{v}^1\|_{4,\Omega}\|\nabla(\boldsymbol{v}^1 - \boldsymbol{v}^2)\|_{2,\Omega}\|\boldsymbol{v}^1 - \boldsymbol{v}^2\|_{4,\Omega}$$
$$\leq C_E(C_F C_I(4))^2\|\nabla(\boldsymbol{v}^1 - \boldsymbol{v}^2)\|^2_{2,\Omega}.$$

Applying the Korn inequality on the left hand side, we finally obtain:

$$2\mu_0 C_K^2\|\nabla(\boldsymbol{v}^1 - \boldsymbol{v}^2)\|^2_{2,\Omega} \leq 2\rho C_E(C_F C_I(4))^2\|\nabla(\boldsymbol{v}^1 - \boldsymbol{v}^2)\|^2_{2,\Omega},$$

from which it follows that $\boldsymbol{v}^1 = \boldsymbol{v}^2$ a.e. in Ω if $C_E < \frac{\mu_0}{\rho}\left(\frac{C_K}{C_F C_I(4)}\right)^2$. This condition will be satisfied for $\|\nabla\boldsymbol{v}_0\|_{3,\widehat{\Omega}}$ small enough, as follows from Remark 1.5. □

Note that the condition (1.77) is independent of $\alpha \in \mathcal{U}_{ad}$. The proof of Theorem 1.4 is completed.

Chapter 2

Shape optimization problem

The aim of this chapter is to formulate a shape optimization problem and to prove the existence of its solution.

2.1 Formulation of the problem

We have shown that under certain assumptions which do not depend on a particular choice of $\Omega(\alpha) \in \mathcal{O}$, there exists at least one weak solution of the state problem $(\mathcal{P}(\alpha))$.

Let $\mathcal{G}$ be the graph of the control-to-state (generally multi-valued) mapping:

$$\mathcal{G} := \{(\alpha, \boldsymbol{v}, p);\ \alpha \in \mathcal{U}_{ad},\ (\boldsymbol{v}, p) \text{ is a weak solution of } (\mathcal{P}(\alpha))\}.$$

Further, let us define the cost functional $J : \mathcal{G} \to \mathbb{R}$ by

$$J : (\alpha, \boldsymbol{v}, p) \mapsto \int_{\tilde{\Gamma}} |v_2 - v_{opt}|^2 \, dS,\ \boldsymbol{v} = (v_1, v_2), \tag{2.1}$$

where $v_{opt} \in L^2(\tilde{\Gamma})$ is a given function representing the desired outlet velocity profile and $\tilde{\Gamma} \subset \Gamma_{out}$. This choice of J reflects the optimization goal formulated in Chapter .

We now formulate the following problem:

Find $(\alpha^*, \boldsymbol{v}^*, p^*) \in \mathcal{G}$ *such that*

$$J(\alpha^*, \boldsymbol{v}^*, p^*) \le J(\alpha, \boldsymbol{v}, p)\ \forall (\alpha, \boldsymbol{v}, p) \in \mathcal{G}. \tag{$\mathbb{P}$}$$

Next we introduce convergence of a sequence of domains.

Definition 2.1. *Let* $\{\Omega(\alpha_n)\}$, $\alpha_n \in \mathcal{U}_{ad}$ *be a sequence of domains. We say that* $\{\Omega(\alpha_n)\}$ *converges to* $\Omega(\alpha)$, *shortly* $\Omega(\alpha_n) \rightsquigarrow \Omega(\alpha)$, *iff* $\alpha_n \rightrightarrows \alpha$ *in* $[0, L]$.

As a direct consequence of the Arzelà–Ascoli theorem we have the following compactness result.

Lemma 2.1. *System* $\mathcal{O}$ *is compact with respect to convergence introduced in Definition* 2.1.

2.2 Uniform estimate of pressure - part II

In Section 1.4.2 we obtained a bound C_p for the norm of the pressure $p := p(\alpha)$, which may possibly depend on α. In order to avoid this dependence we need to estimate the constant C_{div} from Theorem B.VI in Appendix B. To proceed, we will construct subdomains $\Omega_k(\alpha)$, $k = 1, \dots, N$, which are star-shaped[1] with respect to some balls $B_k(\alpha)$ and such that $\Omega(\alpha) = \cup_{k=1}^{N} \Omega_k(\alpha)$.

Let $\alpha \in \mathcal{U}_{ad}$ and let us write $\Omega := \Omega(\alpha)$. We define

$$\Omega_k := \left\{ \boldsymbol{x} \in \Omega; x_1 \in ((k-1)\frac{l}{2}, (k+1)\frac{l}{2}) \right\}, \ k = 1, \dots, N \tag{2.2}$$

(N is the smallest value such that $\Omega_N \neq \emptyset$). The parameter l is determined by the following formula:

$$l := \alpha_{min} \operatorname{tg} \omega = \alpha_{min} \gamma, \tag{2.3}$$

where $\omega := \operatorname{arctg} \gamma$ (see Figure 2.1). Finally we choose the ball B_k such that it is contained in the triangle whose vertices are $[(k-1)\frac{l}{2}, 0]$, $[(k+1)\frac{l}{2}, 0]$, $[k\frac{l}{2}, \frac{\alpha_{min}}{2}]$, e.g. we set the centre of B_k to the centre of gravity of the respective triangle and the radius

$$R_0 = \frac{l}{4} \operatorname{cotg} \frac{\omega}{2}, \tag{2.4}$$

i.e. one half of the radius of the inscribed ball. Consequently the radius R_0 is independent of α and Ω_k is star-shaped with respect to B_k, as it can be seen in Figure 2.1.

Using notation of Theorem B.VI, we have the following estimates:

$$|\Omega_k| \leq l\alpha_{max}, \tag{2.5}$$

$$|\Omega_k \cap D_k| \geq \frac{l}{2}\alpha_{min}, \tag{2.6}$$

$$|D_k \setminus \Omega_k| \leq |D_i| \leq |\widehat{\Omega}|, \tag{2.7}$$

[1] See Appendix B for the definition of a star-shaped domain.

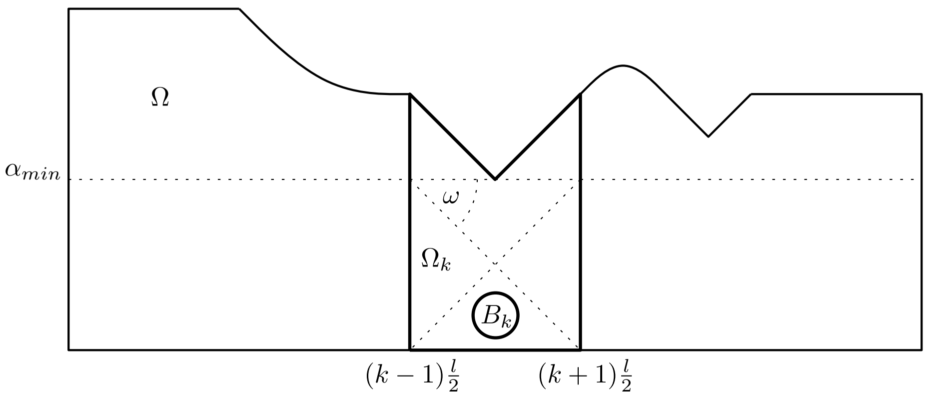

Figure 2.1: Partition of Ω into star-shaped subdomains Ω_k.

which make it possible to estimate $C_{\rm div}$ independently of α:

$$C_{\rm div} \le c_0 C \left(\frac{\operatorname{diam} \widehat{\Omega}}{R_0} \right)^2 \left(1 + \frac{\operatorname{diam} \widehat{\Omega}}{R_0} \right), \tag{2.8}$$

$$C \le (N-2) \left(1 + 2\frac{\alpha_{max}}{\alpha_{min}} \right) \left(1 + \left(\frac{2|\widehat{\Omega}|}{l\alpha_{min}} \right)^{1-1/3} \right). \tag{2.9}$$

The following proposition summarizes this result.

Proposition 2.2. *Theorem B.VI, Corollary B.3 (from Appendix B) and Theorem 1.9 hold for every domain $\Omega(\alpha)$, $\alpha \in \mathcal{U}_{ad}$ with a constant $C_{\rm div} > 0$ independent of α. Consequently, the constant C_E in* (1.10) *is independent of $\alpha \in \mathcal{U}_{ad}$.*

2.3 Existence of an optimal shape

First let us recall that the function $\boldsymbol{v}_0$ which realizes the boundary conditions is the same for all domains $\Omega \in \mathcal{O}$. We now rewrite $(\mathcal{P}(\alpha))$, $\alpha \in \mathcal{U}_{ad}$ using the formulation on the fixed domain $\widehat{\Omega}$:

$$\begin{aligned} 2\mu_0(\mathbb{D}(\tilde{\boldsymbol{v}}(\alpha)), \mathbb{D}(\tilde{\boldsymbol{\varphi}}))_{\widehat{\Omega}} + \langle \tilde{A}_\alpha(\tilde{\boldsymbol{v}}(\alpha)), \tilde{\boldsymbol{\varphi}} \rangle_{\widehat{\Omega}} + \rho(\tilde{v}_j(\alpha) \frac{\partial \tilde{v}_i(\alpha)}{\partial x_j}, \tilde{\varphi}_i)_{\widehat{\Omega}} & \\ + \sigma \int_{\Gamma_{\rm out}} |\tilde{v}_2(\alpha)| \tilde{v}_2(\alpha) \tilde{\varphi}_2 - (\tilde{p}(\alpha), \operatorname{div} \tilde{\boldsymbol{\varphi}})_{\widehat{\Omega}} = 0 \quad \forall \boldsymbol{\varphi} \in W_0(\alpha), & \qquad (\mathcal{P}_{\widehat{\Omega}}(\alpha)) \\ \operatorname{div} \tilde{\boldsymbol{v}}(\alpha) = 0 \text{ a.e. in } \widehat{\Omega}, & \end{aligned}$$

where the symbol $\tilde{}$ stands for the zero extension of functions from $\Omega(\alpha)$ on $\widehat{\Omega}$, $(\cdot,\cdot)_{\widehat{\Omega}}$ denotes the scalar product in $L^2(\widehat{\Omega})$ and

$$\langle \tilde{A}_\alpha(\tilde{\boldsymbol{v}}(\alpha)), \tilde{\boldsymbol{\varphi}}\rangle_{\widehat{\Omega}} := 2\rho \int_{\widehat{\Omega}} \tilde{M}_\alpha^3 |\mathbb{D}(\tilde{\boldsymbol{v}}(\alpha))| \mathbb{D}(\tilde{\boldsymbol{v}}(\alpha)) : \mathbb{D}(\tilde{\boldsymbol{\varphi}}).$$

Further let

$$\widehat{W}(\alpha) := \Big\{ \boldsymbol{v} \in \big(W^{1,2}(\Omega(\alpha))\big)^2;\ \operatorname{div}\ \boldsymbol{v} \in L^3(\Omega(\alpha)),\ M_\alpha |\mathbb{D}(\boldsymbol{v})| \in L^3(\Omega(\alpha)) \Big\}$$

and define

$$\widehat{W}_{\boldsymbol{v}_0}(\alpha) := \Big\{ \boldsymbol{v} \in \widehat{W}(\alpha);\ \boldsymbol{v} \text{ satisfies the Dirichlet conditions } (1.4)_1 - (1.4)_3 \text{ on } \partial\Omega(\alpha) \Big\}.$$

Remark 2.1. *It holds that* $W_{\boldsymbol{v}_0}(\alpha) \subseteq \widehat{W}_{\boldsymbol{v}_0}(\alpha)$. *The question arises, if these spaces are identical. This is in fact the density problem. For the moment we do not know the answer.*

Proposition 2.2 gives the following apriori estimate:

$$\|\nabla \tilde{\boldsymbol{v}}(\alpha)\|_{2,\widehat{\Omega}}^2 + \|\tilde{M}_\alpha |\mathbb{D}(\tilde{\boldsymbol{v}}(\alpha))|\|_{3,\widehat{\Omega}}^3 + \int_{\Gamma_{\text{out}}} |v_2(\alpha)|^3 + \|\tilde{p}(\alpha)\|_{\frac{3}{2},\widehat{\Omega}}^{\frac{3}{2}} \le C_E \tag{2.10}$$

for every $(\alpha, \boldsymbol{v}(\alpha), p(\alpha)) \in \mathcal{G}$ with the constant $C_E := C_E(\mu_0, \rho, \sigma, \|\nabla \boldsymbol{v}_0\|_{3,\widehat{\Omega}}) > 0$ independent of α provided that (1.9) is satisfied.

Theorem 2.3. *Let* $(\alpha_n, \boldsymbol{v}(\alpha_n), p(\alpha_n)) \in \mathcal{G}$, $n = 1, 2, \dots$ *and* $\alpha \in \mathcal{U}_{ad}$ *satisfy*

$$\alpha_n \rightrightarrows \alpha \text{ in } [0, L],\ n \to \infty.$$

Then there exists $\widehat{\boldsymbol{v}} \in \big(W^{1,2}(\widehat{\Omega})\big)^2$, $\widehat{p} \in L^{\frac{3}{2}}(\widehat{\Omega})$ *and a subsequence of* $\{(\tilde{\boldsymbol{v}}_n, \tilde{p}_n)\}$ *(denoted by the same symbol) such that*

$$\begin{aligned} &\tilde{\boldsymbol{v}}_n \rightharpoonup \widehat{\boldsymbol{v}} \text{ in } \big(W^{1,2}(\widehat{\Omega})\big)^2, \\ &\tilde{M}_{\alpha_n} \mathbb{D}(\tilde{\boldsymbol{v}}_n) \rightharpoonup \tilde{M}_\alpha \mathbb{D}(\widehat{\boldsymbol{v}}) \text{ in } \big(L^3(\widehat{\Omega})\big)^{2\times 2}, \\ &\tilde{p}_n \rightharpoonup \widehat{p} \text{ in } L^{\frac{3}{2}}(\widehat{\Omega}),\ n \to \infty. \end{aligned} \tag{2.11}$$

In addition, if we define $\boldsymbol{v}(\alpha) := \widehat{\boldsymbol{v}}_{|\Omega(\alpha)}$ *and* $p(\alpha) := \widehat{p}_{|\Omega(\alpha)}$, *then* $\boldsymbol{v}(\alpha) \in \widehat{W}_{\boldsymbol{v}_0}$ *and* $(\boldsymbol{v}(\alpha), p(\alpha))$ *solves* $(\mathcal{P}_{\widehat{\Omega}}(\alpha))$.

Proof. Let us denote $\tilde{M}_n := \tilde{M}_{\alpha_n}$, $\Omega_n := \Omega(\alpha_n)$, $\langle\cdot,\cdot\rangle_n := \langle\cdot,\cdot\rangle_{\alpha_n}$ etc. We recall (ii) of Lemma 1.2 which yields:

$$\tilde{M}_n \rightrightarrows \tilde{M}_\alpha \text{ in } \widehat{\Omega},\ n \to \infty. \tag{2.12}$$

From the apriori estimate (2.10) it follows that

$$\|\tilde{\boldsymbol{v}}_n\|_{1,2,\widehat{\Omega}} \leq C,\ \|\tilde{M}_n \mathbb{D}(\tilde{\boldsymbol{v}}_n)\|_{3,\widehat{\Omega}} \leq C,\ \|\tilde{p}_n\|_{\frac{3}{2},\widehat{\Omega}} \leq C, \tag{2.13}$$

where $C > 0$ does not depend on n. Therefore we can pass to a subsequence of $\{(\tilde{\boldsymbol{v}}_n, \tilde{p}_n)\}$ (denoted again by the same symbol) such that

$$\begin{array}{lll} \tilde{\boldsymbol{v}}_n \rightharpoonup \widehat{\boldsymbol{v}} & \text{in } \big(W^{1,2}(\widehat{\Omega})\big)^2, \\ \tilde{M}_n \mathbb{D}(\tilde{\boldsymbol{v}}_n) \rightharpoonup \widehat{\mathbb{A}} & \text{in } \big(L^3(\widehat{\Omega})\big)^{2\times 2}, \\ \tilde{p}_n \rightharpoonup \widehat{p} & \text{in } L^{\frac{3}{2}}(\widehat{\Omega}),\ n \to \infty. \end{array} \tag{2.14}$$

The following properties of $\widehat{\boldsymbol{v}}$, $\widehat{p}$ and $\widehat{\mathbb{A}}$ are easily verified:

(i) $\widehat{\boldsymbol{v}} = \mathbf{0}$, $\widehat{p} = 0$ and $\widehat{\mathbb{A}} = \mathbf{0}$ in $\widehat{\Omega} \setminus \overline{\Omega}(\alpha)$;

(ii) $\widehat{\mathbb{A}} = \tilde{M}_\alpha \mathbb{D}(\widehat{\boldsymbol{v}})$ in $\widehat{\Omega}$;

(iii) div $\widehat{\boldsymbol{v}} = 0$ in $\widehat{\Omega}$;

(iv) $\widehat{\boldsymbol{v}}$ satisfies the required Dirichlet boundary conditions on $\partial\Omega(\alpha)$.

We prove (ii). Since $\mathcal{C}^\infty(\overline{\widehat{\Omega}})$ is dense in $L^{\frac{3}{2}}(\widehat{\Omega})$, it is sufficient to show that

$$\int_{\widehat{\Omega}} \tilde{M}_n \mathbb{D}(\tilde{\boldsymbol{v}}_n) : \mathbb{B} \to \int_{\widehat{\Omega}} \tilde{M}_\alpha \mathbb{D}(\widehat{\boldsymbol{v}}) : \mathbb{B},\ n \to \infty$$

holds for every $\mathbb{B} \in \big(\mathcal{C}^\infty(\overline{\widehat{\Omega}})\big)^{2\times 2}$. Indeed:

$$\left| \int_{\widehat{\Omega}} \Big(\tilde{M}_n \mathbb{D}(\tilde{\boldsymbol{v}}_n) : \mathbb{B} - \tilde{M}_\alpha \mathbb{D}(\widehat{\boldsymbol{v}}) : \mathbb{B} \Big) \right| \leq$$

$$\leq \int_{\widehat{\Omega}} |\tilde{M}_n - \tilde{M}_\alpha| |\mathbb{D}(\tilde{\boldsymbol{v}}_n) : \mathbb{B}| + \left| \int_{\widehat{\Omega}} \tilde{M}_\alpha \left(\mathbb{D}(\tilde{\boldsymbol{v}}_n) - \mathbb{D}(\widehat{\boldsymbol{v}}) \right) : \mathbb{B} \right| \to 0,$$

making use of (2.12), $(2.14)_1$ and the fact that $\tilde{M}_\alpha \mathbb{B} \in (L^2(\widehat{\Omega}))^{2\times 2}$.

Let $\boldsymbol{v}(\alpha) := \widehat{\boldsymbol{v}}_{|\Omega(\alpha)}$ and $p(\alpha) := \widehat{p}_{|\Omega(\alpha)}$. Then (i)-(iv) implies that $\boldsymbol{v}(\alpha) \in \widehat{W}_{\boldsymbol{v}_0}(\alpha)$.

Next we prove that $(\boldsymbol{v}(\alpha), p(\alpha))$ solves $(\mathcal{P}_{\widehat{\Omega}}(\alpha))$. We start from the definition of $(\mathcal{P}(\alpha_n))$:

$$2\mu_0(\mathbb{D}(\tilde{\boldsymbol{v}}_n), \mathbb{D}(\tilde{\boldsymbol{\varphi}}))_{\widehat{\Omega}} + \langle \tilde{A}_n(\tilde{\boldsymbol{v}}_n), \tilde{\boldsymbol{\varphi}} \rangle_{\widehat{\Omega}} + \rho(\tilde{v}_{nj} \frac{\partial \tilde{v}_{ni}}{\partial x_j}, \tilde{\varphi}_i)_{\widehat{\Omega}}$$
$$+ \sigma \int_{\Gamma_{\text{out}}} |\tilde{v}_{n2}| \tilde{v}_{n2} \tilde{\varphi}_2 - (\tilde{p}_n, \operatorname{div} \tilde{\boldsymbol{\varphi}})_{\widehat{\Omega}} = 0 \quad \forall \boldsymbol{\varphi} \in W_0(\alpha_n). \tag{2.15}$$

Let $\boldsymbol{\varphi} \in \mathcal{V}_0(\alpha)$ be an arbitrary function. Then $\tilde{\boldsymbol{\varphi}}|_{\Omega_n} \in \mathcal{V}_0(\alpha_n)$ for n sufficiently large so that it can be used as a test function in (2.15). The limit passage in the first, third, fourth and fifth term in (2.15) is a classical one:

$$\begin{aligned} &(\mathbb{D}(\tilde{\boldsymbol{v}}_n), \mathbb{D}(\tilde{\boldsymbol{\varphi}}))_{\widehat{\Omega}} \to (\mathbb{D}(\tilde{\boldsymbol{v}}(\alpha)), \mathbb{D}(\tilde{\boldsymbol{\varphi}}))_{\widehat{\Omega}}, \\ &\int_{\Gamma_{\text{out}}} |\tilde{v}_{n2}| \tilde{v}_{n2} \tilde{\varphi}_2 \to \int_{\Gamma_{\text{out}}} |\tilde{v}_2(\alpha)| \tilde{v}_2(\alpha) \tilde{\varphi}_2, \\ &(\tilde{v}_{nj} \frac{\partial \tilde{v}_{ni}}{\partial x_j}, \tilde{\varphi}_i)_{\widehat{\Omega}} \to (\tilde{v}_j(\alpha) \frac{\partial \tilde{v}_i(\alpha)}{\partial x_j}, \tilde{\varphi}_i)_{\widehat{\Omega}}, \\ &(\tilde{p}_n, \operatorname{div} \tilde{\boldsymbol{\varphi}})_{\widehat{\Omega}} \to (\tilde{p}(\alpha), \operatorname{div} \tilde{\boldsymbol{\varphi}})_{\widehat{\Omega}}, \; n \to \infty. \end{aligned} \tag{2.16}$$

The most difficult is to handle the second term. We will first prove that $\mathbb{D}(\tilde{\boldsymbol{v}}_n) \to \mathbb{D}(\tilde{\boldsymbol{v}}(\alpha))$ pointwise a.e. in $\widehat{\Omega}$, and then use the Vitali theorem to pass to the limit in $\langle \tilde{A}_n(\tilde{\boldsymbol{v}}_n), \tilde{\boldsymbol{\varphi}} \rangle_{\widehat{\Omega}}$.

For any $\varepsilon > 0$ we define the set

$$\Omega_\varepsilon := \{\boldsymbol{x} \in \Omega(\alpha); \;\; \operatorname{dist}(\boldsymbol{x}, \partial\Omega(\alpha)) > \varepsilon\}.$$

Note that for ε small enough $\Omega_\varepsilon \neq \emptyset$ and has a Lipschitz continuous boundary. Let us fix ε and choose $\xi := \xi_\varepsilon \in \mathcal{C}_0^\infty(\overline{\Omega}(\alpha))$ such that $\xi \geq 0$ in $\Omega(\alpha)$ and $\xi_{|\Omega_\varepsilon} \equiv 1$. We will use a special test function of the form $\boldsymbol{\varphi} := \xi(\tilde{\boldsymbol{v}}_{n|\Omega(\alpha)} - \tilde{\boldsymbol{v}}_{m|\Omega(\alpha)})$, where $n, m \in \mathbb{N}$. In what follows we assume that n, m are large enough so that $\operatorname{supp} \boldsymbol{\varphi} \subset \Omega_n \cap \Omega_m$.

Since $\tilde{M}_n \rightrightarrows \tilde{M}_\alpha$ in $\widehat{\Omega}$ and $M_\alpha(\boldsymbol{x}) \geq C_\xi > 0$ for every $\boldsymbol{x} \in \operatorname{supp} \xi$, the estimate (2.10) yields:

$$C_\xi^3 \|\mathbb{D}(\tilde{\boldsymbol{v}}_n)|\|_{3,\operatorname{supp}\xi}^3 \leq \|\tilde{M}_n |\mathbb{D}(\tilde{\boldsymbol{v}}_n)|\|_{3,\operatorname{supp}\xi}^3 \leq C_E, \tag{2.17}$$

therefore $\boldsymbol{\varphi} \in (W_0^{1,3}(\Omega(\alpha)))^2$ which implies immediately that $\boldsymbol{\varphi} \in W_0(\alpha)$ and $\tilde{\boldsymbol{\varphi}}_{|\Omega_n} \in W_0(\alpha_n)$. Let us insert $\tilde{\boldsymbol{\varphi}}_{|\Omega_n}$ into $(\mathcal{P}(\alpha_n))$. Using the fact that

$$\begin{aligned} \mathbb{D}(\boldsymbol{\varphi}) &= \xi \mathbb{D}(\tilde{\boldsymbol{v}}_{n|\Omega(\alpha)} - \tilde{\boldsymbol{v}}_{m|\Omega(\alpha)}) + (\tilde{\boldsymbol{v}}_{n|\Omega(\alpha)} - \tilde{\boldsymbol{v}}_{m|\Omega(\alpha)}) \otimes_s \nabla\xi, \\ \operatorname{div} \boldsymbol{\varphi} &= \xi \operatorname{div}(\tilde{\boldsymbol{v}}_{n|\Omega(\alpha)} - \tilde{\boldsymbol{v}}_{m|\Omega(\alpha)}) + (\tilde{\boldsymbol{v}}_{n|\Omega(\alpha)} - \tilde{\boldsymbol{v}}_{m|\Omega(\alpha)}) \cdot \nabla\xi \\ &= (\tilde{\boldsymbol{v}}_{n|\Omega(\alpha)} - \tilde{\boldsymbol{v}}_{m|\Omega(\alpha)}) \cdot \nabla\xi, \end{aligned}$$

we obtain from (2.15):

$$2\mu_0(\mathbb{D}(\boldsymbol{v}_n),\xi\mathbb{D}(\boldsymbol{v}_n-\boldsymbol{v}_m))_{\operatorname{supp}\xi}+2\rho\int_{\operatorname{supp}\xi}\xi M_n^3|\mathbb{D}(\boldsymbol{v}_n)|\mathbb{D}(\boldsymbol{v}_n):\mathbb{D}(\boldsymbol{v}_n-\boldsymbol{v}_m)$$
$$=-2\mu_0(\mathbb{D}(\boldsymbol{v}_n),(\boldsymbol{v}_n-\boldsymbol{v}_m)\otimes_s\nabla\xi)_{\operatorname{supp}\xi}-2\rho\int_{\operatorname{supp}\xi}M_n^3|\mathbb{D}(\boldsymbol{v}_n)|\mathbb{D}(\boldsymbol{v}_n):((\boldsymbol{v}_n-\boldsymbol{v}_m)\otimes_s\nabla\xi)$$
$$-\rho(v_{nj}\frac{\partial v_{ni}}{\partial x_j},\xi(v_{ni}-v_{mi}))_{\operatorname{supp}\xi}+(p_n,(\boldsymbol{v}_n-\boldsymbol{v}_m)\cdot\nabla\xi)_{\operatorname{supp}\xi}. \quad (2.18)$$

Doing the same for $(\mathcal{P}(\alpha_m))$ and subtracting the two equations leads to:

$$2\mu_0(\mathbb{D}(\boldsymbol{v}_n-\boldsymbol{v}_m),\xi\mathbb{D}(\boldsymbol{v}_n-\boldsymbol{v}_m))_{\operatorname{supp}\xi}$$
$$+2\rho\int_{\operatorname{supp}\xi}\xi(M_n^3|\mathbb{D}(\boldsymbol{v}_n)|\mathbb{D}(\boldsymbol{v}_n)-M_m^3|\mathbb{D}(\boldsymbol{v}_m)|\mathbb{D}(\boldsymbol{v}_m)):\mathbb{D}(\boldsymbol{v}_n-\boldsymbol{v}_m)$$
$$=-2\mu_0(\mathbb{D}(\boldsymbol{v}_n-\boldsymbol{v}_m),(\boldsymbol{v}_n-\boldsymbol{v}_m)\otimes_s\nabla\xi)_{\operatorname{supp}\xi}$$
$$-2\rho\int_{\operatorname{supp}\xi}(M_n^3|\mathbb{D}(\boldsymbol{v}_n)|\mathbb{D}(\boldsymbol{v}_n)-M_m^3|\mathbb{D}(\boldsymbol{v}_m)|\mathbb{D}(\boldsymbol{v}_m)):((\boldsymbol{v}_n-\boldsymbol{v}_m)\otimes_s\nabla\xi)$$
$$-\rho(v_{nj}\frac{\partial v_{ni}}{\partial x_j}-v_{mj}\frac{\partial v_{mi}}{\partial x_j},\xi(v_{ni}-v_{mi}))_{\operatorname{supp}\xi}+(p_n-p_m,(\boldsymbol{v}_n-\boldsymbol{v}_m)\cdot\nabla\xi))_{\operatorname{supp}\xi}. \quad (2.19)$$

The strong convergence of $\tilde{\boldsymbol{v}}_n$ in $L^q(\widehat{\Omega})$, $q\in[1,\infty)$, implies that for any $\delta>0$ and for n,m large enough the right hand side of (2.19) is smaller than δ. The left hand side can be rearranged as follows:

$$2\mu_0(\mathbb{D}(\boldsymbol{v}_n-\boldsymbol{v}_m),\xi\mathbb{D}(\boldsymbol{v}_n-\boldsymbol{v}_m))_{\operatorname{supp}\xi}$$
$$+2\rho\int_{\operatorname{supp}\xi}\xi(M_n^3|\mathbb{D}(\boldsymbol{v}_n)|\mathbb{D}(\boldsymbol{v}_n)-M_m^3|\mathbb{D}(\boldsymbol{v}_m)|\mathbb{D}(\boldsymbol{v}_m)):\mathbb{D}(\boldsymbol{v}_n-\boldsymbol{v}_m)$$
$$=2\mu_0\|\sqrt{\xi}\mathbb{D}(\boldsymbol{v}_n-\boldsymbol{v}_m)\|^2_{2,\operatorname{supp}\xi}+2\rho\int_{\operatorname{supp}\xi}\xi M_n^3(|\mathbb{D}(\boldsymbol{v}_n)|\mathbb{D}(\boldsymbol{v}_n)-|\mathbb{D}(\boldsymbol{v}_m)|\mathbb{D}(\boldsymbol{v}_m)):\mathbb{D}(\boldsymbol{v}_n-\boldsymbol{v}_m)$$
$$+2\rho\int_{\operatorname{supp}\xi}\xi(M_n^3-M_m^3)|\mathbb{D}(\boldsymbol{v}_m)|\mathbb{D}(\boldsymbol{v}_m):\mathbb{D}(\boldsymbol{v}_n-\boldsymbol{v}_m). \quad (2.20)$$

The second term is nonnegative due to monotonicity, while the last term can be estimated as follows:

$$\int_{\operatorname{supp}\xi}\xi(M_n^3-M_m^3)|\mathbb{D}(\boldsymbol{v}_m)|\mathbb{D}(\boldsymbol{v}_m):\mathbb{D}(\boldsymbol{v}_n-\boldsymbol{v}_m)$$
$$\le\|\xi\|_{\infty,\Omega(\alpha)}\|\tilde{M}_n^3-\tilde{M}_m^3\|_{\infty,\widehat{\Omega}}\|\mathbb{D}(\tilde{\boldsymbol{v}}_m)\|^2_{3,\operatorname{supp}\xi}\|\mathbb{D}(\tilde{\boldsymbol{v}}_n-\tilde{\boldsymbol{v}}_m)\|_{3,\operatorname{supp}\xi}\le\delta, \quad (2.21)$$

making use of (2.12) and (2.17). Then (2.20) together with (2.21) give:

$$2\mu_0 \|\mathbb{D}(\boldsymbol{v}_n - \boldsymbol{v}_m)\|^2_{2,\Omega_\varepsilon} \leq 2\mu_0 \|\sqrt{\xi}\mathbb{D}(\boldsymbol{v}_n - \boldsymbol{v}_m)\|^2_{2,\operatorname{supp}\xi} \leq 2\delta,$$

which means that $\mathbb{D}(\boldsymbol{v}_n) \to \mathbb{D}(\boldsymbol{v}(\alpha))$ in $L^3(\Omega_\varepsilon)$. This implies pointwise convergence of $\mathbb{D}(\boldsymbol{v}_n)$ to $\mathbb{D}(\boldsymbol{v}(\alpha))$ a.e. in Ω_ε as well.

For almost all $\boldsymbol{x} \in \Omega(\alpha)$ we can choose $\varepsilon > 0$ such that $\boldsymbol{x} \in \Omega_\varepsilon$, and $\mathbb{D}(\boldsymbol{v}_n)(\boldsymbol{x}) \to \mathbb{D}(\boldsymbol{v}(\alpha))(\boldsymbol{x})$. For $\boldsymbol{x} \in \widehat{\Omega} \setminus \overline{\Omega}(\alpha)$ it is clear that $\mathbb{D}(\boldsymbol{v}_n)(\boldsymbol{x}) = \mathbb{D}(\boldsymbol{v}(\alpha))(\boldsymbol{x}) = 0$ for $n \geq n_0$. Altogether we have:

$$\mathbb{D}(\tilde{\boldsymbol{v}}_n) \to \mathbb{D}(\tilde{\boldsymbol{v}}(\alpha)) \text{ a.e. in } \widehat{\Omega}. \tag{2.22}$$

Now we want to use the Vitali theorem A.V (see Appendix A) to show that

$$\int_{\widehat{\Omega}} \tilde{M}_n^3 |\mathbb{D}(\tilde{\boldsymbol{v}}_n)| \mathbb{D}(\tilde{\boldsymbol{v}}_n) : \mathbb{D}(\tilde{\boldsymbol{\varphi}}) \to \int_{\widehat{\Omega}} \tilde{M}_\alpha^3 |\mathbb{D}(\tilde{\boldsymbol{v}}(\alpha))| \mathbb{D}(\tilde{\boldsymbol{v}}(\alpha)) : \mathbb{D}(\tilde{\boldsymbol{\varphi}})$$

for every $\boldsymbol{\varphi} \in \mathcal{V}_0(\alpha)$. It is sufficient to verify that the integrand converges pointwise a.e. in $\widehat{\Omega}$ (which obviously follows from (2.12) and (2.22)) and that

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall E \subset \widehat{\Omega},\ |E| < \delta : \quad \sup_n \int_E \left| \tilde{M}_n^3 |\mathbb{D}(\tilde{\boldsymbol{v}}_n)| \mathbb{D}(\tilde{\boldsymbol{v}}_n) : \mathbb{D}(\tilde{\boldsymbol{\varphi}}) \right| < \varepsilon.$$

Indeed,

$$\begin{aligned} \int_E \left| \tilde{M}_n^3 |\mathbb{D}(\tilde{\boldsymbol{v}}_n)| \mathbb{D}(\tilde{\boldsymbol{v}}_n) : \mathbb{D}(\tilde{\boldsymbol{\varphi}}) \right| &\leq \|\tilde{M}_n |\mathbb{D}(\tilde{\boldsymbol{v}}_n)|\|^2_{3,\widehat{\Omega}} \|\tilde{M}_n\|_{\infty,\widehat{\Omega}} \|\mathbb{D}(\tilde{\boldsymbol{\varphi}})\|_{3,E} \\ &\leq C \|\mathbb{D}(\tilde{\boldsymbol{\varphi}})\|_{3,E} < \varepsilon, \end{aligned}$$

where we used the apriori estimate (2.10) and (2.12).

This, together with $(2.16)_1$-$(2.16)_3$ gives:

$$\begin{aligned} &2\mu_0 (\mathbb{D}(\tilde{\boldsymbol{v}}(\alpha)), \mathbb{D}(\tilde{\boldsymbol{\varphi}}))_{\widehat{\Omega}} + \langle \tilde{A}_\alpha(\tilde{\boldsymbol{v}}(\alpha)), \tilde{\boldsymbol{\varphi}} \rangle_{\widehat{\Omega}} \\ &\quad + \rho(\tilde{v}_j(\alpha) \frac{\partial \tilde{v}_i(\alpha)}{\partial x_j}, \tilde{\varphi}_i)_{\widehat{\Omega}} + \sigma \int_{\Gamma_{\text{out}}} |\tilde{v}_2(\alpha)| \tilde{v}_2(\alpha) \tilde{\varphi}_2 - (\tilde{p}(\alpha), \operatorname{div} \tilde{\boldsymbol{\varphi}})_{\widehat{\Omega}} = 0, \end{aligned} \tag{2.23}$$

for every $\boldsymbol{\varphi} \in \mathcal{V}_0(\alpha)$ and consequently also for $\boldsymbol{\varphi} \in W_0(\alpha)$. □

Remark 2.2. *Under the assumptions which guarantee uniqueness of the solution to $(\mathcal{P}(\alpha))$, the whole sequence $\{(\tilde{\boldsymbol{v}}_n, \tilde{p}_n)\}$ tends to $(\tilde{\boldsymbol{v}}(\alpha), \tilde{p}(\alpha))$ in the sense of Theorem 2.3.*

Remark 2.3. *The proof of Theorem 2.3 does not guarantee that $\boldsymbol{v}(\alpha) \in W_{\boldsymbol{v}_0}$, thus $(\boldsymbol{v}(\alpha), p(\alpha))$ may not be a solution to $(\mathcal{P}(\alpha))$. However if $W_{\boldsymbol{v}_0}(\alpha) = \widehat{W}_{\boldsymbol{v}_0}(\alpha)$, then automatically $(\boldsymbol{v}(\alpha), p(\alpha))$ solves $(\mathcal{P}(\alpha))$.*

The previous remark leads us to extend the shape optimization problem as follows:

Definition 2.2 (Augmented state problem $(\widehat{\mathcal{P}}(\alpha))$). *Let $\alpha \in \mathcal{U}_{ad}$. A pair $(\boldsymbol{v}, p) := (\boldsymbol{v}(\alpha), p(\alpha)) \in \widehat{W}_{\boldsymbol{v}_0}(\alpha) \times L^{\frac{3}{2}}(\Omega(\alpha))$ is said to be a solution of the augmented state problem $(\widehat{\mathcal{P}}(\alpha))$ iff*

- *$(\boldsymbol{v}, p)$ solves $(\mathcal{P}_{\widehat{\Omega}}(\alpha))$;*
- *$(\boldsymbol{v}, p)$ satisfies the estimate* (2.10).

Clearly any solution of $(\mathcal{P}(\alpha))$ becomes a solution of $(\widehat{\mathcal{P}}(\alpha))$ too. Therefore the statement of Theorem 1.4 can be applied to $(\widehat{\mathcal{P}}(\alpha))$ as well. Moreover, in Theorem 2.3 we can take a sequence $\{(\boldsymbol{v}_n, p_n)\}$ of solutions to $(\widehat{\mathcal{P}}(\alpha_n))$ instead of $(\mathcal{P}(\alpha_n))$, so that the limit $(\boldsymbol{v}(\alpha), p(\alpha))$ is a solution to $(\widehat{\mathcal{P}}(\alpha))$.

Definition 2.3 (Augmented shape optimization problem $(\widehat{\mathbb{P}})$). *Let us define the set*

$$\widehat{\mathcal{G}} := \{(\alpha, \boldsymbol{v}, p);\ \alpha \in \mathcal{U}_{ad},\ (\boldsymbol{v}, p) \text{ is a solution of } (\widehat{\mathcal{P}}(\alpha))\}.$$

A triple $(\alpha^*, \boldsymbol{v}^*, p^*) \in \widehat{\mathcal{G}}$ is said to be a solution of the augmented shape optimization problem $(\widehat{\mathbb{P}})$ iff

$$J(\alpha^*, \boldsymbol{v}^*, p^*) \le J(\alpha, \boldsymbol{v}, p)\ \forall (\alpha, \boldsymbol{v}, p) \in \widehat{\mathcal{G}}.$$

Theorem 2.4. *Problem $(\widehat{\mathbb{P}})$ has a solution.*

Proof. Let $\{(\alpha_n, \boldsymbol{v}_n, p_n)\}$, $(\alpha_n, \boldsymbol{v}_n, p_n) \in \widehat{\mathcal{G}}$, be a minimizing sequence of $(\widehat{\mathbb{P}})$. Without loss of generality we may assume that $\alpha_n \rightrightarrows \alpha^* \in \mathcal{U}_{ad}$ in $[0, L]$. From Theorem 2.3 it follows that there exists an accumulation point $(\alpha^*, \boldsymbol{v}^*, p^*)$ such that $(\alpha^*, \boldsymbol{v}^*_{|\Omega(\alpha^*)}, p^*_{|\Omega(\alpha^*)}) \in \widehat{\mathcal{G}}$. Further, if we define $\boldsymbol{v}^*(\alpha^*) := \boldsymbol{v}^*_{|\Omega(\alpha^*)}$ and $p^*(\alpha^*) := p^*_{|\Omega(\alpha^*)}$, we have:

$$\begin{aligned} q := \inf_{(\alpha, \boldsymbol{v}(\alpha), p(\alpha)) \in \widehat{\mathcal{G}}} J(\alpha, \boldsymbol{v}(\alpha), p(\alpha)) &= \lim_{n \to \infty} J(\alpha_n, \boldsymbol{v}_n, p_n) \\ &= J(\alpha^*, \boldsymbol{v}^*(\alpha^*), p^*(\alpha^*)) \ge q \end{aligned}$$

making use of continuity of J. Hence $(\alpha^*, \boldsymbol{v}^*(\alpha^*), p^*(\alpha^*))$ is an optimal triple for $(\widehat{\mathbb{P}})$. □

Part II

Numerical analysis and computations

Chapter 3

Approximation of the flow problem

In this chapter we describe the finite-element approximation of the state problem ($\mathcal{P}(\alpha)$) and analyze its properties such as existence of discrete solutions and their convergence to a solution of the original problem.

3.1 Definition of the discrete state problem

In what follows we will assume that $\alpha \in \mathcal{U}_{ad}$ is fixed (hence the symbol α will be often dropped) and piecewise linear, so that $\Omega := \Omega(\alpha)$ is a polygonal domain.

Let $\{T_h\}$, $h \to 0+$ be a family of triangulations of Ω and h be the norm of T_h. Throughout the chapter we will assume that the following conditions are satisfied:

(A1) the family $\{T_h\}$ is uniformly regular with respect to h: there is $\theta_0 > 0$ such that $\theta(h) \geq \theta_0$ $\forall h > 0$, where $\theta(h)$ is the minimal interior angle of all triangles from T_h;

(A2) the family $\{T_h\}$ is consistent with the decomposition of $\partial\Omega$ into Γ_{out} and $\partial\Omega \setminus \Gamma_{\text{out}}$.

We now introduce finite-dimensional subspaces of W_0 and $L^{\frac{3}{2}}(\Omega)$, respectively:

$$\begin{aligned} W_{0h} :=\{&\boldsymbol{\varphi}_h \in (\mathcal{C}(\overline{\Omega}))^2;\ \forall K \in T_h\ \boldsymbol{\varphi}_{h|K} \in (P_1(K) \oplus B_3(K))^2, \\ &\varphi_{h1|\partial\Omega} = 0,\ \varphi_{2|\partial\Omega\setminus\Gamma_{\text{out}}} = 0\}, \end{aligned}$$

$$L_h :=\{\psi_h \in \mathcal{C}(\overline{\Omega});\ \forall K \in T_h\ \psi_{h|K} \in P_1(K)\}.$$

Here $P_1(K)$ denotes the set of polynomials in K of 1^{st} order and

$$B_3(K) := \{c\lambda_1(K)\lambda_2(K)\lambda_3(K);\ c \in \mathbb{R}\}$$

is the set of cubic bubble functions in K ($\lambda_1(K), \dots, \lambda_3(K)$ stand for the barycentric coordinates in K).

Definition 3.1. *A pair $(\boldsymbol{v}_h, p_h) \in W \times L_h$ is said to be a solution of the discrete state problem $(\mathcal{P}_h(\alpha))$ iff*

(i) $\boldsymbol{v}_h - \boldsymbol{v}_0 \in W_{0h}$,

(ii) for every $\boldsymbol{\varphi}_h \in W_{0h}$ it holds:

$$2\mu_0(\mathbb{D}(\boldsymbol{v}_h), \mathbb{D}(\boldsymbol{\varphi}_h)) + \rho(v_{hj}\frac{\partial v_{hi}}{\partial x_j}, \varphi_{hi}) + \frac{\rho}{2}((\operatorname{div}\boldsymbol{v}_h)(\boldsymbol{v}_h - \boldsymbol{v}_0), \boldsymbol{\varphi}_h) + \langle A(\boldsymbol{v}_h), \boldsymbol{\varphi}_h\rangle + \sigma\int_{\Gamma_{out}} |v_{h2}|v_{h2}\varphi_{h2} - (p_h, \operatorname{div}\boldsymbol{\varphi}_h) = 0, \quad (3.1)$$

(iii) for every $\psi_h \in L_h$ it holds: $(\psi_h, \operatorname{div}\boldsymbol{v}_h) = 0$.

Let us point out that in contrast to $(\mathcal{P}(\alpha))$, $(\mathcal{P}_h(\alpha))$ contains additionally the term $\frac{\rho}{2}((\operatorname{div}\boldsymbol{v}_h)(\boldsymbol{v}_h - \boldsymbol{v}_0), \boldsymbol{\varphi}_h)$. This is important in order to obtain an apriori estimate for the discrete solutions. In the continuous case, the additional term vanishes due to divergenceless velocity. However, (*iii*) of $(\mathcal{P}_h(\alpha))$ does not guarantee that $\operatorname{div}\boldsymbol{v}_h = 0$ a.e. in Ω.

3.2 Existence of a discrete solution

We will use a technique that is similar to the one presented in Section 1.4 to prove that $(\mathcal{P}_h(\alpha))$ possesses a solution.

Theorem 3.1. *Let the assumptions of Theorem 1.4 hold. Then for every $h > 0$*

(i) there exists a solution of $(\mathcal{P}_h(\alpha))$;

(ii) any solution $(\boldsymbol{v}_h, p_h)$ of $(\mathcal{P}_h(\alpha))$ admits the estimate

$$\|\nabla\boldsymbol{v}_h\|_{2,\Omega}^2 + \|M|\mathbb{D}(\boldsymbol{v}_h)|\|_{3,\Omega}^3 + \|v_{h2}\|_{3,\Gamma_{out}}^3 \le C_E, \quad (3.2)$$

where the constant $C_E := C_E(\mu_0, \rho, \sigma, \|\nabla\boldsymbol{v}_0\|_{3,\widehat{\Omega}}) > 0$ is the same as in Theorem 1.4, especially it is independent of h;

(iii) under the assumption of Lemma 1.12, $\boldsymbol{v}_h$ is unique;

(iv) p_h is uniquely determined by $\boldsymbol{v}_h$.

Theorem 3.1 will be proved in three steps. First we will deal with existence of $\boldsymbol{v}_h$, then for given $\boldsymbol{v}_h$ we will establish p_h and finally uniqueness of $\boldsymbol{v}_h$ and p_h will be discussed.

Let us define the mapping $\operatorname{div}_h : W_{0h} \to L_h^*$ as follows:

$$\langle \operatorname{div}_h \boldsymbol{w}_h, \psi_h \rangle := \int_\Omega \psi_h \operatorname{div} \boldsymbol{w}_h \quad \forall \boldsymbol{w}_h \in W_{0h},\ \psi_h \in L_h.$$

We denote $V_h := \ker \operatorname{div}_h$ and formulate the problem $(\mathcal{P}_{h,\mathrm{div}}(\alpha))$:

Find $\boldsymbol{v}_h \in W$, such that

(i) $\boldsymbol{v}_h - \boldsymbol{v}_0 \in V_h$,

(ii) for every $\boldsymbol{\varphi}_h \in V_h$ it holds:

$$2\mu_0(\mathbb{D}(\boldsymbol{v}_h), \mathbb{D}(\boldsymbol{\varphi}_h)) + \rho(v_{hj}\frac{\partial v_{hi}}{\partial x_j}, \varphi_{hi}) + \frac{\rho}{2}((\operatorname{div} \boldsymbol{v}_h)(\boldsymbol{v}_h - \boldsymbol{v}_0), \boldsymbol{\varphi}_h) + \langle A(\boldsymbol{v}_h), \boldsymbol{\varphi}_h \rangle + \sigma \int_{\Gamma_{out}} |v_{h2}| v_{h2} \varphi_{h2} = 0. \quad (3.3)$$

It is readily seen that for $\boldsymbol{\varphi}_h \in V_h$ the equations (3.1) and (3.3) coincide.

Lemma 3.2. *Under the assumptions of Theorem 1.4, problem $(\mathcal{P}_{h,\mathrm{div}}(\alpha))$ has a solution.*

Proof. We will proceed like in the proof of Theorem 1.7.

Let $N := \dim V_h$, $\{\boldsymbol{\omega}^1, \ldots, \boldsymbol{\omega}^N\}$ be a basis of V_h, and define the mapping $\boldsymbol{P} : \mathbb{R}^N \to \mathbb{R}^N$ as follows: For $s = 1, \ldots, N$, the s-th component $\boldsymbol{P}(\boldsymbol{d})_s$ equals the left hand side of (3.3) with $\boldsymbol{\varphi}_h := \boldsymbol{\omega}^s$, and

$$\boldsymbol{v}_h(\boldsymbol{x}) := \boldsymbol{v}_0(\boldsymbol{x}) + \sum_{r=1}^N d_r \boldsymbol{\omega}^r(\boldsymbol{x}),\ \boldsymbol{d} = (d_1, \ldots, d_N).$$

Then $(\mathcal{P}_{h,\mathrm{div}}(\alpha))$ is equivalent to:

$$\textit{Find } \overline{\boldsymbol{d}} \in \mathbb{R}^N \textit{ such that } \boldsymbol{P}(\overline{\boldsymbol{d}}) = \boldsymbol{0}. \quad (3.4)$$

Using the same technique as in the proof of apriori estimate (1.31) we obtain:

$$\boldsymbol{P}(\boldsymbol{d}) \cdot \boldsymbol{d} \geq C \left(\|\nabla \boldsymbol{v}_h\|_{2,\Omega}^2 + \|M|\mathbb{D}(\boldsymbol{v}_h)|\|_{3,\Omega}^3 + \int_{\Gamma_{\text{out}}} |v_{h2}|^3 \right) \\ - C_E(\|\nabla \boldsymbol{v}_0\|_{3,\widehat{\Omega}}) \geq C(\|\nabla \boldsymbol{v}_h\|_{2,\Omega}^2) - C_E(\|\nabla \boldsymbol{v}_0\|_{3,\widehat{\Omega}}),$$

where $C > 0$ is independent of h and α. Clearly, for $|\boldsymbol{d}|$ large enough, the last expression is positive. From Appendix A, Corollary A.III the existence of $\overline{\boldsymbol{d}} \in \mathbb{R}^N$ solving (3.4) follows. □

Following the technique of proof of Theorem 1.6, one can show that every solution $\boldsymbol{v}_h$ of $(\mathcal{P}_{h,\text{div}}(\alpha))$ satisfies the estimate

$$\|\nabla \boldsymbol{v}_h\|_{2,\Omega}^2 + \|M|\mathbb{D}(\boldsymbol{v}_h)|\|_{3,\Omega}^3 + \|v_{h2}\|_{3,\Gamma_{out}}^3 \leq C_E, \tag{3.5}$$

where C_E is independent of $h > 0$ and $\alpha \in \mathcal{U}_{ad}$.

The following lemma proves the existence of p_h solving $(\mathcal{P}_h(\alpha))$.

Lemma 3.3. *Let $\boldsymbol{v}_h$ be a solution of $(\mathcal{P}_{h,\text{div}}(\alpha))$. Then there exists $p_h \in L_h$ such that $(\boldsymbol{v}_h, p_h)$ solves $(\mathcal{P}_h(\alpha))$.*

Proof. Let us define the mapping $B_h \in W_{0h}^*$:

$$\langle B_h, \boldsymbol{\varphi}_h \rangle := 2\mu_0(\mathbb{D}(\boldsymbol{v}_h), \mathbb{D}(\boldsymbol{\varphi}_h)) + \rho(v_{hj}\frac{\partial v_{hi}}{\partial x_j}, \varphi_{hi}) + \frac{\rho}{2}((\operatorname{div} \boldsymbol{v}_h)(\boldsymbol{v}_h - \boldsymbol{v}_0), \boldsymbol{\varphi}_h) \\ + \langle A(\boldsymbol{v}_h), \boldsymbol{\varphi}_h \rangle + \sigma \int_{\Gamma_{\text{out}}} |v_{h2}| v_{h2} \varphi_{h2} = 0, \ \forall \boldsymbol{\varphi}_h \in W_{0h}. \tag{3.6}$$

Clearly $B_h \in (V_h)^\circ$. From the well known properties of linear mappings of finite dimensional spaces it follows that

$$(V_h)^\circ = (\ker \operatorname{div}_h)^\circ = \mathcal{R}(\operatorname{div}_h').$$

The last equality yields the existence of $p_h \in L_h$ satisfying $\operatorname{div}_h' p_h = B_h$, meaning that

$$\langle \operatorname{div}_h' p_h, \boldsymbol{\varphi}_h \rangle = (p_h, \operatorname{div} \boldsymbol{\varphi}_h) = \langle B_h, \boldsymbol{\varphi}_h \rangle$$

for every $\boldsymbol{\varphi}_h \in W_{0h}$, from which the lemma follows. □

Uniqueness of $\boldsymbol{v}_h$ is proved the same way as in Section 1.4.4. For the uniqueness of p_h, let us assume that $(\boldsymbol{v}_h, p_h^1)$ and $(\boldsymbol{v}_h, p_h^2)$ are two solutions of $(\mathcal{P}_h(\alpha))$. Then, if we insert $(\boldsymbol{v}_h, p_h^1)$ and $(\boldsymbol{v}_h, p_h^2)$ into (3.1) and subtract the two equations, we obtain:

$$\forall \boldsymbol{\varphi}_h \in W_{0h} \quad (p_h^1 - p_h^2, \operatorname{div} \boldsymbol{\varphi}_h) = 0. \tag{3.7}$$

The following lemma shows that (3.7) implies $p_h^1 \equiv p_h^2$.

Lemma 3.4. *Let $q_h \in L_h$ satisfy*

$$(q_h, \operatorname{div} \boldsymbol{\varphi}_h) = 0 \ \forall \boldsymbol{\varphi}_h \in W_{0h}.$$

Then $q_h \equiv 0$.

Proof. Let $K \in T_h$ be arbitrary. We choose $\boldsymbol{\varphi}_h$ to be a bubble function on K such that $\boldsymbol{\varphi}_h \geq 0$. Then the Green theorem yields:

$$(q_h, \operatorname{div} \boldsymbol{\varphi}_h) = \int_K \nabla q_h \cdot \boldsymbol{\varphi}_h.$$

Since $\boldsymbol{\varphi}_h > 0$ inside K and ∇q_h is constant on K, it follows that $\nabla q_{h|K} = 0$. This can be done on each triangle K, thus $q_h = const$. Next we show that $q_h \equiv 0$. Let $\boldsymbol{P}$ be a node of T_h lying inside Γ_{out} and $\boldsymbol{\varphi}_h$ be the piecewise linear Courant basis function associated to $\boldsymbol{P}$, i.e. $\boldsymbol{\varphi}_h(\boldsymbol{P}) = (0,1)^{\mathrm{T}}$ and is zero for all other nodes. Then we again use the Green theorem:

$$0 = (q_h, \operatorname{div} \boldsymbol{\varphi}_h) = \int_{\partial\Omega} q_h \boldsymbol{\varphi}_h \cdot \nu - \int_\Omega \nabla q_h \cdot \boldsymbol{\varphi}_h = -q_h \int_{\Gamma_{\text{out}}} \varphi_{h2}.$$

Since $\int_{\Gamma_{\text{out}}} \varphi_{h2} > 0$, the lemma is proved. ☐

3.3 Convergence of discrete solutions

In this section we will study the relation of $(\boldsymbol{v}_h, p_h)$ and $(\boldsymbol{v}, p)$ in the limit $h \to 0+$. We start with some basic properties.

Lemma 3.5. *The system $\{W_{0h}\}_{h\to 0+}$ is dense in W_0, i.e.*

$$\forall \boldsymbol{\varphi} \in W_0 \ \exists \{\boldsymbol{\varphi}_h\}, \ \boldsymbol{\varphi}_h \in W_{0h} : \ \boldsymbol{\varphi}_h \to \boldsymbol{\varphi} \ \text{in } W_0, \ h \to 0+,$$

and analogously, $\{L_h\}_{h\to 0+}$ is dense in $L^2(\Omega)$.

Proof. We prove only density of $\{W_{0h}\}$ in W_0, as the other statement is well known. Let $\boldsymbol{\varphi} \in W_0$ and $\varepsilon > 0$ be given. Then we can pick $\boldsymbol{\varphi}_\varepsilon \in \mathcal{V}_0$ such that

$$\|\boldsymbol{\varphi} - \boldsymbol{\varphi}_\varepsilon\|_\alpha < \frac{\varepsilon}{2}.$$

If $h > 0$ is small enough, then due to the standard Lagrange interpolation there exists $\boldsymbol{\varphi}_h \in W_{0h}$ such that

$$\|\boldsymbol{\varphi}_\varepsilon - \boldsymbol{\varphi}_h\|_{1,3,\Omega} < \frac{\varepsilon}{2}.$$

Since the norm of $W^{1,3}(\Omega)$ is stronger than the one of W_0, the first statement of the lemma is proved. ☐

The reason, why density of $\{L_h\}$ in $L^2(\Omega)$ (instead of $L^{\frac{3}{2}}(\Omega)$) is needed, will be explained in the proof of Theorem 3.8.

In order to establish uniform boundedness of the discrete pressure, we will need that the following variant of the Babuška-Brezzi inf-sup condition holds:

$$\inf_{q_h \in L_h \setminus \{0\}} \sup_{\boldsymbol{w}_h \in W_{0h} \setminus \{\boldsymbol{0}\}} \frac{(q_h, \operatorname{div} \boldsymbol{w}_h)}{\|q_h\|_{\frac{3}{2},\Omega} \|\boldsymbol{w}_h\|_\alpha} \geq C_{BB}, \tag{3.8}$$

where $C_{BB} > 0$ is a positive constant independent of $h > 0$.

Remark 3.1. *The inf-sup condition has great impact on stability of the finite-element approximation, namely on the error estimates. For our choice of finite element combination (usually called the MINI element) the inf-sup condition has been proved (e.g. in [1]) in case that Dirichlet boundary condition is prescribed on the whole $\partial\Omega$ (i.e. $\Gamma_{out} = \emptyset$):*

$$\inf_{q_h \in L_h/\mathbb{R} \setminus \{0\}} \sup_{\boldsymbol{w}_h \in W_{0h} \setminus \{\boldsymbol{0}\}} \frac{(q_h, \operatorname{div} \boldsymbol{w}_h)}{\|q_h\|_{L^{\frac{3}{2}}(\Omega)/\mathbb{R}} \|\boldsymbol{w}_h\|_{1,3,\Omega}} \geq C_{BB}, \tag{3.9}$$

where C_{BB} is a positive constant independent of h.

It seems to be difficult to prove (3.8) for two reasons:

1. Mixed boundary conditions are present;
2. We are dealing with weighted norm of W_{0h}.

Here we do not prove validity of (3.8). However we at least show that it is enough to prove it with respect to the norm of $W^{1,3}(\Omega)$.

Lemma 3.6. *Assume that there is a constant $C'_{BB} > 0$ such that $\forall\ h > 0$*

$$\inf_{q_h \in L_h \setminus \{0\}} \sup_{\boldsymbol{w}_h \in W_{0h} \setminus \{\boldsymbol{0}\}} \frac{(q_h, \operatorname{div} \boldsymbol{w}_h)}{\|q_h\|_{\frac{3}{2},\Omega} \|\boldsymbol{w}_h\|_{1,3,\Omega}} \geq C'_{BB}. \tag{3.10}$$

Then there is a constant $C_{BB} > 0$ such that $\forall\ h > 0$ (3.8) holds.

Proof. The only difference between (3.10) and (3.8) is in the norm of $\boldsymbol{w}_h$. Thus it is sufficient to realize that for every $\boldsymbol{w}_h \in W_{0h}$ it holds:

$$\begin{aligned} \|\boldsymbol{w}_h\|_\alpha = \|\boldsymbol{w}_h\|_{1,2,\Omega} + \|M|\mathbb{D}(\boldsymbol{w}_h)|\|_{3,\Omega} + \|\operatorname{div} \boldsymbol{w}_h\|_{3,\Omega} \\ \leq (|\Omega| + \|M^3\|_{\infty,\Omega} + 1)\|\boldsymbol{w}_h\|_{1,3,\Omega}, \end{aligned}$$

making use of the Hölder inequality. Then $C_{BB} := \frac{C'_{BB}}{|\Omega| + \|M^3\|_{\infty,\Omega} + 1}$. □

Lemma 3.7. *Let* (3.8) *hold. Then there exists a constant* $C_p > 0$ *independent of* h *such that*

$$\|p_h\|_{\frac{3}{2},\Omega} \leq C_p. \tag{3.11}$$

Proof. Let $(\boldsymbol{v}_h, p_h)$ be a solution of $(\mathcal{P}_h(\alpha))$. The inf-sup condition (3.8) yields:

$$C_{BB}\|p_h\|_{\frac{3}{2},\Omega} \leq \sup_{\boldsymbol{\varphi}_h \in W_{0h}\setminus\{\mathbf{0}\}} \frac{(p_h, \operatorname{div} \boldsymbol{\varphi}_h)}{\|\boldsymbol{\varphi}_h\|_\alpha}. \tag{3.12}$$

Using the definition of $(\mathcal{P}_h(\alpha))$, one can express the right hand side of (3.12) in terms of $\boldsymbol{v}_h$:

$$(p_h, \operatorname{div} \boldsymbol{\varphi}_h) = 2\mu_0(\mathbb{D}(\boldsymbol{v}_h), \mathbb{D}(\boldsymbol{\varphi}_h)) + \rho(v_{hj}\frac{\partial v_{hi}}{\partial x_j}, \varphi_{hi}) + \frac{\rho}{2}((\operatorname{div} \boldsymbol{v}_h)(\boldsymbol{v}_h - \boldsymbol{v}_0), \boldsymbol{\varphi}_h)$$
$$+ \langle A(\boldsymbol{v}_h), \boldsymbol{\varphi}_h\rangle + \sigma \int_{\Gamma_{\text{out}}} |v_{h2}| v_{h2}\varphi_{h2} \leq C\|\boldsymbol{\varphi}_h\|_\alpha,$$

using the apriori estimate (3.5). Since the constants C_{BB} and C are independent of $h > 0$, the lemma is proved. □

Now we are ready to prove the convergence theorem.

Theorem 3.8. *Let the assumptions of Theorem 1.4 and the condition* (3.8) *hold. Let* $\{(\boldsymbol{v}_h, p_h)\}$ *be a sequence of solutions to* $(\mathcal{P}_h(\alpha))$. *Then there exists a subsequence and a limit pair* $(\boldsymbol{v}, p) \in W_{\boldsymbol{v}_0} \times L^{\frac{3}{2}}(\Omega)$ *such that*

$$\boldsymbol{v}_h \to \boldsymbol{v} \textit{ in } W, \tag{3.13a}$$

$$p_h \rightharpoonup p \textit{ in } L^{\frac{3}{2}}(\Omega), \; h \to 0+ \tag{3.13b}$$

and $(\boldsymbol{v}, p)$ *is a solution of* $(\mathcal{P}(\alpha))$.

Proof. Existence of $(\boldsymbol{v}, p) \in (W^{1,2}(\Omega))^2 \times L^{\frac{3}{2}}(\Omega)$ satisfying

$$\boldsymbol{v}_h \rightharpoonup \boldsymbol{v} \text{ in } W^{1,2}(\Omega) \tag{3.14}$$

and (3.13b) follows from (3.5) and (3.11). Moreover, one can easily show the following:

$$M\mathbb{D}(\boldsymbol{v}_h) \rightharpoonup M\mathbb{D}(\boldsymbol{v}) \text{ in } L^3(\Omega),$$
$$A(\boldsymbol{v}_h) \rightharpoonup \overline{A} \text{ in } W^*. \tag{3.15}$$

where $\overline{A} \in W^*$.

Next we prove that $\operatorname{div} \boldsymbol{v} = 0$ a.e. in Ω. Let $\psi \in L^2(\Omega)$. Then we can use Lemma 3.5 to find a sequence $\{\psi_h\}$, $\psi_h \in L_h$, such that

$$\psi_h \to \psi \text{ in } L^2(\Omega).$$

From this and (3.14) we obtain:

$$0 = (\psi_h, \operatorname{div} \boldsymbol{v}_h) \to (\psi, \operatorname{div} \boldsymbol{v}) = 0, \tag{3.16}$$

consequently $\operatorname{div} \boldsymbol{v} = 0$ a.e. in Ω and therefore $\boldsymbol{v} \in W_{\boldsymbol{v}_0}$.

In the rest we make the limit passage in (3.1) and prove strong convergence of $\boldsymbol{v}_h$. Let $\boldsymbol{\varphi} \in W_0$. Using Lemma 3.5 we find a sequence $\{\boldsymbol{\varphi}_h\}$, $\boldsymbol{\varphi}_h \in W_{0h}$ such that

$$\boldsymbol{\varphi}_h \to \boldsymbol{\varphi} \text{ in } W_0. \tag{3.17}$$

Similarly to the proof of Theorem 1.8, we will use the compact imbeddings, (3.13), (3.16) and (3.17) to pass to the limit in the standard terms:

$$\begin{aligned}
(\mathbb{D}(\boldsymbol{v}_h), \mathbb{D}(\boldsymbol{\varphi}_h)) &\to (\mathbb{D}(\boldsymbol{v}), \mathbb{D}(\boldsymbol{\varphi})),\\
(v_{hj}\frac{\partial v_{hi}}{\partial x_j}, \varphi_{hi}) &\to (v_j \frac{\partial v_i}{\partial x_j}, \varphi_i),\\
((\operatorname{div} \boldsymbol{v}_h)(\boldsymbol{v}_h - \boldsymbol{v}_0), \boldsymbol{\varphi}_h) &\to ((\operatorname{div} \boldsymbol{v})(\boldsymbol{v} - \boldsymbol{v}_0), \boldsymbol{\varphi}) = 0,\\
\int_{\Gamma_{\text{out}}} |v_{h2}| v_{h2} \varphi_{h2} &\to \int_{\Gamma_{\text{out}}} |v_2| v_2 \varphi_2,\\
(p_h, \operatorname{div} \boldsymbol{\varphi}_h) &\to (p, \operatorname{div} \boldsymbol{\varphi}).
\end{aligned}$$

We can use this together with (3.15) to obtain for every $\boldsymbol{\varphi} \in W_0$:

$$(\mathbb{D}(\boldsymbol{v}), \mathbb{D}(\boldsymbol{\varphi})) + (v_j \frac{\partial v_i}{\partial x_j}, \varphi_i) + \langle \overline{A}, \boldsymbol{\varphi} \rangle + \int_{\Gamma_{\text{out}}} |v_2| v_2 \varphi_2 - (p, \operatorname{div} \boldsymbol{\varphi}) = 0 \tag{3.18}$$

(here we put $2\mu_0 = \rho = \sigma = 1$ for simplicity).

Finally we use monotonicity of A to show that $\overline{A} = A(\boldsymbol{v})$. Let $\boldsymbol{\varphi} \in W$. Then

$$\begin{aligned}
0 &\le \langle A(\boldsymbol{v}_h) - A(\boldsymbol{\varphi}), \boldsymbol{v}_h - \boldsymbol{\varphi} \rangle\\
&= -(\mathbb{D}(\boldsymbol{v}_h), \mathbb{D}(\boldsymbol{v}_h - \boldsymbol{v}_0)) - (v_{hj}\frac{\partial v_{hi}}{\partial x_j}, v_{hi} - v_{0i}) - \frac{1}{2}((\operatorname{div} \boldsymbol{v}_h)(\boldsymbol{v}_h - \boldsymbol{v}_0), \boldsymbol{v}_h - \boldsymbol{v}_0)\\
&\quad - \int_{\Gamma_{\text{out}}} |v_{h2}| v_{h2} (v_{h2} - v_{02}) + \langle A(\boldsymbol{v}_h), \boldsymbol{v}_0 - \boldsymbol{\varphi} \rangle - \langle A(\boldsymbol{\varphi}), \boldsymbol{v}_h - \boldsymbol{\varphi} \rangle,
\end{aligned} \tag{3.19}$$

making use of (3.1). Letting $h \to 0+$ and using lower semicontinuity of $\|\mathbb{D}(\boldsymbol{v}_h)\|_{2,\Omega}$ and continuity of the remaining terms we obtain:

$$0 \le -(\mathbb{D}(\boldsymbol{v}), \mathbb{D}(\boldsymbol{v}-\boldsymbol{v}_0)) - (v_j \frac{\partial v_i}{\partial x_j}, v_i - v_{0i}) - \int_{\Gamma_{\text{out}}} |v_2| v_2 (v_2 - v_{02}) + \langle \overline{A}, \boldsymbol{v}_0 - \boldsymbol{\varphi} \rangle - \langle A(\boldsymbol{\varphi}), \boldsymbol{v} - \boldsymbol{\varphi} \rangle. \quad (3.20)$$

From (3.18) and (3.20) we arrive at the inequality

$$0 \le \langle \overline{A} - A(\boldsymbol{\varphi}), \boldsymbol{v} - \boldsymbol{\varphi} \rangle, \quad (3.21)$$

which holds for any $\boldsymbol{\varphi} \in W$. Instead of $\boldsymbol{\varphi}$ we insert into (3.21) a function $\boldsymbol{v} \pm \lambda \boldsymbol{\psi}$, where $\lambda > 0$, $\boldsymbol{\psi} \in W$:

$$0 \le \langle \overline{A} - A(\boldsymbol{v} \pm \lambda \boldsymbol{\psi}), \mp \lambda \boldsymbol{\psi} \rangle. \quad (3.22)$$

Dividing this inequality by λ we obtain for $\lambda \to 0+$:

$$0 \le \pm \langle \overline{A} - A(\boldsymbol{v}), \boldsymbol{\psi} \rangle, \quad (3.23)$$

making use of continuity of A. Thus $\overline{A} = A(\boldsymbol{v})$.

To prove strong convergence of $\boldsymbol{v}_h$ we use strong monotonicity of A:

$$\begin{aligned} C\left(\|\mathbb{D}(\boldsymbol{v}_h - \boldsymbol{v})\|_{2,\Omega}^2 + \|M|\mathbb{D}(\boldsymbol{v}_h - \boldsymbol{v})|\|_{3,\Omega}^3\right) \\ \le (\mathbb{D}(\boldsymbol{v}_h - \boldsymbol{v}), \mathbb{D}(\boldsymbol{v}_h - \boldsymbol{v})) + \langle A(\boldsymbol{v}_h) - A(\boldsymbol{v}), \boldsymbol{v}_h - \boldsymbol{v} \rangle \\ = (\mathbb{D}(\boldsymbol{v}_h), \mathbb{D}(\boldsymbol{v}_h - \boldsymbol{v}_0)) + \langle A(\boldsymbol{v}_h), \boldsymbol{v}_h - \boldsymbol{v}_0 \rangle + (\mathbb{D}(\boldsymbol{v}_h), \mathbb{D}(\boldsymbol{v}_0 - \boldsymbol{v})) \\ - (\mathbb{D}(\boldsymbol{v}), \mathbb{D}(\boldsymbol{v}_h - \boldsymbol{v})) + \langle A(\boldsymbol{v}_h), \boldsymbol{v}_0 - \boldsymbol{v} \rangle - \langle A(\boldsymbol{v}), \boldsymbol{v}_h - \boldsymbol{v} \rangle. \quad (3.24) \end{aligned}$$

Terms $(\mathbb{D}(\boldsymbol{v}_h), \mathbb{D}(\boldsymbol{v}_h - \boldsymbol{v}_0)) + \langle A(\boldsymbol{v}_h), \boldsymbol{v}_h - \boldsymbol{v}_0 \rangle$ on the right hand side of (3.24) can be replaced by (3.1). Then due to the weak convergence of $\boldsymbol{v}_h$ and p_h the right hand side vanishes for $h \to 0+$, which yields (3.13a). □

Chapter 4

Approximation of the shape optimization problem

4.1 Parametrization of the discrete shapes

We now introduce two types of discretized domains: *discrete design* and *computational domains*. The boundary Γ_α of discrete design domains are realized by smooth, piecewise quadratic Bézier functions. The optimal discrete design domain is the main output of the computational process according to which a designer makes decisions. On the other hand, our finite element method requires a polygonal computational domain.

Let $\varkappa > 0$ be a discretization parameter, $\Delta_\varkappa : L_1 = a_0 < a_1 < \ldots < a_n = L_1 + L_2$ be an equidistant partition of $[L_1, L_1+L_2]$, $a_i = L_1 + \frac{i}{n} L_2$, $n = n(\varkappa) = \frac{L_2}{\varkappa}$ and $a_{i-1/2}$ be the midpoint of $[a_{i-1}, a_i]$, $i = 1, \ldots, n$. Further let $A_{i-1/2} = (a_{i-1/2}, \alpha_i)$, $\alpha_i \in \mathbb{R}$, $i = 1, \ldots, n$ be *design nodes* and $A_i = \frac{1}{2}(A_{i-1/2} + A_{i+1/2})$ be the midpoint of the segment $[A_{i-1/2}, A_{i+1/2}]$, $i = 1, \ldots, n-1$. In addition let $A_0 = (a_0, H_1)$, $A_n = (a_n, H_2)$, see Figure 4.1. We introduce the set

$$\begin{aligned}\mathcal{U}^\varkappa := \Big\{ s_\varkappa \in C([0, L]);\ & s_{\varkappa|[0,L_1]} = H_1,\ s_{\varkappa|[L_1+L_2,L]} = H_2, \\ & s_{\varkappa|[a_{i-1},a_i]} \text{ is a quadratic Bézier function} \\ & \qquad \text{determined by } \{A_{i-1}, A_{i-1/2}, A_i\},\ i = 1, \ldots, n \Big\}.\end{aligned}$$

In order to define a family of admissible shapes locally realized by Bézier functions, it is necessary to specify $\alpha_i \in \mathbb{R}$ defining the position of $A_{i-1/2}$,

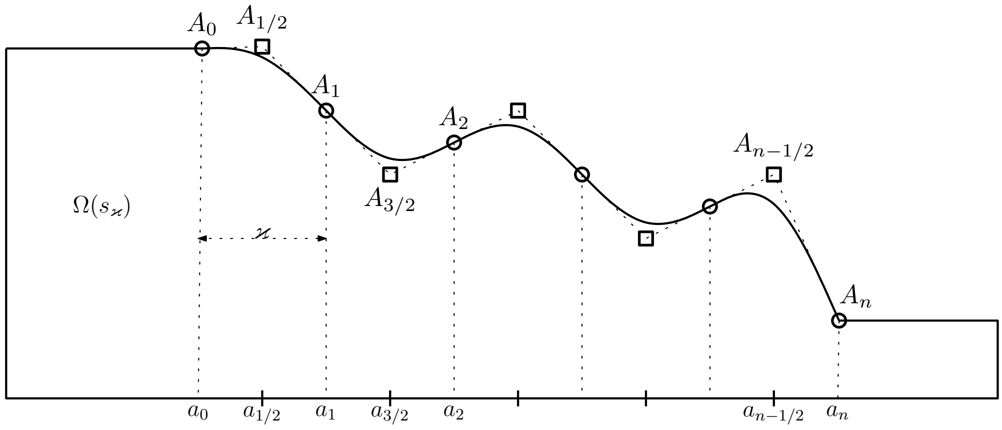

Figure 4.1: Approximation of the boundary of $\Omega(\alpha)$.

$i = 1, \ldots, n$. With the partition $\Delta_\varkappa$ we associate the set $\boldsymbol{U} \subset \mathbb{R}^n$:

$$\boldsymbol{U} = \Big\{ \boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n;\ \alpha_{min} \le \alpha_i \le \alpha_{max},\ i = 1, \ldots, n;$$
$$\frac{|\alpha_{i+1} - \alpha_i|}{\varkappa} \le \gamma,\ i = 1, \ldots, n-1;\ \frac{2|\alpha_0 - H_1|}{\varkappa} \le \gamma,\ \frac{2|\alpha_n - H_2|}{\varkappa} \le \gamma \Big\}.$$

The family of *admissible discretized design domains* is now represented by

$$\mathcal{O}_\varkappa = \{\Omega(s_\varkappa);\ s_\varkappa \in \mathcal{U}_{ad}^\varkappa\},$$

where

$$\mathcal{U}_{ad}^\varkappa = \{s_\varkappa \in \mathcal{U}^\varkappa;\ \text{the design nodes } A_{i-1/2} = (a_{i-1/2}, \alpha_i),\ i = 1, \ldots, n,$$
$$\text{are such that } \boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \boldsymbol{U}\}.$$

Due to properties of the Bézier functions it holds that $\mathcal{U}_{ad}^\varkappa \subset \mathcal{U}_{ad}$.

We now turn to the definition of the computational domains. To this end we introduce another family of partitions $\{\Delta_h\}$, $h \to 0+$, of $[L_1, L_1 + L_2]$ (not necessarily equidistant), whose norm will be denoted by h. Next we will suppose that $h \to 0+$ iff $\varkappa \to 0+$. Let $r_h s_\varkappa$ be the piecewise linear Lagrange interpolant of $s_\varkappa \in \mathcal{U}_{ad}^\varkappa$ on Δ_h. The computational domain related to $\Omega(s_\varkappa)$ will be represented by $\Omega(r_h s_\varkappa)$; i.e. the curved side $\Gamma_{s_\varkappa}$, the graph of $s_\varkappa \in \mathcal{U}_{ad}^\varkappa$, is replaced by its piecewise linear Lagrange approximation $r_h s_\varkappa$ on Δ_h. The system of *computational domains* will be denoted by $\mathcal{O}_{\varkappa h}$ in what follows:

$$\mathcal{O}_{\varkappa h} := \{\Omega(r_h s_\varkappa);\ s_\varkappa \in \mathcal{U}_{ad}^\varkappa\}.$$

Since $\Omega(r_h s_\varkappa)$ is already polygonal, one can construct its triangulation $T_h(s_\varkappa)$ with the norm $h > 0$ and depending on $s_\varkappa \in \mathcal{U}_{ad}^\varkappa$.

Convention. *The domain* $\Omega(r_h s_\varkappa)$ *with a given triangulation* $T_h(s_\varkappa)$ *will be denoted by* $\Omega_h(s_\varkappa)$ *in what follows.*

4.2 Formulation of the discrete problem

Let us define the set

$$\mathcal{G}_{\varkappa h} := \{(s_\varkappa, \boldsymbol{v}_h, p_h);\ s_\varkappa \in \mathcal{U}_{ad}^\varkappa,\ (\boldsymbol{v}_h, p_h) \text{ is a solution of } (\mathcal{P}_h(s_\varkappa))\}.$$

The discretization of $(\mathbb{P})$ then reads as follows:

$$\left\{ \begin{array}{l} \text{Find } (s_\varkappa^*, \boldsymbol{v}_h^*, p_h^*) \in \mathcal{G}_{\varkappa h} \text{ such that} \\ \\ J(s_\varkappa^*, \boldsymbol{v}_h^*, p_h^*) \le J(s_\varkappa, \boldsymbol{v}_h, p_h) \ \ \forall (s_\varkappa, \boldsymbol{v}_h, p_h) \in \mathcal{G}_{\varkappa h}. \end{array} \right. \tag{$\mathbb{P}_{\varkappa h}$}$$

The approximate optimal shape is given by $\Omega(s_\varkappa^*)$.

Next we will analyze the existence of solutions to $(\mathbb{P}_{\varkappa h})$ and their relation to solutions of $(\widehat{\mathbb{P}})$ as $h, \varkappa \to 0+$.

4.3 Existence of solutions

In order to establish the existence results, we have to impose additional assumptions on the family of triangulations $\{T_h(s_\varkappa)\}$, $h, \varkappa \to 0+$, which are listed below.

We will suppose that, for any $h, \varkappa > 0$ fixed, the system $\{T_h(s_\varkappa)\}$, $s_\varkappa \in \mathcal{U}_{ad}^\varkappa$ consists of topologically equivalent triangulations, meaning that

(T1) the triangulation $T_h(s_\varkappa)$ has the same number of nodes and the nodes still have the same neighbors for any $s_\varkappa \in \mathcal{U}_{ad}^\varkappa$;

(T2) the positions of the nodes of $T_h(s_\varkappa)$ depend solely and continuously on variations of the design nodes $\{A_{i-1/2}\}_{i=1}^n$.

For $h, \varkappa \to 0+$ we suppose that

(T3) the family $\{T_h(s_\varkappa)\}$ is uniformly regular with respect to $h, \varkappa$ and $s_\varkappa \in \mathcal{U}_{ad}^\varkappa$: there is $\theta_0 > 0$ such that $\theta(h, s_\varkappa) \ge \theta_0$, $\forall h, \varkappa > 0$, $\forall s_\varkappa \in \mathcal{U}_{ad}^\varkappa$, where $\theta(h, s_\varkappa)$ is the minimal interior angle of all triangles from $T_h(s_\varkappa)$.

Finally, due to the mixed boundary conditions, we suppose that

(T4) the family $\{T_h(s_\varkappa)\}$ is consistent with the decomposition of $\partial\Omega_h(s_\varkappa)$ into Γ_{out} and $\partial\Omega_h(s_\varkappa) \setminus \Gamma_{\text{out}}$.

Let us note that $(T3)-(T4)$ imply the assumptions $(A1)-(A2)$ from the previous chapter.

One can easily show that $(\mathbb{P}_{\varkappa h})$ leads to the following nonlinear programming problem:

$$\begin{cases} \min\limits_{(\boldsymbol{\alpha},\boldsymbol{q}(\boldsymbol{\alpha}))\in \boldsymbol{U}\times\mathbb{R}^m} \mathcal{J}(\boldsymbol{\alpha},\boldsymbol{q}(\boldsymbol{\alpha})) \text{ subject to} \\ \\ \boldsymbol{R}(\boldsymbol{\alpha},\boldsymbol{q}(\boldsymbol{\alpha})) = \boldsymbol{0}, \end{cases} \tag{$\mathbb{P}_n$}$$

where $\mathcal{J}$, $\boldsymbol{R}$, $\boldsymbol{q}(\boldsymbol{\alpha})$ is the algebraic representation of J, $(\mathcal{P}_h(s_\varkappa))$, $(\boldsymbol{v}_h, p_h)$, respectively.

Remark 4.1. *From $(T1)$ it follows that $m := m_1+m_2$, where $m_1 := \dim W_{0h}$ and $m_2 := \dim L_h$, does not depend on $s_\varkappa \in \mathcal{U}_{ad}^{\varkappa}$ or equivalently on $\boldsymbol{\alpha} \in \boldsymbol{U}$. The components of the residual vector $\boldsymbol{R}$ are given by*

$$\begin{aligned} R_k(\boldsymbol{\alpha},\boldsymbol{q}(\boldsymbol{\alpha})) &= 2\mu_0(\mathbb{D}(\boldsymbol{v}_h),\mathbb{D}(\boldsymbol{\varphi}_h^k))_{\Omega_h(s_\varkappa)} + \rho(v_{hj}\frac{\partial v_{hi}}{\partial x_j},\varphi_{hi}^k)_{\Omega_h(s_\varkappa)} \\ &+ \frac{\rho}{2}((\operatorname{div}\boldsymbol{v}_h)(\boldsymbol{v}_h-\boldsymbol{v}_0),\boldsymbol{\varphi}_h^k)_{\Omega_h(s_\varkappa)} + \langle A(\boldsymbol{v}_h),\boldsymbol{\varphi}_h^k\rangle_{\Omega_h(s_\varkappa)} \\ &+ \sigma\int_{\Gamma_{out}}|v_{h2}|v_{h2}\varphi_{h2}^k - (p_h,\operatorname{div}\boldsymbol{\varphi}_h^k)_{\Omega_h(s_\varkappa)},\ k=1,\dots,m_1, \\ R_{m_1+k}(\boldsymbol{\alpha},\boldsymbol{q}(\boldsymbol{\alpha})) &= (\psi_h^k,\operatorname{div}\boldsymbol{v}_h)_{\Omega_h(s_\varkappa)},\ k=1,\dots,m_2, \end{aligned}$$

where

$$\boldsymbol{v}_h := \boldsymbol{v}_h(\boldsymbol{\alpha}) = \boldsymbol{v}_0 + \sum_{k=1}^{m_1} q_k(\boldsymbol{\alpha})\boldsymbol{\varphi}_h^k,$$

$$p_h := p_h(\boldsymbol{\alpha}) = \sum_{k=1}^{m_2} q_{m_1+k}(\boldsymbol{\alpha})\psi_h^k$$

and $\{\boldsymbol{\varphi}_h^k\}$, $\{\psi_h^k\}$ is the Courant basis of $W_{0h}(s_\varkappa)$, $L_h(s_\varkappa)$, respectively. The discrete cost function then reads:

$$\mathcal{J}(\boldsymbol{\alpha},\boldsymbol{q}(\boldsymbol{\alpha})) = \int_{\tilde{\Gamma}}|v_{h2}-v_{opt}|^2.$$

Further we will assume that the inf-sup condition (3.8) holds with a constant C_{BB} independent of $s_\varkappa \in \mathcal{U}_{ad}^{\varkappa}$. Then Lemma 3.7 holds with a constant C_p independent of $\varkappa_n \in \mathcal{U}_{ad}^{\varkappa}$. We recall the apriori estimates:

$$\begin{aligned} \|\nabla\boldsymbol{v}_h\|_{2,\Omega_h(s_\varkappa)}^2 + \|M|\mathbb{D}(\boldsymbol{v}_h)|\|_{3,\Omega_h(s_\varkappa)}^3 + \|v_{h2}\|_{3,\Gamma_{out}}^3 \le C_E, \\ \|p_h\|_{\frac{3}{2},\Omega_h(s_\varkappa)} \le C_p, \end{aligned}$$

where $C_E > 0$ and $C_p > 0$ is independent of $h > 0$ and $s_\varkappa \in \mathcal{U}_{ad}^\varkappa$.

Since $\boldsymbol{U}$ is a bounded subset of $\mathbb{R}^n$, it is compact as well. In what follows we will show that the mapping $\boldsymbol{\alpha} \mapsto \boldsymbol{q}(\boldsymbol{\alpha})$ is continuous.

Lemma 4.1. *Let $\boldsymbol{\alpha}_N \to \boldsymbol{\alpha}$, $N \to \infty$, where $\boldsymbol{\alpha}_N, \boldsymbol{\alpha} \in \boldsymbol{U}$, and let $\boldsymbol{q}(\boldsymbol{\alpha}_N)$ satisfy $\boldsymbol{R}(\boldsymbol{\alpha}_N, \boldsymbol{q}(\boldsymbol{\alpha}_N)) = \boldsymbol{0}$. Then there is a $\boldsymbol{q}(\boldsymbol{\alpha}) \in \mathbb{R}^m$ and a subsequence (denoted identically) such that*

$$\boldsymbol{q}(\boldsymbol{\alpha}_N) \to \boldsymbol{q}(\boldsymbol{\alpha}),\ N \to \infty \tag{4.1}$$

and additionally $\boldsymbol{R}(\boldsymbol{\alpha}, \boldsymbol{q}(\boldsymbol{\alpha})) = \boldsymbol{0}$.

Proof. First we prove (4.1) using the apriori estimates

$$\|\boldsymbol{v}_h(\boldsymbol{\alpha}_N)\|_{1,2,\Omega_h(s_{\varkappa N})} + \|p_h(\boldsymbol{\alpha}_N)\|_{\frac{3}{2},\Omega_h(s_{\varkappa N})} \le C_E.$$

Let us denote by $\Phi(\boldsymbol{\alpha}_N)$, $\Psi(\boldsymbol{\alpha}_N)$ the Gramm matrix of the basis $\{\boldsymbol{\varphi}_h^k(\boldsymbol{\alpha}_N)\}$, $\{\psi_h^k(\boldsymbol{\alpha}_N)\}$, respectively, i.e.

$$\Phi_{kl}(\boldsymbol{\alpha}_N) := (\boldsymbol{\varphi}_h^k(\boldsymbol{\alpha}_N), \boldsymbol{\varphi}_h^l(\boldsymbol{\alpha}_N))_{\Omega_h(s_{\varkappa N})},\ k,l = 1,\dots,m_1,$$

$$\Psi_{kl}(\boldsymbol{\alpha}_N) := (\boldsymbol{\psi}_h^k(\boldsymbol{\alpha}_N), \boldsymbol{\psi}_h^l(\boldsymbol{\alpha}_N))_{\Omega_h(s_{\varkappa N})},\ k,l = 1,\dots,m_2.$$

Then, as a consequence of $(T3)$, these matrices are positive definite with a constant $\beta > 0$, which is independent of $\boldsymbol{\alpha}_N$. Therefore

$$\|\boldsymbol{v}_h(\boldsymbol{\alpha}_N) - \boldsymbol{v}_0\|^2_{1,2,\Omega_h(s_{\varkappa N})} = \boldsymbol{q}_1(\boldsymbol{\alpha}_N) \cdot \Phi(\boldsymbol{\alpha}_N)\boldsymbol{q}_1(\boldsymbol{\alpha}_N) \ge \beta|\boldsymbol{q}_1(\boldsymbol{\alpha}_N)|^2,$$

where $\boldsymbol{q}_1(\boldsymbol{\alpha}_N) := (q_1(\boldsymbol{\alpha}_N), \dots, q_{m_1}(\boldsymbol{\alpha}_N))$. Since

$$|p_h(\boldsymbol{\alpha}_N)| \le \sum_{k=1}^{m_2} |q_{m_1+k}(\boldsymbol{\alpha}_N)|,$$

it also holds:

$$\begin{aligned}\|p_h(\boldsymbol{\alpha}_N)\|^{\frac{3}{2}}_{\frac{3}{2},\Omega_h(s_{\varkappa N})} \ge \frac{\|p_h(\boldsymbol{\alpha}_N)\|^2_{2,\Omega_h(s_{\varkappa N})}}{\sqrt{\sum_{k=1}^{m_2} |q_{m_1+k}(\boldsymbol{\alpha}_N)|}} &\ge \frac{\boldsymbol{q}_2(\boldsymbol{\alpha}_N) \cdot \Psi(\boldsymbol{\alpha}_N)\boldsymbol{q}_2(\boldsymbol{\alpha}_N)}{|\boldsymbol{q}_2(\boldsymbol{\alpha}_N)|^{\frac{1}{2}}} \\ &\ge \beta|\boldsymbol{q}_2(\boldsymbol{\alpha}_N)|^{\frac{3}{2}},\end{aligned}$$

where $\boldsymbol{q}_2(\boldsymbol{\alpha}_N) := (q_{m_1+1}(\boldsymbol{\alpha}_N), \dots, q_m(\boldsymbol{\alpha}_N))$. Altogether we have:

$$|\boldsymbol{q}(\boldsymbol{\alpha}_N)| \le \frac{1}{\beta^{\frac{1}{2}}}\|\boldsymbol{v}_h(\boldsymbol{\alpha}_N) - \boldsymbol{v}_0\|_{1,2,\Omega_h(s_{\varkappa N})} + \frac{1}{\beta^{\frac{2}{3}}}\|p_h(\boldsymbol{\alpha}_N)\|_{\frac{3}{2},\Omega_h(s_{\varkappa N})} \le C,$$

where $C > 0$ is independent of N. Hence there is a subsequence such that (4.1) holds.

In order to show that $\boldsymbol{R}(\boldsymbol{\alpha}, \boldsymbol{q}(\boldsymbol{\alpha})) = \boldsymbol{0}$, it is sufficient to verify that

$$\begin{aligned}\tilde{\boldsymbol{v}}_h(\boldsymbol{\alpha}_N) \rightarrow & \tilde{\boldsymbol{v}}_h(\boldsymbol{\alpha}), \text{ in } (W^{1,3}(\widehat{\Omega}))^2\\ \tilde{p}_h(\boldsymbol{\alpha}_N) \rightarrow & \tilde{p}_h(\boldsymbol{\alpha}), \text{ in } L^{\frac{3}{2}}(\widehat{\Omega}),\ N \to \infty.\end{aligned}$$

This is not difficult, since from $(T2) - (T3)$ it follows that

$$\begin{aligned}\tilde{\boldsymbol{\varphi}}_h^k(\boldsymbol{\alpha}_N) \rightarrow & \tilde{\boldsymbol{\varphi}}_h^k(\boldsymbol{\alpha}), \text{ in } (W^{1,3}(\widehat{\Omega}))^2,\ k = 1, \ldots, m_1,\\ \tilde{\psi}_h^k(\boldsymbol{\alpha}_N) \rightarrow & \tilde{\psi}_h^k(\boldsymbol{\alpha}), \text{ in } L^{\frac{3}{2}}(\widehat{\Omega}),\ k = 1, \ldots, m_2,\ N \to \infty.\end{aligned}$$

Now it is easy to show that $\boldsymbol{R}(\boldsymbol{\alpha}_N, \boldsymbol{q}(\boldsymbol{\alpha}_N)) \to \boldsymbol{R}(\boldsymbol{\alpha}, \boldsymbol{q}(\boldsymbol{\alpha}))$. □

As an easy consequence we obtain the existence of an optimal discrete shape.

Theorem 4.2. *Problem* $(\mathbb{P}_n)$ *(and equivalently* $(\mathbb{P}_{\varkappa h})$*) has a solution.*

Proof. Since $\boldsymbol{U}$ is a compact subset of $\mathbb{R}^n$ and the mapping $\boldsymbol{\alpha} \mapsto \boldsymbol{q}(\boldsymbol{\alpha})$ is continuous, the statement follows directly from the classical Bolzano-Weierstrass theorem. □

4.4 Convergence

Lemma 4.3. *Let* $(s_\varkappa, \boldsymbol{v}_h(s_\varkappa), p_h(s_\varkappa)) \in \mathcal{G}_{\varkappa h},\ h, \varkappa \to 0+,\ s_\varkappa \in \mathcal{U}_{ad}^{\varkappa}$, *and* $\alpha \in \mathcal{U}_{ad}$ *satisfy*

$$s_\varkappa \rightrightarrows \alpha \text{ in } [0, L],\ \varkappa \to 0+.$$

Then there exists $\widehat{\boldsymbol{v}} \in \left(W^{1,2}(\widehat{\Omega})\right)^2$, $\widehat{p} \in L^{\frac{3}{2}}(\widehat{\Omega})$ *and an appropriate subsequence such that*

$$\begin{aligned}\tilde{\boldsymbol{v}}_h(s_\varkappa) &\rightharpoonup \widehat{\boldsymbol{v}} \text{ in } \left(W^{1,2}(\widehat{\Omega})\right)^2,\\ \tilde{M}_{s_\varkappa}\mathbb{D}(\tilde{\boldsymbol{v}}_h(s_\varkappa)) &\rightharpoonup \tilde{M}_\alpha \mathbb{D}(\widehat{\boldsymbol{v}}) \text{ in } \left(L^3(\widehat{\Omega})\right)^{2\times 2},\\ \tilde{p}_h(s_\varkappa) &\rightharpoonup \widehat{p} \text{ in } L^{\frac{3}{2}}(\widehat{\Omega}),\ h, \varkappa \to 0+.\end{aligned} \tag{4.2}$$

In addition, if we define $\boldsymbol{v}(\alpha) := \widehat{\boldsymbol{v}}_{|\Omega(\alpha)}$ *and* $p(\alpha) := \widehat{p}_{|\Omega(\alpha)}$, *then* $\boldsymbol{v}(\alpha) \in \widehat{W}_{\boldsymbol{v}_0}$ *and* $(\boldsymbol{v}(\alpha), p(\alpha))$ *solves* $(\widehat{\mathcal{P}}(\alpha))$.

Proof. We will proceed in the same way as in the proof of Theorem 2.3, considering only several minor changes.

Using the apriori estimates one shows that the sequence $\{\|\boldsymbol{v}_h\|_{s_\varkappa}, \|p_h\|_{\frac{3}{2},\Omega(s_\varkappa)}\}$ is bounded and that (4.2) holds for a proper subsequence. Since $r_h s_\varkappa \rightrightarrows \alpha$ in $[0,L]$ as $h,\varkappa \to 0+$, we easily get that $\boldsymbol{v}(\alpha) := \widehat{\boldsymbol{v}}_{|\Omega(\alpha)} \in \widehat{W}_{\boldsymbol{v}_0}$.

We will focus on the limit passage in $(\mathcal{P}_h(s_\varkappa))$. Let $\boldsymbol{\varphi} \in \mathcal{V}_0(\alpha)$ be given and $\boldsymbol{\varphi}_h$ be its piecewise linear Lagrange interpolant on $T_h(s_\varkappa)$. For $h,\varkappa > 0$ small enough, the graph of $r_h s_\varkappa$ has an empty intersection with supp $\tilde{\boldsymbol{\varphi}}$, which means that $\boldsymbol{\varphi}_h \in W_{0h}(s_\varkappa)$ and it can be used as a test function in $(\mathcal{P}_h(s_\varkappa))$. In addition,

$$\|\tilde{\boldsymbol{\varphi}}_h - \tilde{\boldsymbol{\varphi}}\|_{1,\infty,\widehat{\Omega}} = \|\tilde{\boldsymbol{\varphi}}_h - \tilde{\boldsymbol{\varphi}}\|_{1,\infty,\Omega_h(s_\varkappa)} \le Ch\|\tilde{\boldsymbol{\varphi}}\|_{\mathcal{C}^2(\widehat{\Omega})}, \tag{4.3}$$

where $C > 0$ is a constant that does not depend on $h, \varkappa$ and $s_\varkappa$, as follows from the well-known approximation results and the uniform regularity assumption $(T3)$ on $\{T_h(s_\varkappa)\}$. Now we can pass to the limit in the standard terms:

$$\begin{aligned}
(\mathbb{D}(\tilde{\boldsymbol{v}}_h), \mathbb{D}(\tilde{\boldsymbol{\varphi}}_h))_{\widehat{\Omega}} &\to (\mathbb{D}(\tilde{\boldsymbol{v}}(\alpha)), \mathbb{D}(\tilde{\boldsymbol{\varphi}}))_{\widehat{\Omega}},\\
\int_{\Gamma_{\text{out}}} |\tilde{v}_{h2}|\tilde{v}_{h2}\tilde{\varphi}_{h2} &\to \int_{\Gamma_{\text{out}}} |\tilde{v}_2(\alpha)|\tilde{v}_2(\alpha)\tilde{\varphi}_2,\\
(\tilde{v}_{hj}\frac{\partial \tilde{v}_{hi}}{\partial x_j}, \tilde{\varphi}_{hi})_{\widehat{\Omega}} &\to (\tilde{v}_j(\alpha)\frac{\partial \tilde{v}_i(\alpha)}{\partial x_j}, \tilde{\varphi}_i)_{\widehat{\Omega}},\\
(\tilde{p}_h, \operatorname{div}\tilde{\boldsymbol{\varphi}}_h)_{\widehat{\Omega}} &\to (\tilde{p}(\alpha), \operatorname{div}\tilde{\boldsymbol{\varphi}})_{\widehat{\Omega}},\ \ h,\varkappa \to 0+,
\end{aligned} \tag{4.4}$$

as follows from (4.2) and (4.3).

Finally, in order to show that

$$\langle \tilde{A}_{r_h s_\varkappa}(\tilde{\boldsymbol{v}}_h), \tilde{\boldsymbol{\varphi}}_h\rangle \to \langle \tilde{A}_\alpha(\tilde{\boldsymbol{v}}(\alpha)), \tilde{\boldsymbol{\varphi}}\rangle, \tag{4.5}$$

we use pointwise convergence of $\mathbb{D}(\tilde{\boldsymbol{v}}_h)$ and the Vitali theorem. Using the notation of the proof of Theorem 2.3, we again consider a subdomain $\Omega_\varepsilon \subset \Omega(\alpha)$, $\varepsilon > 0$, and $\xi := \xi_\varepsilon \in \mathcal{C}_0^\infty(\overline{\Omega}(\alpha))$ with the same properties. We construct a test function $\boldsymbol{\varphi} := \xi(\tilde{\boldsymbol{v}}_{h_1|\Omega(\alpha)} - \tilde{\boldsymbol{v}}_{h_2|\Omega(\alpha)})$, where $h_1, h_2 > 0$. Instead of inserting $\boldsymbol{\varphi}$ directly into $(\mathcal{P}_{h_1}(r_h s_\varkappa))$ and $(\mathcal{P}_{h_2}(r_h s_\varkappa))$, we must use the Lagrange interpolant $\boldsymbol{\varphi}_{h_1}$, $\boldsymbol{\varphi}_{h_2}$, respectively. We realize that if $h_1 > 0$ is small enough, then

$$|(\mathbb{D}(\boldsymbol{v}_{h_1}), \mathbb{D}(\boldsymbol{\varphi}_{h_1} - \boldsymbol{\varphi}))_{\operatorname{supp}\xi}| \le \delta,$$

as follows from the Hölder inequality and (4.3). The same can be applied for the remaining terms in (3.1).Therefore we can apply the same process as in (2.18)-(2.21) and obtain:

$$2\mu_0\|\mathbb{D}(\boldsymbol{v}_{h_1} - \boldsymbol{v}_{h_2})\|^2_{2,\Omega_\varepsilon} \le 2\mu_0\|\sqrt{\xi}\mathbb{D}(\boldsymbol{v}_{h_1} - \boldsymbol{v}_{h_2})\|^2_{2,\operatorname{supp}\xi} \le C\delta,$$

for $h_1, h_2 > 0$ small enough, where $C > 0$ is independent of h_1, h_2. Consequently

$$\mathbb{D}(\tilde{\boldsymbol{v}}_h) \to \mathbb{D}(\tilde{\boldsymbol{v}}(\alpha)),\ h, \varkappa \to 0+,$$

a.e. in $\widehat{\Omega}$ and the Vitali theorem yields (4.5). This completes the proof of that $(\boldsymbol{v}(\alpha), p(\alpha))$ solves $(\widehat{\mathcal{P}}(\alpha))$. $\square$

Remark 4.2. *Identically to the continuous case, due to the lack of a density result for $W_0(\alpha)$, we are not able to prove that the limit $\boldsymbol{v}(\alpha)$ belongs to $W_{\boldsymbol{v}_0}(\alpha)$. Therefore the augmented state problem $(\widehat{\mathcal{P}}(\alpha))$ and shape optimization problem $(\widehat{\mathbb{P}})$ is considered instead of $(\mathcal{P}(\alpha))$ and $(\mathbb{P})$, respectively.*

On the basis of the previous lemma we obtain the following result for convergence of discrete optimal solutions.

Theorem 4.4. *Let* (1.77) *be satisfied (i.e. the solutions of $(\mathcal{P}(\alpha))$ and $(\widehat{\mathcal{P}}(\alpha))$, $\alpha \in \mathcal{U}_{ad}$, are unique). Let $\{(s^*_\varkappa, \boldsymbol{v}^*_h, p^*_h)\}$ be a sequence of optimal pairs of $(\mathbb{P}_{\varkappa h})$, $h, \varkappa \to 0+$. Then there is a subsequence of $\{s^*_\varkappa, \boldsymbol{v}^*_h, p^*_h)\}$ such that*

$$s^*_\varkappa \rightrightarrows \alpha^* \text{ in } [0, L], \tag{4.6a}$$

$$\tilde{\boldsymbol{v}}^*_h \rightharpoonup \boldsymbol{v}^* \text{ in } \left(W^{1,2}(\widehat{\Omega})\right)^2, \tag{4.6b}$$

$$\tilde{M}_{s_\varkappa}\mathbb{D}(\tilde{\boldsymbol{v}}^*_h) \rightharpoonup \tilde{M}_\alpha \mathbb{D}(\boldsymbol{v}^*) \text{ in } \left(L^3(\widehat{\Omega})\right)^{2\times 2}, \tag{4.6c}$$

$$\tilde{p}^*_h \rightharpoonup p^* \text{ in } L^{\frac{3}{2}}(\widehat{\Omega}),\ h, \varkappa \to 0+, \tag{4.6d}$$

where $(\alpha^*, \boldsymbol{v}^*_{|\Omega(\alpha^*)}, p^*_{|\Omega(\alpha^*)})$ is an optimal triple for $(\widehat{\mathbb{P}})$. In addition, any accumulation point of $\{s^*_\varkappa, \boldsymbol{v}^*_h, p^*_h)\}$ in the sense of (4.6) *possesses this property.*

Proof. Let $\overline{\alpha} \in \mathcal{U}_{ad}$ be arbitrary. Then there exists a sequence $\{\overline{s}_\varkappa\}$, $\overline{s}_\varkappa \in \mathcal{U}^\varkappa_{ad}$, such that $\overline{s}_\varkappa \rightrightarrows \overline{\alpha}$ in $[0, L]$, $\varkappa \to 0+$, as follows from the well known properties of Bézier functions. From Lemma 4.3 it follows that

$$\tilde{\boldsymbol{v}}_h(\overline{s}_\varkappa) \rightharpoonup \overline{\boldsymbol{v}} \text{ in } \left(W^{1,2}(\widehat{\Omega})\right)^2, \tag{4.7}$$

$$\tilde{M}_{\overline{s}_\varkappa}\mathbb{D}(\tilde{\boldsymbol{v}}_h(\overline{s}_\varkappa)) \rightharpoonup \tilde{M}_\alpha \mathbb{D}(\overline{\boldsymbol{v}}) \text{ in } \left(L^3(\widehat{\Omega})\right)^{2\times 2}, \tag{4.8}$$

$$\tilde{p}_h(\overline{s}_\varkappa) \rightharpoonup \overline{p} \text{ in } L^{\frac{3}{2}}(\widehat{\Omega}),\ h, \varkappa \to 0+, \tag{4.9}$$

where $(\boldsymbol{v}_h(\overline{s}_\varkappa), p_h(\overline{s}_\varkappa))$ are the solutions of $(\mathcal{P}_h(r_h\overline{s}_\varkappa))$ and $(\boldsymbol{v}(\overline{\alpha}), p(\overline{\alpha})) := (\overline{\boldsymbol{v}}_{|\Omega(\overline{\alpha})}, \overline{p}_{|\Omega(\overline{\alpha})})$ is the unique solution of $(\widehat{\mathcal{P}}(\overline{\alpha}))$. Since J is continuous with respect to the convergence in (4.7) and

$$J(s^*_\varkappa, \boldsymbol{v}^*_h, p^*_h) \le J(\overline{s}_\varkappa, \boldsymbol{v}_h(\overline{s}_\varkappa), p_h(\overline{s}_\varkappa)),$$

we have that

$$J(\alpha^*, \boldsymbol{v}^*_{|\Omega(\alpha^*)}, p^*_{|\Omega(\alpha^*)}) \leq J(\overline{\alpha}, \boldsymbol{v}(\overline{\alpha}), p(\overline{\alpha})).$$

Here $\overline{\alpha} \in \mathcal{U}_{ad}$ is arbitrary, hence $(\alpha^*, \boldsymbol{v}^*_{|\Omega(\alpha^*)}, p^*_{|\Omega(\alpha^*)})$ is a solution of $(\widehat{\mathbb{P}})$. □

Remark 4.3. *Let us mention that the state solutions must be unique for the complete convergence result. Otherwise the limit solutions are optimal only in a subclass of $\widehat{\mathcal{G}}$ formed by all possible limits of solutions to $(\mathcal{P}_h(r_h s_\varkappa))$, $h, \varkappa \to 0+$.*

Chapter 5

Numerical realization

In this chapter we present a method of numerical solution of the shape optimization problem. We would like to emphasize that our implementation is not restricted to the particular problem of this thesis but can be applied to a wide range of shape optimization and optimal control problems that can be formulated like $(\mathbb{P}_n)$.

5.1 State problem

We will start with the description of the numerical solution of the discrete state problem $(\mathcal{P}_h(\alpha))$ (see Chapter 3 for definition of $(\mathcal{P}_h(\alpha))$, $\alpha \in \mathcal{U}_{ad}$). For this reason the algebraic form

$$\boldsymbol{R}(\boldsymbol{\alpha}, \boldsymbol{q}(\boldsymbol{\alpha})) = \boldsymbol{0}, \tag{5.1}$$

used in the definition of $(\mathbb{P}_n)$, is more suitable. In this section we assume that $\boldsymbol{\alpha}$ is given and consider (5.1) as a system of m algebraic equations for the vector of unknowns $\boldsymbol{q} := \boldsymbol{q}(\boldsymbol{\alpha}) \in \mathbb{R}^m$. This system is of course nonlinear, therefore a suitable linearization has to be done. We use the Newton-Raphson method:

$$\textit{Given } \boldsymbol{q}_k \in \mathbb{R}^m, \textit{ define } \boldsymbol{q}_{k+1} := \boldsymbol{q}_k - \left(\frac{\partial \boldsymbol{R}}{\partial \boldsymbol{q}}(\boldsymbol{\alpha}, \boldsymbol{q}_k)\right)^{-1} \boldsymbol{R}(\boldsymbol{\alpha}, \boldsymbol{q}_k). \tag{5.2}$$

Let us recall that the sequence $\{\boldsymbol{q}_k\}$, $k = 0, 1, \ldots$, converges provided that the initial guess $\boldsymbol{q}_0$ is close enough to the solution of (5.1). Thus we have to supply a good approximation of $\boldsymbol{q}$ at the beginning. This is usually done by using some other algorithm (e.g. the fixed point iterations) prior to the Newton-Raphson method. The main advantage of this method is that if

$\boldsymbol{R}$ is twice continuously differentiable and the inverse of $\frac{\partial \boldsymbol{R}}{\partial \boldsymbol{q}}(\boldsymbol{\alpha}, \boldsymbol{q}(\boldsymbol{\alpha}))$ exists, then the convergence of (5.2) is at least quadratic. Instead of computing the inverse matrix $\left(\frac{\partial \boldsymbol{R}}{\partial \boldsymbol{q}}(\boldsymbol{\alpha}, \boldsymbol{q}_k)\right)^{-1}$, we solve for every k the linear system

$$\frac{\partial \boldsymbol{R}}{\partial \boldsymbol{q}}(\boldsymbol{\alpha}, \boldsymbol{q}_k)\Delta \boldsymbol{q}_k = \boldsymbol{R}(\boldsymbol{\alpha}, \boldsymbol{q}_k) \tag{5.3}$$

for the unknown $\Delta \boldsymbol{q}_k \in \mathbb{R}^m$ and put $\boldsymbol{q}_{k+1} := \boldsymbol{q}_k - \Delta \boldsymbol{q}_k$. For the solution of (5.3) we use the package SuperLU, which performs an LU decomposition with partial pivoting (see [6] for detailed description).

Remark 5.1. *The analytical form of* (5.3) *can be expressed using Remark 4.1 as follows:*

Find $(\Delta \boldsymbol{v}_h^k, \Delta p_h^k) \in W_{0h} \times L_h$ *such that*

$$\begin{aligned}
&2\mu_0(\mathbb{D}(\Delta \boldsymbol{v}_h^k), \mathbb{D}(\boldsymbol{\varphi}_h^l))_{\Omega_h(s_\varkappa)} + \rho(\Delta v_{hj}^k \frac{\partial v_{hi}^k}{\partial x_j} + v_{hj}^k \frac{\partial \Delta v_{hi}^k}{\partial x_j}, \varphi_{hi}^l)_{\Omega_h(s_\varkappa)} \\
&\quad + \frac{\rho}{2}((\Delta \operatorname{div} \boldsymbol{v}_h^k)(\boldsymbol{v}_h^k - \boldsymbol{v}_0) + (\operatorname{div} \boldsymbol{v}_h^k)\Delta \boldsymbol{v}_h^k, \boldsymbol{\varphi}_h^l)_{\Omega_h(s_\varkappa)} \\
&\quad + 2\rho(M^3 \frac{\mathbb{D}(\boldsymbol{v}_h^k) : \mathbb{D}(\Delta \boldsymbol{v}_h^k)}{|\mathbb{D}(\boldsymbol{v}_h^k)|}\mathbb{D}(\boldsymbol{v}_h^k) + M^3|\mathbb{D}(\boldsymbol{v}_h^k)|\mathbb{D}(\Delta \boldsymbol{v}_h^k), \mathbb{D}(\boldsymbol{\varphi}_h^l))_{\Omega_h(s_\varkappa)} \\
&\quad + 2\sigma \int_{\Gamma_{out}} |v_{h2}^k| \Delta v_{h2}^k \varphi_{h2}^l - (\Delta p_h^k, \operatorname{div} \boldsymbol{\varphi}_h^l)_{\Omega_h(s_\varkappa)} \\
= &\, 2\mu_0(\mathbb{D}(\boldsymbol{v}_h^k), \mathbb{D}(\boldsymbol{\varphi}_h^l))_{\Omega_h(s_\varkappa)} + \rho(v_{hj}^k \frac{\partial v_{hi}^k}{\partial x_j}, \varphi_{hi}^l)_{\Omega_h(s_\varkappa)} \\
&\quad + \frac{\rho}{2}((\operatorname{div} \boldsymbol{v}_h^k)(\boldsymbol{v}_h^k - \boldsymbol{v}_0), \boldsymbol{\varphi}_h^l)_{\Omega_h(s_\varkappa)} + \langle A(\boldsymbol{v}_h^k), \boldsymbol{\varphi}_h^l(s_\varkappa)\rangle_{\Omega_h(s_\varkappa)} \\
&\quad + \sigma \int_{\Gamma_{out}} |v_{h2}^k| v_{h2}^k \varphi_{h2}^l - (p_h^k, \operatorname{div} \boldsymbol{\varphi}_h^l)_{\Omega_h(s_\varkappa)}, \ \ l = 1, \dots, m_1, \\
&\qquad (\psi_h^l, \operatorname{div} \Delta \boldsymbol{v}_h^k)_{\Omega_h(s_\varkappa)} = (\psi_h^l, \operatorname{div} \boldsymbol{v}_h^k)_{\Omega_h(s_\varkappa)}, \ \ l = 1, \dots, m_2,
\end{aligned}$$

where $\boldsymbol{q}_k$, $\Delta\boldsymbol{q}_k$ *and* $(\boldsymbol{v}_h^k, p_h^k)$, $(\Delta\boldsymbol{v}_h^k, \Delta p_h^k)$ *are related by the following formulas:*

$$\boldsymbol{v}_h^k = \boldsymbol{v}_0 + \sum_{l=1}^{m_1} q_{kl}\boldsymbol{\varphi}_h^l,$$

$$p_h^k = \sum_{l=1}^{m_2} q_{k,m_1+l}\psi_h^l,$$

$$\Delta\boldsymbol{v}_h^k = \sum_{l=1}^{m_1} \Delta q_{kl}\boldsymbol{\varphi}_h^l,$$

$$\Delta p_h^k = \sum_{l=1}^{m_2} q_{k,m_1+l}\psi_h^l.$$

Using the previous expressions it can be shown that the matrix $\frac{\partial\boldsymbol{R}}{\partial\boldsymbol{q}}(\boldsymbol{\alpha}, \boldsymbol{q}_k)$ *is of the form*

$$\frac{\partial\boldsymbol{R}}{\partial\boldsymbol{q}}(\boldsymbol{\alpha}, \boldsymbol{q}_k) = \begin{pmatrix} \mathbb{A} & \mathbb{B}^T \\ \mathbb{B} & \mathbb{O} \end{pmatrix}$$

and is positive semi-definite in the following sense:

$$\frac{\partial\boldsymbol{R}}{\partial\boldsymbol{q}}(\boldsymbol{\alpha}, \boldsymbol{q}_k)\boldsymbol{x}\cdot\boldsymbol{x} \geq C\sum_{i=1}^{m_1} x_i^2 \quad \forall\boldsymbol{x}\in\mathbb{R}^m,$$

provided that $\sum_{i=1}^{m_1} q_{ki}^2$ *is small enough.*

In our program we do not implement the analytical system as stated in Remark 5.1. Instead, we only specify the assembly of the residual vector $\boldsymbol{R}(\boldsymbol{\alpha}, \boldsymbol{q}_k)$ in such a way that the matrix of the linearized system (5.3) is obtained automatically.

Let E_h be the partition of $\partial\Omega_h(s_\varkappa)$ into edges, associated with T_h. The residual vector is decomposed into the sum of area and boundary integrals, which are further calculated element by element or edge by edge:

$$\boldsymbol{R}(\boldsymbol{\alpha}, \boldsymbol{q}_k) = \sum_{K\in T_h} \boldsymbol{R}_K(\boldsymbol{\alpha}, \boldsymbol{q}_k) + \sum_{E\in E_h} \boldsymbol{R}_E(\boldsymbol{\alpha}, \boldsymbol{q}_k).$$

The local contributions $\boldsymbol{R}_K$, $\boldsymbol{R}_E$ are calculated using a suitable numerical quadrature. If we write the components of the residual vector in the following form:

$$R_l(\boldsymbol{\alpha}, \boldsymbol{q}_k) = \int_{\Omega_h(s_\varkappa)} f_1(\boldsymbol{v}_h^k, \nabla\boldsymbol{v}_h^k, p_h^k, \boldsymbol{\varphi}_h^l) + \int_{\Gamma_{\mathrm{out}}} g(\boldsymbol{v}_h^k, \boldsymbol{\varphi}_h^l),\ l = 1, \ldots, m_1,$$

$$R_{m_1+l}(\boldsymbol{\alpha}, \boldsymbol{q}_k) = \int_{\Omega_h(s_\varkappa)} f_2(p_h^k, \psi_h^l),\ l = 1, \ldots, m_2,$$

then the algorithm for assembling the residual vector reads:

1. Put $\boldsymbol{R}(\boldsymbol{\alpha}, \boldsymbol{q}_k) := \mathbf{0}$;
2. For every $K \in T_h$:
   - For every quadrature point $\boldsymbol{P}$ in K:
     - Evaluate $\boldsymbol{v}_h^k(\boldsymbol{P})$, $\nabla \boldsymbol{v}_h^k(\boldsymbol{P})$ and $p_h^k(\boldsymbol{P})$;
     - For every base function $\boldsymbol{\varphi}_h^l \in W_{0h}$ that is nonzero on K:
       * Evaluate $\boldsymbol{\varphi}_h^l(\boldsymbol{P})$;
       * Put
$$R_{Kl}(\boldsymbol{\alpha}, \boldsymbol{q}_k) := R_{Kl}(\boldsymbol{\alpha}, \boldsymbol{q}_k) \\ + w(K, \boldsymbol{P}) f_1(\boldsymbol{v}_h^k(\boldsymbol{P}), \nabla \boldsymbol{v}_h^k(\boldsymbol{P}), p_h^k(\boldsymbol{P}), \boldsymbol{\varphi}_h^l(\boldsymbol{P}));$$
     - For every base function $\psi_h^l \in L_h$ that is nonzero on K:
       * Evaluate $\psi_h^l(\boldsymbol{P})$;
       * Put
$$R_{K,m_1+l}(\boldsymbol{\alpha}, \boldsymbol{q}_k) := R_{K,m_1+l}(\boldsymbol{\alpha}, \boldsymbol{q}_k) \\ + w(K, \boldsymbol{P}) f_2(\boldsymbol{v}_h^k(\boldsymbol{P}), \psi_h^l(\boldsymbol{P}));$$
3. For every $E \in E_h$ such that $E \subset \Gamma_{\text{out}}$:
   - For every quadrature point $(\boldsymbol{P})$ in E:
     - Evaluate $\boldsymbol{v}_h^k(\boldsymbol{P})$;
     - For every base function $\boldsymbol{\varphi}_h^l$ that is nonzero on E:
       * Evaluate $\boldsymbol{\varphi}_h^l(\boldsymbol{P})$;
       * Put $R_{El}(\boldsymbol{\alpha}, \boldsymbol{q}_k) := R_{El}(\boldsymbol{\alpha}, \boldsymbol{q}_k) + w(E, \boldsymbol{P}) g(\boldsymbol{v}_h^k(\boldsymbol{P}), \boldsymbol{\varphi}_h^l(\boldsymbol{P}))$.

Here $w(K, \boldsymbol{P})$ and $w(E, \boldsymbol{P})$ stands for the respective quadrature weight. The evaluation of $\boldsymbol{v}_h^k$, $\nabla \boldsymbol{v}_h^k$, p_h^k, $\boldsymbol{\varphi}_h^l$, ψ_h^l in the particular quadrature points will contain latent evaluation of their partial derivatives with respect to $\boldsymbol{\alpha}$ and $\boldsymbol{q}$. We will describe this technique (called the automatic differentiation) thoroughly below in Section 5.3.

Finally we write schematically the algorithm for the numerical solution of the state problem:

1. Choose the tolerance for the residuum $r_{max} > 0$ and the max. number of nonlinear iterations $k_{max} \in \mathbb{N}$.
2. Choose $\boldsymbol{q}_0 \in \mathbb{R}^m$.

3. For $k = 0, \dots, k_{max} - 1$:
   - Compute $\boldsymbol{b}_k := \boldsymbol{R}(\boldsymbol{\alpha}, \boldsymbol{q}_k)$ and put $\mathbb{A}_k := \frac{\partial \boldsymbol{R}(\boldsymbol{\alpha}, \boldsymbol{q}_k)}{\partial \boldsymbol{q}}$.
   - Solve $\mathbb{A}_k \Delta \boldsymbol{q}_k = \boldsymbol{b}_k$ and put $\boldsymbol{q}_{k+1} := \boldsymbol{q}_k - \Delta \boldsymbol{q}_k$, $r_k := |\boldsymbol{b}_k|$.
   - If $r_k < r_{max}$ then go to 4.
4. If $r_k < r_{max}$ then put $\boldsymbol{q} := \boldsymbol{q}_k$, otherwise report error.

5.2 Shape optimization problem

We solve numerically the mathematical programming problem $(\mathbb{P}_n)$ with a suitable choice of n. Since the function to be minimized is smooth, we will use a gradient based minimization algorithm. Gradient based methods usually require significantly less function evaluations than methods which work only with function values. On the other hand, we have to provide sufficiently accurate cost function gradient. This is usually done by performing the sensitivity analysis, which can be quite involved in shape optimization, since one must deal with shape derivatives (see [22] for more details on the shape sensitivity analysis). In our approach we provide a simple algebraic sensitivity analysis of the discrete cost function. Most of the tedious derivatives will be calculated by means of the automatic differentiation.

For the numerical minimization itself we use the following packages:

- KNITRO - a robust tool for many types of smooth optimization problems. KNITRO has 3 algorithms for dealing with the constraints: interior point method [23], interior CG method [5, 2] and active set method [3, 4].
- NAG C library - the function e04wdc is intended for smooth optimization and uses the sequential quadratic programming [8].

Both packages provide a Fortran/C interface that allows to supply arbitrary routines for the cost function and gradient evaluation. A comparison of the packages and the obtained results can be found within the example computations in Section 5.4.

The value of the cost function $\mathcal{J}$ will be calculated in the following order:

$$\boldsymbol{\alpha} \mapsto \boldsymbol{q}(\boldsymbol{\alpha}) \mapsto \mathcal{J}(\boldsymbol{\alpha}, \boldsymbol{q}(\boldsymbol{\alpha})).$$

Here and later we assume that the first mapping is single-valued, i.e. the state problem has a unique solution, so that $\mathcal{J}(\boldsymbol{\alpha}, \boldsymbol{q}(\boldsymbol{\alpha}))$ and $\nabla \mathcal{J}(\boldsymbol{\alpha}, \boldsymbol{q}(\boldsymbol{\alpha}))$ have sense. Hence we can define

$$\mathfrak{J}(\boldsymbol{\alpha}) := \mathcal{J}(\boldsymbol{\alpha}, \boldsymbol{q}(\boldsymbol{\alpha})).$$

Note that the condition (1.77) guaranteeing uniqueness of the solutions to the discrete state problems is independent of $\boldsymbol{\alpha}$ and h.

Let us describe how the gradient $\nabla\mathfrak{J}(\boldsymbol{\alpha})$ will be calculated. The easiest to implement possibility is approximation by difference quotients:

$$\frac{\partial\mathfrak{J}(\boldsymbol{\alpha})}{\partial\alpha_k} \approx \frac{\mathfrak{J}(\boldsymbol{\alpha}+\delta\boldsymbol{e}_k^n)-\mathfrak{J}(\boldsymbol{\alpha})}{\delta},$$

where $\boldsymbol{e}_k^n$ stands for the k-th canonical base vector of $\mathbb{R}^n$ and $\delta > 0$ is a suitable difference parameter. However, this method is inexact (depending sensitively on the choice of δ) and time-consuming, especially when dealing with nonlinear state problem, since one gradient approximation requires $n+1$ cost function evaluations.

Instead we will implement exact gradient evaluation with help of the adjoint equation technique which avoids differentiation of the control-to-state mapping:

$$\frac{\partial\mathfrak{J}}{\partial\alpha_k}(\boldsymbol{\alpha}) = \frac{\partial\mathcal{J}}{\partial\alpha_k}(\boldsymbol{\alpha},\boldsymbol{q}(\boldsymbol{\alpha})) - \boldsymbol{p}\cdot\left(\frac{\partial\boldsymbol{R}}{\partial\alpha_k}(\boldsymbol{\alpha},\boldsymbol{q}(\boldsymbol{\alpha}))\right),\ k=1,\dots,n, \qquad (5.4)$$

where $\boldsymbol{p} := \boldsymbol{p}(\boldsymbol{\alpha},\boldsymbol{q}(\boldsymbol{\alpha}))$ is the solution of the adjoint equation:

$$\left(\frac{\partial\boldsymbol{R}}{\partial\boldsymbol{q}}(\boldsymbol{\alpha},\boldsymbol{q}(\boldsymbol{\alpha}))\right)^{\mathrm{T}}\boldsymbol{p} = \frac{\partial\mathcal{J}}{\partial\boldsymbol{q}}(\boldsymbol{\alpha},\boldsymbol{q}(\boldsymbol{\alpha})). \qquad (5.5)$$

In this method we have to solve only one additional linear problem for $\boldsymbol{p}$ in spite of the fact that the state problem is nonlinear. Let us also notice that for our cost function it holds that

$$\frac{\partial\mathcal{J}}{\partial\alpha_k}(\boldsymbol{\alpha},\boldsymbol{q}) = 0,\ k=1,\dots,n.$$

However, the partial derivatives $\frac{\partial\boldsymbol{R}}{\partial\boldsymbol{\alpha}}$, $\frac{\partial\boldsymbol{R}}{\partial\boldsymbol{q}}$, $\frac{\partial\mathcal{J}}{\partial\boldsymbol{q}}$ must be supplied to (5.4) and (5.5). Their hand-coding is in most cases elaborate and error-prone, requiring an additional algebraic sensitivity analysis. We compute them with the aid of automatic differentiation, which will be described in Section 5.3.

For the computations of the state problem for different $\boldsymbol{\alpha}\in\boldsymbol{U}$ (or $s_\varkappa \in \mathcal{U}_{ad}^\varkappa$, equivalently) we need to construct triangulations of $\Omega_h(s_\varkappa)$ that satisfy $(T1)-(T4)$. We use an approach which exploits the geometrical properties of $\Omega_h(s_\varkappa)$: We choose a suitable $\overline{s}_\varkappa \in \mathcal{U}_{ad}^\varkappa$ and create a triangulation $T_h(\overline{s}_\varkappa)$. Then, given $s_\varkappa \in \mathcal{U}_{ad}^\varkappa$, we define the triangulation of $\Omega_h(s_\varkappa)$ from $T_h(\overline{s}_\varkappa)$ in such a way, that every node $(x(\overline{s}_\varkappa), y(\overline{s}_\varkappa))$ of $T_h(\overline{s}_\varkappa)$ is shifted in the vertical direction:

$$\begin{aligned} x(s_\varkappa) &:= x(\overline{s}_\varkappa), \\ y(s_\varkappa) &:= y(\overline{s}_\varkappa)\frac{s_\varkappa(x(s_\varkappa))}{\overline{s}_\varkappa(x(s_\varkappa))}. \end{aligned} \tag{5.6}$$

Due to the properties of $\mathcal{U}_{ad}^{\varkappa}$ the assumptions $(T1) - (T4)$ are then valid.

The evaluation of $\mathfrak{J}$ and $\nabla\mathfrak{J}$ can be summarized in the following algorithm:

1. Given $\boldsymbol{\alpha} \in \boldsymbol{U}$, solve the state problem and obtain $\boldsymbol{q}(\boldsymbol{\alpha})$;
2. Evaluate $\mathfrak{J}(\boldsymbol{\alpha}) := \mathcal{J}(\boldsymbol{\alpha}, \boldsymbol{q}(\boldsymbol{\alpha}))$;
3. Solve the adjoint equation (5.5) and obtain $\boldsymbol{p}(\boldsymbol{\alpha})$;
4. Evaluate $\nabla\mathfrak{J}$ using (5.4).

5.3 Automatic differentiation

Automatic differentiation (AD) is a technique for augmenting computer programs with calculations of derivatives. It exploits the fact that every program (including our finite element code) executes a sequence of elementary arithmetic operations. By applying the chain rule of differentiation to these operations accurate derivatives of arbitrary orded can be obtained automatically. The principles and description of a simple implementation of the AD can be found in [12], for more extensive study see e.g. [9]. Some applications of the AD in shape optimization are contained in [18].

5.3.1 Introduction

Most problems of numerical mathematics can be considered as the evaluation of a nonlinear function $\boldsymbol{f} : \mathbb{R}^M \to \mathbb{R}^N$ at a point $\boldsymbol{\xi} \in \mathbb{R}^M$. The corresponding computer program then can be written as follows:

$$
\begin{aligned}
&\texttt{do i} = 1, \ldots, \texttt{M} \\
&\qquad \texttt{x}_\texttt{i} = \xi_\texttt{i} \\
&\texttt{end do} \\
&\texttt{do i} = \texttt{M} + 1, \ldots, \texttt{K} \\
&\qquad \texttt{x}_\texttt{i} = \phi_\texttt{i}(\texttt{x}_1, \ldots, \texttt{x}_{\texttt{i}-1}) \\
&\texttt{end do} \\
&\texttt{do i} = 1, \ldots, \texttt{N} \\
&\qquad \texttt{f}_\texttt{i} = \texttt{x}_{\texttt{K}-\texttt{N}+\texttt{i}} \\
&\texttt{end do}
\end{aligned}
\tag{5.7}
$$

The first M variables $\texttt{x}_1, \ldots, \texttt{x}_\texttt{M}$ are called the independent variables and the last N ones are the output variables. The elementary functions ϕ_i usually depend only on some of the variables $\texttt{x}_{\texttt{j}_1}, \ldots, \texttt{x}_{\texttt{j}_{\texttt{n}_\texttt{i}}}$. Therefore we define $I_i := \{\texttt{j}_1, \ldots, \texttt{j}_{\texttt{n}_\texttt{i}}\}$, $\boldsymbol{x}_{I_i} := (\texttt{x}_{\texttt{j}_1}, \ldots, \texttt{x}_{\texttt{j}_{\texttt{n}_\texttt{i}}})$ and write $\phi_i(\boldsymbol{x}_{I_i}) := \phi_\texttt{i}(\texttt{x}_1, \ldots, \texttt{x}_{\texttt{i}-1})$. In practice, ϕ_i stands for one of the standard unary or binary arithmetic operations (+, -, *, /, sin, exp, etc.), thus $|I_i| \le 2$.

If we want to differentiate the output variables with respect to the independent ones, then the chain rule of differentiation yields:

$$
\frac{\partial \texttt{x}_\texttt{i}}{\partial \texttt{x}_\texttt{j}} = \begin{cases} \sum_{k=1}^{i-1} \frac{\partial \phi_i}{\partial \texttt{x}_\texttt{k}} \frac{\partial \texttt{x}_\texttt{k}}{\partial \texttt{x}_\texttt{j}}, & j < i \le K, \\ 1, & j = i, \\ 0, & j > i. \end{cases}
$$

The derivatives $\frac{\partial \phi_i}{\partial \texttt{x}_\texttt{k}}$ are known a priori, thus we have a system of linear equations for the unknowns $\frac{\partial \texttt{x}_\texttt{i}}{\partial \texttt{x}_\texttt{j}}$. There are several possibilities how to solve it.

5.3.2 Forward and reverse method

Let us restrict to evaluation of one output variable (i.e. $N = 1$) for simplicity. The program (5.7) can be completed by the calculations of the derivatives using either the forward method:

$$
\begin{aligned}
&\texttt{do i} = 1, \ldots, \texttt{M} \\
&\qquad \texttt{x}_\texttt{i} = \xi_\texttt{i} \\
&\qquad \nabla \texttt{x}_\texttt{i} = \boldsymbol{e}^\texttt{i} \\
&\texttt{end do} \\
&\texttt{do i} = \texttt{M} + 1, \ldots, \texttt{K} \\
&\qquad \texttt{x}_\texttt{i} = \phi_\texttt{i}(\boldsymbol{x}_{\texttt{I}_\texttt{i}}) \\
&\qquad \nabla \texttt{x}_\texttt{i} = \sum_{\texttt{j} \in \texttt{I}_\texttt{i}} \frac{\partial \phi_\texttt{i}}{\partial \texttt{x}_\texttt{j}} \nabla \texttt{x}_\texttt{j} \\
&\texttt{end do} \\
&\texttt{f} = \texttt{x}_\texttt{K} \\
&\nabla \texttt{f} = \nabla \texttt{x}_\texttt{K}
\end{aligned}
\tag{5.8}
$$

or alternatively using the reverse method:

$$
\begin{aligned}
&\texttt{do i} = 1, \ldots, \texttt{M} \\
&\qquad \texttt{x}_\texttt{i} = \xi_\texttt{i} \\
&\texttt{end do} \\
&\texttt{do i} = \texttt{M} + 1, \ldots, \texttt{K} \\
&\qquad \texttt{x}_\texttt{i} = \phi_\texttt{i}(\boldsymbol{x}_{\texttt{I}_\texttt{i}}) \\
&\texttt{end do} \\
&\texttt{f} = \texttt{x}_\texttt{K} \\
&\texttt{do i} = 1, \ldots, \texttt{K} - 1 \\
&\qquad \overline{\texttt{x}}_\texttt{i} = 0 \\
&\texttt{end do} \\
&\overline{\texttt{x}}_\texttt{K} = 1 \\
&\texttt{do i} = \texttt{K}, \texttt{K} - 1, \ldots, \texttt{M} + 1 \\
&\qquad \overline{\texttt{x}}_\texttt{j} = \overline{\texttt{x}}_\texttt{j} + \frac{\partial \phi_\texttt{i}}{\partial \texttt{x}_\texttt{j}} \overline{\texttt{x}}_\texttt{i} \quad \forall \texttt{j} \in \texttt{I}_\texttt{i} \\
&\texttt{end do} \\
&\nabla \texttt{f} = (\overline{\texttt{x}}_\texttt{i})_{\texttt{i}=1}^{\texttt{M}}
\end{aligned}
\tag{5.9}
$$

These two methods correspond to evaluation of the derivatives starting from the first or from the last variable, respectively. The forward method is easy to implement and is suitable when $N >> 1$, i.e. if many output derivatives have to be computed. However its computational effort and memory consumption grows linearly with M. On the other hand, if we have only one output variable, then the reverse method seems to be more efficient, but the implementation is quite tricky because one has to store the full call graph during the program execution.

As in our program we will need to differentiate a large amount of variables (in particular the components of the residual vectors and the cost function), i.e. $N = m + 1$, and the number of independent variables is relatively small ($M = n$), we will focus on the forward method.

5.3.3 Implementation

In our program we use the forward method of the AD, which is implemented in the following way.

We define a composite data type `CVar`, which holds the value of a real variable and its partial derivatives with respect to the independent variables $\mathtt{x}_1, \ldots, \mathtt{x}_\mathtt{M}$. We can now represent the variables $\mathtt{x}_1, \ldots, \mathtt{x}_\mathtt{K}$ with this data type, and implement the functions ϕ_i such that they compute both the results $\mathtt{x}_\mathtt{i}$ and their partial derivatives.

A unique global identification number is assigned to each independent variable. In the following we assume that the identification number of $\mathtt{x}_\mathtt{i}$ is i. The object of type `CVar` representing a generic real variable `x` includes a vector of integers `ind` and a vector of real numbers `d`. The vector `ind` holds the set $\mathtt{nz}(\mathtt{x}) = \{\mathtt{i} \mid \partial\mathtt{x}/\partial\mathtt{x}_\mathtt{i} \neq 0\}$ in increasing order, and `d` holds the values $\mathtt{d}(\mathtt{j}) = \partial\mathtt{x}/\partial\mathtt{x}_{\mathtt{ind(j)}}$.

We initialize the partial derivatives $\partial\mathtt{x}_\mathtt{i}/\partial\mathtt{x}_\mathtt{i} = 1$ for the objects representing the independent variables, and apply the automatic differentiation procedure at every step of the computation.

We use the operator overloading property of C++ to "hide" the implementation of the AD from the user. The elementary operators are overloaded such that the user can simply write for example $\mathtt{y} = \mathtt{x}_1 + \mathtt{x}_2$, where `y`, $\mathtt{x}_1$ and $\mathtt{x}_2$ are of type `CVar`, and the compiler takes care of calling the appropriate function implementing the AD.

5.3.4 Application to shape optimization

We want our program to calculate $\frac{\partial \boldsymbol{R}}{\partial \boldsymbol{\alpha}}$, $\frac{\partial \boldsymbol{R}}{\partial \boldsymbol{q}}$, $\frac{\partial \mathcal{J}}{\partial \boldsymbol{\alpha}}$ and $\frac{\partial \mathcal{J}}{\partial \boldsymbol{q}}$ with help of the AD. Thus we set the nodal values of the FEM solution $q_1, \ldots, q_m$ and the design parameters $\alpha_1, \ldots, \alpha_n$ to be the independent variables of the type `CVar`. Of course, the intermediate and output variables have to be of the type `CVar` as well. Then, after assembling the residual and performing the cost functional evaluation, the required derivatives are obtained in the associated vectors.

Possible source of inefficiency lies in the fact that typically the number of degrees of freedom m is large, so that many partial derivatives might have to be computed. But since we use a FEM discretization, each residual component depends only on few degrees of freedom. The same holds also for all the intermediate variables generated in the execution chain of the residual assembly. Further the cost function $\mathcal{J}$ can be decomposed to a sum over edges:

$$\mathcal{J} = \sum_{E \in E_h} \mathcal{J}_E, \tag{5.10}$$

so that $\mathcal{J}_E$ again depends only on a small number of d.o.f. and design parameters. Therefore one has to differentiate the components of $\boldsymbol{R}$ and $\mathcal{J}_E$ only with respect to a small number of the independent variables. The final term $\mathcal{J}$ in (5.10) is obtained using the standard assembly, without any differentiation.

The AD is also applied in calculation of the positions of the computational mesh nodes and their derivatives with respect to the design parameters. Therefore the node coordinates have to be stored in variables of the type `CVar`. Since the node positions are given by a simple formula (5.6), the nodal sensitivities are obtained automatically.

The proposed method is quite easy to implement and efficient in terms of computational effort and memory. It is restricted neither to a specific type of finite elements nor to a specific state and shape optimization problem. Changing the problem parameters or structure can be done easily without need to recalculate any partial derivatives.

5.4 Example computations

We end up with several numerical examples. Let us note that the parameters used in the following computations do not correspond to any real industrial application.

5.4.1 State problem

Traditionally the paper machine header has been designed with a linearly tapered header. We use this header design to test the state problem solver. The computational domain is 9.5 m long and 1 m wide (see Figure 5.1). This domain is partitioned to a uniform triangular mesh consisting of 8000

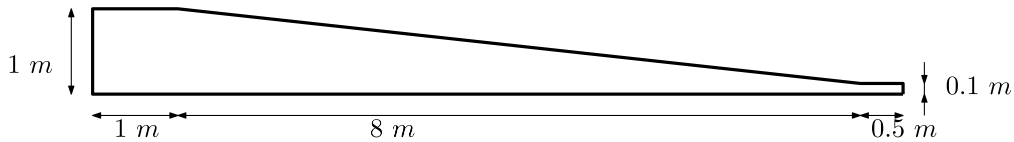

Figure 5.1: Dimensions of the computational header.

triangles, so that the resulting vector of degrees of freedom $\boldsymbol{q}$ has 28663 components.

The pulp is modelled as an incompressible fluid with the laminar viscosity $\mu_0 = 10^{-3}\ Pa.s$ and the density $\rho = 10^3\ kg/m^3$. The inlet and outlet velocity profiles are chosen as follows:

$$\boldsymbol{v}_{D|\{0\}\times(0,H_1)} = (4(1-(\frac{2x_2}{H_1}-1)^8),0)\ m/s,$$
$$\boldsymbol{v}_{D|\{L\}\times(0,H_2)} = (1-(\frac{2x_2}{H_2}-1)^8,0)\ m/s.$$

It remains to specify the outflow suction parameter σ. We have performed computations with several values of σ between 0 and 10^5.

If we define the kinematic viscosity $\nu := \mu_0/\rho$, then ν^{-1} gives the Reynolds number 10^6 in case of standard Navier-Stokes equations. This usually requires the use of a stabilized numerical scheme. However our turbulence model produces enough turbulent viscosity so that the state problem can be solved without any additional stabilization. As the initial approximation we chose a solution of a similar problem with higher viscosity, which was possible to obtain starting from zero. The stopping criterion for the residuum is $r_{max} = 10^{-9}$. The nonlinear loop usually stopped after 2 to 10 iterations, each of which took about 7.1 s on AMD Opteron 246 with 2 GB RAM. The direct solver SuperLU was efficient enough for this problem size, requiring only 20% of one nonlinear iteration time, while the rest was spent on the residual assembly (see Table 5.1 for the details).

m	Residual assembly	Linear system solution	Total
7333	1.4 s	0.2 s	1.6 s
28663	5.7 s	1.4 s	7.1 s
63993	12.8 s	6.6 s	19.4 s
253983	56.9 s	65.8 s	122.7 s

Table 5.1: Time demands for one nonlinear iteration.

In Figure 5.2 one can compare the influence of σ on the outflow velocity profile. In spite of the theoretical result, the computation seems to be stable

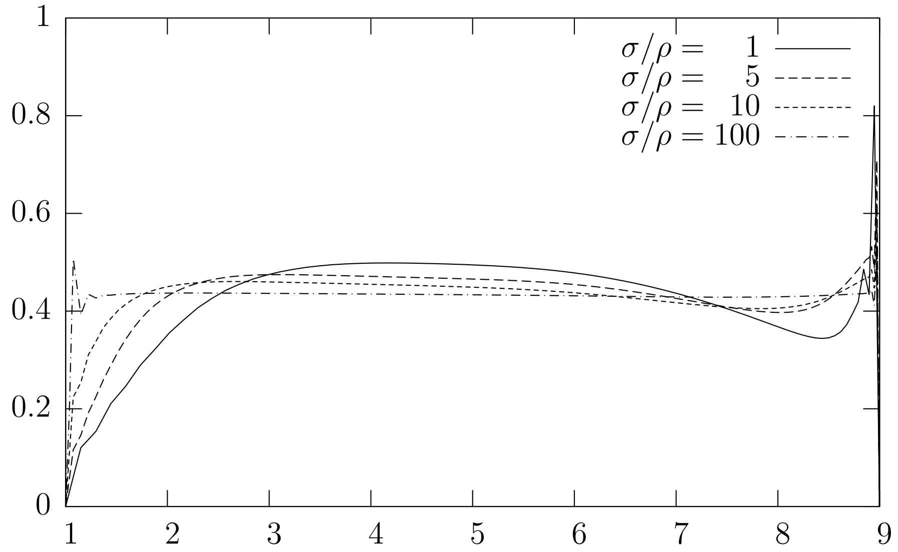

Figure 5.2: Outflow velocity $v_{2|\Gamma_{\mathrm{out}}}$ depending on σ.

even in cases when $\sigma/\rho < \frac{1}{2}$. One can notice that at the right end of Γ_{out} a high outflow velocity gradient occurs. The plots of the solution in Figure 5.3 reveal that even the pressure changes rapidly at that point, presumably due to the change of boundary conditions.

5.4.2 Shape optimization problem

The traditional linearly tapered header serves as a starting point for the shape optimization. The number of design parameters is set to $n = 20$. Due to the well known properties of the Bézier functions the derivative of $s_\varkappa \in \mathcal{U}_{ad}^\varkappa$ can be estimated as follows:

$$|s'_\varkappa| \leq \frac{\alpha_{max} - \alpha_{min}}{\varkappa} = \frac{\alpha_{max} - \alpha_{min}}{L_2} n \quad \forall s_\varkappa \in \mathcal{U}_{ad}^\varkappa.$$

Therefore for reasonably small n the constraint γ from the definition of $\mathcal{U}_{ad}$ can be dropped from the computation. We then obtain a nonlinear optimization problem with simple bounds. We set $\alpha_{max} = H_1$ and $\alpha_{min} = H_2$. The boundary segment on which the cost function is evaluated is $\tilde{\Gamma} = (1.5, 8.5)$. The outflow suction coefficient σ has the value 10^3 in what follows.

We run the computation repeatedly with different target velocity profiles. In the first case we used a constant target velocity $v_{opt} = -0.443\ m/s$. We have tested two numerical optimization packages in order to compare the obtained results and performance. All parameters were left default, only

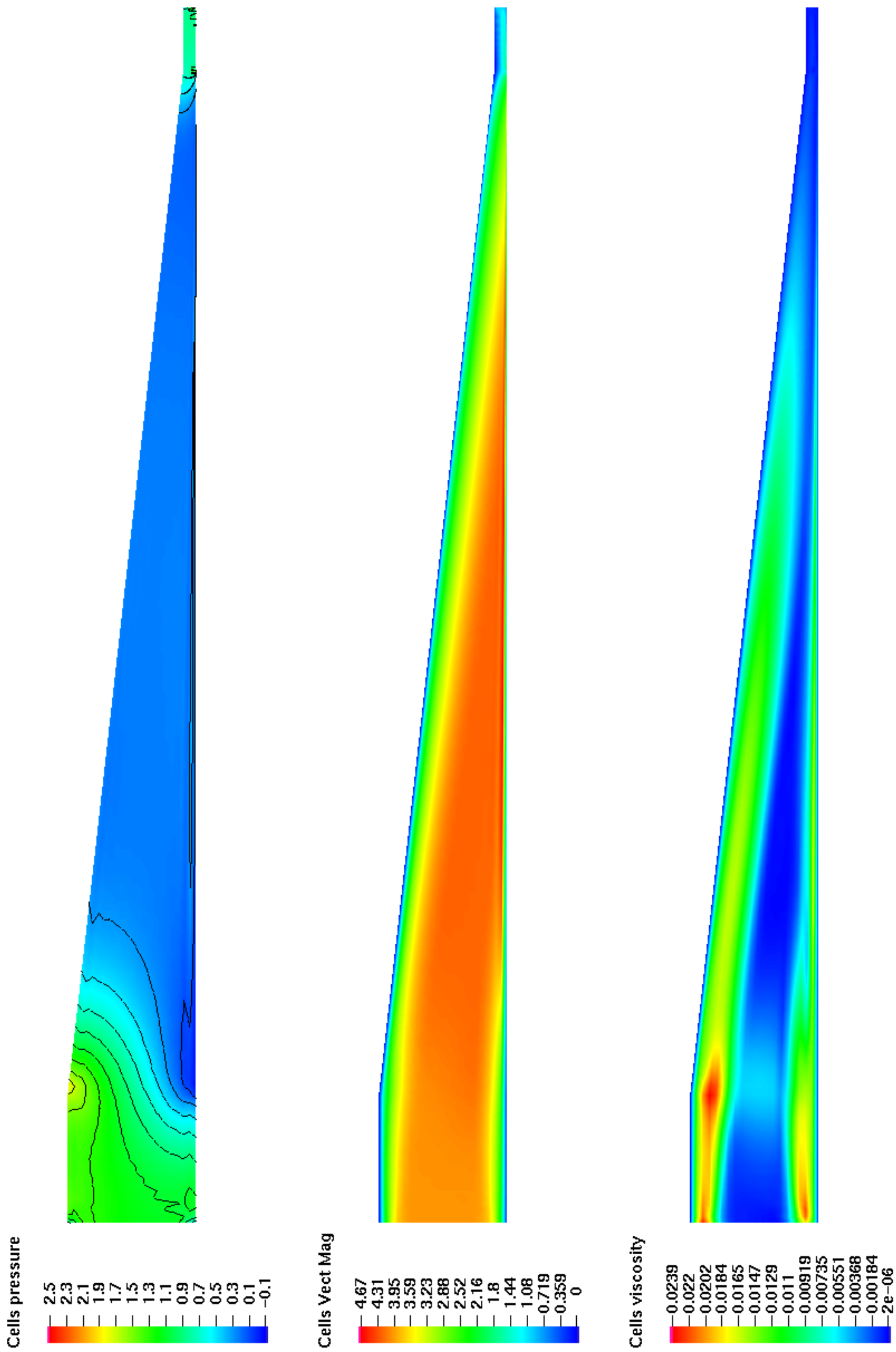

Figure 5.3: Solution of the state problem (for $\sigma = 10^3$): pressure p, velocity magnitude $|\boldsymbol{v}|$, dynamic viscosity μ.

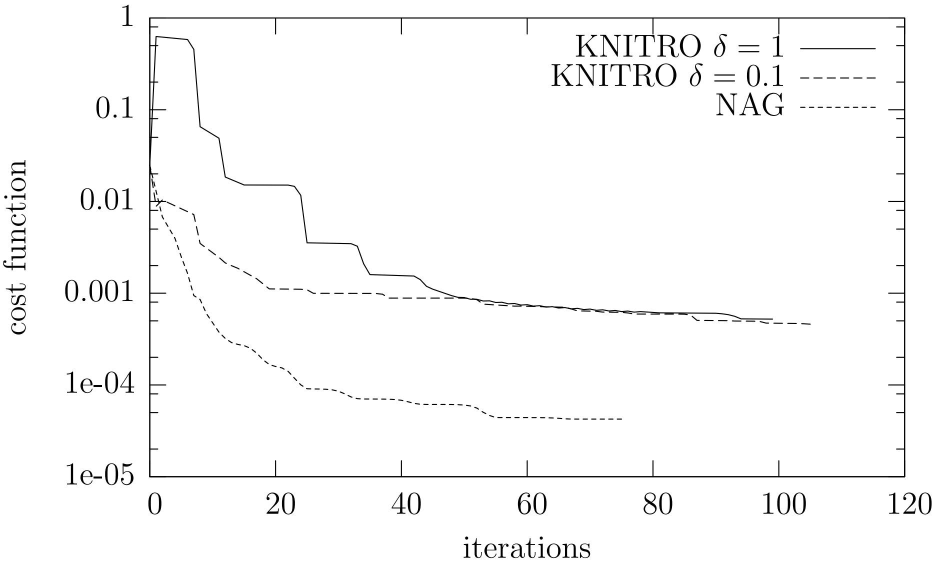

Figure 5.4: Convergence history of the used optimization algorithms.

in case of KNITRO solver we tried several values of the initial trust region parameter δ. Both packages, KNITRO and NAG, converged apparently to the same shape. However NAG seemed to be superior in terms of required cost function and gradient evaluations. KNITRO solver ended in all cases after approximately 100 major iterations, achieving the KKT optimality error smaller than 10^{-3}. On the other hand, NAG C library needed 73 major iterations to yield the optimality error smaller than 2×10^{-6}. The value of the cost function decreased from 2.5×10^{-2} to 4.2×10^{-5} in case of NAG. In Figure 5.4 the convergence history of all algorithms is compared. The obtained optimal shapes and outlet velocity profiles are depicted in Figure 5.5 and in Figure 5.6, respectively.

In the second case a function

$$v_{opt} = -0.65 \sin\left(\frac{x - L_1}{L_2}\pi\right) \ m/s$$

was chosen as the target outlet velocity. Here the process ended after 44 major iterations using NAG and the cost function value decreased from 8.7×10^{-2} to 1.1×10^{-3}. The optimal shapes and velocities for the non-constant target are shown in Figure 5.7 and in Figure 5.8, respectively.

There is no reason to assume that the cost function is convex, therefore the found minima are possibly only local. However, all the used algorithms converged to very similar shapes that are close to the one obtained in [12], where a different method was applied. Thus, there is a chance that the final

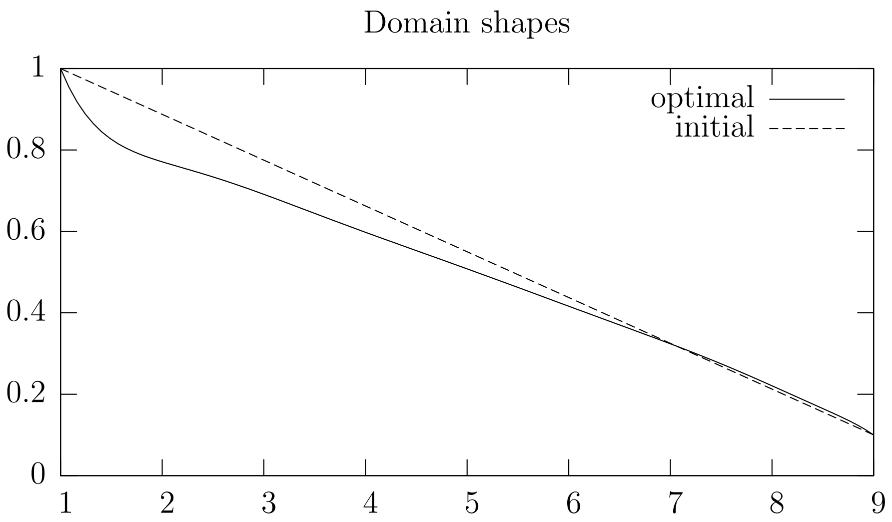

Figure 5.5: Initial and optimized shape (constant target velocity).

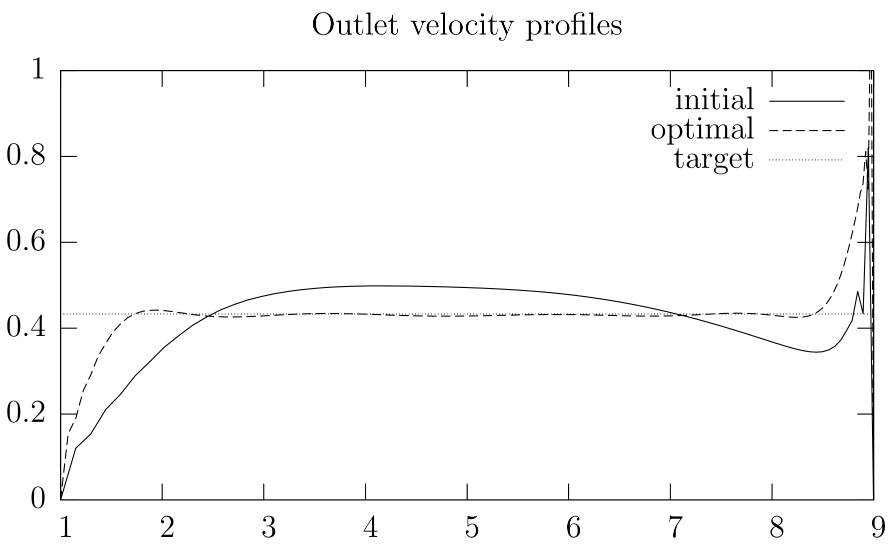

Figure 5.6: Initial and optimized outlet velocity (constant target velocity).

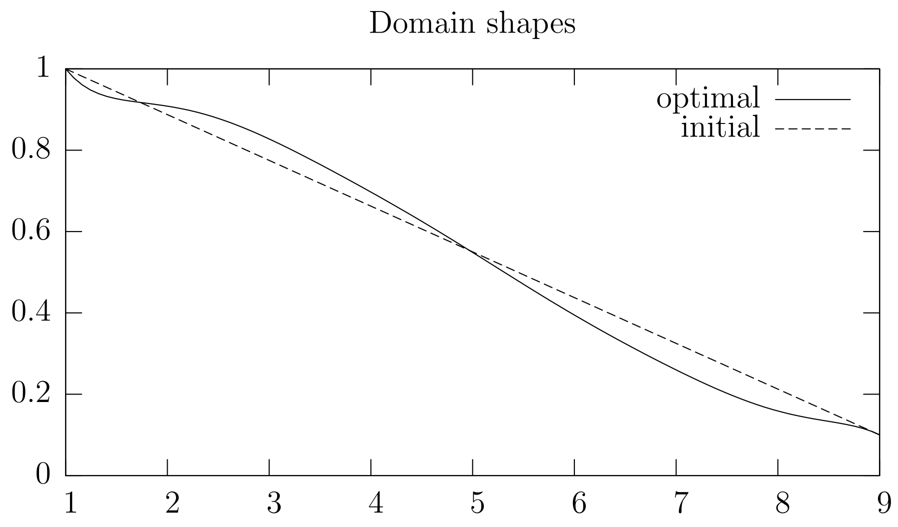

Figure 5.7: Initial and optimized shape (non-constant target velocity).

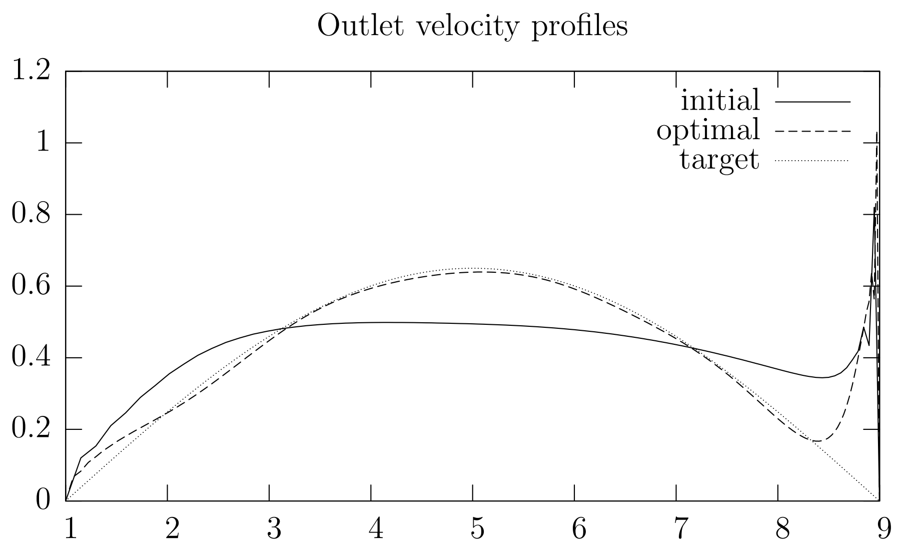

Figure 5.8: Initial and optimized outlet velocity (nonconstant target velocity).

design is close to the global minimum. In any case, for practical purposes it is usually sufficient to find a local minimum which improves the initial state.

One can see in Figure 5.9 that the difference between the initial and optimized shapes is not too big. This indicates that the cost function is very sensitive with respect to shape variations. In spite of this fact, the proposed examples reveal that it is possible to control the outflow velocity and consequently the quality of produced paper by appropriate change of the header geometry.

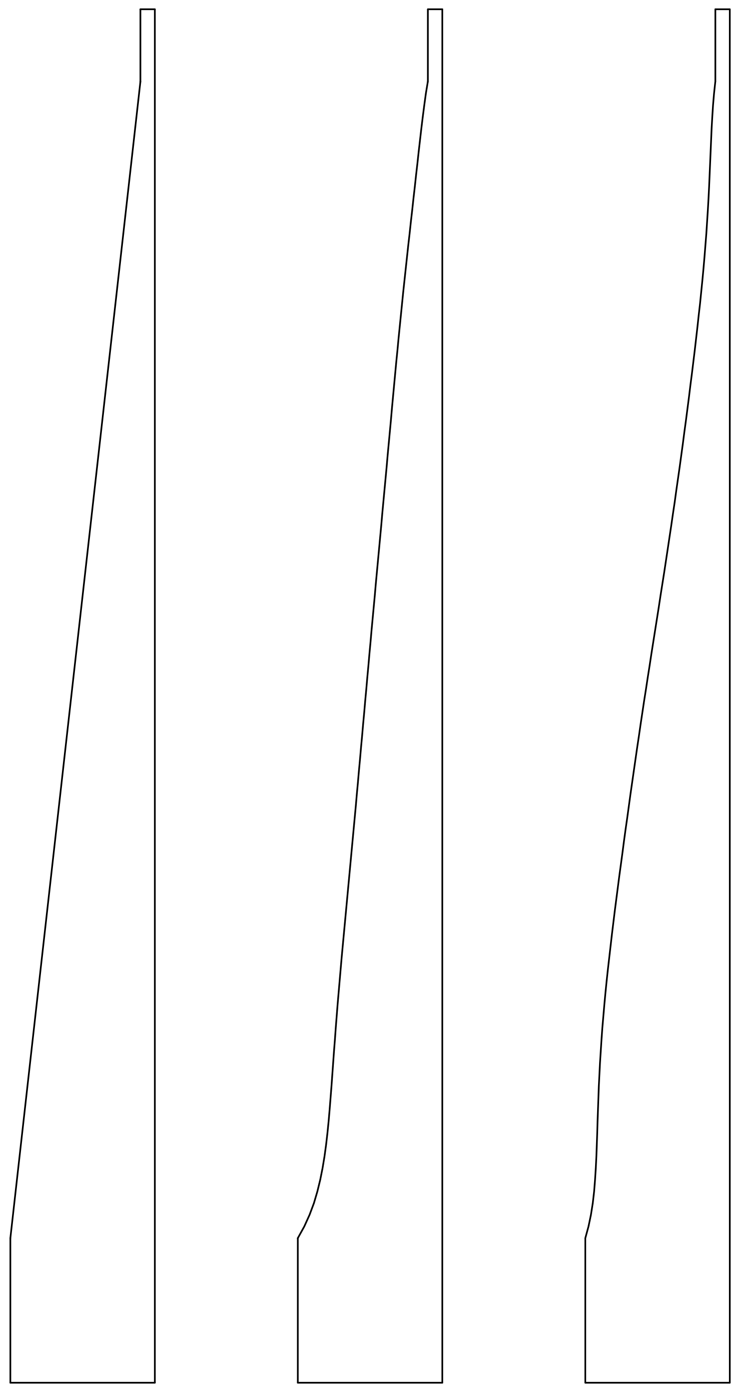

Figure 5.9: Shapes of the header: Initial, optimized for constant target velocity and optimized non-constant target velocity.

Conclusion

The thesis consists of 2 parts. In the first one the mathematical properties of the fluid flow and shape optimization problem are studied. Chapter 1 deals with the existence proof for the generalised steady-state Navier–Stokes system. In Chapter 2 the shape optimization problem with the Navier–Stokes system as a state constraint is studied.

Due to an algebraic turbulence model the weak formulation of the state problem involves the weighted Sobolev spaces. The existence and uniqueness of a solution is proved with a constraint imposed on the model parameters by using energy estimates, the monotone operator theory and the Galerkin method. The analysis of the state problem shares many similarities with the techniques presented in [13, 14, 15] and [19].

The proof of the continuous dependence of solutions on boundary variations is the key result in the shape optimization part. This property is proved provided that the problem is reformulated using broader function spaces. This requirement is due to the lack of a density result for the used weighted spaces.

The second part of the thesis is devoted to approximation and numerical realization of the problem formulated beforehand: In Chapter 3 a finite element discretization of the flow problem is studied. The existence of discrete solutions and their convergence to a solution of the continuous problem is proved, the later under the assumption that the inf-sup condition is valid. Chapter 4 presents an approximation of the domain boundary, existence of discrete optimal shapes and their convergence to a solution of the reformulated shape optimization problem is established. The results of these two chapters are based on the theory obtained in Part I.

Finally, an algorithm for the numerical solution is described in Chapter 5. The proposed method takes the advantage of the automatic differentiation which significantly simplifies and speeds up the computer program. The example computations show that very accurate results can be obtained and that the mathematical modelling together with numerical analysis can bring a significant contribution to the paper making engineering.

Appendix A

Auxiliary tools

Theorem A.I (Young's inequality). *Let* $a, b \geq 0$, $r, s > 1$, $\frac{1}{r} + \frac{1}{s} = 1$. *Then*

$$ab \leq \frac{a^r}{r} + \frac{b^s}{s}. \tag{A.1}$$

Theorem A.II (Brouwer's fixed-point theorem). *Let* B *denote a closed ball in* $\mathbb{R}^d$ *and* $\boldsymbol{P} : B \to B$ *be a continuous mapping. Then there exists a point* $\boldsymbol{x} \in B$ *such that* $\boldsymbol{P}(\boldsymbol{x}) = \boldsymbol{x}$.

Corollary A.III. *Let* $\boldsymbol{P} : \mathbb{R}^d \to \mathbb{R}^d$ *be continuous and let for some* $R > 0$

$$\boldsymbol{P}(\boldsymbol{x}) \cdot \boldsymbol{x} > 0 \ \forall \boldsymbol{x} \in \mathbb{R}^d, |\boldsymbol{x}| = R.$$

Then there exists a point $\boldsymbol{x}_0 \in B_R$ *such that* $\boldsymbol{P}(\boldsymbol{x}_0) = \boldsymbol{0}$, *where* B_R *is the closed ball of radius* R.

Theorem A.IV (Arzelà-Ascoli). *Let* (S, ρ) *be a compact metric space and* $\mathcal{C}(S)$ *be the Banach space of real- or complex-valued continuous functions* f *in* S *normed by* $\|f\| = \sup_{s \in S} |f(s)|$. *Then a sequence* $\{f_n\} \subset \mathcal{C}(S)$ *is relatively compact in* $\mathcal{C}(S)$ *if the following two conditions are satisfied:*

(i) f_n *is equibounded, i.e.* $\sup_{n \geq 1} \sup_{s \in S} |f_n(s)| < \infty$,

(ii) f_n *is equicontinuous, i.e.*

$$\lim_{\delta \searrow 0} \sup_{\substack{n \geq 1 \\ \rho(s,s') < \delta}} |f_n(s) - f_n(s')| = 0.$$

Proof. See Yosida [24], Chapter III.3. □

Theorem A.V (Vitali). *Let Ω be a bounded domain in $\mathbb{R}^d$ and $f_n, f : \Omega \to \mathbb{R}$ satisfy:*

(i) $f_n \to f$ *a.e. in* Ω*;*

(ii) $\forall \varepsilon > 0\ \exists \delta > 0\ \forall E \subset \Omega : \ |E| < \delta \Rightarrow \sup_n \int_E |f_n| < \varepsilon.$

Then

$$\lim_{n\to\infty} \int_\Omega f_n = \int_\Omega f.$$

The following Lemma presents usual assumptions on constitutive laws for non-Newtonian fluids and enables us to show strong monotonicity of the leading nonlinear term.

Lemma A.VI. *Let $S : \mathbb{R}^{d\times d} \to \mathbb{R}^{d\times d}$ be a matrix function with the following properties:*

(i) S *is continuously differentiable in* $\mathbb{R}^{d\times d}$*,*

(ii) $S(0) = 0$*,*

(iii) for some $r \in (1,\infty)$ *there exists a constant* $C_1 > 0$ *such that*

$$\frac{\partial S(\mathbb{A})}{\partial \mathbb{A}} : (\mathbb{B} \otimes \mathbb{B}) \geq C_1 |\mathbb{A}|^{r-2} |\mathbb{B}|^2 \ \forall \mathbb{A}, \mathbb{B} \in \mathbb{R}^{d\times d}.$$

Then S is monotone in the following sense: There exists a constant $C_2 >$ such that

$$(S(\mathbb{A}) - S(\mathbb{B})) : (\mathbb{A} - \mathbb{B}) \geq C_2 |\mathbb{A} - \mathbb{B}|^r,$$

holds for all $\mathbb{A}, \mathbb{B} \in \mathbb{R}^{d\times d}$*.*

Proof. See [16], Lemma 1.19. □

Appendix B

Properties of the Sobolev spaces

In what follows we assume that Ω is a bounded domain in $\mathbb{R}^d$ with the Lipschitz boundary $\partial\Omega$. We denote for k a non-negative integer and $r \in [1,\infty)$ the Sobolev space

$$W^{k,r}(\Omega) := \left\{ v \in L^r(\Omega);\ D^\alpha v \in L^r(\Omega),\ |\alpha| \le k \right\}$$

with the norm

$$\|v\|_{k,r,\Omega} := \left(\sum_{|\alpha|\le k} \|D^\alpha v\|_{r,\Omega}^r \right)^{1/r},$$

where $L^r(\Omega)$ is the Lebesgue space endowed with the norm $\|\cdot\|_{r,\Omega}$.

B.1 Basic imbeddings and inequalities

Theorem B.I (Hölder's inequality). *Let $r, s \in (1,\infty)$ be such that $\frac{1}{r} + \frac{1}{s} = 1$, $f \in L^r(\Omega)$ and $g \in L^s(\Omega)$. Then $fg \in L^1(\Omega)$ and*

$$\|fg\|_{1,\Omega} \le \|f\|_{r,\Omega} \|g\|_{s,\Omega}.$$

Theorem B.II (Imbedding theorem). *Let $r \in (1,d)$ and $s \in [1, \frac{dr}{d-r}]$, or $r = d$ and $s \in [1,\infty)$. Then there exists a positive constant $C_I := C_I(\Omega, r, s)$ such that for all $v \in W^{1,r}(\Omega)$ it holds:*

$$\|v\|_{s,\Omega} \le C_I \|v\|_{1,r,\Omega}.$$

For $s < \frac{dr}{d-r}$ this imbedding is compact.

We denote by $\operatorname{Tr} v$ the trace of $v \in W^{1,r}(\Omega)$ on $\partial\Omega$. The symbol $L^r(\partial\Omega)$ stands for the Lebesgue space of traces with the norm $\|\cdot\|_{r,\partial\Omega}$.

Theorem B.III (Trace theorem). *Let $r \in (1,d)$ and $s \in [1, \frac{dr-r}{d-r}]$. Then there exists a positive constant $C_{Tr} := C_{Tr}(\Omega, r, s)$ such that for all $v \in W^{1,r}(\Omega)$ it holds:*

$$\|\operatorname{Tr} v\|_{s,\partial\Omega} \le C_{Tr}\|v\|_{1,r,\Omega}.$$

For $s < \frac{dr-r}{d-r}$ the operator $\operatorname{Tr} : W^{1,r}(\Omega) \to L^s(\partial\Omega)$ is compact.

Theorem B.IV (Friedrichs' inequality). *Let $r \in (1,\infty)$ and Γ be a non-empty and open part of $\partial\Omega$. Then there exists a positive constant $C_F := C_F(\Omega, \Gamma, r)$ such that for all $v \in W^{1,r}(\Omega)$ it holds:*

$$\|v\|_{1,r,\Omega} \le C_F\left(\|v\|_{r,\Gamma} + \|\nabla v\|_{r,\Omega}\right).$$

In particular, if $\operatorname{Tr} v = 0$ on Γ then

$$\|v\|_{1,r,\Omega} \le C_F\|\nabla v\|_{r,\Omega}.$$

Theorem B.V (Green's theorem). *Let $u \in W^{1,r}(\Omega)$, $v \in W^{1,s}(\Omega)$, $\frac{1}{r} + \frac{1}{s} = 1$ and $i \in \{1, \dots, d\}$. Then*

$$\int_\Omega u\frac{\partial v}{\partial x_i} = \int_{\partial\Omega} u \; v \; \nu_i - \int_\Omega \frac{\partial u}{\partial x_i} v,$$

where ν_i denotes the i-th component of the unit outward normal vector to $\partial\Omega$.

Theorem B.1 (Korn's inequality). *Let $r \in (1,\infty)$, Γ be a non-empty open part of $\partial\Omega$ and Ω' be a non-empty open subset of Ω. Then there exist positive constants $C_K := C_K(\Omega, \Gamma, r)$, $C'_K := C'_K(\Omega, \Gamma, r)$ and $C''_K := C''_K(\Omega, \Omega', r)$ such that*

(i) for every $\boldsymbol{u} \in \left(W^{1,r}(\Omega)\right)^d$ with $\boldsymbol{u}_{|\Gamma} = \boldsymbol{0}$ it holds:

$$C_K\|\nabla \boldsymbol{u}\|_{r,\Omega} \le \|\mathbb{D}(\boldsymbol{u})\|_{r,\Omega}; \tag{B.1}$$

(ii) for every $\boldsymbol{u} \in \left(W^{1,r}(\Omega)\right)^d$ it holds:

$$C'_K\|\boldsymbol{u}\|_{1,r,\Omega} \le \|\mathbb{D}(\boldsymbol{u})\|_{r,\Omega} + \|\boldsymbol{u}\|_{r,\Gamma}. \tag{B.2}$$

(iii) for every $\boldsymbol{u} \in \left(W^{1,r}(\Omega)\right)^d$ it holds:

$$C''_K\|\boldsymbol{u}\|_{1,r,\Omega} \le \|\mathbb{D}(\boldsymbol{u})\|_{r,\Omega} + \|\boldsymbol{u}\|_{r,\Omega'}. \tag{B.3}$$

Proof. We start by proving (*ii*). Due to [16], Theorem 1.10 the following inequality

$$C'\|\boldsymbol{u}\|_{1,r,\Omega} \leq \|\boldsymbol{u}\|_{r,\Omega} + \|\mathbb{D}(\boldsymbol{u})\|_{r,\Omega} \tag{B.4}$$

holds for every $\boldsymbol{u} \in (W^{1,r}(\Omega))^d$ with a constant $C' := C'(\Omega, r) > 0$. Thus it is enough to show that for all $\boldsymbol{u} \in (W^{1,r}(\Omega))^d$

$$\|\boldsymbol{u}\|_{r,\Omega} \leq C''(\|\mathbb{D}(\boldsymbol{u})\|_{r,\Omega} + \|\boldsymbol{u}\|_{r,\Gamma})$$

with some constant $C'' > 0$. Let us assume for contradiction that there is a sequence $\{\boldsymbol{u}_n\} \subset (W^{1,r}(\Omega))^d$, $\|\boldsymbol{u}_n\|_{r,\Omega} = 1$ such that

$$1 > n(\|\mathbb{D}(\boldsymbol{u}_n)\|_{r,\Omega} + \|\boldsymbol{u}_n\|_{r,\Gamma}).$$

Then $\|\mathbb{D}(\boldsymbol{u}_n)\|_{r,\Omega} \to 0$ and $\{\boldsymbol{u}_n\}$ is bounded in $W^{1,r}(\Omega)$, as follows from (B.4). Thus a subsequence converges weakly to some $\boldsymbol{u}$ in $(W^{1,r}(\Omega))^d$, where $\mathbb{D}(\boldsymbol{u}) = 0$ a.e. in Ω and $\operatorname{Tr} \boldsymbol{u}_{|\Gamma} = \mathbf{0}$. From this it follows that $\boldsymbol{u} = \mathbf{0}$ a.e. in Ω (see e.g. proof of Theorem 1.10 in [16]). On the other hand, the compact imbedding of $W^{1,r}(\Omega)$ into $L^r(\Omega)$ yields $\|\boldsymbol{u}\|_{r,\Omega} = 1$, which is a contradiction.

The statements (*i*) and (*iii*) can be proved analogously. □

Lemma B.2. *Let* $\boldsymbol{u} \in (W^{1,r}(\Omega))^d$, $r \in [1, \infty)$. *Then*

$$\|\mathbb{D}(\boldsymbol{u})\|_{r,\Omega} \leq \|\nabla \boldsymbol{u}\|_{r,\Omega}. \tag{B.5}$$

Proof. From the triangle inequality we obtain

$$\|\mathbb{D}(\boldsymbol{u})_{ij}\|_{r,\Omega} \leq \frac{1}{2}\left(\left\|\frac{\partial u_i}{\partial x_j}\right\|_{r,\Omega} + \left\|\frac{\partial u_j}{\partial x_i}\right\|_{r,\Omega} \right)$$

and therefore

$$\|\mathbb{D}(\boldsymbol{u})\|_{r,\Omega} = \left(\sum_{i,j} \|\mathbb{D}(\boldsymbol{u})_{ij}\|_{r,\Omega}^r \right)^{1/r} \leq \sum_{i,j} \|\mathbb{D}(\boldsymbol{u})_{ij}\|_{r,\Omega} \leq \|\nabla \boldsymbol{u}\|_{r,\Omega}. \tag{B.6}$$

B.2 Solvability of the divergence equation

In this section we will recall fundamental results for solving the problem:

Given $f \in L^q(\Omega)$ *and* $\boldsymbol{\mathcal{A}} \in (W^{1,q}(\Omega))^d$, $1 < q < \infty$, *satisfying*

$$\int_\Omega f = \int_{\partial\Omega} \boldsymbol{\mathcal{A}} \cdot \boldsymbol{\nu},$$

find $\boldsymbol{\varphi} \in [W^{1,q}(\Omega)]^d$ *such that* *(i)* $\operatorname{div} \boldsymbol{\varphi} = f$ *in* Ω,

(ii) $\operatorname{Tr} \boldsymbol{\varphi} = \operatorname{Tr} \boldsymbol{\mathcal{A}}$ *on* $\partial\Omega$,

(iii) $\|\boldsymbol{\varphi}\|_{1,q,\Omega} \leq C_{\text{div}} (\|f\|_{q,\Omega} + \|\boldsymbol{\mathcal{A}}\|_{1,q,\Omega})$ *with a constant* $C_{\text{div}} > 0$ *that is independent of* f *and* $\boldsymbol{\mathcal{A}}$.

Generally, this problem is not solvable. However for a certain class of domains it is possible to construct a solution explicitely.

Definition B.1. *We call a bounded domain* $\Omega \subset \mathbb{R}^d$ *to be star-shaped (or star-like), if there exists* $\overline{\boldsymbol{x}} \in \Omega$ *and a continuous positive function* h *defined on the unit sphere such that*

$$\Omega = \left\{ \boldsymbol{x} \in \mathbb{R}^d;\ |\boldsymbol{x} - \overline{\boldsymbol{x}}| < h\left(\frac{\boldsymbol{x} - \overline{\boldsymbol{x}}}{|\boldsymbol{x} - \overline{\boldsymbol{x}}|} \right) \right\}. \tag{B.7}$$

Similarly, we say that Ω *is star-shaped with respect to a set* $B \subset \Omega$ *if* Ω *is star-shaped w.r.t. each point of* B.

Remark B.1. *It is known (see [7], Chapter II, Lemma 3.2) that every bounded domain* Ω *with locally Lipschitz continuous boundary is a finite union of domains which are star-shaped w.r.t. some balls and whose boundaries are locally Lipschitz continuous as well.*

Theorem B.VI. *Let* $\Omega \subset \mathbb{R}^d$ *be a bounded domain in* $\mathbb{R}^d$, $d \geq 2$, *such that*

$$\Omega = \cup_{k=1}^{N} \Omega_k, N \geq 1,$$

where each Ω_k *is star-shaped with respect to some open ball* B_k *with* $\overline{B_k} \subset \Omega_k$. *Let* $1 < q < \infty$. *Then there exists a constant* $C_{\text{div}} > 0$ *such that for any* $f \in L^q(\Omega)$ *with* $\int_\Omega f = 0$ *there exists at least one* $\boldsymbol{\varphi} \in (W_0^{1,q}(\Omega))^d$ *such that*

(i) $\operatorname{div} \boldsymbol{\varphi} = f$ *a.e. in* Ω,

(ii) $\|\boldsymbol{\varphi}\|_{1,q,\Omega} \leq C_{\text{div}} \|f\|_{q,\Omega}$.

Furthermore, the constant C_{div} *admits the following estimate:*

$$C_{\text{div}} \leq c_0 C \left(\frac{\operatorname{diam} \Omega}{R_0} \right)^d \left(1 + \frac{\operatorname{diam} \Omega}{R_0} \right), \tag{B.8}$$

where R_0 *is the smallest radius of the balls* B_k, $c_0 := c_0(d, q)$ *and*

$$C = \max_{k=1,\dots,N-1} \left(1 + \frac{|\Omega_k|}{|\Omega_k \cap D_k|} \right) \prod_{i=1}^{k-1} \left(1 + |F_i|^{1/q-1} |D_i \setminus \Omega_i|^{1-1/q} \right), \tag{B.9}$$

with $D_i = \cup_{s=i+1}^{N} \Omega_s$ *and* $F_i = \Omega_i \cap D_i$.

For the proof see Galdi [7], Chapter III.3, Theorem 3.1.

Corollary B.3. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with the Lipschitz continuous boundary and $1 < q < \infty$. Then there exists a constant $C_{\mathrm{div}} > 0$ such that for any $f \in L^q(\Omega)$ and $\boldsymbol{\mathcal{A}} \in (W^{1,q}(\Omega))^d$ with $\int_\Omega f = \int_{\partial\Omega} \boldsymbol{\mathcal{A}} \cdot \boldsymbol{\nu}$ there exists at least one $\boldsymbol{\varphi} \in (W^{1,q}(\Omega))^d$ such that*

(i) $\operatorname{div} \boldsymbol{\varphi} = f$ *a.e. in* Ω,

(ii) $\operatorname{Tr} \boldsymbol{\varphi} = \operatorname{Tr} \boldsymbol{\mathcal{A}}$ *on* $\partial\Omega$,

(iii) $\|\boldsymbol{\varphi}\|_{1,q,\Omega} \leq C_{\mathrm{div}} \left(\|f\|_{q,\Omega} + \|\boldsymbol{\mathcal{A}}\|_{1,q,\Omega}\right)$.

Proof. From Theorem B.VI it follows that there exists $\tilde{\boldsymbol{\varphi}} \in (W_0^{1,q}(\Omega))^d$ such that $\operatorname{div} \tilde{\boldsymbol{\varphi}} = f - \operatorname{div} \boldsymbol{\mathcal{A}}$ in Ω and

$$\|\tilde{\boldsymbol{\varphi}}\|_{1,q,\Omega} \leq C_{\mathrm{div}} \left(\|f\|_{q,\Omega} + \|\operatorname{div} \boldsymbol{\mathcal{A}}\|_{q,\Omega}\right).$$

Then it is sufficient to set $\boldsymbol{\varphi} := \tilde{\boldsymbol{\varphi}} + \boldsymbol{\mathcal{A}}$. □

B.3 Some properties of weighted Sobolev Spaces

Lemma B.4. *$W(\alpha)$, defined by (1.5), is a separable reflexive Banach space.*

Proof. We will use the fact that a closed subspace of a separable reflexive space is also separable and reflexive (for details see e.g. Schechter [21], Chapter 8). Let us define the space

$$S := \left(L^2(\Omega(\alpha))\right)^2 \times \left(L^2(\Omega(\alpha))\right)^{2\times 2} \times \left(L^3(\Omega(\alpha))\right)^{2\times 2} \times L^3(\Omega(\alpha))$$

which is a separable reflexive Banach space with the norm

$$\|(\boldsymbol{v}, \mathbb{W}, \mathbb{Z}, y)\|_S := \|\boldsymbol{v}\|_{2,\Omega(\alpha)} + \|\mathbb{W}\|_{2,\Omega(\alpha)} + \|\mathbb{Z}\|_{3,\Omega(\alpha)} + \|y\|_{3,\Omega(\alpha)}, \qquad (\boldsymbol{v}, \mathbb{W}, \mathbb{Z}, y) \in S.$$

Further define the mapping $\mathcal{I} : W(\alpha) \to S$ by the formula

$$\mathcal{I}(\boldsymbol{v}) := (\boldsymbol{v}, \nabla \boldsymbol{v}, M_\alpha \mathbb{D}(\boldsymbol{v}), \operatorname{div} \boldsymbol{v}).$$

Then $\mathcal{I}$ is an isomorphism of $W(\alpha)$ onto $S_\alpha := \mathcal{I}(W(\alpha))$ and

$$\forall \boldsymbol{v} \in W(\alpha) \ \|\mathcal{I}(\boldsymbol{v})\|_S = \|\boldsymbol{v}\|_\alpha.$$

We show that S_α is a closed subspace of S. Let $\{\boldsymbol{v}_n\} \subset W(\alpha)$ and $\mathcal{I}(\boldsymbol{v}_n) \to (\boldsymbol{v}, \mathbb{W}, \mathbb{Z}, y)$ in S. Then clearly $\mathbb{W} = \nabla \boldsymbol{v}$ and $y = \operatorname{div} \boldsymbol{v}$ in $\Omega(\alpha)$. Moreover

$$\forall f \in L^{\frac{3}{2}}(\Omega(\alpha))\ \forall i,j = 1,2 \int_{\Omega(\alpha)} M_\alpha \mathbb{D}(\boldsymbol{v}_n)_{ij} f \, dx \to \int_{\Omega(\alpha)} \mathbb{Z}_{ij} f \, dx,\ \mathbb{Z} = (\mathbb{Z}_{ij})_{i,j=1}^2.$$

Because $\nabla \boldsymbol{v}_n \to \nabla \boldsymbol{v}$ in $L^2(\Omega(\alpha))$ and $M_\alpha \in L^\infty(\Omega(\alpha))$, also

$$\int_{\Omega(\alpha)} M_\alpha \mathbb{D}(\boldsymbol{v}_n)_{ij} \varphi \, dx \to \int_{\Omega(\alpha)} M_\alpha \mathbb{D}(\boldsymbol{v})_{ij} \varphi \, dx\ \forall \varphi \in \mathcal{C}^\infty(\overline{\Omega(\alpha)}).$$

Since $\mathcal{C}^\infty(\overline{\Omega(\alpha)})$ is dense in $L^{3/2}(\Omega(\alpha))$, we have $\mathbb{Z} = M_\alpha \mathbb{D}(\boldsymbol{v})$ in $\Omega(\alpha)$. Finally, for any $\delta > 0$ there exists $\boldsymbol{v}_n \in \{\boldsymbol{v}_n\}$ and $\boldsymbol{\varphi}_n \in \mathcal{V}(\alpha)$ such that

$$\|\boldsymbol{v} - \boldsymbol{v}_n\|_\alpha \le \delta/2,$$

$$\|\boldsymbol{v}_n - \boldsymbol{\varphi}_n\|_\alpha \le \delta/2.$$

From this and the triangle inequality we have

$$\|\boldsymbol{v} - \boldsymbol{\varphi}_n\|_\alpha \le \delta,$$

meaning that $\boldsymbol{v} \in W(\alpha)$ and $(\boldsymbol{v}, \mathbb{W}, \mathbb{Z}, y) = \mathcal{I}(\boldsymbol{v}) \in S_\alpha$. $\square$

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