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UPPER BOUNDS FOR THE EIGENVALUES
OF COMPACT LINEAR OPERATORS
IN A PREORDERED BANACH SPACE

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ABSTRACT. We establish new efficient upper bounds for the spectral radius of a completely continuous operator in a Banach space equipped by a suitable preordering (e. g., that generated by a solid wedge). The operator considered is assumed to admit a majorant preserving the preordering and, generally speaking, may not leave the given wedge invariant.

1. INTRODUCTION

Many concrete problems of various nature, where the question of uniqueness of a solution is essential, as is well-known, lead one to the study of regular values of a certain bounded linear operator. Since it is always natural to try to get some useful information on the base of as little initial data as possible, is appears that estimates of the spectral radius of an operator which are derived from certain relations involving its value on a single element only, should be of the best imaginable efficiency.

There is a vast literature devoted to this kind of estimates of spectra of linear operators that are positive with respect to a cone in a Banach space (see, e. g., [1,2]). The main idea of such statements goes back to some results of Perron, Jentzsch, Uryson, Collatz, and M. Krein (see, e. g., [1–4]).

The conditions imposed on the operator and the space where it acts vary as well as the assumed properties of the chosen element do. For example, the spectral radius of A admits the estimate $r(A) \geq \gamma$, where γ is a given positive constant, whenever A is a bounded linear operator leaving invariant a cone K and such that $Ag - \gamma g \in K$ with some $g \in (K - K) \setminus (-K)$. On the other hand, $r(A)$ satisfies the inequality

$$r(A) \leq \gamma \qquad (1.1)$$

if $A(K) \subseteq K$, K is a solid normal cone, and the inclusion

$$\gamma g - Ag \in K \qquad (1.2)$$

is true for some interior element g of K [1]. It is natural to find out that obtaining the upper bounds for the spectral radius is more difficult, that a relation of type (1.2) implies (1.1) only under additional conditions on A and K, and that these additional conditions are stronger than those guaranteeing a similar estimate from below.

Relation (1.2) is known to imply estimate (1.1) under various assumptions on A and K (see, e. g., [1, §5.6]). The essential limitation, however, is that

the given linear operator is almost always (with the exception for [1, Theorem 5.3]; see Remark 4.7) assumed to preserve a certain cone, which means that, in applications, certain functions related to the problem considered (e. g., the coefficients of a differential equation or the kernel of an integral operator) should be sign-constant or monotone, or possess some other similar property. Such conditions, in spite of their considerable generality, are nevertheless not satisfied in many important cases. It is desirable to have some techniques for estimating the spectra of linear operators in the "non-positive" cases, in particular, when studying the integral equations associated to boundary value problems determined by mappings which are not monotone in any natural sense.

In this paper, we show that the aforementioned limitation can be overcome in a certain way for linear mappings majorized by linear operators preserving a preordering which may not be a partial ordering. More precisely, we obtain an efficient upper bound for the spectral radius of a completely continuous linear mapping $A : X \to X$ representable in the form

$$A = A_1 - A_2, \tag{1.3}$$

where X is a Banach space with a wedge K, and the operators A_1 and A_2 leave K invariant. The proof of the result mentioned (namely, Theorem 4.1) uses an inequality satisfied by the so-called K-substantial eigenvalues of A and established in Section 3. Note that the wedge K, generally speaking, may not be a cone.

Our present study of completely continuous linear operators in a preordered Banach space is motivated mainly by the related problems arising in the theory of functional differential equations. To a boundary value problem for a linear functional differential equation, a compact linear operator is usually associated and, therefore, the compactness condition in this paper appears to be rather natural from this point of view.

The paper is organised as follows. Section 2 contains some definitions, both classical and new ones, and a number of preliminary results.In Section 3, we establish an estimate for the so-called K-substantial eigenvalues (see Definition 2.41) of a bounded linear operator in a Banach space with a wedge K. The main Theorem 4.1 of Section 4 provides a convenient upper bound for the spectral radius of a completely continuous linear operator vanishing on the blade of the wedge K containing elements that are strongly positive in the sense of Definition 2.12. Finally, in the last Section 7, Theorem 4.1 is applied to obtain conditions sufficient for the solvability of certain integro-functional equations.

2. Wedges in Banach spaces and operators preserving them

In this section, we recall the basic definitions related to wedges in Banach spaces, introduce some notation and definitions, and establish a number of statements relied upon in the subsequent sections. Throughout the rest of this paper, X is a Banach space over the field $\mathbb{R}$.

2.1. **Wedges and related preorderings.** A closed subset K of X is said to be a *wedge* (see, e. g., [2]) if

$$\alpha_1 K + \alpha_2 K \subseteq K \tag{2.1}$$

for all $\{\alpha_1, \alpha_2\} \subset [0, +\infty)$, where, as usual, $\alpha_1 K + \alpha_2 K := \left\{\alpha_1 x_1 + \alpha_2 x_2 \mid \{x_1, x_2\} \subset K\right\}$.

In what follows, we assume implicitly that the wedge K is proper, i. e., is different from both the singleton $\{0\}$ and the entire space X, for there is no meaningful theory in those two extreme cases.

Remark 2.1. In the original terminology introduced by M. Krein [4], a closed set satisfying condition (2.1) is called a *linear semigroup*.

The following standard definition introduces a natural preordering in a space X with a wedge K.

Definition 2.2. The relation $x_1 \leqq_K x_2$ is said to be satisfied if, and only if $x_2 - x_1 \in K$.

We also write $x_1 \geqq_K x_2$ if, and only if $x_2 \leqq_K x_1$. Note that the relations $x_1 \leqq_K x_2$ and $x_1 \geqq_K x_2$, generally speaking, do not imply the equality $x_1 = x_2$.

Definition 2.3. The set $K \cap (-K)$ is referred to as the *blade* [2] of the wedge K.

For the sake of brevity, we shall denote the blade of the wedge K by the symbol $K^\diamond$:

$$K^\diamond := \{x \in X \mid x \geqq_K 0 \wedge x \leqq_K 0\}. \tag{2.2}$$

Remark 2.4. It is obvious from condition (2.1) and definition (2.2) that the blade of an arbitrary wedge K is a closed linear subset of K. One can readily show that $K^\diamond$ coincides with the maximal linear subspace contained in K.

Definition 2.5. We write $x_1 \square_K x_2$ if, and only if either $x_1 \leqq_K x_2$ or $x_1 \geqq_K x_2$.

The relation $\square_K$ is obviously reflexive and symmetric.

2.2. **Measurable elements of a Banach space.** Let f be an element from X, β be a real constant, and $X_{K,\beta}(f)$ be the set defined as follows:

$$X_{K,\beta}(f) := \{x \in X \mid -\beta f \leqq_K x \leqq_K \beta f\}. \tag{2.3}$$

2.2.1. *Basic properties of the sets $X_{K,\beta}(f)$.*

Lemma 2.6. *Let β be a fixed real number. Then an element x from X belongs to the set $X_{K,\beta}(f)$ if, and only if $-x \in X_{K,\beta}(f)$.*

Proof. Due to the symmetry of the left-hand and right-hand terms, the inequality

$$-\beta f \leqq_K x \leqq_K \beta f \tag{2.4}$$

is equivalent to the relation

$$-\beta f \leqq_K -x \leqq_K \beta f, \tag{2.5}$$

whereas the latter means that $-x \in X_{K,\beta}(f)$. $\square$

Lemma 2.7. *The following assertions are true:*

(i) $X_{K,0}(f) = K^\diamond$ *for all* $f \in X$;

(ii) $X_{K,\beta}(0) = K^\diamond$ *for any* $\beta \in \mathbb{R}$;

(iii) *The set* $X_{K,\beta}(f)$, *where* $\beta \neq 0$ *and* $f \neq 0$, *is non-empty if, and only if* $\beta f \geqq_K 0$;

(iv) *If* $\beta f \geqq_K 0$, *then* $X_{K,\beta}(f) \supseteq K^\diamond$;

(v) *If* $f \in K^\diamond$, *then* $X_{K,\beta}(f) = K^\diamond$ *for all* $\beta \in \mathbb{R}$;

(vi) $X_{K,\beta}(f) \setminus K^\diamond \neq \varnothing$ *if, and only if* $\beta f \geqq_K 0$ *and* $\beta f \notin K^\diamond$.

Proof. Assertions (i) and (ii) are obvious from (2.3). Let us verify assertion (iii). Indeed, let x belong to $X_{K,\beta}(f)$. This is true if, and only if (2.4) holds or, which is the same (see Lemma 2.6), relation (2.5) is satisfied. Combining (2.4) and (2.5) and using property (2.1) of K, we obtain

$$-2\beta f \leqq_K 0 \leqq_K 2\beta f,$$

i. e., $\beta f \geqq_K 0$. Conversely, if $\beta f \geqq_K 0$, then, in particular,

$$-\beta f \leqq_K \beta f \leqq_K \beta f.$$

This means that (2.4) is satisfied with $x = \beta f$, i. e., $\beta f \in X_{K,\beta}(f)$.

To prove assertion (iv), it is sufficient to note that if $\beta f \geqq_K 0$, then (2.4) is true for all elements x satisfying the relation $0 \leqq_K x \leqq_K 0$.

Let $f \in K^\diamond$ be arbitrary. By (iv), we have $K^\diamond \subset X_{K,\beta}(f)$ for all $\beta \in \mathbb{R}$. On the other hand, if $x \in X_{K,\beta}(f)$, then, according to (2.4), we obtain $x \in K^\diamond$ because βf is also an element of $K^\diamond$. Thus, assertion (v) is true.

Finally, assertion (vi) is obvious from (iii), (iv), and (v).   □

Assertions (i) and (ii) of Lemma 2.7 show that there is no much sense to consider the sets $X_{K,\beta}(f)$ with $\beta f = 0$ because, in that case, they consist solely of those elements of X which are 0-measurable with respect to K in the sense of Definition 2.8 given below.

2.2.2. *The definition of f-measurability.*

Definition 2.8. An element x from X is said to be *f-measurable with respect to K* if there exists a real constant β such that $x \in X_{K,\beta}(f)$.

In other words, x is f-measurable with respect to the wedge K whenever (2.4) holds for some β.

Remark 2.9. Definition 2.8 differs from a similar notion introduced in [4] because the negative values of β are allowed in (2.4). For the purposes of this paper, the definition mentioned seems to be advantageous due to the need to consider complexifications (see Section 2.5 below). Note also that, according to Definition 2.8, the set of f-measurable elements is never empty (see Proposition 2.11).

Definition 2.10. For every fixed $f \in X$, the set of all the elements of X that are f-measurable with respect to K will be denoted by $X_K(f)$.

Clearly, $X_K(f) := \bigcup_{\beta \in \mathbb{R}} X_{K,\beta}(f)$. Moreover, it follows from Lemma 2.7 that, in fact,

$$X_K(f) = \bigcup_{\beta \in \mathbb{R}:\, \beta f \geqq_K 0} X_{K,\beta}(f) \tag{2.6}$$

for any $f \in X$.

Proposition 2.11. *For any $f \in X$, the set $X_K(f)$ is a linear manifold containing $K^\diamond$. Furthermore, $X_K(f) \neq K^\diamond$ if, and only if the element f is such that $f \square_K 0$ and $f \notin K^\diamond$.*

Proof. The set $X_K(f)$ obviously satisfies the condition

$$\alpha_1 X_K(f) + \alpha_2 X_K(f) \subset X_K(f)$$

for all $\{\alpha_1, \alpha_2\} \subset [0, +\infty)$ and, therefore, Lemma 2.6 guarantees that it is a linear manifold.

According to Definition 2.10 and assertion (i) of Lemma 2.7, we have $X_K(f) \supset X_{K,0}(f) = K^\diamond$. Furthermore, equality (2.6) yields

$$X_K(f) \setminus K^\diamond = \bigcup_{\beta \in \mathbb{R}: \ 0 \neq \beta f \geqq_K 0} X_{K,\beta}(f) \setminus K^\diamond. \tag{2.7}$$

However, assertion (vi) of Lemma 2.7 guarantees that the condition $f \notin K^\diamond$ is necessary and sufficient for the union in the right-hand side of (2.7) to contain non-empty sets.   □

2.3. **Strict inequalities.** Given a wedge $K \subseteq X$ and a linear manifold H in X, we introduce the following binary relation on X.

Definition 2.12. For $\{f_1, f_2\} \subset X$, we write $f_1 \boxdot_{K;H} f_2$ if, and only if the inclusion

$$X_K(f_2 - f_1) \supseteq H$$

is satisfied.

One can readily verify that the equality

$$X_K(-f) = X_K(f)$$

holds for any f and, hence, the relation introduced by Definition 2.12 is symmetric, i. e., $f_1 \boxdot_{K;H} f_2$ if, and only if $f_2 \boxdot_{K;H} f_1$. It is also easy to see that $f_1 \boxdot_{K;H_1} f_2$ implies $f_1 \boxdot_{K;H_2} f_2$ whenever $H_1 \supseteq H_2$.

Lemma 2.13. *For an arbitrary f from X, the relation*

$$f \boxdot_{K;X_K(f)} 0 \tag{2.8}$$

is true.

Proof. By Proposition 2.11, the set $X_K(f)$ is a linear manifold in X. According to Definition 2.12, relation (2.8) is equivalent to the inclusion $X_K(f) \supseteq X_K(f)$ and, hence, is always satisfied.   □

In the case where $H = X$, we drop the corresponding subscript in the above notation and, instead of $f_1 \boxdot_{K;X} f_2$, we write $f_1 \boxdot_K f_2$:

Definition 2.14. For $\{f_1, f_2\} \subset X$, we write $f_1 \boxdot_K f_2$ if, and only if $X_K(f_2 - f_1) = X$.

The above definition allows one to introduce the following

Definition 2.15. Two elements f_1 and f_2 are said to be in the relation $f_1 >_{K;H} f_2$ (resp., $f_1 <_{K;H} f_2$) if they satisfy the conditions $f_1 \boxdot_{K;H} f_2$ and $f_1 \geqq_K f_2$ (resp., $f_1 \leqq_K f_2$).

By analogy with Definition 2.14, we introduce

Definition 2.16. Two elements f_1 and f_2 are said to be in the relation $f_1 >_K f_2$ (resp., $f_1 <_K f_2$) if they satisfy the conditions $f_1 \ \Box_K \ f_2$ and $f_1 \geqq_K f_2$ (resp., $f_1 \leqq_K f_2$).

The fulfilment of the relations described by Definition 2.16 is verified most easily in the case of a solid wedge.

Definition 2.17. A wedge is said to be *solid* [4] if its interior is non-empty.

Following [4], we write $x_1 \ll_K x_2$ (resp., $x_1 \gg_K x_2$) if, and only if the difference $x_2 - x_1$ (resp., $x_1 - x_2$) lies in the interior of K.

Lemma 2.18. *If K is a solid wedge in X and an element $f \in X$ is such that $f \gg_K 0$, then f satisfies the relation*

$$f >_K 0. \tag{2.9}$$

Proof. A statement equivalent to equality (2.14) for f lying in the interior of K is well-known, e. g., from [4, 5]. $\square$

Remark 2.19. When K is a minihedral cone in X [4] (and, hence, the partial ordering $\leqq_K$ makes X into a vector lattice [6]), an element u possessing the property $u >_K 0$ is called a *strong unit* [5, Definition XIII.1.5]. In this case, the condition $X_K(u) = X$ means that the element u satisfies Axiom V from [7].

Remark 2.20. Relation (2.9), generally speaking, does not imply that $f \gg_K 0$. For example, in the space $L^\infty([0, 1])$ of essentially bounded functions endowed with the usual norm and partial ordering [5], relation (2.9) is true, e. g., for f equal almost everywhere to 1. However, the set of functions non-negative almost everywhere on $[0, 1]$ has empty interior in $L^\infty([0, 1])$.

For suitable linear manifolds H, the condition

$$f >_{K;H} 0$$

may be regarded as a certain "strong positivity" an element $f \geqq_K 0$. The word "suitable" here means that, roughly speaking, there should not be too many strongly positive elements. For instance, there is no much sense to study the case where

$$H \subseteq K^\diamond \tag{2.10}$$

because, by virtue of Proposition 2.11, the inclusion

$$\bigcap_{f \in X} X_K(f) \supseteq K^\diamond$$

is always true and, hence, under condition (2.10), the relation $f >_{K;K^\diamond} 0$ is satisfied by an arbitrary element f from X. On the other hand, certain undesirable classes of vectors f (e. g., $f = 0$ or, more generally, f satisfying the relation $0 \leqq_K f \leqq_K 0$), that are unlikely candidates for strongly positive elements, should also be excluded from consideration. These considerations lead us to the following

Proposition 2.21. *Let H be a linear manifold in X such that*

$$H \nsubseteq K^\diamond \tag{2.11}$$

and f be an element of X such that either the relation $f \ \square_K \ 0$ is not true or $0 \leqq_K f \leqq_K 0$. Then the relation

$$f \ \boxminus_{K;H} \ 0 \tag{2.12}$$

is not satisfied.

Proof. Indeed, let, on the contrary, relation (2.12) holds. According to Definition 2.12, this means that $H \subseteq X_K(f)$ and, therefore, in view of condition (2.11), the set $X_K(f)$ contains some elements not belonging to $K^\diamond$. It then follows from Proposition 2.11 that f should satisfy the relations $f \ \square_K \ 0$ and $f \notin K^\diamond$, contrary to the assumption. □

In other words, Proposition 2.21 means that a strongly positive element should always be comparable with zero and cannot be positive and negative simultaneously. This agrees well with the intuitive idea of the strict inequality.

2.4. **The mappings $n_{K,f} : X_K(f) \rightarrow [0,+\infty)$.** Taking a glance at Definition 2.10, we see that the non-negative number

$$n_{K,f}(x) := \inf \left\{ |\beta| \mid \beta \in (-\infty, +\infty) \text{ and } x \in X_{K,\beta}(f) \right\} \tag{2.13}$$

is well-defined for an arbitrary x from $X_K(f)$. It is also convenient to put $n_{K,f}(x) := +\infty$ for all $x \in X \setminus X_K(f)$. Thus, $n_{K,f}(x) < +\infty$ if, and only if x is f-measurable with respect to K.

Remark 2.22. One can show that, for any $f \in X$, the mapping $n_{K,f} : X_K(f) \rightarrow [0,+\infty)$ is a seminorm on the linear manifold $X_K(f)$. This seminorm is a norm if, and only if K is a cone [4, 8], i. e., if the blade of K is trivial. The mapping mentioned is defined on the entire space X if, and only if

$$X_K(f) = X, \tag{2.14}$$

which property, in contrast to the poorest case where

$$X_K(f) = K^\diamond,$$

may be regarded as a reflection of a reasonable choice of an element $f \geqq_K 0$. By Lemma 2.18, condition (2.14) is satisfied if $f \ \gg_K \ 0$. It may happen, however, that (2.14) does not hold for any f from X (e. g., if X is the Banach space of the Lebesgue integrable functions on a bounded interval $[a,b]$ and K is the cone of integrable functions $[a,b] \rightarrow \mathbb{R}$ that are non-negative almost everywhere on $[a,b]$).

In the case where K is a solid cone and $f \ \gg_K \ 0$, formula (2.13) determines the so-called f-norm [1, 4]

$$\|x\|_f = \inf \{\beta \in [0,+\infty) \mid \text{ relation (2.4) is true}\} \tag{2.15}$$

of an arbitrary element x from X. Functional (2.15) is also used in [7] in studies of vector lattices.

It is clear from (2.13) that $n_{K,f}(0) = 0$ independently of the choice of f. Moreover, the following lemma holds.

Lemma 2.23. *Let $f \in X$. Then an element $x \in X$ satisfies the equality*

$$n_{K,f}(x) = 0 \tag{2.16}$$

if, and only if $x \in K^\diamond$.

Proof. Let $f \in X$ and let x be an element from the corresponding (non-empty) set $X_K(f)$. In view of Proposition 2.11, we can suppose that $f \square_K 0$. Then, clearly, $\sigma f \geqq_K 0$ for some $\sigma \in \{-1, 1\}$.

Let $x \in K^\diamond$. The element x belongs to the blade of K if, and only if

$$0 \leqq_K x \leqq_K 0, \tag{2.17}$$

which means that (2.4) is true with an arbitrary constant β such that $\operatorname{sign} \beta = \sigma$. In particular,

$$-\frac{\sigma}{k} f \leqq_K x \leqq_K \frac{\sigma}{k} f$$

for all $k \in \mathbb{N}$. Taking (2.13) into account, we conclude that $0 \leq n_{K,f}(x) \leq \inf_{k \in \mathbb{N}} k^{-1} = 0$, i. e., relation (2.16) holds.

Conversely, if x satisfies equality (2.16), then there exists a sequence $(\beta_k)_{k=1}^{+\infty} \subset (-\infty, +\infty)$ such that $\lim_{k \to +\infty} \beta_k = 0$ and, for all $k \geq 1$,

$$-\beta_k f \leqq_K x \leqq_K \beta_k f. \tag{2.18}$$

Passing to the limit as $k \to +\infty$ in relation (2.18) or, which is the same, in the inclusion

$$\{\beta_k f - x, \, \beta_k f + x\} \subset K$$

and taking into account the fact that K is a closed set, we arrive at relation (2.17). □

2.5. **Complexification of a wedge.** In the sequel, the complex counterparts of some of the notions defined above will be needed. Throughout this section, where the related notions are introduced, we fix a real Banach space X and wedge K in X.

2.5.1. *Basic issues.* The complexification (see, e. g., [9], Chapter XIII, §2) of a real Banach space $\langle X, \|\cdot\| \rangle$ is convenient to be interpreted as the complex Banach space $\hat{X}$ of formal sums $x + iy$, $\{x, y\} \subset X$, $i^2 = -1$, equipped with the linear operations

$$\begin{aligned} (x_1 + iy_1) + (x_2 + iy_2) &:= x_1 + x_2 + i(y_1 + y_2), \\ (\nu + i\mu)(x + iy) &:= \nu x - \mu y + i(\mu x + \nu y), \end{aligned} \tag{2.19}$$

where $\{x_1, x_2, y_1, y_2, x, y\} \subset X$, $\{\nu, \mu\} \subset \mathbb{R}$, and the norm

$$\|x + iy\| := \max_{\theta \in [-\pi, \pi]} \|x \cos \theta + y \sin \theta\|, \qquad \{x, y\} \subset X. \tag{2.20}$$

The same technique allows one to define a natural complexification of an arbitrary wedge in a real Banach space.

Definition 2.24. The set

$$\hat{K} := \{x + iy \mid x \in K \wedge y \in K\} \tag{2.21}$$

will be referred to as the *complexification* of a wedge K in a Banach space X over $\mathbb{R}$.

It is easy to verify that the set $\hat{K}$, represented alternatively as $\hat{K} = K + iK$, is closed with respect to norm (2.20) and forms a wedge in $\hat{X}$ in the sense that

$$\alpha_1 \hat{K} + \alpha_2 \hat{K} \subset \hat{K} \quad \text{for all } \{\alpha_1, \alpha_2\} \subset [0, +\infty). \tag{2.22}$$

By analogy with Sections 2.1 and 2.3, one can extend the binary relations $\leqq_K$ and $\ll_K$ to $\hat{X}^2$ in a natural way. More precisely, given two elements $\{z_1, z_2\} \subset \hat{X}$, we shall write $z_1 \geqq_{\hat{K}} z_2$ (resp., $z_1 \gg_{\hat{K}} z_2$) if, and only if $z_1 - z_2 \in \hat{K}$ (resp., $z_1 - z_2$ is an interior element of $\hat{K}$). Similarly, the relation $\square_{\hat{K}}$ is natural to be defined by putting $z_1 \square_{\hat{K}} z_2$ if, and only if the elements z_1 and z_2 satisfy at least one of the relations $z_1 \geqq_{\hat{K}} z_2$ and $z_1 \leqq_{\hat{K}} z_2$. The blade $(\hat{K})^\diamond$ of $\hat{K}$ is natural to be defined as the set of all those z from $\hat{X}$ for which both relations $z \geqq_{\hat{K}} 0$ and $z \leqq_{\hat{K}} 0$ are true, i. e.,

$$(\hat{K})^\diamond = \hat{K} \cap (-\hat{K}).$$

It is obvious that

$$(\hat{K})^\diamond = K^\diamond + iK^\diamond. \tag{2.23}$$

The complexification $\hat{K}$ of a real wedge K inherits its main characteristic properties. For example, $\hat{K}$ is solid if, and only if K possesses this property.

2.5.2. *Measurability of complex elements.* Let $g \in \hat{X}$ and $\lambda \in \mathbb{C}$. Similarly to formula (2.3), one can define the set $\hat{X}_{\hat{K},\lambda}(g) \subset \hat{X}$ by putting

$$\hat{X}_{\hat{K},\lambda}(g) := \left\{ z \in \hat{X} \mid z \geqq_{\hat{K}} -\lambda g \wedge z \leqq_{\hat{K}} \lambda g \right\} \tag{2.24}$$

and introduce the following

Definition 2.25. An element $z \in \hat{X}$ is said to be *g-measurable with respect to the wedge $\hat{K}$* if, and only if it belongs to the set

$$\hat{X}_{\hat{K}}(g) := \bigcup_{\lambda \in \mathbb{C}} \hat{X}_{\hat{K},\lambda}(g). \tag{2.25}$$

Definition 2.25 may be regarded as a natural extension of Definition 2.8 to the complex case. For example, analogues of Lemma 2.6 and Proposition 2.11 are true for sets (2.24) and, just as in the real case, zero belongs to the set $\hat{X}_{\hat{K},\lambda}(g)$ for arbitrary $\lambda \in \mathbb{C}$ and $g \in \hat{X}$. Further properties of sets (2.24) are described by Lemma 2.31 below.

Remark 2.26. Analogues of sets (2.24) and the related objects can also be introduced in the case where $\hat{K}$ is replaced by some other set possessing property (2.22), not necessarily constructed according to formula (2.21). Such more general complex wedges are however not needed for our purposes.

A convenient characterisation of the property introduced in Definition 2.25 is provided by the following

Lemma 2.27. *Let* $\{x, y, f\} \subset X$. *Then the element* $x + iy$ *is* $\hat{f}$-measurable with respect to $\hat{K}$ if, and only if there exist some $r \in [0, +\infty)$ and $\omega \in [-\pi, \pi]$ such that the relations

$$-rf \sin \omega \leqq_K x \leqq_K rf \sin \omega, \tag{2.26}$$

$$-rf \cos \omega \leqq_K y \leqq_K rf \cos \omega \tag{2.27}$$

are satisfied.

Here and everywhere in the sequel, we write $\hat{f} = f + if$ for any f from X.

Proof. By virtue of relations (2.24) and (2.25), the element $x + iy$ is $\hat{f}$-measurable if, and only if there exist some $\varrho \in [0, +\infty)$ and $\theta \in [-\pi, \pi]$ for which

$$-\varrho\, e^{i\theta} \hat{f} \leqq_{\hat{K}} x + iy \leqq_{\hat{K}} \varrho\, e^{i\theta} \hat{f}. \tag{2.28}$$

According to (2.19), we have

$$e^{i\theta} \hat{f} = (\cos\theta + i\sin\theta)(f + if) = (\cos\theta - \sin\theta) f + i(\sin\theta + \cos\theta) f$$

$$= \sqrt{2}\left[\sin\left(\frac{\pi}{4} - \theta\right) + i\cos\left(\frac{\pi}{4} - \theta\right)\right] f. \tag{2.29}$$

Therefore, in view of definition (2.21) of the set $\hat{K}$, the relation (2.28) is equivalent to the system of order inequalities

$$-\varrho\,\sqrt{2}f\sin\left(\frac{\pi}{4} - \theta\right) \leqq_K x \leqq_K \varrho\,\sqrt{2}f\sin\left(\frac{\pi}{4} - \theta\right), \tag{2.30}$$

$$-\varrho\,\sqrt{2}f\cos\left(\frac{\pi}{4} - \theta\right) \leqq_K y \leqq_K \varrho\,\sqrt{2}f\cos\left(\frac{\pi}{4} - \theta\right), \tag{2.31}$$

which, obviously, has form (2.26), (2.27) with $r := \varrho\,\sqrt{2}$ and

$$\omega := \begin{cases} \frac{\pi}{4} - \theta & \text{if } -\pi \leq \theta \leq -\frac{3\pi}{4}, \\ -\frac{7\pi}{4} - \theta & \text{if } -\frac{3\pi}{4} < \theta \leq \pi. \end{cases}$$

It is clear that the above relation between the pairs (ϱ, θ) and (r, ω) is one-to-one. $\qquad\square$

Remark 2.28. Definition 2.25 reduces to Definition 2.8 in the real case. Indeed, let $\sigma f \in \hat{K}$ with some $\sigma \in \{-1, 1\}$. Lemma 2.27 characterises the $\hat{f}$-measurability of the element $x = x + i0$ with respect to $\hat{K}$ in terms of the existence of $(r, \omega) \in [0, +\infty) \times [-\pi, \pi]$ such that $\sigma\cos\omega \geq 0$ and relation (2.26) is true. However, the property mentioned means that (2.4) is satisfied with $\beta := r\sin\omega$.

It is natural to find out that the $\hat{f}$-measurability of an element $x + iy$ with respect to $\hat{K}$ is equivalent to the f-measurability of its real and imaginary parts, x and y.

Lemma 2.29. *Let $\{x, y, f\} \subset X$. Then the element $x + iy$ is $\hat{f}$-measurable with respect to $\hat{K}$ if, and only if both x and y are f-measurable with respect to K.*

Proof. The f-measurability of x and y, on the assumption that $x + iy \in \hat{X}_{\hat{K}}(\hat{f})$, is a consequence of Lemma 2.27. Conversely, if $\{x, y\} \subset X_K(f)$, then, according to Definition 2.8, there exist some real α and β such that

$$-\alpha f \leqq_K x \leqq_K \alpha f, \tag{2.32}$$

$$-\beta f \leqq_K y \leqq_K \beta f. \tag{2.33}$$

Let us put

$$\omega := \begin{cases} \frac{\pi}{4}\operatorname{sign}\alpha & \text{if } \beta \geq 0, \\ \frac{3\pi}{4}\operatorname{sign}\alpha & \text{if } \beta < 0 \end{cases}$$

and $r := \sqrt{2}\max\{|\alpha|, |\beta|\}$. Then, as is easy to see,

$$\sin\omega = \frac{\operatorname{sign}\alpha}{\sqrt{2}}, \qquad \cos\omega = \frac{\operatorname{sign}\beta}{\sqrt{2}},$$

and, therefore, relations (2.32) and (2.33) imply that (2.26) and (2.27) are satisfied with the above values of ω and r. It remains to refer to Lemma 2.27.

$\square$

It turns out that all the sets $\hat{X}_{\hat{K}}(\hat{f})$, where $\hat{f} = f + \mathrm{i}f$, are invariant under rotations. More precisely, the following statement is true.

Lemma 2.30. *Let $f \in X$ and $\{x, y\} \subset X_K(f)$. Then, for an arbitrary $\phi \in [-\pi, \pi]$, the element $\mathrm{e}^{\mathrm{i}\phi}(x + \mathrm{i}y)$ is $\hat{f}$-measurable with respect to $\hat{K}$.*

Proof. It will suffice to consider the case where $f \square_K 0$. By assumption, $\{x, y\} \subset X_K(f)$ and, hence, in view of Lemma 2.29, the element $x + \mathrm{i}y$ is $\hat{f}$-measurable with respect to $\hat{K}$, where $\hat{f} = f + \mathrm{i}f$. Lemma 2.27 guarantees the existence of an $\omega \in [-\pi, \pi]$ such that relations (2.26) and (2.27) are satisfied. Multiplying both parts of (2.26) by $|\cos \phi|$ and $|\sin \phi|$ and taking Lemma 2.6 into account, we obtain, respectively, the relations

$$-fr|\cos \phi| \sin \omega \leqq_K \pm x \cos \phi \leqq_K fr|\cos \phi| \sin \omega$$

and

$$-fr|\sin \phi| \sin \omega \leqq_K \pm x \sin \phi \leqq_K fr|\sin \phi| \sin \omega,$$

where the symbol "$\pm$" means that the inequality is satisfied with both signs of the corresponding term. Similarly, multiplying both parts of (2.27) by $|\cos \phi|$ and $|\sin \phi|$, we get

$$-fr|\cos \phi| \cos \omega \leqq_K \pm y \cos \phi \leqq_K fr|\cos \phi| \cos \omega$$

and

$$-fr|\sin \phi| \cos \omega \leqq_K \pm y \sin \phi \leqq_K fr|\sin \phi| \cos \omega.$$

Therefore, by choosing the appropriate signs in the relations above and summing the corresponding terms, we obtain

$$\begin{aligned}
x \cos \phi - y \sin \phi &\leqq_K rf[|\cos \phi| \sin \omega + |\sin \phi| \cos \omega], \\
x \cos \phi - y \sin \phi &\geqq_K -rf[|\cos \phi| \sin \omega + |\sin \phi| \cos \omega]
\end{aligned} \tag{2.34}$$

and

$$\begin{aligned}
y \cos \phi + x \sin \phi &\leqq_K rf[|\cos \phi| \cos \omega + |\sin \phi| \sin \omega], \\
y \cos \phi + x \sin \phi &\geqq_K -rf[|\cos \phi| \cos \omega + |\sin \phi| \sin \omega].
\end{aligned} \tag{2.35}$$

It is supposed that $f \square_K 0$, and, therefore, $\sigma f \geqq_K 0$ for some $\sigma \in \{-1, 1\}$. Since neither $|\cos \phi| \sin \omega + |\sin \phi| \cos \omega$ nor $|\cos \phi| \cos \omega + |\sin \phi| \sin \omega$ takes values outside the interval $[-2, 2]$, relations (2.34) and (2.35) yield

$$2\sigma rf \leqq_K x \cos \phi - y \sin \phi \leqq_K 2\sigma rf, \tag{2.36}$$

$$2\sigma rf \leqq_K y \cos \phi + x \sin \phi \leqq_K 2\sigma rf. \tag{2.37}$$

Let us put

$$\theta_\sigma := \begin{cases} \frac{\pi}{4} & \text{if } \sigma = 1, \\ \frac{-3\pi}{4} & \text{if } \sigma = -1. \end{cases}$$

Then $\sin \theta_\sigma = \cos \theta_\sigma = \sigma 2^{-\frac{1}{2}}$ and, therefore, relations (2.36) and (2.37) can be brought to the form

$$\varrho f \sin \theta_\sigma \leqq_K x \cos \phi - y \sin \phi \leqq_K \varrho f \sin \theta_\sigma,$$

$$\varrho f \cos \theta_\sigma \leqq_K y \cos \phi + x \sin \phi \leqq_K \varrho f \cos \theta_\sigma,$$

where $\varrho := 2r\sqrt{2}$. Applying now Lemma 2.27 and taking into account the formula

$$e^{i\phi}(x + iy) = x\cos\phi - y\sin\phi + i(y\cos\phi + x\sin\phi), \qquad (2.38)$$

we conclude that the element $e^{i\phi}(x + iy)$ is $\hat{f}$-measurable.   □

The next lemma summarises several properties of sets (2.24) referred to in the sequel.

Lemma 2.31. *The following assertions are true:*

(i) $\hat{K}, 0(\hat{f}) = \hat{X}_{\hat{K},\lambda}(0) = K^\diamond + iK^\diamond$ *for all* $f \in X$ *and* $\lambda \in \mathbb{C}$;

(ii) $\hat{X}_{\hat{K},\lambda}(g) = -\hat{X}_{\hat{K},\lambda}(g)$ *for any* $g \in \hat{X}$;

(iii) $\alpha^{-1}\hat{X}_{\hat{K},|\alpha|\lambda}(g) = \hat{X}_{\hat{K},\lambda}(g)$ *for any* $g \in \hat{X}$ *and* $\alpha \in \mathbb{R} \setminus \{0\}$;

(iv) *For* $\lambda \neq 0$ *and* $g \neq 0$, *the set* $\hat{X}_{\hat{K},\lambda}(g)$ *is non-empty if, and only if* $\lambda g \in \hat{K}$;

(v) *If* $\lambda g \in \hat{K}$, *then* $\hat{X}_{\hat{K},\lambda}(g) \supseteq K^\diamond + iK^\diamond$;

(vi) $\hat{X}_{\hat{K},\lambda}(g) \setminus (K^\diamond + iK^\diamond) \neq \varnothing$ *if, and only if* $\lambda g \in \hat{K} \setminus (K^\diamond + iK^\diamond)$.

(vii) $\bigcap_{g \in \hat{X}} \hat{X}_{\hat{K}}(g) \supseteq K^\diamond + iK^\diamond$;

(viii) $\hat{X}_{\hat{K}}(g) \neq K^\diamond + iK^\diamond$ *if, and only if* $g \in [\hat{K} \cup (-\hat{K})] \setminus (K^\diamond + iK^\diamond)$.

Proof. This statement is established similarly to Lemmata 2.6 and 2.7 and Propositions 2.11 and 2.11 from Section 2.2.

Let us prove, e. g., assertion (iii). Indeed, let $\alpha \neq 0$. By virtue of (ii), an element z belongs to the set $\hat{X}_{\hat{K},|\alpha|\lambda}$ if, and only if

$$-|\alpha|\lambda g \leqq_{\hat{K}} z\,\text{sign}\,\alpha \leqq_{\hat{K}} |\alpha|\lambda g,$$

or, which is the same,

$$-\lambda g \leqq_{\hat{K}} \frac{z}{\alpha} \leqq_{\hat{K}} \lambda g. \qquad (2.39)$$

However, (2.39) means nothing but the inclusion $\alpha^{-1}z \in \hat{X}_{\hat{K},\lambda}(g)$.   □

2.5.3. *The mappings* $n_{\hat{K},g} : \hat{X} \to [0, +\infty]$. Similarly to the case of the original real space X, the $\hat{f}$-measurability of elements of $\hat{X}$ with respect to the complexification $\hat{K}$ of a wedge K in X can be characterised by a certain non-linear functional. More precisely, given $z \in \hat{X}$ and $g \in \hat{X}$, we put

$$n_{\hat{K},g}(z) := \sqrt{2}\inf\left\{|\lambda| \mid \lambda \in \mathbb{C} \wedge z \in \hat{X}_{\hat{K},\lambda}(g)\right\} \qquad (2.40)$$

if z is g-measurable with respect to $\hat{K}$, and $n_{\hat{K},g}(z) := +\infty$ for $z \notin \hat{X}_{\hat{K}}(g)$. Here, we retain the same letter, n, as in the real case (cf. Section 2.2) in order not to complicate the notation unnecessarily.

Lemma 2.32. *For any* $g \in \hat{X}$, *the functional* $n_{\hat{K},g} : \hat{X}_{\hat{K}}(g) \to [0, +\infty)$ *is homogeneous in the sense that*

$$n_{\hat{K},g}(\gamma z) = |\gamma| n_{\hat{K},g}(z)$$

for all $z \in \hat{X}_{\hat{K}}(g)$ *and* $\gamma \in \mathbb{R}$.

Proof. Let us fix some $z \in \hat{X}_{\hat{K}}(g)$ and $\gamma \in \mathbb{R}$, $\gamma \neq 0$. According to formula (2.40), we have

$$n_{\hat{K},g}(\gamma z) = \sqrt{2}\inf\left\{|\lambda| \mid \lambda \in \mathbb{C} \wedge \gamma z \in \hat{X}_{\hat{K},\lambda}(g)\right\}. \qquad (2.41)$$

Applying assertion (iii) of Lemma 2.31 with $\alpha = 1/\gamma$, we conclude that an element γz belongs to $\hat{X}_{\hat{K},\lambda}(g)$ if, and only if $z \in \hat{X}_{\hat{K},\lambda|\gamma|^{-1}}(g)$. Therefore, equality (2.41) can be rewritten as

$$
\begin{aligned}
n_{\hat{K},g}(\gamma z) &= \sqrt{2}\inf\left\{|\lambda| \mid \lambda \in \mathbb{C} \wedge \gamma z \in \hat{X}_{\hat{K},\lambda}(g)\right\} \\
&= \sqrt{2}\inf\left\{|\lambda| \mid \lambda \in \mathbb{C} \wedge z \in \hat{X}_{\hat{K},\lambda/|\gamma|}(g)\right\} \\
&= |\gamma|\sqrt{2}\inf\left\{\frac{\lambda}{|\gamma|} \mid \lambda \in \mathbb{C} \wedge z \in \hat{X}_{\hat{K},\lambda/|\gamma|}(g)\right\} \\
&= |\gamma|\sqrt{2}\inf\left\{|\mu| \mid \mu \in \mathbb{C} \wedge z \in \hat{X}_{\hat{K},\mu}(g)\right\} = |\gamma|n_{\hat{K},g}(z),
\end{aligned}
$$

as required.   □

We are interested mainly in elements of $\hat{X}$ that are $\hat{f}$-measurable with respect to $\hat{K}$ for a suitably chosen f from X (actually, from $[K \cup (-K)] \setminus K^{\diamond}$ because otherwise, by Proposition 2.11, there are no f-measurable elements outside $K^{\diamond}$). In this case, it is convenient to use the following formulae for computation of value (2.40).

Lemma 2.33. *Let $f \in X$ and let $\{x, y\} \subset X$ be some elements f-measurable with respect to K. Then the formulae*

$$
n_{\hat{K},\hat{f}}(x + \mathrm{i}y) = \sqrt{2}\inf\{\varrho \in [0, +\infty) \mid \exists\theta \in [-\pi, \pi] : (2.30) \text{ and } (2.31) \text{ hold}\}
$$
$$(2.42)$$

and

$$
n_{\hat{K},\hat{f}}(x + \mathrm{i}y) = \inf\{r \in [0, +\infty) \mid \exists\omega \in [-\pi, \pi] : (2.26) \text{ and } (2.27) \text{ hold}\} \quad (2.43)
$$

are true.

Proof. By Lemma 2.29, the element $x + \mathrm{i}y$ is $\hat{f}$-measurable with respect to $\hat{K}$ and, therefore, the value of $n_{\hat{K},g}(x + \mathrm{i}y)$ is finite. In view of formula (2.29) established in the proof of Lemma 2.27, the relation

$$
-\lambda\hat{f} \leqq_{\hat{K}} x + \mathrm{i}y \leqq_{\hat{K}} \lambda\hat{f}
$$

with $\lambda = \varrho\,\mathrm{e}^{\mathrm{i}\theta}$ is equivalent to the system of order inequalities (2.30), (2.31). Therefore, definition (2.40) of the mapping $n_{\hat{K},\hat{f}}$ yields the required equality (2.42).

Formula (2.43) is a consequence of (2.42). Indeed, as is shown in the proof of Lemma 2.27, there is a one-to-one correspondence between systems (2.30), (2.31) and (2.26), (2.27), with $r/\varrho = \sqrt{2}$, and it suffices to use Lemma 2.32.   □

The best constant r in relations (2.26) and (2.27) satisfied by the respective components x and y of an $\hat{f}$-measurable element $x + \mathrm{i}y$ is determined by the value of functional (2.40). More precisely, we have

Lemma 2.34. *Let $\{f, x, y\} \subset X$ and*

$$
n_{\hat{K},\hat{f}}(x + \mathrm{i}y) =: r < +\infty.
$$

Then there exists an $\omega \in [-\pi, \pi]$ such that relations (2.26) and (2.27) are satisfied.

Proof. The definition of the functional $n_{\hat{K},f}$ and Lemmata 2.27 and 2.33 yield the existence of sequences $(r_k)_{k=1}^{+\infty} \subset [0, +\infty)$ and $(\omega_k)_{k=1}^{+\infty} \subset [-\pi, \pi]$ such that $\lim_{k \to +\infty} r_k = r$ and the relations

$$-r_k f \sin \omega_k \leqq_K x \leqq_K r_k f \sin \omega_k, \qquad (2.44)$$

$$-r_k f \cos \omega_k \leqq_K y \leqq_K r_k f \cos \omega_k \qquad (2.45)$$

are true for all $k \in \mathbb{N}$. The compact real sequence $(\omega_k)_{k=1}^{+\infty}$ contains a subsequence $(\omega_{k_j})_{j=1}^{+\infty}$ convergent to a number $\omega \in [-\pi, \pi]$. Putting $k = k_j$ in (2.44) and (2.45), passing to the limit as $j \to +\infty$, and taking into account the fact that K is a closed subset of X, we arrive at relations (2.26) and (2.27). $\square$

The following statement is an extension of Lemma 2.23 to the complex case.

Lemma 2.35. *Let $f \in X$ and $z \in \hat{X}$. Then $n_{\hat{K},f}(z) = 0$ if, and only if $z \in K^\diamond + iK^\diamond$.*

Proof. Let $z = x + iy$, where $\{x, y\} \subset K^\diamond$. According to formula (2.43) of Lemma 2.33, we have

$$n_{\hat{K},f}(x + iy) = \inf \{r \in [0, +\infty) \mid \exists \omega \in [-\pi, \pi] : (2.26) \text{ and } (2.27) \text{ hold}\}$$

$$\leq \inf \Big\{r \in [0, +\infty) \mid \exists \{\sigma, \kappa\} \subset \{-1, 1\} : -\sigma r f \leqq_K x \sqrt{2} \leqq_K \sigma r f$$

$$\text{and} - \kappa r f \leqq_K y \sqrt{2} \leqq_K \kappa r f \Big\}$$

$$\leq \inf \Big\{r \in [0, +\infty) \mid -\sigma r f \leqq_K x \sqrt{2} \leqq_K \sigma r f \text{ with some } \sigma \in \{-1, 1\}\Big\}$$

$$= \inf \Big\{|\alpha| \mid \alpha \in \mathbb{R} \wedge -\alpha f \leqq_K x \sqrt{2} \leqq_K \alpha f\Big\}$$

$$= n_{K,f}(x \sqrt{2}). \qquad (2.46)$$

By virtue of Lemma 2.32, we have $n_{K,f}(x \sqrt{2}) = \sqrt{2} \, n_{K,f}(x)$. However, in view of Lemma 2.23, $n_{K,f}(x) = 0$ and, therefore, by (2.46), the non-negative number $n_{\hat{K},f}(x + iy)$ is equal to 0.

Assume now that $n_{\hat{K},f}(x + iy) = 0$. By virtue of Lemma 2.34, there exists an $\omega \in [-\pi, \pi]$ such that relations (2.26) and (2.27) are satisfied with $r = 0$, i. e., $0 \leqq_K x \leqq_K 0$ and $0 \leqq_K y \leqq_K 0$. This means that $\{x, y\} \subset K^\diamond$. $\square$

Formula (2.40) allows one to construct a natural extension $n_{\hat{K},f} : \hat{X} \to [0, +\infty]$ of the mapping $X \ni x \mapsto n_{K,f}(x)$ given by relation (2.13). More precisely, the following statement is true.

Proposition 2.36. *Let $f \in X$. Then the equality*

$$n_{\hat{K},f}(x) = n_{K,f}(x) \qquad (2.47)$$

is true for all $x \in X$.

Proof. First of all, we note that it suffices to consider the case where $f \,\square_K\, 0$ because in the contrary case, by Proposition 2.11, we have $X_K(f) = K^\diamond$ and, therefore, in view of Lemmata 2.23 and 2.35, both $n_{K,f}$ and $n_{\hat{K},f}$ vanish on the set $X_K(f)$.

For the sake of definiteness, we assume that $f \geqq_K 0$. Setting $y = 0$ in formula (2.43) of Lemma 2.33, we obtain

$$n_{\hat{K},\hat{f}}(x) = \inf\{r \in [0, +\infty) \mid \exists \omega \in [-\pi, \pi] : \cos \omega \geq 0 \text{ and } (2.26) \text{ holds}\}$$

$$= \inf\left\{r \in [0, +\infty) \mid \exists \omega \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] : (2.26) \text{ holds}\right\}. \tag{2.48}$$

Since the mapping $\sin : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1]$ is a bijection, we see that (2.48) can be rewritten in the form

$$n_{\hat{K},\hat{f}}(x) = \inf\{r \in [0, +\infty) \mid \exists h \in [-1, 1] : -rhf \leqq_K x \leqq_K rhf\}$$

$$= \inf\{|\alpha| \in [0, +\infty) \mid -\alpha f \leqq_K x \leqq_K \alpha f\},$$

which, by virtue of (2.13), proves that equality (2.47) is true for all x from $X_K(f)$. In the case where x is not f-measurable with respect to K, by Lemma 2.29, both values are equal to $+\infty$. □

2.5.4. *Measuring rotated elements.* In the sequel, we need to compute the values of the functional $n_{\hat{K},\hat{f}}$ on elements of the form $e^{it}(x + iy)$, where $t \in [-\pi, \pi]$ is arbitrary and $\{x, y\} \subset X_K(f)$ with some f satisfying the condition $f \square_K 0$.

Definition 2.37. Given an $f \in X$, we put

$$R_{\hat{K},\hat{f}}(x + iy) := \inf_{t \in [-\pi,\pi]} n_{\hat{K},\hat{f}}(e^{it}(x + iy)) \tag{2.49}$$

if $\{x, y\} \subset X$ are f-measurable with respect to K, and set formally $R_{\hat{K},\hat{f}}(x + iy) := +\infty$ in the contrary case.

It follows from Lemma 2.30 that the right-hand side of (2.49) is finite for arbitrary $\{x, y\} \subset X_K(f)$ and $t \in [-\pi, \pi]$ and, thus, Definition 2.37 makes sense.

Lemma 2.38. *Let $f \in X$ and $\{x, y\} \subset X_K(f)$. Then, for an arbitrary $\phi \in [-\pi, \pi]$, the equality*

$$R_{\hat{K},\hat{f}}(e^{i\phi}(x + iy)) = R_{\hat{K},\hat{f}}(x + iy) \tag{2.50}$$

is true.

Proof. According to formula (2.49), we have

$$R_{\hat{K},\hat{f}}(e^{i\phi}(x + iy)) = \inf_{t \in [-\pi,\pi]} n_{\hat{K},\hat{f}}(e^{i\phi}e^{it}(x + iy))$$

$$= \inf_{t \in [-\pi,\pi]} n_{\hat{K},\hat{f}}(e^{i(t+\phi)}(x + iy)). \tag{2.51}$$

Let us take an arbitrary $\phi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and put

$$\phi_t := \begin{cases} t + \phi + \pi & \text{if } t + \phi < -\pi, \\ t + \phi & \text{if } -\pi \leq t + \phi \leq \pi, \\ t + \phi - \pi & \text{if } t + \phi > \pi \end{cases}$$

for all t from $[-\pi, \pi]$. It is clear that $\{e^{i\phi_t} \mid t \in [-\pi, \pi]\} = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$ for any ϕ. Therefore, equality (2.51) yields

$$
\begin{aligned}
R_{\hat{K}, f}(e^{i\phi}(x + iy)) &= \inf_{t \in [-\pi, \pi]} n_{\hat{K}, f}(e^{i\phi_t}(x + iy)) \\
&= \inf_{t \in [-\pi, \pi]} n_{\hat{K}, f}(e^{it}(x + iy)) = R_{\hat{K}, f}(x + iy).
\end{aligned}
$$

Applying formula (2.50) sequentially, we prove that it is true with arbitrary values of ϕ from $[-\pi, \pi]$. □

Together with Lemma 2.38, the next statement is a basic tool in the proof of Theorem 3.1 from Section 3.

Lemma 2.39. *Let $f \in X$, $\{x, y\} \subset X_K(f)$, and*

$$
R := R_{\hat{K}, f}(x + iy). \tag{2.52}
$$

Then there exist some $\{\theta_*, \omega_*\} \subset [-\pi, \pi]$ such that the relations

$$
-Rf \sin \omega_* \leqq_K x \cos \theta_* - y \sin \theta_* \leqq_K Rf \sin \omega_*, \tag{2.53}
$$

$$
-Rf \cos \omega_* \leqq_K x \sin \theta_* + y \cos \theta_* \leqq_K Rf \cos \omega_* \tag{2.54}
$$

are true. Moreover, if $R > 0$, then the inequalities

$$
-(R - \epsilon)f \sin \tilde{\omega} \leqq_K x \cos \tilde{\theta} - y \sin \tilde{\theta} \leqq_K (R - \epsilon)f \sin \tilde{\omega}, \tag{2.55}
$$

$$
-(R - \epsilon)f \cos \tilde{\omega} \leqq_K x \sin \tilde{\theta} + y \cos \tilde{\theta} \leqq_K (R - \epsilon)f \cos \tilde{\omega} \tag{2.56}
$$

are not satisfied with any $\{\tilde{\theta}, \tilde{\omega}\} \subset [-\pi, \pi]$ and $\epsilon \in (0, R)$.

Proof. Let us fix some $\{x, y\} \subset X_K(f)$ and define R by (2.52). Then

$$
R = \inf_{t \in [-\pi, \pi]} r_t, \tag{2.57}
$$

where $r_t := n_{\hat{K}, f}(e^{it}(x + iy))$ for all $t \in [-\pi, \pi]$. By virtue of Lemma 2.30, we have $0 \leq R < +\infty$.

Taking Lemma 2.34 and formula (2.38) into account, we conclude that, with any $t \in [-\pi, \pi]$, one can associate an $\omega_t \in [-\pi, \pi]$ for which

$$
-r_t f \sin \omega_t \leqq_K x \cos t - y \sin t \leqq_K r_t f \sin \omega_t \tag{2.58}
$$

and

$$
-r_t f \cos \omega_t \leqq_K y \cos t + x \sin t \leqq_K r_t f \cos \omega_t. \tag{2.59}
$$

By virtue of (2.57), there exists a sequence $(t_m)_{m=1}^{+\infty} \subset [-\pi, \pi]$ such that

$$
\lim_{m \to +\infty} r_{t_m} = R. \tag{2.60}
$$

Being bounded, this sequence contains a subsequence convergent to a certain $\theta_* \in [-\pi, \pi]$. We can assume, without loss of generality, that such a subsequence has already been selected and, thus, in addition to (2.60), we have

$$
\lim_{m \to +\infty} t_m = \theta_*. \tag{2.61}
$$

On the other hand, the sequence $(\omega_{t_m})_{m=1}^{+\infty} \subset [-\pi, \pi]$ is also bounded and, therefore, there exists a sequence $(m_j)_{j=1}^{+\infty} \subset \mathbb{N}$ such that $\lim_{j \to +\infty} \omega_{t_{m_j}} = \omega_*$ with a certain $\omega_* \in [-\pi, \pi]$. Setting $t = t_{m_j}$ in (2.58) and (2.59), passing to the

limit as j tends to $+\infty$, and using (2.60), (2.61), and the fact that K is a closed set, we arrive at relations (2.53) and (2.54) with the above values of θ_* and ω_*.

Assume now that $R > 0$ and relations (2.55) and (2.56) are satisfied with some ϵ, $0 \leq \epsilon < R$, and $\{\tilde{\theta}, \tilde{\omega}\} \subset [-\pi, \pi]$. Due to formula (2.43) of Lemma 2.33 and equality (2.38) from the proof of Lemma 2.30, relations (2.55) and (2.56) imply that $r_t \leq R - \epsilon$ for all $t \in [-\pi, \pi]$, whence

$$\inf_{t \in [-\pi,\pi]} r_t \leq R - \epsilon. \tag{2.62}$$

However, by (2.57), inequality (2.62) yields $R \leq R - \epsilon$ and, therefore, $\epsilon = 0$. $\qquad \square$

The property described by Lemma 2.35 is also true for functional (2.49).

Lemma 2.40. *Let $f \in X$ and $z \in \hat{X}$. Then $R_{\hat{K},\hat{f}}(z) = 0$ if, and only if $z \in K^\diamond + iK^\diamond$.*

Proof. The inclusion $z \in K^\diamond + iK^\diamond$ means that the element z is 0-measurable with respect to $\hat{K}$. In this case, Lemma 2.30 guarantees that so does the element $e^{it}z$ with any t from $[-\pi, \pi]$ and, hence, by Lemma 2.35, $n_{\hat{K},\hat{f}}(e^{it}z) = 0$ for all $t \in [-\pi, \pi]$. Relation (2.49) then yields $R_{\hat{K},\hat{f}}(z) = 0$.

Conversely, if $R_{\hat{K},\hat{f}}(x + iy) = 0$, then, by Lemma 2.39, there exists some θ from $[-\pi, \pi]$ such that

$$0 \leqq_K x \cos\theta - y \sin\theta \leqq_K 0,$$
$$0 \leqq_K x \sin\theta + y \cos\theta \leqq_K 0.$$

According to formula (2.38), this means that the element $e^{i\theta}(x + iy)$ is 0-measurable with respect to $\hat{K}$ and, thus, by Lemma 2.30, so does the element $x + iy$. $\qquad \square$

2.6. **Operators vanishing on the blade of a wedge.** For the sake of brevity, we introduce the following definition [10].

Definition 2.41. We say that an eigenvalue λ of a bounded linear operator $A : X \to X$ is *substantial* with respect to the wedge K (or, shortly, *K-substantial*) if λ is non-zero and at least one eigenvector not belonging to $K^\diamond + iK^\diamond$ corresponds to it.

As usual (see, e. g., [8]), by a complex eigenvalue $\lambda \in \mathbb{C}$ of a bounded linear operator $A : X \to X$ acting in a real Banach space X, the eigenvalue of its complexification $\hat{A} = A + iA : \hat{X} \to \hat{X}$ is meant, where

$$\hat{A}(x + iy) := Ax + iAy \tag{2.63}$$

for all $\{x, y\} \subset X$.

Example 2.42. All the eigenvalues of a bounded linear operator $A : X \to X$ are substantial with respect to an arbitrary cone in X.

We devote our present study mostly to the linear operators $A : X \to X$ vanishing on the blade of a proper wedge K, i. e., such that

$$K^\diamond \subseteq \ker A. \tag{2.64}$$

Example 2.43. If K is a cone, then condition (2.64) is satisfied in an obvious way for every linear operator $A : X \to X$.

In the general case, the restrictiveness of condition (2.64) imposed on the operator A grows with the "width" of $K^\diamond$.

Example 2.44. Let us consider the set

$$C_{\Omega,\sigma}\left([a,b],\mathbb{R}\right) = \left\{ x \in C\left([a,b],\mathbb{R}\right) \mid \sigma x\left([a,b]\setminus\Omega\right) \subseteq [0,+\infty)\right\},$$

where Ω is a certain subset of $[a,b]$ such that $[a,b]\setminus\Omega$ is closed, and $\sigma \in \{-1,1\}$. The set $C_{\Omega,\sigma}\left([a,b],\mathbb{R}\right)$ is obviously a closed wedge in the Banach space $C\left([a,b],\mathbb{R}\right)$ of all the continuous scalar functions on the bounded interval $[a,b]$. This wedge is solid because, as one can show, its interior is constituted by the continuous functions $x : [a,b] \to \mathbb{R}$ such that $\sigma x\left([a,b]\setminus\Omega\right) \subseteq (0,+\infty)$.

Consider the operator $A : C\left([a,b],\mathbb{R}\right) \to C\left([a,b],\mathbb{R}\right)$ given by the formula

$$(Ax)\,(t) = \int_\tau^t k\,(t,s)\,x\,(\omega(s))\,ds, \qquad t \in [a,b], \tag{2.65}$$

in which $\omega : [a,b] \to [a,b]$ is a measurable function, whereas the function $k : [a,b] \times [a,b] \to \mathbb{R}$ is continuous in the first variable and Lebesgue integrable in the second one. The operator A vanishes on the blade of the wedge $C_{\Omega,\sigma}\left([a,b],\mathbb{R}\right)$ when ω satisfies the condition

$$\omega\left([a,b]\right) \subseteq [a,b]\setminus\Omega. \tag{2.66}$$

Indeed, the blade of $C_{\Omega,\sigma}\left([a,b],\mathbb{R}\right)$ consists of those continuous functions $x : [a,b] \to \mathbb{R}$ such that

$$x(t) = 0 \quad \text{for all } t \in [a,b]\setminus\Omega. \tag{2.67}$$

If ω is such that condition (2.66) holds, then Ax is equal identically to zero for every function x satisfying condition (2.67), i. e., the relation $x \in \left(C_{\Omega,\sigma}\left([a,b],\mathbb{R}\right)\right)^\diamond$ implies that $Ax = 0$. This means that (2.64) is true for $K = C_{\Omega,\sigma}\left([a,b],\mathbb{R}\right)$ and A given by (2.65).

Our interest to the property described by condition (2.64) is motivated by the following statement.

Lemma 2.45. *Assume that $A : X \to X$ is a linear operator vanishing on the blade of a wedge $K \subseteq X$. Then every non-zero eigenvalue of A is K-substantial.*

Proof. Let λ be an arbitrary non-zero eigenvalue of A. Then there exists some non-zero element w from $\hat{X}$ such that

$$\lambda w = \hat{A}w. \tag{2.68}$$

Assume that, on the contrary, λ is not K-substantial and, therefore, according to Definition 2.41, every eigenvector w in (2.68) belongs to $K^\diamond + iK^\diamond$. By virtue of inclusion (2.64), this yields $\hat{A}w = 0$, and, hence, by (2.68), $w = 0$, which is impossible because w is an eigenvector of $\hat{A}$. The contradiction obtained proves our lemma.   □

Assumption (2.64) may seem to be unnecessarily strong because, in fact, it guarantees that not only *some* eigenvectors corresponding to non-zero eigenvalues of A do not belong to the blade of $\hat{K}$ but *all* such eigenvectors

possess this property. Note however that, in the theorems of Sections 3 and 4, condition (2.64) cannot be dropped even in the two-dimensional case (see Example 2.43).

3. An upper bound for K-substantial eigenvalues

The following theorem provides an upper bound for K-substantial eigenvalues of a sufficiently wide class of linear operators in a real Banach space X.

Theorem 3.1. *Let K be a proper wedge in X and $A_1 : X \to X$, $A_2 : X \to X$ be bounded linear operators such that*

$$A_1(K) \cup A_2(K) \subseteq K. \tag{3.1}$$

Assume also that the relation

$$A_1 f + A_2 f \leq_K \alpha f \tag{3.2}$$

is true with some $\alpha \in [0, +\infty)$ and $f \in X$ for which

$$f >_{K;H} 0, \tag{3.3}$$

where H is a certain linear manifold in X satisfying the inclusion

$$H \supseteq \operatorname{im}(A_1 - A_2). \tag{3.4}$$

Then every K-substantial eigenvalue λ of the operator $A_1 - A_2$ admits the estimate

$$|\lambda| \leq \alpha. \tag{3.5}$$

In (3.1), (3.4), and similar relations, we use the standard notation $A(M) := \{Ax \mid x \in M\}$, $M \subset X$.

Proof of Theorem 3.1. Let $\lambda = \varrho\,\mathrm{e}^{\mathrm{i}\theta}$, $\varrho \in (0, +\infty)$, be a K-substantial eigenvalue of the complexification $\hat{A} = \hat{A}_1 - \hat{A}_2$ of the operator

$$A := A_1 - A_2. \tag{3.6}$$

In view of Definition 2.41 and equality (2.23), there exists an element $w = x + \mathrm{i}y$ such that $\{x, y\} \subset X$, $\{x, y\} \not\subset K^\diamond$, and equality (2.68) holds.

We divide the present proof into several parts.

CLAIM 1. *The element w is $\hat{f}$-measurable with respect to $\hat{K}$.*

Indeed, equality (2.68) means that

$$\varrho w = \mathrm{e}^{-\mathrm{i}\theta}\hat{A}w. \tag{3.7}$$

According to formulae (2.38) and (2.63), we have

$$\mathrm{e}^{-\mathrm{i}\theta}\hat{A}w = Ax \cos\theta + Ay \sin\theta + \mathrm{i}(Ay \cos\theta - Ax \sin\theta)$$

and, therefore, (3.7) can be rewritten as the system

$$\varrho x = Ax \cos\theta + Ay \sin\theta, \tag{3.8}$$

$$\varrho y = Ay \cos\theta - Ax \sin\theta. \tag{3.9}$$

By virtue of assumption (3.4), it follows from (3.8) and (3.9) that x and y both lie in H (to prove this, it suffices to use the linearity of the set H). However, according to Definition 2.12, condition (3.3) means that all the elements from

H are f-measurable with respect to K and, hence, by Lemma 2.29, $x + \mathrm{i}y$ is $(f + \mathrm{i}f)$-measurable with respect to $\hat{K}$.

CLAIM 2. *The number*

$$R := R_{\hat{K}, \hat{f}}(x + \mathrm{i}y) \tag{3.10}$$

is strictly positive.

Indeed, by Claim 1 and Lemma 2.38, the right-hand side of (3.10) is a finite number. Since $\{x, y\} \not\subset K^{\diamond}$, Lemma 2.35 yields $R > 0$.

CLAIM 3. *The elements x and y satisfy the equalities*

$$Ax = \varrho(x \cos \theta - y \sin \theta), \tag{3.11}$$

$$Ay = \varrho(x \sin \theta + y \cos \theta). \tag{3.12}$$

According to formula (2.38), system (3.11), (3.12) is an equivalent form of relation (2.68) satisfied by w.

CLAIM 4. *There exist some ω_* and t_* from $[-\pi, \pi]$ such that (2.53) and (2.54) are true for x and y, and there do not exist any $\{\tilde{\omega}, \tilde{\theta}\} \subset [-\pi, \pi]$ for which the relations*

$$-rf \sin \tilde{\omega} \leqq_K x \cos \tilde{\theta} - y \sin \tilde{\theta} \leqq_K rf \sin \tilde{\omega}, \tag{3.13}$$

$$-rf \cos \tilde{\omega} \leqq_K x \sin \tilde{\theta} + y \cos \tilde{\theta} \leqq_K rf \cos \tilde{\omega} \tag{3.14}$$

would be satisfied with $r \in (0, R)$.

This statement is an immediate consequence of formula (3.10) and Lemma 2.39.

CLAIM 5. *There is an Ω from $[-\pi, \pi]$ such that the relations*

$$-\alpha Rf \sin \Omega \leqq_K Ax \cos \theta_* - Ay \sin \theta_* \leqq_K \alpha Rf \sin \Omega, \tag{3.15}$$

$$-\alpha Rf \cos \Omega \leqq_K Ax \sin \theta_* + Ay \cos \theta_* \leqq_K \alpha Rf \cos \Omega, \tag{3.16}$$

are true, where A is given by (3.6).

In view of assumption (3.1), both operators A_1 and A_2 preserve order inequalities. Therefore, relations (2.53) and (2.54), together with Lemma 2.6, yield

$$-RA_j f \sin \omega_* \leqq_K \sigma[A_j x \cos \theta_* - A_j y \sin \theta_*] \leqq_K RA_j f \sin \omega_*, \tag{3.17}$$

$$-RA_j f \cos \omega_* \leqq_K \varkappa[A_j x \sin \theta_* + A_j y \cos \theta_*] \leqq_K RA_j f \cos \omega_* \tag{3.18}$$

for all $j = 1, 2$ and $\{\sigma, \varkappa\} \subset \{-1, 1\}$. Summing the two relations obtained from (3.17) with $j = 1$, $\sigma = 1$ and $j = 2$, $\sigma = -1$, respectively, we obtain

$$-R(A_1+A_2)f \sin \omega_* \leqq_K (A_1-A_2)x \cos \theta_* - (A_1-A_2)y \sin \theta_* \leqq_K R(A_1+A_2)f \sin \omega_*,$$

i. e.,

$$-R(A_1 + A_2)f \sin \omega_* \leqq_K Ax \cos \theta_* - Ay \sin \theta_* \leqq_K R(A_1 + A_2)f \sin \omega_*. \tag{3.19}$$

In a similar manner, putting in (3.18) $j = 1$, $\varkappa = 1$ and $j = 2$, $\varkappa = -1$ and summing the resulting two relations, we get

$$-R(A_1 + A_2)f \cos \omega_* \leqq_K Ax \sin \theta_* + Ay \cos \theta_* \leqq_K R(A_1 + A_2)f \cos \omega_*. \tag{3.20}$$

Let us consider the following four cases.

Case 1. $\sin \omega_* \geq 0$ *and* $\cos \omega_* \geq 0$.

Using assumption (3.2) in relations (3.19), (3.20) and putting $\Omega := \omega_*$, we arrive immediately at (3.15), (3.16).

Case 2. $\sin \omega_* \geq 0$ *and* $\cos \omega_* < 0$.

Recall that, by assumption, $f \geqq_K 0$ and, due to condition (3.1), $A_1 f + A_2 f \geqq_K 0$. In view of assertion (iii) of Lemma 2.7, relation (3.20) and Claim 2 imply that, in this case,

$$0 \leqq_K Ax \sin \theta_* + Ay \cos \theta_* \leqq_K 0. \tag{3.21}$$

whereas the term $(A_1 + A_2)f \sin \omega_*$, by virtue of (3.2), admits the estimate

$$(A_1 + A_2)f \sin \omega_* \leqq_K \alpha f.$$

Therefore, (3.15) and (3.16) are satisfied with $\Omega := \frac{\pi}{2}$.

Case 3. $\sin \omega_* < 0$ *and* $\cos \omega_* \geq 0$.

Relation (3.19) now yields

$$0 \leqq_K Ax \cos \theta_* - Ay \sin \theta_* \leqq_K 0 \tag{3.22}$$

and, similarly to Case 2, we conclude that (3.15) and (3.16) hold with $\Omega := 0$.

Case 4. $\sin \omega_* < 0$ *and* $\cos \omega_* < 0$.

A reasoning analogous to those presented above show that, in this case, system (3.19), (3.20) has form (3.21), (3.22) and, therefore, relations (3.15) and (3.16) are satisfied both with $\Omega = \frac{\pi}{2}$ and $\Omega = 0$. This proves our Claim 5.

Having established the facts above, we now turn to the proof of estimate (3.5).

According to Claim 3, the components x and y of the eigenvector w of $\hat{A}$ satisfy equalities (3.11) and (3.12). Therefore,

$$\begin{aligned}
Ax \cos \theta_* - Ay \sin \theta_* &= \varrho(x \cos \theta - y \sin \theta) \cos \theta_* - \varrho(x \sin \theta + y \cos \theta) \sin \theta_* \\
&= \varrho[\cos \theta \sin \theta_* - \sin \theta \sin \theta_*]x \\
&\quad - \varrho[\sin \theta \cos \theta_* + \cos \theta \sin \theta_*]y \\
&= \varrho x \cos(\theta + \theta_*) - \varrho y \sin(\theta + \theta_*)
\end{aligned}$$

and, similarly,

$$\begin{aligned}
Ax \sin \theta_* + Ay \cos \theta_* &= \varrho(x \cos \theta - y \sin \theta) \sin \theta_* + \varrho(x \sin \theta + y \cos \theta) \cos \theta_* \\
&= \varrho[\cos \theta \sin \theta_* + \sin \theta \cos \theta_*]x \\
&\quad + \varrho[\cos \theta \cos \theta_* - \sin \theta \sin \theta_*]y \\
&= \varrho x \sin(\theta + \theta_*) + \varrho y \cos(\theta + \theta_*).
\end{aligned}$$

Applying these formulae to the corresponding expressions in (3.15) and (3.16) and taking the inequality $\varrho > 0$ into account, we obtain

$$-\frac{\alpha R}{\varrho} f \sin \Omega \leqq_K x \cos(\theta + \theta_*) - y \sin(\theta + \theta_*) \leqq_K \frac{\alpha R}{\varrho} f \sin \Omega, \tag{3.23}$$

$$-\frac{\alpha R}{\varrho} f \cos \Omega \leqq_K x \sin(\theta + \theta_*) + y \cos(\theta + \theta_*) \leqq_K \frac{\alpha R}{\varrho} f \cos \Omega. \tag{3.24}$$

System (3.23), (3.24), obviously, has form (3.13), (3.14) with $\tilde{\theta} := \theta + \theta_*$, $\tilde{\omega} := \Omega$, and $r := \alpha R / \varrho$. In view of Claim 4, it now follows that

$$\frac{\alpha R}{\varrho} \geq R,$$

whence, by Claim 2, we arrive at the inequality $\varrho \leq \alpha$. Recalling that $\varrho = |\lambda|$, we conclude that the required estimate (3.5) holds.   □

The assumption that f should lie in K and be different from zero in Theorem 3.1 is motivated by Proposition 2.21.

Remark 3.2. A linear operator $A : X \to X$ admits representation in form (1.3), where A_1 and A_2 are linear mappings preserving K, if and only if there exists a linear operator $B : X \to X$ such that

$$B(K) \subseteq K \tag{3.25}$$

and

$$Ax \leqq_K Bx \qquad \text{for all } x \in K. \tag{3.26}$$

Indeed, (1.3) implies that

$$Ax \leqq_K A_1 x \leqq_K A_1 x + A_2 x$$

for all x such that $x \geqq_K 0$ and, therefore, one can set $B := A_1 + A_2$. Conversely, it follows from (3.25) and (3.26) that the operator $A_2 := B - A$ preserves the wedge K and, thus, it remains to put $A_1 := B$ in (1.3).

It should be noted that, in the case where the space X is infinite-dimensional, one cannot claim that every bounded linear operator $A : X \to X$ admits representation (1.3) with bounded linear mappings $A_1 : X \to X$ and $A_2 : X \to X$ preserving K. In particular, in the case where K is a cone which does not possess the property of normality, the classical Theorem 2 of [11] ensures the existence of a continuous (even finite-dimensional) linear operator $A : X \to X$ that cannot be represented in form (1.3) with bounded $A_k : X \to X$, $k = 1, 2$, satisfying condition (3.1).

4. A THEOREM ON THE SPECTRAL RADIUS

The following statement appears to be rather useful in studies of the solvability of various linear equations with compact operators.

Theorem 4.1. *Let X be a Banach space over the field $\mathbb{R}$, K be a proper wedge in X, and $A_1 : X \to X$, $A_2 : X \to X$ be completely continuous linear operators leaving the wedge K invariant and satisfying the condition*

$$K^\diamond \subseteq \ker(A_1 - A_2). \tag{4.1}$$

In addition, assume that relation (3.2) is satisfied with some constant $\alpha \in [0, +\infty)$ and element $f \in X$ such that (3.3) holds with a certain linear manifold $H \subseteq X$ for which inclusion (3.4) is true.

Then the spectral radius of the operator $A_1 - A_2$ admits the estimate

$$r(A_1 - A_2) \leq \alpha \tag{4.2}$$

Proof. It follows from the Riesz–Schauder theory (see, e. g., [12]) that, due to the complete continuity of the operator $A_1 - A_2$, its spectrum consists of countably many eigenvalues.

Assumption (4.1), by virtue of Lemma 2.45, implies that every non-zero eigenvalue of $A_1 - A_2$ is K-substantial. Therefore, under the conditions assumed, Theorem 3.1 can be applied.

Application of Theorem 3.1 guarantees that an arbitrary non-zero eigenvalue λ of the operator $A_1 - A_2$ admits estimate (3.5). Considering the least

upper bound of $|\lambda|$ in the left-hand side of relation (3.5) with respect to all the non-zero eigenvalues λ of $A_1 - A_2$, we arrive immediately at inequality (4.2). □

Remark 4.2. The complete continuity of $A_1 - A_2$ in the proof of Theorem 4.1 is used only to guarantee that this operator has discrete spectrum.

Condition (3.3), as follows from Lemma 2.18, is satisfied, in particular, for an element f belonging to the interior of a solid wedge. In this case, Theorem 4.1 implies the following statement:

Theorem 4.3. *Let $K \subset X$ be a solid wedge and $A_1 : X \to X$, $A_2 : X \to X$ be completely continuous linear operators leaving the wedge K invariant, satisfying condition* (4.1) *and such that relation* (3.2) *is true with some $\alpha \in [0, +\infty)$ and $f \in X$, $f \gg_K 0$. Then the spectral radius of the operator $A_1 - A_2$ admits estimate* (4.2).

Proof. It suffices to take Lemma 2.18 into account and apply Theorem 4.1 with $H := X$. □

Remark 4.4. Condition (4.1) for the operator $A : X \to X$ in Theorems 4.1 and 4.3 is automatically satisfied when K is a cone.

As a particular case, Theorem 4.3 contains the following statement.

Corollary 4.5. *If $A : X \to X$ is a completely continuous linear operator leaving invariant a solid wedge K, and, moreover, satisfying condition* (2.64) *and the relation*

$$Af \leqq_K \alpha f \tag{4.3}$$

with some $\alpha \in [0, +\infty)$ and $f \in X$ such that $f \gg_K 0$, then the estimate

$$r(A) \leq \alpha \tag{4.4}$$

is true.

Proof. Corollary 4.5 is a consequence of Theorem 4.3 with $A_1 = A$ and $A_2 = 0$. □

Theorem 4.3 also implies an analogue of Corollary 4.5 for the "negative" operators.

Corollary 4.6. *Let $A : X \to X$ be a completely continuous linear operator such that $A(-K) \subset K$ and, moreover, the relation*

$$Af \geqq_K -\alpha f$$

be satisfied with some $\alpha \in [0, +\infty)$ and $f \in X$ possessing the property $f \gg_K 0$. Then the spectral radius of A admits estimate (4.4).

Proof. It suffices to put $A_1 = 0$ and $A_2 = -A$ in Theorem 4.3. □

Remark 4.7. In the case where K is a solid and normal cone, the assertion of Theorem 4.3 can also be proved by using Theorems 5.3 and 5.5 of [1].

Remark 4.8. In the case where $A : X \to X$ is a completely continuous linear operator leaving invariant a normal and solid cone K, Corollary 4.5 contains, in particular, assertions (a) and (b) of [1, Theorem 5.5].

4.1. **The case of an f-bounded operator.** Corollary 4.11 established below is an example of application of Theorem 4.1 with H different from X. Prior to its formulation, we introduce a definition.

Definition 4.9. Let K be a wedge in X and f be an element from X. An operator $A : X \to X$ is said to be *f-bounded* with respect to K if, for every $x \in X$, there exists a constant $\beta \in (-\infty, +\infty)$ such that

$$-\beta f \leqq_K Ax \leqq_K \beta f.$$

In other words, A is f-bounded if the element Ax is f-measurable for all x. It follows from Proposition 2.11 that, in the pathological cases where $0 \leqq_K f \leqq_K 0$ or f is incomparable with zero, every operator A which is f-bounded with respect to K has the property $\mathrm{im}\, A \subseteq K^{\diamond}$.

Remark 4.10. An operator f-bounded with respect to K is, in particular, f-bounded from above in the sense of the definition from [8, Chapter 2, §1]. The converse statement is not true.

Corollary 4.11. *Let $f \geqq_K 0$ be a given element and $A_k : X \to X$, $k = 1, 2$, be completely continuous linear operators preserving the wedge K, satisfying condition (4.1), and f-bounded with respect to K.*

Then the existence of a non-negative constant α for which relation (3.2) is satisfied implies estimate (4.2) for the spectral radius of the operator $A_1 - A_2$.

Proof. Setting $H := X_K(f)$, we see that condition (3.4) is satisfied due to the f-boundedness of A_1 and A_2 with respect to K. By Lemma 2.13, f satisfies relation (2.8) and, hence,

$$f >_{K;X_K(f)} 0.$$

Therefore, condition (3.3) holds with our choice of H, and it remains to apply Theorem 4.1.   □

5. A remark on condition (4.1)

As is seen from the proof of Theorem 4.1, the applicability of statements on K-substantial eigenvalues is guaranteed by condition (4.1). It is natural to expect that estimating the spectrum of an operator on the base of assumptions of type (3.2) is not possible any more if one admits the existence of a non-zero eigenvalue which is not K-substantial, and imposes no additional conditions on A.

The following example [10] shows that the assumption on the fulfilment of condition (2.64) in Theorem 4.1 is essential and, generally speaking, cannot be omitted.

Example 5.1. Let us consider the set

$$K = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : x_1 \geq 0, \; x_2 \in \mathbb{R} \right\}. \tag{5.1}$$

Obviously, K is a solid wedge in $X := \mathbb{R}^2$, and the blade of K has the form

$$K^{\diamond} = \left\{ \begin{pmatrix} 0 \\ c \end{pmatrix} : c \in \mathbb{R} \right\}.$$

It is not difficult to verify that the linear operator A given by the matrix $A = \left(\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right)$ leaves invariant the set K given by (5.1) if, and only if

$$a_{11} \geq 0, \qquad a_{12} = 0. \tag{5.2}$$

Furthermore, one can show that, under condition (5.2), A vanishes on $K^{\diamond}$ if, and only if

$$a_{22} = 0. \tag{5.3}$$

A vector $f = \left(\begin{smallmatrix} f_1 \\ f_2 \end{smallmatrix}\right)$ belongs to the interior of K if, and only if

$$f_1 > 0. \tag{5.4}$$

whereas the corresponding condition (4.3) is equivalent to the inequality $\alpha f_1 \geq a_{11} f_1$, which, in view of (5.4), means that

$$\alpha \geq a_{11}. \tag{5.5}$$

If condition (2.64) or, which is the same in our case, equality (5.3) is violated, then assumption (4.3), generally speaking, cannot guarantee the validity of the estimate $r(A) \leq \alpha$ for the spectral radius of A. Indeed, it is clear from (5.2) that $r(A) = \max\{a_{11}, |a_{22}|\}$ and, hence,

$$r(A) \geq |a_{22}|. \tag{5.6}$$

However, if the inequalities

$$|a_{22}| > \alpha \geq a_{11} \geq 0 \tag{5.7}$$

hold, then the assertion of Corollary 4.5 in the case considered would have the form $r(A) \leq \alpha$, which is impossible in view of (5.6) and (5.7).

Thus, condition (2.64) Corollary 4.5 (and, therefore, condition (4.1) in Theorem 4.1), generally speaking, cannot be dropped.

6. Unique solvability of linear equations

Theorem 4.1 allows one to obtain efficient conditions under which the linear equation

$$x = A_1 x - A_2 x + q, \tag{6.1}$$

where A_1 and A_2 are linear operators, possesses a unique solution for an arbitrary element q from X.

Corollary 6.1. *Let X be a real Banach space, $K \subset X$ be a wedge, and $A_i : X \to X$, $i = 1, 2$, be completely continuous linear operators leaving K invariant and satisfying condition (4.1). In addition, assume that relation (3.2) is satisfied with some constant $\alpha \in [0, 1)$ and element $f \in X$ such that (3.3) holds with a certain linear manifold $H \subseteq X$ for which inclusion (3.4) is true.*

Then equation (6.1) is uniquely solvable for arbitrary $q \in X$, and the solution x of equation (6.1) is represented by the convergent Neumann series

$$x = \sum_{k=0}^{+\infty} (A_1 - A_2)^k q. \tag{6.2}$$

Proof. It suffices to notice that, by virtue of Theorem 4.1, the conditions assumed guarantee that the spectrum of the operator $A_1 - A_2$ is contained in the interior of the unit disk in $\mathbb{C}$. $\qquad \square$

In the cases where the fact of convergence of series (6.2) is unimportant, one may prefer to use the following

Corollary 6.2. *Let K be a proper wedge in a real Banach space X and $A_i : X \to X$, $i = 1, 2$, be bounded linear operators leaving K invariant and satisfying condition (4.1). Assume that relation (3.2) is true, where $\alpha \in [0, 1)$ and $f \in X$ is an element for which (3.3) holds with some linear manifold $H \subseteq X$ satisfying inclusion (3.4).*
   Then the homogeneous equation

$$x = A_1 x - A_2 x \tag{6.3}$$

has no non-trivial solutions. If, moreover, the operators A_1 and A_2 are such that

$$1_X - A_1 - A_2 \text{ is a Fredholm operator of index } 0, \tag{6.4}$$

then equation (6.1) is uniquely solvable for an arbitrary $q \in X$.

The symbol 1_X here stands for the identity operator in X.

Proof. In view of assumption (4.1) and Lemma 2.45, it follows from Theorem 3.1 that operator (3.6) has no eigenvalues outside the open interval $(-1, 1)$ and, in particular, the number 1 is not an eigenvalue for the operator mentioned. Therefore, zero is the unique solution of the homogeneous equation (6.3). The unique solvability of equation (6.1) for any q is guaranteed by condition (6.4).   □

Corollary 6.2 allows one to obtain the following statement.

Corollary 6.3. *Let K be a proper wedge in a real Banach space X and $A_i : X \to X$, $i = 1, 2$, be bounded linear operators leaving K invariant, possessing property (6.4), and satisfying the condition*

$$K^{\diamond} \subseteq \ker A_1 \cap \ker A_2. \tag{6.5}$$

Assume also that relation (3.2) is true, where $\alpha \in [0, 1)$ and $f \in X$ is an element for which inequality (3.3) holds with some linear manifold $H \subseteq X$ satisfying inclusion (3.4).
   Then the equation

$$x = \sigma_1 A_1 x + \sigma_2 A_2 x + q, \tag{6.6}$$

is uniquely solvable for arbitrary $q \in X$ and $\{\sigma_1, \sigma_2\} \subset \{-1, 1\}$.

Proof. Let us define the operators $\tilde{A}_i : X \to X$, $i = 1, 2$, by putting

$$\tilde{A}_1 := \frac{1 + \sigma_1}{2} A_1 + \frac{1 + \sigma_2}{2} A_2 \tag{6.7}$$

and

$$\tilde{A}_2 := \frac{1 - \sigma_1}{2} A_1 + \frac{1 - \sigma_2}{2} A_2. \tag{6.8}$$

One can verify that the relations

$$\sigma_1 A_1 + \sigma_2 A_2 = \tilde{A}_1 - \tilde{A}_2$$

and

$$A_1 + A_2 = \tilde{A}_1 + \tilde{A}_2 \tag{6.9}$$

are true. It is also easy to see from (6.7) and (6.8) that both operators $\tilde{A}_1$ and $\tilde{A}_2$ leave invariant the wedge K.

Assumption (6.5) guarantees the fulfilment of the inclusion

$$K^{\diamond} \subseteq \ker(\tilde{A}_1 - \tilde{A}_2).$$

Moreover, by virtue of (6.9), condition (3.2) can be rewritten as

$$\tilde{A}_1 f + \tilde{A}_2 f \leqq_K \alpha f.$$

We have thus shown that Corollary 6.2 can be applied with A_1 and A_2 replaced by $\tilde{A}_1$ and $\tilde{A}_2$, respectively. $\qquad\square$

7. An example

We illustrate the idea of the results above on an example. Let us consider the problem on the continuous solutions of the equation

$$x(t) = \int_0^1 h(t,s) \, x(\omega(s)) \, ds + q(t), \quad t \in [0,1], \tag{7.1}$$

where $q : [0,1] \to \mathbb{R}$ is continuous, the function $\omega : [0,1] \to [0,1]$ is measurable, $h(t,\cdot) : [0,1] \to \mathbb{R}$ is Lebesgue integrable for all $t \in [0,1]$, and $h(\cdot,s) : [0,1] \to \mathbb{R}$ is continuous for almost every s from $[0,1]$.

Theorem 7.1. *Let there exist a non-negative continuous function $\psi : [0,1] \to \mathbb{R}$ such that*

$$\operatorname*{vrai\,max}_{t \in [0,1] \backslash \Gamma_{\psi,\omega}} \frac{1}{\psi(\omega(t))} \int_0^1 |h(\omega(t),s)| \, ds < +\infty \tag{7.2}$$

and

$$\operatorname*{vrai\,max}_{t \in [0,1] \backslash \Gamma_{\psi,\omega}} \frac{1}{\psi(\omega(t))} \int_0^1 |h(\omega(t),s)| \, \psi(\omega(s)) \, ds < 1, \tag{7.3}$$

where

$$\Gamma_{\psi,\omega} := \{t \in [0,1] \mid \psi(\omega(t)) = 0\}. \tag{7.4}$$

Then equation (7.1) has a unique solution for any continuous function $q : [0,1] \to \mathbb{R}$.

Proof. Equation (7.1) can obviously be rewritten in form (6.1), where the operators A_i, $i = 1,2$, in the Banach space $X := C([0,1], \mathbb{R})$ of all the continuous scalar functions on $[0,1]$ are introduced by the formulae

$$(A_i x)(t) := \int_0^1 \max\{(-1)^{i+1} h(t,s), 0\} \, x(\omega(s)) \, ds, \quad t \in [0,1], \ i = 1,2, \tag{7.5}$$

for any continuous $x : [0,1] \to \mathbb{R}$. Clearly, each of these operators leaves invariant the wedge

$$K_{\omega} := \{u \in C([0,1], \mathbb{R}) \mid u(\omega(t)) \geq 0 \text{ for a. e. } t \in [0,1]\}.$$

It is easy to show that operators (7.5) are completely continuous.

Let $C_{\psi,\omega}$ be the set of all the continuous functions $x : [0,1] \to \mathbb{R}$ satisfying the condition

$$\operatorname*{vrai\,max}_{t \in [0,1] \backslash \Gamma_{\psi,\omega}} \frac{|x(\omega(t))|}{\psi(\omega(t))} < +\infty.$$

It is easy to see that $C_{\psi,\omega}$ is a linear manifold in $C([0,1],\mathbb{R})$. Assumption (7.2) and formula (7.5) imply that, for any x from $C([0,1],\mathbb{R})$ and almost every $t \in [0,1]$, the estimate

$$|(A_1 x)(\omega(t)) - (A_2 x)(\omega(t))| \leq \psi(\omega(t)) \Delta \max_{\xi \in [0,1]} |x(\xi)|$$

is true, where

$$\Delta := \operatorname*{vrai\,max}_{t \in [0,1] \backslash \Gamma_{\psi,\omega}} \frac{1}{\psi(\omega(t))} \int_0^1 |h(\omega(t),s)|\,ds.$$

This means that condition (3.4) is satisfied with the above definitions of A_1 and A_2 and $H := C_{\psi,\omega}$.

The function ψ satisfies the condition

$$\psi >_{K_\omega;\, C_{\psi,\omega}} 0, \tag{7.6}$$

which means that (3.3) holds with $f := \psi$ and $K := K_\omega$. Indeed, according to Definition 2.15, relation (7.6) means that, for any x from $C_{\psi,\omega}$, there exists a constant $\beta \geq 0$ such that

$$|x(\omega(t))| \leq \beta \psi(\omega(t))$$

at almost every point t from the interval $[0,1]$. However, the above property is an immediate consequence of the definition of the set $C_{\psi,\omega}$.

Finally, inequality (7.3) guarantees that

$$\int_0^1 |h(\omega(t),s)|\,\psi(\omega(s))\,ds \leq \alpha \psi(\omega(t)), \quad t \in [0,1], \tag{7.7}$$

where the constant α equal to the value at the left-hand side of (7.3) is less than 1. In view of (7.5), relation (7.7) can be rewritten in form (3.2) for $K := K_\omega$. Applying Theorem 4.1, we conclude that 1 is a regular value for the operator (3.6) corresponding to the given problem. □

The above theorem implies, for example, the following statement.

Corollary 7.2. *Assume that, for certain $\tau \in [0,1]$ and $\gamma \geq 0$, the functions $h : [0,1]^2 \to \mathbb{R}$ and $\omega : [0,1] \to [0,1]$ satisfy the conditions*

$$\operatorname*{vrai\,max}_{t \in [0,1] \backslash \omega^{-1}(\tau)} \frac{1}{|\omega(t) - \tau|^\gamma} \int_0^1 |\omega(s) - \tau|^\gamma\,|h(\omega(t),s)|\,ds < 1 \tag{7.8}$$

and

$$\operatorname*{vrai\,max}_{t \in [0,1] \backslash \omega^{-1}(\tau)} \frac{1}{|\omega(t) - \tau|^\gamma} \int_0^1 |h(\omega(t),s)|\,ds < +\infty. \tag{7.9}$$

Then equation (7.1) has a unique solution for any continuous function $q : [0,1] \to \mathbb{R}$.

Proof. It suffices to apply Theorem 7.1 with

$$\psi(t) := |t - \tau|^\gamma, \qquad t \in [0,1],$$

in which case set (7.4) is given by the formula $\Gamma_{\psi,\omega} = \{t \in [0,1] \mid \omega(t) = \tau\}$.
                                                                                    □

Remark 7.3. If the function $\omega : [0,1] \to [0,1]$ possesses the property

$$\operatorname*{vrai\,min}_{t \in [0,1]} |\omega(t) - \tau| > 0, \tag{7.10}$$

then condition (7.9) of Corollary 7.2 is a consequence of assumption (7.8). Indeed, it follows from (7.8) and (7.10) that

$$\operatorname*{vrai\,max}_{t \in [0,1] \backslash \omega^{-1}(\tau)} \frac{1}{|\omega(t) - \tau|^{\gamma}} \int_0^1 |h(\omega(t), s)|\, ds$$

$$\leq \frac{1}{\epsilon^{\gamma}} \operatorname*{vrai\,max}_{t \in [0,1] \backslash \omega^{-1}(\tau)} \frac{1}{|\omega(t) - \tau|^{\gamma}} \int_0^1 |\omega(s) - \tau|^{\gamma} |h(\omega(t), s)|\, ds < \frac{1}{\epsilon^{\gamma}},$$

where $\epsilon := \operatorname*{vrai\,min}_{t \in [0,1]} |\omega(t) - \tau|$, and, therefore, relation (7.9) is true.

For instance, in the case of the equation

$$x(t) = \int_0^1 h(t,s)\, x(s^{\alpha})\, ds + q(t), \quad t \in [0,1], \tag{7.11}$$

where $\alpha \in (0, +\infty)$ and $q : [0,1] \to \mathbb{R}$ is continuous, we have

Corollary 7.4. *Assume that there exist some* $\tau \in [0,1]$ *and* $\gamma \in [0, +\infty)$ *for which*

$$\sup_{t \in [0,1] \backslash \{\tau\}} \frac{1}{|t - \tau|^{\gamma}} \int_0^1 |s^{\alpha} - \tau|^{\gamma} |h(t, s)|\, ds < 1. \tag{7.12}$$

Then equation (7.11) *is uniquely solvable for any continuous function* $q : [0,1] \to \mathbb{R}$.

Proof. Obviously, assumption (7.12) implies that

$$\sup_{t \in [0,1] \backslash \{\sqrt[\alpha]{\tau}\}} \frac{1}{|t^{\alpha} - \tau|^{\gamma}} \int_0^1 |s^{\alpha} - \tau|^{\gamma} |h(t^{\alpha}, s)|\, ds < 1,$$

which means that condition (7.8) is satisfied with

$$\omega(t) := t^{\alpha}, \qquad t \in [0,1]. \tag{7.13}$$

Moreover, in view of Remark 7.3, inequality (7.9) is also true in this case because function (7.13) has property (7.10). Thus, Corollary 7.2 can be applied. □

Remark 7.5. Condition (7.8) of Corollary 7.2 is unimprovable in the sense that the corresponding non-strict inequality

$$\sup_{t \in [0,1] \backslash \{\tau\}} \frac{1}{|t - \tau|^{\gamma}} \int_0^1 |s^{\alpha} - \tau|^{\gamma} |h(t, s)|\, ds \leq 1 \tag{7.14}$$

does not guarantee the unique solvability of equation (7.11) for all continuous q. In order to show this, it is sufficient to consider the simplest functional equation

$$x(t) = x(\theta) + q(t), \qquad t \in [0,1], \tag{7.15}$$

where θ is a given point from $[0,1]$ and $q : [0,1] \to \mathbb{R}$ is a continuous function. Obviously, equation (7.15) can be rewritten as (7.1) with $\omega(s) := \theta$ and $h(t,s) := 1$ for all t and almost every s from $[0,1]$. Equation (7.15) has no solutions continuous for any continuous $q : [0,1] \to \mathbb{R}$ satisfying the

inequality $q(\theta) \neq 0$. Nevertheless, the corresponding condition (7.14) is true in the form of an equality with an arbitrary non-negative γ. Note that, for $\gamma = 1$, one can also refer to the example of equation (7.20) from Remark 7.5.

The argument given above also shows that the strict inequality (7.8) in Corollary 7.2 cannot be replaced by the inequality

$$\operatorname*{vrai\,max}_{t \in [0,1] \setminus \omega^{-1}(\tau)} \frac{1}{|\omega(t) - \tau|^\gamma} \int_0^1 |\omega(s) - \tau|^\gamma \, |h(\omega(t), s)| \, ds \leq 1.$$

The following statement gives somewhat simpler but more restrictive conditions sufficient for the solvability of equation (7.11).

Corollary 7.6. *Equation* (7.11) *has a unique continuous solution for any continuous q, provided that the inequality*

$$\operatorname*{vrai\,max}_{s \in [0,1]} \sup_{t \in [0,1] \setminus \{\tau\}} \left| \frac{h(t,s)}{t - \tau} \right| < \frac{\alpha + 1}{2\alpha\tau^{1 + \frac{1}{\alpha}} - (\alpha + 1)\tau + 1} \tag{7.16}$$

is satisfied with some $\tau \in [0,1]$.

Proof. In view of (7.16), there exists a $\delta \in [0,1)$ such that

$$\left| \frac{h(t,s)}{t - \tau} \right| \leq \frac{\delta(\alpha + 1)}{2\alpha\tau^{1 + \frac{1}{\alpha}} - (\alpha + 1)\tau + 1} \tag{7.17}$$

for almost every $s \in [0,1]$ and all $t \in [0,1] \setminus \{\tau\}$. Using estimate (7.17) and taking into account the identity

$$\int_0^1 |\xi^\alpha - \tau| \, d\xi = \frac{2\alpha\tau^{1 + \frac{1}{\alpha}} + 1}{\alpha + 1} - \tau, \tag{7.18}$$

we conclude that

$$\frac{1}{|t - \tau|} \int_0^1 |\xi^\alpha - \tau| \, |h(t, \xi)| \, d\xi \leq \int_0^1 |\xi^\alpha - \tau| \, d\xi \operatorname*{vrai\,max}_{s \in [0,1]} \left| \frac{h(t,s)}{t - \tau} \right| \leq \delta < 1$$

for every t different from τ, i. e., condition (7.12) is satisfied with $\gamma := 1$. Applying Corollary 7.4, we obtain the required assertion. $\qquad\square$

Corollary 7.6 implies, in particular, that equation (7.11) is uniquely solvable if

$$\operatorname*{vrai\,max}_{s \in [0,1]} \sup_{t \in (0,1]} t^{-1} |h(t,s)| < \alpha + 1. \tag{7.19}$$

It should be noted that (7.19) is weaker than the condition

$$\operatorname*{vrai\,max}_{s \in [0,1]} \sup_{t \in (0,1]} t^{-1} |h(t,s)| < 1,$$

which is obtained by using the standard techniques (e. g., [1, Theorem 5.5] with $E = C([0,1], \mathbb{R})$, K defined as the cone of non-negative functions, $y_0 \equiv 1$, and the operator A given by the expression in the right-hand side of (7.11)).

Remark 7.7. None of conditions (7.16) and (7.19) can be weakened. Indeed, consider the equation

$$x(t) = \frac{(\alpha + 1) |t - \tau|}{2\alpha\tau^{1 + \frac{1}{\alpha}} - \tau(\alpha + 1) + 1} \int_0^1 x(s^\alpha) \, ds, \qquad t \in [0,1], \tag{7.20}$$

where $\alpha \in (0, +\infty)$ and $\tau \in [0,1]$ are arbitrary constants. Obviously, equation (7.20) has form (7.11) with

$$h(t,s) := \frac{(\alpha + 1)\,|t - \tau|}{2\alpha\tau^{1+\frac{1}{\alpha}} - \tau\,(\alpha + 1) + 1} \qquad (7.21)$$

for all t and almost every s from $[0,1]$. Moreover, due to formulae (7.18) and (7.21), we have

$$\underset{s\in[0,1]}{\text{vrai max}}\ \underset{t\in[0,1]\backslash\{\tau\}}{\sup}\left|\frac{h(t,s)}{t - \tau}\right| = \frac{\alpha + 1}{2\alpha\tau^{1+\frac{1}{\alpha}} - (\alpha + 1)\,\tau + 1}.$$

However, the homogeneous equation (7.20) has the non-trivial solution

$$x(t) = \frac{(\alpha + 1)\,|t - \tau|}{2\alpha\tau^{1+\frac{1}{\alpha}} - \tau\,(\alpha + 1) + 1}, \qquad t \in [0,1].$$

Thus, we see that condition (7.16) is unimprovable. In order to show the optimality of condition (7.19), it is sufficient to put $\tau := 0$ in (7.20).

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