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# EMBEDDINGS OF BESSEL-POTENTIAL-TYPE SPACES

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# EMBEDDINGS OF BESSEL-POTENTIAL-TYPE SPACES

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Abstract. We survey, comment and complement our results on embeddings of Bessel potential spaces modelled upon Lorentz-Karamata spaces. Target spaces in our embeddings are either Lorentz-Karamata spaces or generalized Hölder spaces. In particular, we characterize sharp embeddings of spaces in question and we prove the non-compactness of such embeddings. As consequences of our results, we get both growth envelopes and continuous envelopes of Bessel potential spaces modelled upon Lorentz-Karamata ones. We also show how to modify parameters of target spaces to arrive to compact embeddings. Finally, we establish necessary and sufficient conditions both for continuous and compact embeddings. As a result, we can see that the modification mentioned above is optimal.

## 1. Introduction

Classical Bessel potential spaces $H^{\sigma,p}(\mathbb{R}^n) = H^\sigma L^p(\mathbb{R}^n)$, introduced in [3] and [9], have played a significant role in mathematical analysis and in applications for many years (cf. [46], [49], [1], etc.). These spaces are modelled upon the scale of Lebesgue spaces $L^p(\mathbb{R}^n)$ and they coincide with the Sobolev spaces $W^{k,p}(\mathbb{R}^n) = W^k L^p(\mathbb{R}^n)$ when $\sigma = k \in \mathbb{N}$ and $p \in (1, +\infty)$. However, it has gradually become clear that to handle some situations (especially limiting ones) a more refined tuning is desirable. For this purpose, one needs to replace the Lebesgue scale of spaces by a scale of spaces which can be more finely tuned. For example, to obtain estimates of degenerate elliptic differential operators with coefficients having singular behaviour, Edmunds and Triebel (cf. [22], [23]) replaced the $L^p$ spaces by the spaces $L^p(\log L)^q$ of Zygmund type. The same replacement enables to obtain interesting results concerning smoothness properties of orientation–preserving maps (see [37] for references and an account of work in this direction).

In a series of recent papers [15]-[18] a systematic research of embeddings of Bessel potential spaces with order of smoothness $\sigma \geq 1$ modelled upon generalized Lorentz-Zygmund (GLZ) spaces was carried out. The authors of those papers established embeddings of such spaces either into GLZ-spaces or into Hölder-type spaces $C^{0,\lambda(\cdot)}(\overline{\Omega})$ and showed that their results are sharp (within the given scale of target spaces) and fail to be compact. They also clarified the role of the logarithmic terms involved in the quasi-norms of the spaces mentioned. This role proved to be important especially in limiting cases. In particular, they obtained refinements of the Sobolev embedding theorems, Trudinger's limiting embedding as well as embeddings of Sobolev spaces into $\lambda(\cdot)$-Hölder continuous functions including the result of Brézis and Wainger about almost Lipschitz continuity of elements of the (fractional) Sobolev space $H^{1+n/p,p}(\mathbb{R}^n)$ (cf. [8]). For a survey of these results we refer to [42].

Although GLZ-spaces form an important scale of spaces containing, for example, Zygmund classes $L^p(\log L)^\alpha$, Orlicz spaces of multiple exponential type, Lorentz spaces $L^{p,q}$, Lebesgue

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spaces $L^p$, etc., GLZ-spaces are a particular case of more general spaces, namely the Lorentz-Karamata (LK) spaces.

The embeddings mentioned above were extended in [40]-[41] to the case when Bessel potential spaces are modelled upon LK-spaces. Since Neves considered more general targets (besides LK-spaces and Hölder-type spaces also generalized Hölder spaces), in several cases he obtained improvements of embeddings from [15]-[18]. The sharpness and the non-compactness of these embeddings were proved in [26] and [27]. (An account of the principal embedding results involving Bessel potential spaces modelled upon LK-spaces is also given in [14].) We should mention that in [26] and [27] we followed [31] and used a more general definition of LK-spaces; in contrast to [40]-[41], [21] and [14], we did not require a symmetry (with respect to the point 1) of slowly varying functions given on $(0, +\infty)$.

In [26], where target spaces of our embeddings are LK-spaces, main ingredients of proofs of embeddings are the O'Neil inequality for convolutions and convenient Hardy-type inequalities. The sharpness and non-compactness of embeddings in question are proved by means of suitable sequences of test functions. The methods is based on results and techniques of [16] and [18].

Note that in [27] one of main steps in the proof of continuous embeddings of Bessel potential spaces with order of smoothness $\sigma \geq 1$ into generalized Hölder spaces consists in the application of Stein's inequality (cf. [6, Exercise 12(b), p. 430] or [13]). This inequality states that a function u, such that the norm of its distributional gradient $|\nabla u|$ belongs locally to the Lorentz space $L^{n,1}(\mathbb{R}^n)$, can be redefined on a set of measure zero so that the modulus of smoothness $\omega(u, \cdot)$ of $u$ satisfies the inequality

$$\omega(u,t) \precsim \int_0^{t^n} s^{\frac{1}{n}-1} |\nabla u|^*(s)\, ds \quad \text{for all} \quad t \in (0,1) \tag{1.1}$$

(here $|\nabla u|^*$ denotes the non-increasing rearrangement of $|\nabla u|$). As in [26], appropriate sequences of test functions are used to prove the sharpness and non-compactness of embeddings in question. The approach is based on results and techniques of [18].

In [19] and [20] the authors analysed the situation when the order of smoothness is less than one. In such a case one cannot use the method in which inequality (1.1) and a lifting argument (based on [17, Lemma 4.1] or [27, Lemma 4.5], which extend the Calderón result [9, Theorem 7]) are applied to reduce the superlimiting case to the sublimiting one, and a new approach was used. The authors of those papers established embeddings of such spaces into Hölder-type spaces $C^{0,\lambda(\cdot)}(\overline{\Omega})$ and showed that their results concerning nonlimiting cases are sharp (within the given scale of target spaces) and fail to be compact. Results of [20] were extended in [29], where Bessel potential spaces were modelled upon LK-spaces.

In [19] also another question is treated. Namely, it is shown how to get compact embeddings of Bessel-potential-type spaces into Lorentz-type spaces from sharp continuous ones in sublimiting and limiting cases. While in the situation of classical Sobolev embeddings this can be achieved by restricting the parameter on the power-type level, in our general situation the same effect is caused by an appropriate modification of slowly varying function involved in the target space.

In [30] we established necessary and sufficient conditions for embeddings of Bessel potential spaces $H^\sigma X(\mathbb{R}^n)$ with order of smoothness less than one, modelled upon rearrangement invariant Banach function spaces $X(\mathbb{R}^n)$, into generalized Hölder spaces $\Lambda^{\lambda(\cdot)}_{\infty,r}(\overline{\Omega}), 0 < r \leq +\infty$. (We refer to Section 2 for precise definitions. Note also that the space $\Lambda^{\lambda(\cdot)}_{\infty,\infty}(\overline{\Omega})$ coincides with the space $C^{0,\lambda(\cdot)}(\overline{\Omega})$ mentioned above.) For this purpose, we derived a convenient replacement of (1.1). Namely, if $\sigma \in (0,1)$, $X = X(\mathbb{R}^n)$ is a rearrangement invariant Banach function space and the Bessel potential kernel $g_\sigma$ belongs to the associate space of $X$, then we proved that

$$\omega(f * g_\sigma, t) \precsim \int_0^{t^n} s^{\frac{\sigma}{n}-1} f^*(s)\, ds \quad \text{for all} \quad t \in (0,1) \quad \text{and every} \quad f \in X, \tag{1.2}$$

where $f^*$ denotes the non-increasing rearrangement of $f$. Moreover, estimate (1.2) is sharp in the sense that

$$\omega(\overline{f} * g_\sigma, t) \succsim \int_0^{t^n} s^{\frac{\sigma}{n}-1} f^*(s)\, ds \quad \text{for all} \quad t \in (0,1) \quad \text{and every} \quad f \in X, \tag{1.3}$$

where

$$\overline{f}(x) = f^*(\beta_n |x|^n)\chi_{\{y\in\mathbb{R}^n:\, y_1>0\}\cap B(0,1)}(x), \qquad x = (x_1, \cdots, x_n) \in \mathbb{R}^n,$$

and $\beta_n$ is the volume of the unit ball in $\mathbb{R}^n$. Inequalities (1.2) and (1.3) enabled us to show that the continuous embedding of the Bessel potential space $H^\sigma X(\mathbb{R}^n)$ into the generalized Hölder space $\Lambda^{\mu(\cdot)}_{\infty,r}(\overline{\mathbb{R}^n})$ is equivalent to the condition that

$$g_\sigma \text{ belongs to the associate space of } X \tag{1.4}$$

and to the boundedness of the Hardy-type operator

$$H \,:\overline{X} \longrightarrow L_r((0,1); t^{-1/r}(\mu(t))^{-1}), \tag{1.5}$$

where the operator $H$ is defined by

$$(Hf)(t) := \int_0^{t^n} s^{\frac{\sigma}{n}-1} f^*(s)\, ds, \tag{1.6}$$

$\overline{X}$ denotes the representation space of $X$ and $L_r((0,1); t^{-1/r}(\mu(t))^{-1})$ is the weighted Lebesgue space over the interval $(0,1)$.

Furthermore, we characterized compact subsets of generalized Hölder spaces $\Lambda^{\mu(\cdot)}_{\infty,r}(\overline{\Omega}), 0 < r < +\infty$, with a bounded domain $\Omega$ in $\mathbb{R}^n$ and then we derived necessary and sufficient conditions for compact embeddings of Bessel potential spaces $H^\sigma X(\mathbb{R}^n)$ into generalized Hölder spaces $\Lambda^{\mu(\cdot)}_{\infty,r}(\overline{\Omega}), 0 < r < +\infty$. To this end, we made use of local versions of inequalities (1.2) and (1.3) to show that the compactness of the embedding in question is equivalent to (1.4) and to the compactness of the Hardy-type operator (1.5). (Note that if $r = +\infty$, then our conditions are sufficient; under some additional assumptions, they are also necessary.)

Finally, we applied our results to the case when $X(\mathbb{R}^n)$ is the Lorentz-Karamata space $L_{p,q;b}(\mathbb{R}^n)$. Applications cover both the superlimiting case when $p > n/\sigma$ and the limiting case when $p = n/\sigma$. Our results extend and improve those of [19], [20] and [29]. For instance, taking the slowly varying function (involved in the definition of the Lorentz-Karamata space) of logarithmic type and using Example 5.7 (i) (with $\sigma = k \in \mathbb{N}$, $k < n$, and $\beta = 0$) and Remark 5.8 (ii) below (cf. also [19, pp. 230, 231]), we obtain an interesting result which has no analogue in the classical theory of embeddings of Sobolev-Orlicz spaces. Namely, the Sobolev-Orlicz space $W^k L^{\frac{n}{k}}(\log L)^\alpha(\mathbb{R}^n)$, $k \in \mathbb{N}$ and $k < n$, (the Sobolev space modelled upon the Orlicz space $L^{\frac{n}{k}}(\log L)^\alpha(\mathbb{R}^n) \equiv L_\Phi(\mathbb{R}^n)$, where the Young function satisfies $\Phi(t) = [t(1 + |\log t|)^\alpha]^{n/k}$, $t > 0$) is continuously embedded into the $\lambda(\cdot)$–Hölder class $C^{0,\lambda(\cdot)}(\overline{\mathbb{R}^n})$ with

$$\lambda(t) = (1 + |\log t|)^{-\alpha+1-k/n}, \qquad t > 0, \tag{1.7}$$

that is,

$$W^k L^{\frac{n}{k}}(\log L)^\alpha(\mathbb{R}^n) \hookrightarrow C^{0,\lambda(\cdot)}(\overline{\mathbb{R}^n}) \tag{1.8}$$

provided that $\alpha > 1 - k/n$ (the function $\lambda(t)$ tends to 0 as $t \to 0_+$ more slowly than any function $t^\varepsilon$ with $\varepsilon > 0$). This complements [17, Corollary 4.6] and illustrates the important role of the logarithmic term $(\log L)^\alpha$ involved in the Sobolev–Orlicz space $W^kL^{\frac{n}{k}}(\log L)^\alpha(\mathbb{R}^n)$. (By the classical result, the Sobolev space $W^{k,\frac{n}{k}}(\mathbb{R}^n) \equiv W^kL^{\frac{n}{k}}(\mathbb{R}^n)$, $k \in \mathbb{N}$ and $k < n$, is not even continuously embedded into the space $L^\infty(\Omega)$ for any subdomain $\Omega \subset \mathbb{R}^n$.) If $k = 1$ and $\mathbb{R}^n$ is replaced by a bounded domain $\Omega \subset \mathbb{R}^n$, then such a result also follows from [11, Theorem 3.15]. (Note that [11, Theorem 3.15] is stronger than [2, Theorem 8.40].) Embedding (1.8) with $\lambda$ from (1.7) should be also compared with the following corollary of [17, Theorem 4.11] (which extends the result of [8] about "almost Lipschitz continuity"):

$$W^{k+1}L^{\frac{n}{k}}(\log L)^\alpha(\mathbb{R}^n) \hookrightarrow C^{0,\lambda(\cdot)}(\overline{\mathbb{R}^n}), \tag{1.9}$$

with

$$\lambda(t) = t\,(1 + |\log t|)^{-\alpha+1-k/n}, \quad t > 0,$$

if $k \in \mathbb{N}$, $k < n$, and $\alpha < 1 - k/n$.

Although embeddings (1.8) and (1.9) are sharp (within the given scale of target spaces), they are consequences of more precise embeddings (5.16) and (4.65), respectively, mentioned below (recall that, by Remark 5.8 (ii) below, embedding (5.16) holds for $\sigma \in (0, n)$).

Here we present a survey of some results proved in [26], [19], [27] and [30]. We also comment and complement these results. Now we are able to look at them from a uniform point of view. In particular, now we are able to characterize compact embeddings in all considered cases. We also extend some results of [30] to the case when $\sigma \in (0, n)$ (cf. Section 6 below). The survey is organized as follows. Section 2 contains notation, definitions and preliminary assertions. Sections 3 and 4 involve main results on embeddings of Bessel potential spaces, modelled upon Lorentz-Karamata spaces, from [26] (partially also from [19]) and [27], respectively. While in Section 3 target spaces of our embeddings are Lorentz-Karamata spaces, in Section 4 generalized Hölder spaces serve as targets of embeddings. Note also that in Section 4 the Bessel potential spaces with order of smoothness greater or equal one are considered. In contrast to Section 4, in Section 5 we investigate the case when order of smoothness of Bessel potential spaces belongs to the interval $(0, 1)$. The approach of Section 5 is also more general than that of Section 4 since in Section 5 Bessel potential spaces are modelled upon rearrangement invariant Banach function spaces $X(\mathbb{R}^n)$. Embeddings of Bessel potential spaces modelled upon Lorentz-Karamata spaces are then obtained as particular cases. Assertions of Section 5 are proved in [30]. We close this survey with Section 6 (Concluding remarks), where we show that some results of Section 5 can be extended to the case when order of smoothness belongs to the interval $(0, n)$.

## 2. Notation and preliminaries

As usual, $\mathbb{R}^n$ denotes the Euclidean $n$-dimensional space. Throughout the paper $\mu_n$ is the $n$-dimensional Lebesgue measure in $\mathbb{R}^n$ and $\Omega$ is a $\mu_n$-measurable subset of $\mathbb{R}^n$. We denote by $\chi_\Omega$ the characteristic function of $\Omega$ and put $|\Omega|_n = \mu_n(\Omega)$. The family of all extended scalar-valued (real or complex) $\mu_n$-measurable functions on $\Omega$ is denoted by $\mathcal{M}(\Omega)$. The *non-increasing rearrangement* of $f \in \mathcal{M}(\Omega)$ is the function $f^*$ defined by

$$f^*(t) := \inf\{\lambda \geq 0 : |\{x \in \Omega : |f(x)| > \lambda\}|_n \leq t\} \quad \text{for all} \quad t \geq 0.$$

By $f^{**}$ we denote the maximal function of $f^*$ given by $f^{**}(t) := t^{-1}\int_0^t f^*(\tau)\,d\tau$, $t > 0$.

Given a rearrangement-invariant Banach function space (r. i. BFS) $X$, the associate space is denoted by $X'$. For general facts about rearrangement-invariant Banach function spaces we refer to [6].

Let $X$ and $Y$ be two (quasi-)Banach spaces. We say that $X$ *coincides* with $Y$ (and write $X = Y$) if $X$ and $Y$ are equal in the algebraic and topological sense (their (quasi-)norms are equivalent). The symbol $X \hookrightarrow Y$ or $X \hookrightarrow\hookrightarrow Y$ means that $X \subset Y$ and the natural embedding of $X$ in $Y$ is continuous or compact, respectively.

For two non-negative expressions (i.e. functions or functionals) $\mathcal{A}$, $\mathcal{B}$, the symbol $\mathcal{A} \precsim \mathcal{B}$ (or $\mathcal{A} \succsim \mathcal{B}$) means that $\mathcal{A} \le c\,\mathcal{B}$ (or $c\,\mathcal{A} \ge \mathcal{B}$), where $c$ is a positive constant independent of appropriate quantities involved in $\mathcal{A}$ and $\mathcal{B}$. If $\mathcal{A} \precsim \mathcal{B}$ and $\mathcal{A} \succsim \mathcal{B}$, we write $\mathcal{A} \approx \mathcal{B}$ and say that $\mathcal{A}$ and $\mathcal{B}$ are equivalent. Throughout the paper we use the abbreviation LHS($*$) (RHS($*$)) for the left- (right-) hand side of the relation ($*$). We adopt the convention that $a/{+\infty} = 0$ and $a/0 = +\infty$ for all $a > 0$. If $p \in (0, +\infty]$, the conjugate number $p'$ is given by $1/p + 1/p' = 1$. Note that $p'$ is negative if $p \in (0, 1)$. The symbol $\|.\|_{p;(c,d)}$, $p \in (0, +\infty]$, stands for the usual $L^p$-(quasi-)norm on the interval $(c, d) \subseteq \mathbb{R}^n$. For $\rho \in (0, +\infty)$ and $x \in \mathbb{R}^n$, $B(x, \rho) = B_n(x, \rho)$ denotes the open ball in $\mathbb{R}^n$ of radius $\rho$ and centre $x$. By $\beta_n$ we mean the volume of the unit ball in $\mathbb{R}^n$.

Following [31], we say that a positive and Lebesgue-measurable function $b$ is *slowly varying* on $(0, +\infty)$, and write $b \in SV(0, +\infty)$, if, for each $\epsilon > 0$, $t^\epsilon b(t)$ is equivalent to a non-decreasing function on $(0, +\infty)$ and $t^{-\epsilon} b(t)$ is equivalent to a non-increasing function on $(0, +\infty)$. The family of all slowly varying functions includes not only powers of iterated logarithms and the broken logarithmic functions of [24], but also such functions as $t \to \exp\left(|\log t|^a\right)$, $a \in (0, 1)$. (The last mentioned function has the interesting property that it tends to infinity more quickly than any positive power of the logarithmic function.)

Some basic properties of slowly varying functions are mentioned in the following lemma. (We refer to [31, Proposition 2.2] for properties (i)-(iii); property (iv) is a simple consequence of the definition.)

**Lemma 2.1.** *Let $b$, $b_1$ and $b_2$ belong to $SV(0, +\infty)$. Then*

(i) *$b_1 b_2 \in SV(0, +\infty)$ and $b^r \in SV(0, +\infty)$ for each $r \in \mathbb{R}$;*

(ii) *given positive numbers $\varepsilon$ and $\kappa$, there are positive constants $c_\varepsilon$ and $C_\varepsilon$ such that*

$$c_\varepsilon \min\{\kappa^{-\varepsilon}, \kappa^{\varepsilon}\} b(t) \le b(\kappa t) \le C_\varepsilon \max\{\kappa^{-\varepsilon}, \kappa^{\varepsilon}\} b(t) \quad \text{for all } t > 0;$$

(iii) *if $\alpha > 0$ and $q \in (0, \infty]$, then, for all $t > 0$,*

$$\left\| \tau^{\alpha - 1/q} b(\tau) \right\|_{q,(0,t)} \approx t^\alpha b(t) \quad \text{and} \quad \left\| \tau^{-\alpha - 1/q} b(\tau) \right\|_{q,(t,\infty)} \approx t^{-\alpha} b(t);$$

(iv) *if $\alpha > 0$, then*

$$t^\alpha b(t) \to 0 \quad \text{as} \quad t \to 0_+.$$

We can see from Lemma 2.1 (iii) that any $b \in SV(0, +\infty)$ is equivalent to a $\widetilde{b} \in SV(0, +\infty)$ which is continuous on $(0, +\infty)$. Consequently, without loss of generality, we can assume that all slowly varying functions in question are continuous on $(0, +\infty)$. More properties and examples of slowly varying functions can be found in [50, Chapter V, p. 186], [7], [21], [35], [40] and [31].

Let $p, q \in (0, +\infty]$, $b \in SV(0, +\infty)$ and let $\Omega$ be a measurable subset of $\mathbb{R}^n$. The *Lorentz-Karamata* (LK) space $L_{p,q;b}(\Omega)$ is defined to be the set of all functions $f \in \mathcal{M}(\Omega)$ such that

$$\|f\|_{p,q;b;\Omega} := \|t^{1/p - 1/q}\, b(t)\, f^*(t)\|_{q;(0,+\infty)} < +\infty. \tag{2.1}$$

If $\Omega = \mathbb{R}^n$, we simply write $\|\cdot\|_{p,q;b}$ instead of $\|\cdot\|_{p,q;b;\mathbb{R}^n}$.

When $0 < p < +\infty$, the Lorentz-Karamata space $L_{p,q;b}(\Omega)$ contains the characteristic function of every measurable subset of $\Omega$ with finite measure and hence, by linearity, every $\mu_n$-simple function. When $p = +\infty$, the Lorentz-Karamata space $L_{p,q;b}(\Omega)$ is different from the trivial space if and only if $\|\, t^{1/p - 1/q} b(t)\, \|_{q;(0,1)} < +\infty$.

The Lorentz-Karamata spaces $L_{p,q;b}$ (introduced in [21] in the case when the function $b$ is symmetrical with respect to the point 1) form an important scale of spaces. They are particular cases (if $q \in [1,\infty)$) of the classical Lorentz spaces $\Lambda^q(w)$ introduced by Lorentz (cf. [34], [12]). On the other hand, if $m \in \mathbb{N}$, $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_m) \in \mathbb{R}^m$ and

$$b(t) = \boldsymbol{\ell}^{\boldsymbol{\alpha}}(t) := \prod_{i=1}^{m} l_i^{\alpha_i}(t) \quad \text{for all} \quad t > 0, \tag{2.2}$$

where, for $t>0$, $l_1(t) := 1+|\log t|$, $l_i(t) := l_1(l_{i-1}(t))$ if $i > 1$, then the LK-space $L_{p,q;b}(\Omega)$ is the generalized Lorentz-Zygmund space $L_{p,q,\boldsymbol{\alpha}}(\Omega)$ introduced in [17] and endowed with the (quasi-)norm $\|f\|_{p,q;\boldsymbol{\alpha};\Omega}$, which in turn becomes the Lorentz-Zygmund space $L^{p,q}(\log L)^{\alpha_1}(\Omega)$ of Bennett and Rudnick (see [5], [6]) when $m = 1$. If $\boldsymbol{\alpha} = (0, \dots, 0)$, we obtain the Lorentz space $L^{p,q}(\Omega)$ endowed with the (quasi-)norm $\|.\|_{p,q;\Omega}$, which is just the Lebesgue space $L^p(\Omega)$ equipped with the (quasi-)norm $\|.\|_{p;\Omega}$ when $p = q$; if $p = q$ and $m = 1$, we obtain the Zygmund space $L^p(\log L)^{\alpha_1}(\Omega)$ endowed with the (quasi-)norm $\|.\|_{p;\alpha_1;\Omega}$.

If $b(t) = l_1(t)^{\alpha} l_2(t)^{\beta}$, $t > 0$, $\alpha, \beta \in \mathbb{R}$, then we also write $L^{p,q}(\log L)^{\alpha}(\log\log L)^{\beta}(\Omega)$ instead of $L_{p,q;b}(\Omega)$.

By a *Young function* $\Phi$ we mean a continuous, non-negative, strictly increasing and convex function on $[0, +\infty)$ such that

$$\lim_{t\to 0+} \Phi(t)/t = \lim_{t\to+\infty} t/\Phi(t) = 0.$$

The symbol $L_\Phi(\Omega)$ is used to denote the corresponding *Orlicz space* equipped with the Luxemburg norm $\|\cdot\|_\Phi$.

Orlicz spaces and Lorentz-Karamata spaces are different scales of spaces with a non-trivial intersection. For example, assume that $b$ is given by (2.2). If $1 < p < \infty$, then

$$L_{p,p;b}(\Omega) = L_\Phi(\Omega), \tag{2.3}$$

where the Young function $\Phi$ satisfies (cf. [44])

$$\Phi(t) = [t\, b(t)]^p \quad \text{for} \ \ t \in [0,\infty). \tag{2.4}$$

When $p = 1$, $\mu_n(\Omega) < \infty$ and either $\alpha_1 > 0$, or $\alpha_1 = 0$ and $\alpha_2 > 0$, $\dots$, or $\alpha_1 = \dots = \alpha_{m-1} = 0$ and $\alpha_m > 0$, then (2.3) with $\Phi$ from (2.4) remains true. Moreover, if $\mu_n(\Omega) < \infty$, $\alpha_m = -a$ where $a > 0$, and, if $m > 1$, $\alpha_i = 0$ for $i = 1, \dots, m-1$, then (cf. [44])

$$L_{\infty,\infty;b}(\Omega) = \mathrm{EXP}_m\, L^{1/a}(\Omega) := L_\Phi(\Omega) \tag{2.5}$$

(if $m = 1$, we omit the index $m$ in (2.5)), where

$$\Phi(t) = (\underbrace{\exp\circ\exp\circ\ldots\circ\exp}_{m\,\text{times}})(t^{1/a}) \quad \text{for all large} \ \ t.$$

Note also that if $b$ is given by (2.2) and $p \neq q$, then the space $L_{p,q;b}(\Omega)$ does not coincide with an Orlicz space (cf. [44, p. 444]).

The *Bessel kernel* $g_\sigma$, $\sigma > 0$, is defined as that function on $\mathbb{R}^n$ whose Fourier transform is $\widehat{g_\sigma}(\xi) = (2\pi)^{-n/2}(1+|\xi|^2)^{-\sigma/2}$, $\xi \in \mathbb{R}^n$, where the Fourier transform $\hat{f}$ of a function $f$ is given by $\hat{f}(\xi) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-i\xi\cdot x} f(x)\, dx$.

Let us summarize the basic properties of the Bessel kernel $g_\sigma$:

(2.6) $g_\sigma$ is a positive, integrable function which is analytic except at the origin;

(2.7) $g_\sigma$ is radially decreasing;

$$g_\sigma(x) \le c_1|x|^{\sigma-n}e^{-c_2|x|} \ \text{for} \ 0 < \sigma < n \ \text{and all} \ x \in \mathbb{R}^n\setminus\{0\}; \tag{2.8}$$

$$g_\sigma(x) \approx |x|^{\sigma-n} \ \text{as} \ |x| \to 0 \ \ \text{if} \ \ 0 < \sigma < n; \tag{2.9}$$

(2.10) $$\left|\frac{\partial}{\partial x_j} g_\sigma(x)\right| \le c|x|^{\sigma-n-1} \text{ for } 0<\sigma\le n+1,\ j\in\{1,\dots,n\} \text{ and all } x\in \mathbb{R}^n\backslash\{0\};$$

(2.11) $$g_\sigma^*(t) \precsim t^{(\sigma-n)/n} e^{-ct^{1/n}} \text{ for } 0<\sigma<n \text{ and all } t>0$$

($c, c_1$ and $c_2$ are positive constants).

Property (2.7) follows from formula (26) in [46, Chapter V]. For the proof of (2.6), (2.8)-(2.10) see [4], for (2.11) see [15].

Let $\sigma>0$ and let $X = X(\mathbb{R}^n) = X(\mathbb{R}^n, \mu_n)$ be a r. i. Banach function space endowed with the norm $\|\cdot\|_X$. The *Bessel potential* space $H^\sigma X(\mathbb{R}^n)$ is defined by

(2.12) $$H^\sigma X(\mathbb{R}^n) := \{u : u = f * g_\sigma,\ f\in X(\mathbb{R}^n)\}$$

and is equipped with the norm

(2.13) $$\|u\|_{H^\sigma X} := \|f\|_X.$$

Note that, given $f\in X$, the convolution $u = f*g_\sigma$ is well defined and finite $\mu_n$-a.e. on $\mathbb{R}^n$ since the measure space $(\mathbb{R}^n,\mu_n)$ is resonant and so (cf. [6, Theorem II.6.6]) $X \hookrightarrow L^1(\mathbb{R}^n)+L^\infty(\mathbb{R}^n)$.

If $p\in(1,+\infty]$, $q\in[1,+\infty]$ and $b\in SV(0,+\infty)$, then the space $L_{p,q;b}(\mathbb{R}^n)$ coincides with a r. i. Banach function space $X(\mathbb{R}^n)$ (the (quasi-)norm (2.1) is equivalent to the norm $\|t^{1/p-1/q}\ b(t)\ f^{**}(t)\|_{q;(0,+\infty)}$). Consequently, if $\sigma>0$, $p\in(1,+\infty]$, $q\in[1,+\infty]$ and $b\in SV(0,+\infty)$, then $H^\sigma L_{p,q;b}(\mathbb{R}^n) := H^\sigma X(\mathbb{R}^n)$ is the usual Bessel potential space modelled upon the Lorentz-Karamata space $L_{p,q;b}(\mathbb{R}^n)$, which is equipped with the (quasi-)norm

(2.14) $$\|u\|_{\sigma;p,q;b} := \|f\|_{p,q;b}.$$

When $m\in\mathbb{N}$, $\boldsymbol{\alpha} = (\alpha_1,\dots,\alpha_m)\in\mathbb{R}^m$ and $b = \boldsymbol{\ell}^{\boldsymbol{\alpha}}$, we obtain the logarithmic Bessel potential space $H^\sigma L_{p,q;\boldsymbol{\alpha}}(\mathbb{R}^n)$, endowed with the (quasi-)norm $\|u\|_{\sigma;p,q;b}$ and considered in [17]. Note that if $\boldsymbol{\alpha} = (0,\dots,0)$, $H^\sigma L_{p,p;\boldsymbol{\alpha}}(\mathbb{R}^n)$ is simply the (fractional) Sobolev space $H^{\sigma,p}(\mathbb{R}^n)$ of order $\sigma$.

When $k\in\mathbb{N}$, $p,q\in(1,+\infty)$ and $b\in SV(0,+\infty)$, then

$$H^k L_{p,q;b}(\mathbb{R}^n) = \{u : D^\alpha u \in L_{p,q;b}(\mathbb{R}^n) \text{ if } |\alpha|\le k\},$$

and

$$\|u\|_{k;p,q;b} \approx \sum_{|\alpha|\le k} \|D^\alpha u\|_{p,q;b} \quad \text{for all } u\in H^k L_{p,q;b}(\mathbb{R}^n)$$

according to [27, Lemma 4.5] and [41, Theorem 5.3].

Let $\Omega$ be a domain in $\mathbb{R}^n$. The space of all scalar-valued (real or complex), bounded and continuous functions on $\Omega$ is denoted by $C_B(\Omega)$ and it is equipped with the $L^\infty(\Omega)$-norm. For each $h\in\mathbb{R}^n$, let $\Omega_h = \{x\in\Omega : x+h\in\Omega\}$ and let $\Delta_h$ be the difference operator given on scalar functions $f$ on $\Omega$ by $(\Delta_h f)(x) = f(x+h)-f(x)$ for all $x\in\Omega_h$. The *modulus of smoothness* of a function $f$ in $C_B(\Omega)$ is defined by

$$\omega(f,t) := \sup_{|h|\le t} \|\Delta_h f|L_\infty(\Omega_h)\| \quad \text{for all } t\ge 0.$$

If

$$\widetilde{\omega}(f,t) := \omega(f,t)/t \quad \text{for all } t>0,$$

then $\widetilde{\omega}(f,.)$ is equivalent to a non-increasing function on $(0,+\infty)$. The function $f\in C_B(\Omega)$ is uniformly continuous on $\Omega$ if and only if $\omega(f,t)\to\omega(f,0)=0$ as $t\to 0_+$. We refer to [6, pp. 331–333] and to [12, pp. 40 – 50] for more details.

Let $\Omega$ be a domain in $\mathbb{R}^n$. By $C(\overline{\Omega})$ we mean the subspace of $C_B(\Omega)$ of all bounded and uniformly continuous functions on $\Omega$. A subset $S$ of $C(\overline{\Omega})$ is *equicontinuous* if and only if

$$\sup_{f\in S}\omega(f,t)\to 0 \quad \text{as} \quad t\to 0_+.$$

Let $r\in(0,+\infty]$ and let $\mathcal{L}_r$ be the class of all continuous functions $\lambda:(0,1]\to(0,+\infty)$ which are increasing on some interval $(0,\delta)$, with $\delta=\delta_\lambda\in(0,1]$, and satisfy

$$\lim_{t\to 0_+}\lambda(t)=0$$

and

$$\left\|t^{-1/r}\frac{t}{\lambda(t)}\right\|_{r;(0,\delta)}<+\infty. \tag{2.15}$$

When $r=+\infty$, we simply write $\mathcal{L}$ instead of $\mathcal{L}_r$.

If $\lambda\in\mathcal{L}_r$, one can easy see that $\lambda$ is equivalent to a continuous increasing function on the interval (0,1]. Consequently, without loss of generality, we can assume that all elements of $\mathcal{L}_r$ are continuous increasing functions on the interval (0,1].

Let $r\in(0,+\infty]$, $\lambda\in\mathcal{L}_r$ and let $\Omega$ be a domain in $\mathbb{R}^n$. The *generalized Hölder* space $\Lambda^{\lambda(\cdot)}_{\infty,r}(\overline{\Omega})$ consists of all those functions $f\in C_B(\Omega)$ for which the quasi-norm

$$\|f|\Lambda^{\lambda(\cdot)}_{\infty,r}(\overline{\Omega})\|:=\|f|L_\infty(\Omega)\|+\left\|t^{-1/r}\frac{\omega(f,t)}{\lambda(t)}\right\|_{r;(0,1)} \tag{2.16}$$

is finite. Standard arguments show that the space $\Lambda^{\lambda(\cdot)}_{\infty,r}(\overline{\Omega})$ is complete (cf. [38, Theorem 3.1.4]). If (2.15) does not hold, then the space $\Lambda^{\lambda(\cdot)}_{\infty,r}(\overline{\Omega})$ contains only constant functions.

The space $\Lambda^{\lambda(\cdot)}_{\infty,\infty}(\overline{\Omega})$ coincides (cf. [39, Proposition 3.5]) with the space $C^{0,\lambda(\cdot)}(\overline{\Omega})$ defined by

$$\|f|C^{0,\lambda(\cdot)}(\overline{\Omega})\|:=\sup_{x\in\Omega}|f(x)|+\sup_{\substack{x,y\in\Omega\\ 0<|x-y|\le 1}}\frac{|f(x)-f(y)|}{\lambda(|x-y|)}<+\infty.$$

If $\lambda(t)=t$, $t\in(0,1]$, and $\Omega=\mathbb{R}^n$, then $\Lambda^{\lambda(\cdot)}_{\infty,\infty}(\overline{\Omega})$ coincides with the space $Lip(\mathbb{R}^n)$ of the Lipschitz functions. If $\lambda(t)\equiv t^\alpha$, $\alpha\in(0,1]$, then the space $\Lambda^{\lambda(\cdot)}_{\infty,r}(\overline{\Omega})$ coincides with the space $C^{0,\alpha,r}(\overline{\Omega})$ introduced in [2].

The next lemma shows that we could define the generalized Hölder space $\Lambda^{\lambda(\cdot)}_{\infty,r}(\overline{\Omega})$ as a subspace of $C(\overline{\Omega})$ rather than a subspace of $C_B(\Omega)$.

**Lemma 2.2** ([30, Lemma 2.4])**.** *Let $r\in(0,+\infty]$, $\lambda\in\mathcal{L}_r$ and let $\Omega$ be a domain in $\mathbb{R}^n$. Then*

$$\Lambda^{\lambda(\cdot)}_{\infty,r}(\overline{\Omega})\hookrightarrow C(\overline{\Omega}).$$

Let $\Omega$ be a measurable subset of $\mathbb{R}^n$. We denote by $B(\Omega)$ the set of all scalar-valued functions (real or complex) which are bounded on $\Omega$ and we equip this set with the norm

$$\|f\|_{B(\Omega)}:=\sup\{|f(x)|:\,x\in\Omega\}.$$

The following lemma is related to [17, Lemma 4.5].

**Lemma 2.3** ([30, Lemma 3.3])**.** *Let $X=X(\mathbb{R}^n)$ be a r. i. BFS and let $\Omega$ be a domain in $\mathbb{R}^n$. Suppose that $\sigma>0$ and let $g_\sigma$ be the Bessel kernel. Then*

$$H^\sigma X(\mathbb{R}^n)\hookrightarrow B(\Omega) \tag{2.17}$$

*if and only if*

$$\|g_\sigma\|_{X'}<+\infty. \tag{2.18}$$

We shall investigate embeddings of the form

$$H^\sigma X(\mathbb{R}^n) \hookrightarrow Y(\Omega), \tag{2.19}$$

where $\sigma > 0$, $X = X(\mathbb{R}^n)$ is a r. i. Banach function space, $\Omega$ is a domain in $\mathbb{R}^n$ and $Y(\Omega)$ is a convenient Banach space of functions defined on $\Omega$. Note that embedding (2.19) means that the mapping $u \mapsto u|_\Omega$ from $H^\sigma X(\mathbb{R}^n)$ into $Y(\Omega)$ is continuous. Note also that in the whole paper we use the symbol $u$ both for the function $u$ and its restriction to $\Omega$.

To describe compact embeddings of Bessel potential spaces modelled upon Lorentz-Karamata spaces into generalized Hölder spaces, in [30] we proved the following assertion, where we characterized totally bounded subsets of the space $\Lambda^{\mu(\cdot)}_{\infty,r}(\overline{\Omega})$ with a bounded domain $\Omega \subset \mathbb{R}^n$.

**Theorem 2.4** ([30, Theorem 7.1])**.** *Let $r \in (0, +\infty)$, $\mu \in \mathcal{L}_r$ and let $\Omega$ be a bounded domain in $\mathbb{R}^n$. Then $\mathcal{S} \subset \Lambda^{\mu(\cdot)}_{\infty,r}(\overline{\Omega})$ is totally bounded if and only if $\mathcal{S}$ is bounded in $\Lambda^{\mu(\cdot)}_{\infty,r}(\overline{\Omega})$ and*

$$\sup_{u\in\mathcal{S}} \|t^{-1/r}[\mu(t)]^{-1}\omega(u,t)\|_{r;(0,\xi)} \to 0 \quad as \quad \xi \to 0_+. \tag{2.20}$$

**Remark 2.5** ([30, Remark 7.2])**.** (i) In Theorem 2.4 the implication

$$\mathcal{S} \subset \Lambda^{\mu(\cdot)}_{\infty,r}(\overline{\Omega}) \text{ is bounded and (2.20) holds} \implies \mathcal{S} \text{ is totally bounded in } \Lambda^{\mu(\cdot)}_{\infty,r}(\overline{\Omega}) \tag{2.21}$$

remains true even if $r = +\infty$. (This can be seen from the proof of Theorem 2.4.)

(ii) If $r = +\infty$ in Theorem 2.4, then the reverse implication to (2.21) holds provided that we assume that $\mathcal{S} \subset \Lambda^{\mu(\cdot),0}_{\infty,\infty}(\overline{\Omega})$. Here $\Lambda^{\mu(\cdot),0}_{\infty,\infty}(\overline{\Omega})$ is a subspace of $\Lambda^{\mu(\cdot)}_{\infty,\infty}(\overline{\Omega})$ consisting of those functions $u$ which satisfy

$$\lim_{\delta\to 0_+} \|[\mu(t)]^{-1}\, \omega(u,t)\|_{\infty;(0,\delta)} = 0.$$

(This follows from the necessity part of the proof of Theorem 2.4.)

(iii) Summarizing what we have said, we arrive at the following result.

*Let $u \in \mathcal{L}$ and let $\Omega$ be a bounded domain in $\mathbb{R}^n$. Then $\mathcal{S} \subset \Lambda^{\mu(\cdot),0}_{\infty,\infty}(\overline{\Omega})$ is totally bounded in $\Lambda^{\mu(\cdot)}_{\infty,\infty}(\overline{\Omega})$ if and only if $\mathcal{S}$ is bounded in $\Lambda^{\mu(\cdot)}_{\infty,\infty}(\overline{\Omega})$ and*

$$\sup_{u\in\mathcal{S}} \|[\mu(t)]^{-1}\, \omega(u,t)\|_{\infty;(0,\xi)} \to 0 \quad as \quad \xi \to 0_+.$$

## 3. Embeddings into Lorentz-Karamata spaces

In this section we present embeddings of Bessel-potential-type spaces into Lorentz-Karamata spaces, which extend and slightly improve those of [16], [18] and complement those of [40]. Our main results state that such embeddings are sharp and fail to be compact. We also show how to modify parameters of target spaces to arrive to compact embeddings.

The first theorem concerns the sublimiting case when $\sigma \in (0, n)$ and $1 < p < n/\sigma$. Part (i) of this theorem extends [18, Theorem 3.1] and [40, Theorem 5.1] and corresponds to the Sobolev-type embedding. The second and the third parts concern the (local) sharpness of the embedding from part (i) while the fourth part states that the embedding from part (i) is not compact. In part (v) it is shown that this embedding becomes compact if the parameters of the target space are restricted in a proper way.

**Theorem 3.1** (cf. [26, Theorem 3.1])**.** *Let $\sigma \in (0, n)$, $1 < p < n/\sigma$, $q \in [1, +\infty]$, $r \in [q, +\infty]$, $1/p_\sigma = 1/p - \sigma/n$ and let $b \in SV(0, +\infty)$. Let $\Omega \subset \mathbb{R}^n$ be a nonempty domain.*

(i) *Then*

$$H^\sigma L_{p,q;b}(\mathbb{R}^n) \hookrightarrow L_{p_\sigma,r;b}(\mathbb{R}^n). \tag{3.1}$$

(ii) *Let* $P \in [p_\sigma, +\infty]$ *and let* $\tilde{b} \in SV(0,+\infty)$. *Suppose that either*

$$P > p_\sigma \tag{3.2}$$

*or*

$$P = p_\sigma \quad \text{and} \quad \overline{\lim_{t\to 0_+}} \frac{\tilde{b}(t)}{b(t)} = +\infty. \tag{3.3}$$

*Then the embedding*

$$H^\sigma L_{p,q;b}(\mathbb{R}^n) \hookrightarrow L_{P,r;\tilde{b}}(\Omega) \tag{3.4}$$

*does not hold.*

(iii) *Let* $\overline{q} \in (0,q)$. *Then the embedding*

$$H^\sigma L_{p,q;b}(\mathbb{R}^n) \hookrightarrow L_{p_\sigma,\overline{q};b}(\Omega)$$

*fails.*

(iv) *The embedding*

$$H^\sigma L_{p,q;b}(\mathbb{R}^n) \hookrightarrow L_{p_\sigma,r;b}(\Omega) \tag{3.5}$$

*is not compact.*

(v) *Let* $\Omega$ *be bounded and let* $\tilde{b} \in SV(0,+\infty)$. *Suppose that either*

$$P \in (0, p_\sigma) \tag{3.6}$$

*or*

$$P = p_\sigma \quad \text{and} \quad \lim_{t\to 0_+} \frac{\tilde{b}(t)}{b(t)} = 0. \tag{3.7}$$

*Then*

$$H^\sigma L_{p,q;b}(\mathbb{R}^n) \hookrightarrow\hookrightarrow L_{P,r;\tilde{b}}(\Omega). \tag{3.8}$$

**Remark 3.2** ( [26, Remarks 3.2–3.6])**.** (i) By Theorem 3.1 (ii), all embeddings (3.1) are sharp with respect to the first and third parameters of the target space. (This is why we consider all embeddings (3.1) in Theorem 3.1 and not only optimal embedding (3.10) mentioned below.)

(ii) As

$$L_{p_\sigma,r;b}(\mathbb{R}^n) \hookrightarrow L_{p_\sigma,s;b}(\mathbb{R}^n) \quad \text{if} \quad 0 < r < s < +\infty, \tag{3.9}$$

among embeddings (3.1) the embedding

$$H^\sigma L_{p,q;b}(\mathbb{R}^n) \hookrightarrow L_{p_\sigma,q;b}(\Omega), \tag{3.10}$$

with $\Omega = \mathbb{R}^n$, is optimal. (Note that embedding (3.9) can be proved analogously as the classical embedding $L_{p_\sigma,r}(\mathbb{R}^n) \hookrightarrow L_{p_\sigma,s}(\mathbb{R}^n)$ if $0 < r < s < +\infty$.)

(iii) Theorem 3.1 (iii) shows that embedding (3.10) is also sharp with respect to the second parameter.

(iv) By Theorem 3.1 (i), embedding (3.10) is continuous and, by Theorem 3.1 (iv), this embedding is not compact. Moreover, Theorem 3.1 (iv) also shows that we cannot arrive to a compact embedding if we replace the target space $L_{p_\sigma,q;b}(\Omega)$ in (3.10) by a larger space $L_{p_\sigma,r;b}(\Omega)$ with $r > q$.

(v) Put $X := H^\sigma L_{p,q;b}(\mathbb{R}^n)$. By Theorem 3.1 (i),

$$\sup_{t>0} t^{1/p_\sigma} b(t) f^*(t) \precsim \|f\|_X \quad \text{for all} \quad f \in X,$$

and, by Theorem 3.1 (ii), the inequality

$$\sup_{t>0} t^{1/p_\sigma} \tilde{b}(t) f^*(t) \precsim \|f\|_X$$

does not hold for all $f \in X$ if $\tilde{b} \in SV(0,+\infty)$ satisfies

$$\overline{\lim_{t\to 0_+}} \frac{\tilde{b}(t)}{b(t)} = +\infty.$$

If we use an analogue of terminology from [32] or [48], this means that the function $[t^{1/p_\sigma} b(t)]^{-1}$, $t > 0$, is the *growth envelope function* of the space $H^\sigma L_{p,q;b}(\mathbb{R}^n)$. Using also Theorem 3.1 (iii), we can see that the couple

$$([t^{1/p_\sigma} b(t)]^{-1}, q)$$

is the *growth envelope* of the space $H^\sigma L_{p,q;b}(\mathbb{R}^n)$.

**Remark 3.3.** One can easily verify that the assumptions of part (ii) of Theorem 3.1 are equivalent to: Let $P \in (0,+\infty]$, $\tilde{b} \in SV(0,+\infty)$ and let

$$\overline{\lim_{t\to 0_+}} \frac{t^{1/P}\, \tilde{b}(t)}{t^{1/p_\sigma}\, b(t)} = +\infty. \tag{3.11}$$

Similarly, the assumptions of part (v) Theorem 3.1 can be rewritten as: Let $\Omega$ be bounded, $P \in (0,+\infty]$, $\tilde{b} \in SV(0,+\infty)$ and let

$$\lim_{t\to 0_+} \frac{t^{1/P}\, \tilde{b}(t)}{t^{1/p_\sigma}\, b(t)} = 0. \tag{3.12}$$

Note also that, by Lemma 2.1 (iii), $t^{1/\rho} b(t) \approx \|\tau^{1/\rho - 1/r} b(\tau)\|_{r;(0,\,t)}$ for all $t \in (0,+\infty)$ when $0 < \rho < +\infty$, $0 < r \le +\infty$ and $b \in SV(0,+\infty)$. Thus, the function $t \mapsto t^{1/\rho} b(t)$ is (equivalent) to the fundamental function of the Lorentz-Karamata space $L_{\rho,r;b}$ (we refer to [6] for this notion).

**Proof of Theorem 3.1.** Since $p_\sigma < +\infty$, parts (i)–(iv) coincide with parts (i)–(iv) of [26, Theorem 3.1]. Thus, we refer to [26] for proofs of these statements. The proof of part (v) of Theorem 3.1 is similar to that of [19, Theorem 3.1]. □

**Corollary 3.4.** *Let all the assumptions of* Theorem 3.1 *be satisfied. Let* $P \in (0,+\infty]$ *and* $\tilde{b} \in SV(0,+\infty)$.

(i) *Let* $|\Omega|_n < +\infty$. *Then*

$$H^\sigma L_{p,q;b}(\mathbb{R}^n) \hookrightarrow L_{P,r;\tilde{b}}(\Omega) \tag{3.13}$$

*if and only if*

$$\overline{\lim_{t\to 0_+}} \frac{t^{1/P}\, \tilde{b}(t)}{t^{1/p_\sigma}\, b(t)} < +\infty. \tag{3.14}$$

(ii) *Let* $\Omega$ *be bounded. Then*

$$H^\sigma L_{p,q;b}(\mathbb{R}^n) \hookrightarrow\hookrightarrow L_{P,r;\tilde{b}}(\Omega) \tag{3.15}$$

*if and only if*

$$\lim_{t\to 0_+} \frac{t^{1/P}\, \tilde{b}(t)}{t^{1/p_\sigma}\, b(t)} = 0. \tag{3.16}$$

Proof. (i) Since singularities of slowly varying functions $b$ and $\tilde{b}$ at the interval $[0,|\Omega|_n]$ are only those at 0, part (i) of the corollary is a consequence of parts (i) and (ii) of Theorem 3.1 (see also Remark 3.3).

(ii) The implication (3.16) $\Rightarrow$ (3.15) holds by Theorem 3.1 (v) (see also Remark 3.3). The converse implication follows by contradiction from estimates [26, (5.4)], [26, (5.6)] and the second displayed estimate in Step 5 of the proof of [26, Theorem 3.1] (cf. also [26, (5.10)]). □

The next theorem represents an analogue of Theorem 3.1 and concerns the limiting case when $p = n/\sigma$. Part (i) of this theorem extends [16, Theorem 3.1 and Theorem 3.2].

**Theorem 3.5** (cf. [26, Theorem 3.7])**.** *Let $\sigma \in (0,n)$, $q \in (1,+\infty]$, $r \in [q,+\infty]$ and let $b \in SV(0,+\infty)$ be such that*

$$\| t^{-1/q'}[b(t)]^{-1}\|_{q';(0,1)} = +\infty. \tag{3.17}$$

*Suppose that $\Omega \subset \mathbb{R}^n$ is a nonempty domain with $|\Omega|_n < +\infty$ and that $b_{qr} \in SV(0,+\infty)$ satisfies*

$$b_{qr}(t) := [b(t)]^{-q'/r}\left(\int_t^2 \tau^{-1}[b(\tau)]^{-q'}\,d\tau\right)^{-1/q'-1/r} \quad \text{for all} \quad t \in (0,1]. \tag{3.18}$$

(i) *Then*

$$H^\sigma L_{n/\sigma,q;b}(\mathbb{R}^n) \hookrightarrow L_{\infty,r;b_{qr}}(\Omega). \tag{3.19}$$

(ii) *If a function $\tilde{b} \in SV(0,+\infty)$ is such that*

$$\overline{\lim_{t\to 0_+}} \frac{\|\tau^{-1/r}\tilde{b}(\tau)\|_{r;(0,t)}}{\|\tau^{-1/r}b_{qr}(\tau)\|_{r;(0,t)}} = +\infty, \tag{3.20}$$

*then the embedding*

$$H^\sigma L_{n/\sigma,q;b}(\mathbb{R}^n) \hookrightarrow L_{\infty,r;\tilde{b}}(\Omega) \tag{3.21}$$

*does not hold.*

(iii) *Let $\overline{q} \in (0,q)$. Then the embedding*

$$H^\sigma L_{n/\sigma,q;b}(\mathbb{R}^n) \hookrightarrow L_{\infty,\overline{q};b_{q\overline{q}}}(\Omega)$$

*fails, where $b_{q\overline{q}}$ is again defined by* (3.18) *with $r$ replaced by $\overline{q}$.*

(iv) *The embedding*

$$H^\sigma L_{n/\sigma,q;b}(\mathbb{R}^n) \hookrightarrow L_{\infty,r;b_{qr}}(\Omega) \tag{3.22}$$

*is not compact.*

(v) *Let $\Omega$ be bounded and let $\tilde{b} \in SV(0,+\infty)$ be such that*

$$\lim_{t\to 0_+} \frac{\|\tau^{-1/r}\tilde{b}(\tau)\|_{r;(0,t)}}{\|\tau^{-1/r}b_{qr}(\tau)\|_{r;(0,t)}} = 0. \tag{3.23}$$

*Then*

$$H^\sigma L_{n/\sigma,q;b}(\mathbb{R}^n) \hookrightarrow\hookrightarrow L_{\infty,r;\tilde{b}}(\Omega). \tag{3.24}$$

Proof. Parts (i)–(iv) coincide with parts (i)–(iv) of [26, Theorem 3.7]. Thus, we refer to [26] for proofs of these statements. The proof of part (v) of Theorem 3.5 makes use of the same main ideas as that of [19, Theorem 3.4]. However, to estimate analogues of quantities $\mathcal{J}_1, \mathcal{J}_2$

and $\mathcal{J}_3$ ($\mathcal{J}_1, \mathcal{J}_2$ and $\mathcal{J}_3$ are introduced in the proof of [19, Theorem 3.4]), one has to employ the estimate

$$\|\tau^{-1/r} b_{qr}(\tau)\|_{r;(0,t)} \approx \left(\int_t^2 \tau^{-1}[b(\tau)]^{-q'}\, d\tau\right)^{-1/q'} = b_{q\infty}(t) \quad \text{for all} \quad t \in (0,1) \tag{3.25}$$

(which follows from (3.18) and (3.17)) and some further properties of slowly varying functions. Therefore, it is much shorter to proceed as follows:

Put $Y := L_{\infty,r,\tilde{b}}(\Omega)$, $X = L_{p,q,b}(\mathbb{R}^n)$ and $K := \{u \in H^\sigma X : ||u||_{H^\sigma X} \le 1\}$. As in the proof of [19, Theorem 3.4], it is sufficient to verify that, given $\varepsilon > 0$, there is $\delta > 0$ such that

$$\|\, u\, \chi_M \,\|_Y \le \varepsilon \quad \text{for all} \quad u \in K \quad \text{and every} \quad M \subset \Omega \quad \text{with} \quad |M|_n < \delta.$$

Let $\varepsilon > 0$. Assumption (3.23) implies that there is $\delta > 0$ such that

$$B_1(\delta) := \sup_{0<R<\delta} \frac{\|\tau^{-1/r}\tilde{b}(\tau))\|_{r;(0,R)}}{\|\tau^{-1/r} b_{qr}(\tau)\|_{r;(0,R)}} < \varepsilon. \tag{3.26}$$

Assume that $u \in K$ and let $M \subset \Omega$ satisfy $|M|_n < \delta$. Then

$$\|\, u\, \chi_M \,\|_Y \le \|t^{-1/r}\tilde{b}(t)\, u^*(t)\|_{r;(0,\delta)}.$$

Moreover, by theorem on embeddings of classical Lorentz spaces (cf., e.g., [10, Theorem 3.1]) and by (3.26),

$$\|t^{-1/r}\tilde{b}(t)\, u^*(t)\|_{r;(0,\delta)} \le B_1(\delta)\, \|t^{-1/r} b_{qr}(t)\, u^*(t)\|_{r;(0,\delta)} < \varepsilon\, \|u\|_{\infty,r;b_{qr}}.$$

Finally, by Theorem 3.5 (i),

$$\|u\|_{\infty,r;b_{qr}} \precsim \|u\|_{H^\sigma X} \le 1.$$

Combining estimates mentioned above, we arrive at $\|\, u\, \chi_M \,\|_Y \precsim \varepsilon$ for all $u \in K$ and every $M \subset \Omega$ with $|M|_n < \delta$. □

**Corollary 3.6.** *Let all the assumptions of* Theorem 3.5 *be satisfied. Let* $\tilde{b} \in SV(0,+\infty)$.

(i) *Let* $|\Omega|_n < +\infty$. *Then*

$$H^\sigma L_{n/\sigma,q;b}(\mathbb{R}^n) \hookrightarrow L_{\infty,r;\tilde{b}}(\Omega) \tag{3.27}$$

*if and only if*

$$\overline{\lim_{t\to 0_+}} \frac{\|\tau^{-1/r}\tilde{b}(\tau)\|_{r;(0,t)}}{\|\tau^{-1/r} b_{qr}(\tau)\|_{r;(0,t)}} < +\infty. \tag{3.28}$$

(ii) *Let* $\Omega$ *be bounded. Then*

$$H^\sigma L_{n/\sigma,q;b}(\mathbb{R}^n) \hookrightarrow\hookrightarrow L_{\infty,r;\tilde{b}}(\Omega) \tag{3.29}$$

*if and only if*

$$\overline{\lim_{t\to 0_+}} \frac{\|\tau^{-1/r}\tilde{b}(\tau)\|_{r;(0,t)}}{\|\tau^{-1/r} b_{qr}(\tau)\|_{r;(0,t)}} = 0. \tag{3.30}$$

Proof. (i) Singularities of slowly varying functions $b$ and $\tilde{b}$ at the interval $[0, |\Omega|_n]$ are only those at 0. Thus, part (i) of the corollary is a consequence of parts (i) and (ii) of Theorem 3.5 and [10, Theorem 3.1]. Indeed, Theorem 3.5 (i), (3.28) and [10, Theorem 3.1] imply that

$$H^\sigma L_{n/\sigma,q;b}(\mathbb{R}^n) \hookrightarrow L_{\infty,r;b_{qr}}(\Omega) \hookrightarrow L_{\infty,r;\tilde{b}}(\Omega).$$

On the other hand, if (3.28) is not satisfied, then, by Theorem 3.5 (ii), embedding (3.27) fails.

(ii) The implication (3.30) $\Rightarrow$ (3.29) holds by Theorem 3.5 (v). The converse implication follows by contradiction from estimates [26, (5.16)], [26, (5.22)] and (3.25) (cf. also [26, (5.23)]). $\square$

**Remark 3.7** (cf. [26, Remarks 3.8–3.18])**.** (i) Theorem 3.5 (i) holds without assumption (3.17). However, if $q \in (1,+\infty)$ and $\| t^{-1/q'}[b(t)]^{-1}\|_{q';(0,1)} < +\infty$, then

$$H^\sigma L_{n/\sigma,q;b}(\mathbb{R}^n) \hookrightarrow C_B(\mathbb{R}^n),$$

c.f. [41, Proposition 5.6].

(ii) Assume that all the assumptions of Theorem 3.5 are satisfied. If $r \in [q,+\infty]$, then the embedding

$$H^\sigma L_{n/\sigma,q;b}(\mathbb{R}^n) \hookrightarrow L_{\infty,r;\tilde{b}}(\Omega) \tag{3.31}$$

with $\tilde{b} = b_{qr}$ is *sharp* with respect to the parameter $\tilde{b} \in SV(0,+\infty)$, that is, the target space $L_{\infty,r;\tilde{b}}(\Omega)$ in (3.31) and the space $L_{\infty,r;b_{qr}}(\Omega)$ (i.e., the target space in (3.31) with $\tilde{b} = b_{qr}$) satisfy $L_{\infty,r;b_{qr}}(\Omega) \hookrightarrow L_{\infty,r;\tilde{b}}(\Omega)$. Indeed, the last embedding is equivalent to

$$\|t^{-1/r}\tilde{b}(t)\, f^*(t)\|_{r;(0,|\Omega|_n)} \precsim \|t^{-1/r}b_{qr}(t)\, f^*(t)\|_{r;(0,|\Omega|_n)} \quad \text{for all} \quad f \in L_{\infty,r;b_{qr}}(\Omega).$$

This inequality holds (cf., e.g., [10, Theorem 3.1]) if

$$\sup_{x\in(0,|\Omega|_n))} \|t^{-1/r}\tilde{b}(t)\|_{r;(0,|\Omega|_n)}/\|t^{-1/r}b_{qr}(t)\|_{r;(0,|\Omega|_n)} < +\infty,$$

which is equivalent to (3.28). The result follows from Corollary 3.6 (i) since (3.28) is satisfied when (3.31) holds.

(iii) The target spaces in (3.19) form a scale $\{L_{\infty,r;b_{qr}}(\Omega)\}_{r=q}^{+\infty}$ whose endpoint spaces with $r = +\infty$ and $r = q$ are of particular interest. The former endpoint space $L_{\infty,\infty;b_{q\infty}}(\Omega)$ corresponds to the target space in Trudinger's-type embedding while the latter endpoint space $L_{\infty,q;b_{qq}}(\Omega)$ corresponds to the target space in the Brézis-Wainger-type embedding. Since the spaces $\{L_{\infty,r;b_{qr}}(\Omega)\}_{r=q}^{+\infty}$ satisfy

$$L_{\infty,r;b_{qr}}(\Omega) \hookrightarrow L_{\infty,s;b_{qs}}(\Omega) \quad \text{if} \quad q \le r \le s \le +\infty, \tag{3.32}$$

the embedding

$$H^\sigma L_{n/\sigma,q;b}(\mathbb{R}^n) \hookrightarrow L_{\infty,q;b_{qq}}(\Omega) \tag{3.33}$$

is optimal. (Making use of estimate (3.25), one can prove (3.32) analogously as the embedding $L_{p,r}(\Omega) \hookrightarrow L_{p,s}(\Omega)$, $0 < p < +\infty$, $0 < r \le s \le +\infty$. Another way how to verify (3.32) is to apply, e.g., [10, Theorem 3.1].)

(iv) Theorem 3.5 (iii) shows that embedding (3.33) is also sharp with respect to the second parameter.

(v) By Theorem 3.5 (i), embedding (3.33) is continuous and, by Theorem 3.5 (iv), this embedding is not compact. Moreover, Theorem 3.5 (iv) also shows that we cannot arrive to a compact embedding if we replace the target space $L_{\infty,q;b_{qq}}(\Omega)$ in (3.33) by a larger space $L_{\infty,r;b_{qr}}(\Omega)$ with $r > q$.

(vi) Put $X := H^\sigma L_{n/\sigma,q;b}(\mathbb{R}^n)$. By Theorem 3.5 (i),

$$\sup_{t>0} b_{q\infty}(t) f^*(t) \precsim \|f\|_X \quad \text{for all} \quad f \in X,$$

and, by Theorem 3.5 (ii) (cf. also part (vii) of this remark), the inequality

$$\sup_{t>0} \tilde{b}(t) f^*(t) \precsim \|f\|_X$$

does not hold for all $f \in X$ if $\tilde{b} \in SV(0,+\infty)$ satisfies

$$\overline{\lim_{t\to 0_+}} \frac{\tilde{b}(t)}{b_{q\infty}(t)} = +\infty.$$

This means that the function $[b_{q\infty}(t)]^{-1} = \left(\int_t^2 \tau^{-1}[b(\tau)]^{-q'}d\tau\right)^{1/q'}$, $t \in (0,1]$, is the *growth envelope function* of the space $H^\sigma L_{n/\sigma,q;b}(\mathbb{R}^n)$. Using also Theorem 3.5 (iii), we can see that the couple

$$\left(\left(\int_t^2 \tau^{-1}[b(\tau)]^{-q'}d\tau\right)^{1/q'}, q\right)$$

is the *growth envelope* of the space $H^\sigma L_{n/\sigma,q;b}(\mathbb{R}^n)$.

(vii) Let $r \in [q,+\infty]$. By (3.25),

$$\overline{\lim_{t\to 0_+}} \frac{\|\tau^{-1/r}\tilde{b}(\tau)\|_{r;(0,t)}}{\|\tau^{-1/r}b_{qr}(\tau)\|_{r;(0,t)}} \approx \overline{\lim_{t\to 0_+}} \frac{\|\tau^{-1/r}\tilde{b}(\tau)\|_{r;(0,t)}}{b_{q\infty}(t)}. \tag{3.34}$$

Together with the estimate

$$\begin{aligned} \|\tau^{-1/r}\tilde{b}(\tau)\|_{r;(0,t)} &= \|\tau^{1-1/r}[\tau^{-1}\tilde{b}(\tau)]\|_{r;(0,t)} \\ &\gtrsim t^{-1}\tilde{b}(t)\|\tau^{1-1/r}\|_{r;(0,t)} \approx \tilde{b}(t) \quad \text{for all} \quad t \in (0,1), \end{aligned} \tag{3.35}$$

this shows that condition (3.20) holds if

$$\overline{\lim_{t\to 0_+}} \frac{\tilde{b}(t)}{b_{q\infty}(t)} = +\infty. \tag{3.36}$$

(viii) Let $r = +\infty$ and let $\tilde{b}$ be equivalent to a non-decreasing function on some interval $(0,\delta)$, $\delta \in (0,1)$. Then, for all $t \in (0,\delta)$,

$$\|\tau^{-1/r}\tilde{b}(\tau)\|_{r;(0,t)} = \|\tilde{b}(\tau)\|_{\infty;(0,t)} \approx \tilde{b}(t).$$

Applying this estimate in (3.34), we can see that (3.20) is now equivalent to (3.36).

(ix) Let $r \in [q,+\infty)$ and let

$$\|\tau^{-1/r}\tilde{b}(\tau)\|_{r;(0,T)} < +\infty \quad \text{for some} \quad T \in (0,1). \tag{3.37}$$

Then $\|\tau^{-1/r}\tilde{b}(\tau)\|_{r;(0,t)} \to 0$ as $t \to 0_+$. Since also, by (3.25) and (3.17), $\|\tau^{-1/r}b_{qr}(\tau)\|_{r;(0,t)} \to 0$ as $t \to 0_+$, a convenient version of L'Hospital's rule implies that

$$\overline{\lim_{t\to 0_+}} \frac{\|\tau^{-1/r}\tilde{b}(\tau)\|_{r;(0,t)}}{\|\tau^{-1/r}b_{qr}(\tau)\|_{r;(0,t)}} = \overline{\lim_{t\to 0_+}} \frac{\tilde{b}(t)}{b_{qr}(t)} \tag{3.38}$$

provided that the last limit exists.

**Remark 3.8.** It follows from Remark 3.3 that conditions (3.2), (3.3) are of the same form as condition (3.20). The same concerns conditions (3.6), (3.7) and condition (3.23).

**Examples 3.9** (cf. [28, Examples 3.1]). *Let $\sigma \in (0,n)$, $p = n/\sigma$, $q \in (1,+\infty]$ and let $r \in [q,+\infty]$. Suppose that $\Omega \subset \mathbb{R}^n$ is a nonempty domain with $|\Omega|_n < +\infty$.*

**1.** *Let $\alpha, \beta \in \mathbb{R}$ and let $b \in SV(0,+\infty)$ be defined by*

$$b(t) = l_1(t)^\alpha \ l_2(t)^\beta, \quad t > 0.$$

**(i)** *If* $\alpha < 1/q'$ *and* $\beta \in \mathbb{R}$*, then*

$$H^\sigma L^{p,q}(\log L)^\alpha(\log\log L)^\beta(\mathbb{R}^n) \hookrightarrow L_{\infty,q;b_{qq}}(\Omega) \hookrightarrow L_{\infty,r;b_{qr}}(\Omega) \hookrightarrow L_{\infty,\infty;b_{q\infty}}(\Omega) \tag{3.39}$$

*with*

$$\begin{aligned} b_{qq}(t) &\approx l_1(t)^{\alpha-1}\ l_2(t)^\beta, \quad t \in (0,1]; \\ b_{qr}(t) &\approx l_1(t)^{\alpha-1/q'-1/r}\ l_2(t)^\beta, \quad t \in (0,1]; \\ b_{q\infty}(t) &\approx l_1(t)^{\alpha-1/q'}\ l_2(t)^\beta, \quad t \in (0,1]. \end{aligned}$$

*Note* (*cf.* [25, Lemma 2.2]) *that* $L_{\infty,\infty;b_{q\infty}}(\Omega)$ *coincides with the Orlicz space* $L_\Phi(\Omega)$*, where*

$$\Phi(t) = \exp(t^{1/\gamma}(\log t)^{\beta/\gamma}) \quad \text{for all large } t, \quad \gamma = 1/q' - \alpha.$$

**(ii)** *If* $\alpha = 1/q'$ *and* $\beta < 1/q'$*, then*

$$H^\sigma L^{p,q}(\log L)^\alpha(\log\log L)^\beta(\mathbb{R}^n) \hookrightarrow L_{\infty,q;b_{qq}}(\Omega) \hookrightarrow L_{\infty,r;b_{qr}}(\Omega) \hookrightarrow L_{\infty,\infty;b_{q\infty}}(\Omega) \tag{3.40}$$

*with*

$$\begin{aligned} b_{qq}(t) &\approx l_1(t)^{-1/q}\ l_2(t)^{\beta-1}, \quad t \in (0,1]; \\ b_{qr}(t) &\approx l_1(t)^{-1/r}\ l_2(t)^{\beta-1/q'-1/r}, \quad t \in (0,1]; \\ b_{q\infty}(t) &\approx l_2(t)^{\beta-1/q'}, \quad t \in (0,1]. \end{aligned}$$

*Recall that* (*see* (2.5)) $L_{\infty,\infty;b_{q\infty}}(\Omega) = \mathrm{EXP}_2\, L^{1/\gamma}(\Omega)$*, where* $\gamma = 1/q' - \beta$.
**(iii)** *If* $\alpha = 1/q'$ *and* $\beta = 1/q'$*, then*

$$H^\sigma L^{p,q}(\log L)^\alpha(\log\log L)^\beta(\mathbb{R}^n) \hookrightarrow L_{\infty,q;b_{qq}}(\Omega) \hookrightarrow L_{\infty,r;b_{qr}}(\Omega) \hookrightarrow L_{\infty,\infty;b_{q\infty}}(\Omega) \tag{3.41}$$

*with*

$$\begin{aligned} b_{qq}(t) &\approx l_1(t)^{-1/q}\ l_2(t)^{-1/q}\ l_3(t)^{-1}, \quad t \in (0,1]; \\ b_{qr}(t) &\approx l_1(t)^{-1/r}\ l_2(t)^{-1/r}\ l_3(t)^{-1/q'-1/r}, \quad t \in (0,1]; \\ b_{q\infty}(t) &\approx l_3(t)^{-1/q'}, \quad t \in (0,1]. \end{aligned}$$

*Recall that* (*see* (2.5)) $L_{\infty,\infty;b_{q\infty}}(\Omega) = \mathrm{EXP}_3\, L^{q'}(\Omega)$.
**(iv)** *If either* $\alpha > 1/q'$ *or* $\alpha = 1/q'$ *and* $\beta > 1/q'$*, then*

$$H^\sigma L^{p,q}(\log L)^\alpha(\log\log L)^\beta(\mathbb{R}^n) \hookrightarrow C_B(\mathbb{R}^n).$$

**2.** *Let* $\alpha \in (0,1)$*,* $\beta \in \mathbb{R}$ *and let* $b \in SV(0,+\infty)$ *be defined by*

$$b(t) = l_1(t)^{-(\alpha-1)/q'} \exp(\beta\, l_1(t)^\alpha), \quad t > 0.$$

**(i)** *If* $\beta < 0$*, then*

$$H^\sigma L_{p,q;b}(\mathbb{R}^n) \hookrightarrow L_{\infty,q;b_{qq}}(\Omega) \hookrightarrow L_{\infty,r;b_{qr}}(\Omega) \hookrightarrow L_{\infty,\infty;b_{q\infty}}(\Omega) \tag{3.42}$$

*with*

$$\begin{aligned} b_{qq}(t) &\approx l_1(t)^{(\alpha-1)/q} \exp(\beta\, l_1(t)^\alpha), \quad t \in (0,1]; \\ b_{qr}(t) &\approx l_1(t)^{(\alpha-1)/r} \exp(\beta\, l_1(t)^\alpha), \quad t \in (0,1]; \\ b_{q\infty}(t) &\approx \exp(\beta\, l_1(t)^\alpha), \quad t \in (0,1]. \end{aligned}$$

**(ii)** *If* $\beta > 0$*, then*

$$H^\sigma L_{p,q;b}(\mathbb{R}^n) \hookrightarrow C_B(\mathbb{R}^n).$$

## 4. Embeddings into generalized Hölder spaces: the case $\sigma \geq 1$

In this section we present embeddings of Bessel-potential-type spaces into generalized Hölder spaces , which extend and improve those of [18] and complement those of [41]. Our main results state that such embeddings are sharp and fail to be compact. We also show how to modify parameters of target spaces to arrive to compact embeddings.

Part (i) of the following theorem improves and extends [18, Theorem 3.2] and [41, Theorem 5.10] and discusses embeddings of Bessel potential spaces modelled upon Lorentz-Karamata spaces into generalized Hölder spaces in the sublimiting case. The second and the third parts concern the sharpness of the embedding from part (i) while the fourth part states that the embedding from part (i) is not compact. In part (v) it is shown that this embedding becomes compact if the parameters of the target space are restricted in a proper way.

**Theorem 4.1** (cf. [27, Theorem 3.1]). *Let $\sigma \in [1, n+1)$, $\max\{1, n/\sigma\} < p < n/(\sigma - 1)$, $q \in (1, +\infty)$, $r \in [q, +\infty]$ and let $b \in SV(0, +\infty)$. Suppose that $\Omega \subset \mathbb{R}^n$ is a nonempty domain. Let $\lambda : (0, 1] \to (0, +\infty)$ be defined by*

$$\lambda(t) = t^{\sigma - n/p}[b(t^n)]^{-1}, \quad t \in (0, 1]. \tag{4.1}$$

*(Note that $\lambda \in \mathcal{L}_r$ for any $r \in [1, +\infty]$.)*

(i) *Then*

$$H^\sigma L_{p,q;b}(\mathbb{R}^n) \hookrightarrow \Lambda^{\lambda(\cdot)}_{\infty,r}(\overline{\mathbb{R}^n}). \tag{4.2}$$

(ii) *If a function $\mu \in \mathcal{L}_r$ satisfies*

$$\overline{\lim_{t \to 0_+}} \frac{\lambda(t)}{\mu(t)} = +\infty, \tag{4.3}$$

*then the embedding*

$$H^\sigma L_{p,q;b}(\mathbb{R}^n) \hookrightarrow \Lambda^{\mu(\cdot)}_{\infty,r}(\overline{\Omega}) \tag{4.4}$$

*does not hold.*

(iii) *Let $\overline{q} \in (0, q)$. Then the embedding*

$$H^\sigma L_{p,q;b}(\mathbb{R}^n) \hookrightarrow \Lambda^{\lambda(\cdot)}_{\infty,\overline{q}}(\overline{\Omega}) \tag{4.5}$$

*fails.*

(iv) *The embedding*

$$H^\sigma L_{p,q;b}(\mathbb{R}^n) \hookrightarrow \Lambda^{\lambda(\cdot)}_{\infty,r}(\overline{\Omega})$$

*is not compact.*

(v) *Let $\Omega$ be bounded and let $\mu \in \mathcal{L}_r$ be such that*

$$\lim_{t \to 0_+} \frac{\lambda(t)}{\mu(t)} = 0. \tag{4.6}$$

*Then*

$$H^\sigma L_{p,q;b}(\mathbb{R}^n) \hookrightarrow\hookrightarrow \Lambda^{\mu(\cdot)}_{\infty,r}(\overline{\Omega}). \tag{4.7}$$

Proof. (i) Part (i) coincides with part (i) of [27, Theorem 3.1]. Thus, we refer to [27] for the proof of this statement.

(ii) Part (ii) is proved in [27] (see the proof of [27, Theorem 3.1 (ii)]) provided that condition (4.3) is replaced by

$$\lim_{t \to 0_+} \frac{\frac{t}{\lambda(t)}}{\left\| \tau^{-1/r} \frac{\tau}{\mu(\tau)} \right\|_{r;(0,t)}} = 0. \tag{4.8}$$

First, checking the proof of [27, Theorem 3.1 (ii)] (cf. lines 3–6 on page 316 in [27]), we see that this theorem continue to hold if we replace assumption (4.8) by

$$\liminf_{t\to 0_+} \frac{\frac{t}{\lambda(t)}}{\left\|\tau^{-1/r}\frac{\tau}{\mu(\tau)}\right\|_{r;(0,t)}} = 0. \tag{4.9}$$

Second, we prove that (4.9) is equivalent to (4.3) . Indeed, since (4.9) can be rewritten as

$$\overline{\lim_{t\to 0_+}} \frac{\left\|\tau^{-1/r}\frac{\tau}{\mu(\tau)}\right\|_{r;(0,t)}}{\frac{t}{\lambda(t)}} = +\infty, \tag{4.10}$$

it is sufficient to prove that (4.10) is equivalent to (4.3). Instead of this, it is enough to verify that

$$\overline{\lim_{t\to 0_+}} \frac{\lambda(t)}{\mu(t)} < +\infty \tag{4.11}$$

if and only if

$$\overline{\lim_{t\to 0_+}} \frac{\left\|\tau^{-1/r}\frac{\tau}{\mu(\tau)}\right\|_{r;(0,t)}}{\frac{t}{\lambda(t)}} < +\infty. \tag{4.12}$$

To this end, first we assume that (4.11) holds. Then there is $\delta \in (0,\, 1)$ such that $\lambda(t)/\mu(t) \lesssim 1$ for all $t \in (0,\, \delta)$, which implies that $1/\mu(t) \lesssim 1/\lambda(t)$ for all $t \in (0,\, \delta)$. Using this estimate, (4.1), the fact that $1-\sigma+n/p > 0$ and Lemma 2.1 (iii), we obtain

$$\begin{aligned} \left\|\tau^{-1/r}\frac{\tau}{\mu(\tau)}\right\|_{r;(0,t)} &\lesssim \left\|\tau^{-1/r}\frac{\tau}{\lambda(\tau)}\right\|_{r;(0,t)} \\ &= \left\|\tau^{-1/r}\tau^{1-\sigma+n/p}\,[b(\tau^n)]\right\|_{r;(0,t)} \\ &= t^{1-\sigma+n/p}\,[b(t^n)] \\ &= \frac{t}{\lambda(t)} \quad \text{for all } t \in (0,\, \delta), \end{aligned} \tag{4.13}$$

and (4.12) follows.

Suppose now that (4.12) holds. Then there is $\delta \in (0,\, 1)$ such that

$$\frac{\left\|\tau^{-1/r}\frac{\tau}{\mu(\tau)}\right\|_{r;(0,t)}}{\frac{t}{\lambda(t)}} \lesssim 1 \quad \text{for all } t \in (0,\, \delta). \tag{4.14}$$

Since the function $1/\mu$ is non-increasing on $(0,\, 1)$, we obtain

$$\left\|\tau^{-1/r}\frac{\tau}{\mu(\tau)}\right\|_{r;(0,t)} \geq \frac{1}{\mu(t)}\left\|\tau^{1-1/r}\right\|_{r;(0,t)} \approx \frac{t}{\mu(t)} \quad \text{for all } t \in (0,\, 1).$$

This and (4.14) imply that

$$1 \gtrsim \frac{\frac{t}{\mu(t)}}{\frac{t}{\lambda(t)}} = \frac{\lambda(t)}{\mu(t)} \quad \text{for all } t \in (0,\, \delta),$$

and (4.11) follows.

(iii) Part (iii) coincides with part (iii) of [27, Theorem 3.1]. Thus, we refer to [27] for the proof of this statement.

(iv) Part (iv) is the same as part (iv) of [27, Theorem 3.1]. Thus, we refer to [27] for the proof of this statement.

(v) Put $X := L_{p,q,b}(\mathbb{R}^n)$ and $\mathcal{S} := \{u \in H^\sigma X : ||u||_{H^\sigma X} \le 1\}$. By Theorem 2.4 and Remark 2.5 (i), it is enough to prove that

$$\mathcal{S} \quad \text{is bounded in} \quad \Lambda^{\mu(\cdot)}_{\infty,r}(\overline{\Omega}) \tag{4.15}$$

and

$$\sup_{u\in\mathcal{S}} \|t^{-1/r}[\mu(t)]^{-1}\omega(u,t)\|_{r;(0,\xi)} \to 0 \quad \text{as} \quad \xi \to 0_+. \tag{4.16}$$

First, by Theorem 4.1 (i), (4.2) holds. Thus, to verify (4.15), it is sufficient to show that

$$\Lambda^{\lambda(\cdot)}_{\infty,r}(\overline{\Omega}) \hookrightarrow \Lambda^{\mu(\cdot)}_{\infty,r}(\overline{\Omega}). \tag{4.17}$$

Since, for any $f \in C_B(\Omega)$, $\widetilde{\omega}(f,.)$ is equivalent to a non-increasing function, we see from (2.16) that (4.17) holds if

$$L_{\infty,r;t/\lambda(t)}(0,1) \hookrightarrow L_{\infty,r;t/\mu(t)}(0,1). \tag{4.18}$$

However, by [10, Theorem 3.1], embedding (4.18) holds when

$$\sup_{x\in(0,1)} \left\|t^{-1/r}\, t/\mu(t)\right\|_{r;(0,x)} \Big/ \left\|t^{-1/r}\, t/\lambda(t)\right\|_{r;(0,x)} < +\infty. \tag{4.19}$$

Take $\varepsilon > 0$. By (4.6), there is $\delta \in (0,1)$ such that

$$1/\mu(t) \le \varepsilon/\lambda(t) \quad \text{for all} \quad t \in (0,\, \delta). \tag{4.20}$$

Using this estimate, one can show (cf. 4.13) that

$$\left\|t^{-1/r}\, t/\mu(t)\right\|_{r;(0,x)} \lesssim \varepsilon\, x/\lambda(x) \quad \text{for all} \quad x \in (0,\, \delta). \tag{4.21}$$

Moreover, by (4.1) and Lemma 2.1 (iii),

$$\left\|t^{-1/r}\, t/\lambda(t)\right\|_{r;(0,x)} \approx x/\lambda(x) \quad \text{for all} \quad x \in (0,1). \tag{4.22}$$

Thus, (4.19) is a consequence of (4.21), (4.22) and the fact that the functions $\mu$ and $\lambda$ have singularities only at 0.

To prove (4.16), take $\varepsilon > 0$. Then, by (4.6), there is $\delta \in (0,1)$ such that (4.20) holds. If $\xi \in (0,\delta)$, then (4.20) and Theorem 4.1 (i) imply that

$$\sup_{u\in\mathcal{S}} \|t^{-1/r}[\mu(t)]^{-1}\omega(u,t)\|_{r;(0,\xi)} \le \varepsilon \sup_{u\in\mathcal{S}} \|t^{-1/r}[\lambda(t)]^{-1}\omega(u,t)\|_{r;(0,\xi)} \precsim \varepsilon \sup_{u\in\mathcal{S}} \|u\|_{H^\sigma X} \le \varepsilon$$

and (4.16) follows. □

**Corollary 4.2.** *Let all the assumptions of* Theorem 4.1 *be satisfied. Suppose that* $\mu \in \mathcal{L}_r$.

(i) *Then*

$$H^\sigma L_{p,q;b}(\mathbb{R}^n) \hookrightarrow \Lambda^{\mu(\cdot)}_{\infty,r}(\overline{\mathbb{R}^n}) \tag{4.23}$$

*if and only if*

$$\overline{\lim_{t\to 0_+}}\, \frac{\lambda(t)}{\mu(t)} < +\infty. \tag{4.24}$$

(ii) *Let* $\Omega \subset \mathbb{R}^n$ *be bounded and let* $r < +\infty$. *Then*

$$H^\sigma L_{p,q;b}(\mathbb{R}^n) \hookrightarrow\hookrightarrow \Lambda^{\mu(\cdot)}_{\infty,r}(\overline{\Omega}) \tag{4.25}$$

*if and only if*

$$\lim_{t\to 0_+} \frac{\lambda(t)}{\mu(t)} = 0. \tag{4.26}$$

Proof. Part (i) of the corollary is a consequence of parts (i) and (ii) of Theorem 4.1. Indeed, by Theorem 4.1 (i), embedding (4.2) holds. Moreover,

$$\Lambda^{\lambda(\cdot)}_{\infty,r}(\overline{\mathbb{R}^n}) \hookrightarrow \Lambda^{\mu(\cdot)}_{\infty,r}(\overline{\mathbb{R}^n})$$

(this embedding is a consequence of (4.24) – cf. the proof of (4.17)).

On the other hand, if (4.24) is not satisfied, then, by Theorem 4.1 (ii), embedding (4.23) fails.

(ii) The implication (4.26) $\Rightarrow$ (4.25) holds by Theorem 4.1 (v). The converse implication follows by contradiction. Indeed, assume that (4.25) holds and

$$\overline{\lim_{t\to 0_+}} \frac{\lambda(t)}{\mu(t)} > 0. \tag{4.27}$$

Consider the functions $u_s$, $s \in (0, 1/4)$, given by [27, (4.18)] (with $\mathcal{G}$ from [27, (5.4)]). By [27, (5.6)],

$$\|u\|_{\sigma;p,q;b} \lesssim 1 \quad \text{for all} \quad s \in (0, 1/4). \tag{4.28}$$

Take $S \in (0, 1/4)$ fixed. An analogue of [27, (5.13)] (with $\lambda$ replaced by $\mu$) reads as

$$\|(u_s - u_S)|\Lambda^{\mu(\cdot)}_{\infty,r}(\overline{\Omega})\| \gtrsim \frac{\lambda(s)}{ks} \|t^{-1/r}\, t/\mu(t)\|_{r;(0,ks)} \tag{4.29}$$

for all sufficiently small positive $s$. Since

$$\|t^{-1/r}\, t/\mu(t)\|_{r;(0,ks)} \geq \|t^{1-1/r}\|_{r;(0,ks)} /\mu(ks) \approx ks/\mu(ks) \quad \text{for all} \quad s \in (0, 1/4),$$

we obtain from (4.29) that

$$\|(u_s - u_S)|\Lambda^{\mu(\cdot)}_{\infty,r}(\overline{\Omega})\| \gtrsim \frac{\lambda(s)}{\mu(ks)} \approx \frac{\lambda(s)}{\mu(s)} \tag{4.30}$$

for all sufficiently small positive $s$. However, (4.30), (4.27) and (4.28) contradict (4.25). □

**Remark 4.3** (cf. [30, Remark 7.5])**.** (i) In Corollary 4.2 (ii) the implication (4.26) $\Longrightarrow$ (4.25) remains true even if $r = +\infty$. This follows from Theorem 4.1 (v).

(ii) We see from Remark 2.5 (iii) that if we assume additionally in Corollary 4.2 (ii) that $r = +\infty$ and the space $X(\mathbb{R}^n) := L_{p,q;b}(\mathbb{R}^n)$ and $\mu \in \mathcal{L}$ are such that

$$H^\sigma X(\mathbb{R}^n) \hookrightarrow \Lambda^{\mu(\cdot),0}_{\infty,\infty}(\overline{\Omega}), \tag{4.31}$$

then (4.25) is equivalent to (4.26).

(iii) For example, (4.31) (with any r.i. BFS $X(\mathbb{R}^n)$) is satisfied provided that

(4.32) the Schwartz space $\mathscr{S}(\mathbb{R}^n)$ is dense in $H^\sigma X(\mathbb{R}^n)$,

$$H^\sigma X(\mathbb{R}^n) \hookrightarrow \Lambda^{\mu(\cdot)}_{\infty,\infty}(\overline{\Omega}), \tag{4.33}$$

$$\lim_{t\to 0_+} t/\mu(t) = 0. \tag{4.34}$$

Indeed, given $u \in H^\sigma X(\mathbb{R}^n)$ and $\varepsilon > 0$, there is $v \in \mathscr{S}(\mathbb{R}^n)$ such that $\|u - v\|_{H^\sigma X} < \varepsilon$. Moreover, $\omega(v,t) \le ct$ for all $t \in (0,1)$, where $c = c(v)$ is a positive constant. Thus, using also (4.33), we obtain

$$\begin{aligned} \|[\mu(t)]^{-1}\omega(u,t)\|_{\infty;(0,\delta)} &\le \|[\mu(t)]^{-1}\omega(u-v,t)\|_{\infty;(0,\delta)} + \|[\mu(t)]^{-1}\omega(v,t)\|_{\infty;(0,\delta)} \\ &\precsim \|u-v\|_{H^\sigma X} + c\|t/\mu(t)\|_{\infty;(0,\delta)} \\ &\le \varepsilon + c\|t/\mu(t)\|_{\infty;(0,\delta)} \quad \text{for all} \quad \delta \in (0,1). \end{aligned}$$

Together with (4.34), this implies (4.31).

For instance, (4.32) holds if

$$\text{the Schwartz space } \mathscr{S}(\mathbb{R}^n) \text{ is dense in } X(\mathbb{R}^n). \tag{4.35}$$

Indeed, this is a consequence of (2.12), (2.13), the fact that the mapping $h \mapsto g_\sigma * h$ maps $\mathscr{S}(\mathbb{R}^n)$ on $\mathscr{S}(\mathbb{R}^n)$, and (4.35).

In particular, (4.35) is satisfied provided that the r. i. BFS $X(\mathbb{R}^n)$ has absolutely continuous norm (cf. [17, Remark 3.13]).

**Remark 4.4** (cf. [27, Remark 3.1 (i)–(iv)])**.** (i) If $r = +\infty$, then (4.2) yields

$$H^\sigma L_{p,q;b}(\mathbb{R}^n) \hookrightarrow C^{\lambda(\cdot)}(\overline{\mathbb{R}^n}).$$

(ii) By Theorem 4.1, all embeddings (4.2) are sharp with respect to the parameter $\lambda$ of the target space. (This is why we consider all embeddings (4.2) and not only optimal embedding (4.37) mentioned below.)

(iii) As

$$\Lambda^{\lambda(\cdot)}_{\infty,r}(\overline{\mathbb{R}^n}) \hookrightarrow \Lambda^{\lambda(\cdot)}_{\infty,s}(\overline{\mathbb{R}^n}) \quad \text{if} \quad 0 < r < s \le +\infty, \tag{4.36}$$

among embeddings (4.2) the embedding

$$H^\sigma L_{p,q;b}(\mathbb{R}^n) \hookrightarrow \Lambda^{\lambda(\cdot)}_{\infty,q}(\overline{\mathbb{R}^n}) \tag{4.37}$$

is optimal. (Note that embedding (4.36) can be proved analogously as ([26, (3.6)]) if one replaces the role of $f^*(t)$ by the role of $\widetilde{\omega}(f,t)$. Another way how to verify (4.36) is to apply, e.g., [10, Theorem 3.1] and the fact that, for any $f \in C_B(\overline{\mathbb{R}^n})$, $\widetilde{\omega}(f,.)$ is equivalent to a non-increasing function.)

(iv) By Theorem 4.1 (i), embedding (4.37) is continuous and, by Theorem 4.1 (iv), this embedding is not compact. Moreover, Theorem 4.1 (iv) also shows that we cannot arrive to a compact embedding if we replace the target space $\Lambda^{\lambda(\cdot)}_{\infty,q}(\overline{\mathbb{R}^n})$ in (4.37) by a larger space $\Lambda^{\lambda(\cdot)}_{\infty,r}(\overline{\mathbb{R}^n})$ with $r > q$ or even by $\Lambda^{\lambda(\cdot)}_{\infty,r}(\overline{\Omega})$, where $\Omega$ is a bounded domain in $\mathbb{R}^n$.

(v) Put $X := H^\sigma L_{p,q;b}(\mathbb{R}^n)$ and $r = +\infty$. By Theorem 4.1 (i),

$$\sup_{t\in(0,1)} \frac{\widetilde{\omega}(f,t)}{\lambda(t)/t} \precsim \|f\|_X \quad \text{for all} \quad f \in X,$$

and, by Theorem 4.1 (i) and (ii), the inequality

$$\sup_{t\in(0,1)} \frac{\widetilde{\omega}(f,t)}{\mu(t)/t} \precsim \|f\|_X$$

does not hold for all $f \in X$ if $\mu \in \mathcal{L}$ satisfies (4.3). If we use an analogue of terminology from [32] and [48], this means that the function $\frac{\lambda(t)}{t} = t^{\sigma-n/p-1}[b(t^n)]^{-1}$, $t \in (0,1]$, is the *continuous envelope function* of the space $H^\sigma L_{p,q;b}(\mathbb{R}^n)$. Using also Theorem 4.1 (iii), we can see that the couple

$$(t^{\sigma-n/p-1}[b(t^n)]^{-1}, q)$$

is the *continuous envelope* of the space $H^\sigma L_{p,q;b}(\mathbb{R}^n)$.

The next result is an analogue of Theorem 4.1 and concerns the limiting case when $p = n/(\sigma-1)$. Part (i) of this theorem is an extension of [18, Theorem 3.3].

**Theorem 4.5** (cf. [27, Theorem 3.2]). *Let $\sigma \in (1, n+1)$, $q \in (1,+\infty)$, $r \in [q,+\infty]$ and let $b \in SV(0,+\infty)$ be such that*

$$\| t^{-1/q'}[b(t)]^{-1}\|_{q';(0,1)} = +\infty. \tag{4.38}$$

*Suppose that $\Omega \subset \mathbb{R}^n$ is a nonempty domain and that $\lambda_{qr} \in \mathcal{L}_r$ is defined by*

$$\lambda_{qr}(t) = t\,[b(t^n)]^{q'/r} \left(\int_{t^n}^2 \tau^{-1}[b(\tau)]^{-q'}\,d\tau\right)^{1/q'+1/r}, \quad t \in (0,1]. \tag{4.39}$$

(i) *Then*

$$H^\sigma L_{n/(\sigma-1),q;b}(\mathbb{R}^n) \hookrightarrow \Lambda^{\lambda_{qr}(\cdot)}_{\infty,r}(\overline{\mathbb{R}^n}). \tag{4.40}$$

(ii) *If a function $\mu \in \mathcal{L}_r$ satisfies*

$$\overline{\lim_{t\to 0_+}} \frac{\left\|\tau^{-1/r}\frac{\tau}{\mu(\tau)}\right\|_{r;(0,t)}}{\left\|\tau^{-1/r}\frac{\tau}{\lambda_{qr}(\tau)}\right\|_{r;(0,t)}} = +\infty, \tag{4.41}$$

*then the embedding*

$$H^\sigma L_{n/(\sigma-1),q;b}(\mathbb{R}^n) \hookrightarrow \Lambda^{\mu(\cdot)}_{\infty,r}(\overline{\Omega}) \tag{4.42}$$

*does not hold.*

(iii) *Let $\overline{q} \in (0,q)$. Then the embedding*

$$H^\sigma L_{n/(\sigma-1),q;b}(\mathbb{R}^n) \hookrightarrow \Lambda^{\lambda_{q\overline{q}}(\cdot)}_{\infty,\overline{q}}(\overline{\Omega}) \tag{4.43}$$

*fails, where $\lambda_{q\overline{q}}$ is again defined by (4.39) with $r$ replaced by $\overline{q}$.*

(iv) *The embedding*

$$H^\sigma L_{n/(\sigma-1),q;b}(\mathbb{R}^n) \hookrightarrow \Lambda^{\lambda_{qr}(\cdot)}_{\infty,r}(\overline{\Omega})$$

*is not compact.*

(v) *Let $\Omega$ be bounded and let $\mu \in \mathcal{L}_r$ be such that*

$$\lim_{t\to 0_+} \frac{\left\|\tau^{-1/r}\frac{\tau}{\mu(\tau)}\right\|_{r;(0,t)}}{\left\|\tau^{-1/r}\frac{\tau}{\lambda_{qr}(\tau)}\right\|_{r;(0,t)}} = 0. \tag{4.44}$$

*Then*

$$H^\sigma L_{p,q;b}(\mathbb{R}^n) \hookrightarrow\hookrightarrow \Lambda^{\mu(\cdot)}_{\infty,r}(\overline{\Omega}). \tag{4.45}$$

Proof. (i) Part (i) coincides with part (i) of [27, Theorem 3.2]. Thus, we refer to [27] for the proof of this statement.

(ii) Part (ii) is proved in [27] (see the proof of [27, Theorem 3.2 (ii)]) provided that condition (4.41) is replaced by

$$\lim_{t\to 0_+} \frac{\left\|\tau^{-1/r}\frac{\tau}{\lambda_{qr}(\tau)}\right\|_{r;(0,t)}}{\left\|\tau^{-1/r}\frac{\tau}{\mu(\tau)}\right\|_{r;(0,t)}} = 0. \tag{4.46}$$

Checking the proof of [27, Theorem 3.2 (ii)] (cf. estimates [27, (5.29)] and [27, (5.30)], we see that this theorem continue to hold if we replace assumption (4.46) by

$$\liminf_{t\to 0_+} \frac{\left\|\tau^{-1/r}\frac{\tau}{\lambda_{qr}(\tau)}\right\|_{r;(0,t)}}{\left\|\tau^{-1/r}\frac{\tau}{\mu(\tau)}\right\|_{r;(0,t)}} = 0.$$

However, the last condition is equivalent to (4.41).

(iii) Part (iii) coincides with part (iii) of [27, Theorem 3.2]. Thus, we refer to [27] for the proof of this statement.

(iv) Part (iv) is the same as part (iv) of [27, Theorem 3.2]. Thus, we refer to [27] for the proof of this assertion.

(v) By Theorem 2.4 and Remark 2.5 (i), it is enough to prove (4.15) and (4.16).

First, by Theorem 4.5 (i), embedding (4.40) holds. Thus, to verify (4.15), it is sufficient to prove (4.17) with $\lambda$ replaced by $\lambda_{qr}$. As in the proof of Theorem 4.1 (v), (4.17) with $\lambda$ replaced by $\lambda_{qr}$ is a consequence of (4.19) with $\lambda$ replaced by $\lambda_{qr}$. However, this variant of (4.19) follows from (4.44) and the fact that the functions $\mu$ and $\lambda_{qr}$ have singularities only at 0.

To prove (4.16), take $\varepsilon > 0$. Then, by (4.44), there is $\delta \in (0,1)$ such that

$$\|\tau^{-1/r}\tau/\mu(\tau)\|_{r;(0,t)} \le \varepsilon\|\tau^{-1/r}\tau/\lambda_{qr}(\tau)\|_{r;(0,t)} \quad \text{for all} \quad t \in (0,\, \delta). \tag{4.47}$$

Together with [10, Theorem 3.1] and the fact that, for any $f \in C_B(\Omega)$, $\widetilde{\omega}(f,.)$ is equivalent to a non-increasing function, this implies that

$$\sup_{u\in\mathcal{S}} \|\tau^{-1/r}[\mu(\tau)]^{-1}\omega(u,\tau)\|_{r;(0,\delta)} \precsim \varepsilon \sup_{u\in\mathcal{S}} \|\tau^{-1/r}[\lambda_{qr}(\tau)]^{-1}\omega(u,\tau)\|_{r;(0,\delta)}. \tag{4.48}$$

Let $\xi \in (0,\delta)$. Then (4.48) and Theorem 4.5 (i) imply that

$$\sup_{u\in\mathcal{S}} \|t^{-1/r}[\mu(t)]^{-1}\omega(u,t)\|_{r;(0,\xi)} \precsim \varepsilon \sup_{u\in\mathcal{S}} \|t^{-1/r}[\lambda_{qr}(t)]^{-1}\omega(u,t)\|_{r;(0,\delta)} \precsim \varepsilon \sup_{u\in\mathcal{S}} \|u\|_{H^\sigma X} \le \varepsilon$$

and (4.16) follows. □

**Corollary 4.6.** *Let all the assumptions of* Theorem 4.5 *be satisfied. Suppose that* $\mu \in \mathcal{L}_r$.

(i) *Then*

$$H^\sigma L_{p,q;b}(\mathbb{R}^n) \hookrightarrow \Lambda^{\mu(.)}_{\infty,r}(\overline{\mathbb{R}^n}) \tag{4.49}$$

*if and only if*

$$\overline{\lim_{t\to 0_+}} \frac{\left\|\tau^{-1/r}\frac{\tau}{\mu(\tau)}\right\|_{r;(0,t)}}{\left\|\tau^{-1/r}\frac{\tau}{\lambda_{qr}(\tau)}\right\|_{r;(0,t)}} < +\infty. \tag{4.50}$$

(ii) *Let* $\Omega \subset \mathbb{R}^n$ *be bounded and let* $r < +\infty$. *Then*

$$H^\sigma L_{p,q;b}(\mathbb{R}^n) \hookrightarrow\hookrightarrow \Lambda^{\mu(.)}_{\infty,r}(\overline{\Omega}) \tag{4.51}$$

*if and only if*

$$\lim_{t\to 0_+} \frac{\left\|\tau^{-1/r}\frac{\tau}{\mu(\tau)}\right\|_{r;(0,t)}}{\left\|\tau^{-1/r}\frac{\tau}{\lambda_{qr}(\tau)}\right\|_{r;(0,t)}} = 0. \tag{4.52}$$

Proof. Part (i) of the corollary is a consequence of parts (i) and (ii) of Theorem 4.5. Indeed, by Theorem 4.5 (i), embedding (4.40) holds. Moreover,

$$\Lambda^{\lambda_{qr}(\cdot)}_{\infty,r}(\overline{\mathbb{R}^n}) \hookrightarrow \Lambda^{\mu(\cdot)}_{\infty,r}(\overline{\mathbb{R}^n})$$

(this embedding is a consequence of (4.50) – cf. the proof of (4.17)).

On the other hand, if (4.50) is not satisfied, then, by Theorem 4.5 (ii), embedding (4.49) fails.

(ii) The implication (4.52) $\Rightarrow$ (4.51) holds by Theorem 4.5 (v). The converse implication follows by contradiction. Indeed, assume that (4.51) holds and

$$\overline{\lim_{t\to 0_+}} \frac{\left\| \tau^{-1/r} \frac{\tau}{\mu(\tau)} \right\|_{r;(0,t)}}{\left\| \tau^{-1/r} \frac{\tau}{\lambda_{qr}(\tau)} \right\|_{r;(0,t)}} > 0. \tag{4.53}$$

Consider the functions $u_s$, $s \in (0,1/4)$, given by [27, (4.18)] (with $\mathcal{G}$ from [27, (5.21)]). By [27, (5.26)],

$$\|u\|_{\sigma;p,q;b} \lesssim 1 \quad \text{for all} \quad s \in (0,1/4). \tag{4.54}$$

Take $S \in (0,1/4)$ fixed. An analogue of [27, (5.32)] (with $\lambda_r$ replaced by $\mu$) reads as

$$\|(u_s \,-\, u_S)|\Lambda^{\mu(\cdot)}_{\infty,r}(\overline{\Omega})\| \gtrsim B(s)\, \|t^{-1/r}\, t/\mu(t)\|_{r;(0,s\varepsilon)}, \tag{4.55}$$

for all sufficiently small positive $s$, where $B(s) := \left(\int_{s^n}^2 \tau^{-1}[b(\tau)]^{-q'}d\tau\right)^{1/q'}$. Since $B(s) \approx B(\varepsilon s)$ and $B(s) \approx \|t^{-1/r}\, t/\lambda_{q,r}(t)\|^{-1}_{r;(0,s)}$ for all $s \in (0,1/4)$ (cf. [27, (5.30)]), (4.55) implies that

$$\|(u_s \,-\, u_S)|\Lambda^{\mu(\cdot)}_{\infty,r}(\overline{\Omega})\| \gtrsim \frac{\left\| \tau^{-1/r} \frac{\tau}{\mu(\tau)} \right\|_{r;(0,s\varepsilon)}}{\left\| \tau^{-1/r} \frac{\tau}{\lambda_{qr}(\tau)} \right\|_{r;(0,s\varepsilon)}} \tag{4.56}$$

for all sufficiently small positive $s$. However, (4.56), (4.53) and (4.54) contradict (4.51). $\square$

**Remark 4.7.** (i) In Corollary 4.6 (ii) the implication (4.52) $\Longrightarrow$ (4.51) remains true even if $r = +\infty$. This follows from Theorem 4.5 (v).

(ii) We see from Remark 2.5 (iii) that if we assume additionally in Corollary 4.6 (ii) that $r = +\infty$ and the space $X(\mathbb{R}^n) := L_{p,q;b}(\mathbb{R}^n)$ and $\mu \in \mathcal{L}$ are such that (4.31) holds, then (4.51) is equivalent to (4.52).

(iii) Concerning (4.31), we refer to Remark 4.3 (iii).

**Remark 4.8** (cf. [27, Remark 3.2 (i)–(vii)])**.** (i) Part (i) of Theorem 4.5 holds without assumption (4.38). However, if $\|\, t^{-1/q'}[b(t)]^{-1}\|_{q';(0,1)} < +\infty$, then

$$H^\sigma L_{n/(\sigma-1),q;b}(\mathbb{R}^n) \hookrightarrow Lip(\mathbb{R}^n),$$

c.f. [41, Theorem 5.12].

(ii) Assume that all the assumptions of Theorem 4.5 are satisfied. If $r \in [q,+\infty]$, then the embedding

$$H^\sigma L_{p,q;b}(\mathbb{R}^n) \hookrightarrow \Lambda^{\mu(\cdot)}_{\infty,r}(\overline{\mathbb{R}^n}) \tag{4.57}$$

with $\mu = \lambda_{qr}$ is *sharp* with respect to the parameter $\mu \in \mathcal{L}_r$, that is, the target space $\Lambda^{\mu(\cdot)}_{\infty,r}(\overline{\mathbb{R}^n})$ in (4.57) and the target space $\Lambda^{\lambda_{qr}(\cdot)}_{\infty,r}(\overline{\mathbb{R}^n})$ (i.e. the target space in (4.57) with $\mu = \lambda_{qr}$) satisfy

$$\Lambda^{\lambda_{qr}(\cdot)}_{\infty,r}(\overline{\mathbb{R}^n}) \hookrightarrow \Lambda^{\mu(\cdot)}_{\infty,r}(\overline{\mathbb{R}^n}). \tag{4.58}$$

Indeed, as in the proof of Theorem 4.1 (v), embedding (4.58) holds when (4.18) with $\lambda$ replaced by $\lambda_{qr}$ is satisfied. However, by [10, Theorem 3.1], this variant of (4.18) is a consequence of (4.19) with $\lambda$ replaced by $\lambda_{qr}$. Finally, (4.57) and Corollary 4.6 (i) imply (4.50), which, in turn, yields the mentioned variant of (4.19).

(iii) The target spaces in (4.40) form a scale $\{\Lambda^{\lambda_{qr}(.)}_{\infty,r}(\overline{\mathbb{R}^n})\}_{r=q}^{+\infty}$ whose endpoint spaces with $r = +\infty$ and $r = q$ are of particular interest. The former endpoint space $\Lambda^{\lambda_{q\infty}(.)}_{\infty,\infty}(\overline{\mathbb{R}^n})$ corresponds to the target space in the Brézis-Wainger-type embedding while the latter endpoint space $\Lambda^{\lambda_{qq}(.)}_{\infty,q}(\overline{\mathbb{R}^n})$ corresponds to the target space in the Triebel-type embedding. Since the spaces $\{\Lambda^{\lambda_{qr}(.)}_{\infty,r}(\overline{\mathbb{R}^n})\}_{r=q}^{+\infty}$ satisfy

$$\Lambda^{\lambda_{qr}(.)}_{\infty,r}(\overline{\mathbb{R}^n}) \hookrightarrow \Lambda^{\lambda_{qs}(.)}_{\infty,s}(\overline{\mathbb{R}^n}) \quad \text{if} \quad q \le r \le s \le +\infty, \tag{4.59}$$

the embedding (4.40) with $r = q$, that is,

$$H^\sigma L_{n/(\sigma-1),q;b}(\mathbb{R}^n) \hookrightarrow \Lambda^{\lambda_{qq}(.)}_{\infty,q}(\overline{\mathbb{R}^n}). \tag{4.60}$$

is optimal. (The proof of (4.59) is analogous to the proof of ([26, (3.14)]) if one replaces the role of $f^*(t)$ by the role of $\widetilde{\omega}(f,t)$.)

(iv) By Theorem 4.5 (i), embedding (4.60) is continuous and, by Theorem 4.5 (iv), this embedding is not compact. Moreover, Theorem 4.5 (iv) also shows that we cannot arrive to a compact embedding if we replace the target space $\Lambda^{\lambda_{qq}(.)}_{\infty,q}(\overline{\mathbb{R}^n})$ in (4.60) by a larger space $\Lambda^{\lambda_{qr}(.)}_{\infty,r}(\overline{\Omega})$ with $r > q$.

(v) Put $X = H^\sigma L_{n/(\sigma-1),q;b}(\mathbb{R}^n)$ and $r = +\infty$. By Theorem 4.5 (i),

$$\sup_{t\in(0,1)} \frac{\widetilde{\omega}(f,t)}{\lambda_{q\infty}(t)/t} \precsim \|f\|_X \quad \text{for all} \quad f \in X,$$

and, by Theorem 4.5 (i) and (ii) (cf. also part (vi) of this remark), the inequality

$$\sup_{t\in(0,1)} \frac{\widetilde{\omega}(f,t)}{\mu(t)/t} \precsim \|f\|_X$$

does not hold for all $f \in X$ if $\mu \in \mathcal{L}$ satisfies

$$\overline{\lim_{t\to 0_+}} \frac{\lambda_{q\infty}(t)}{\mu(t)} = +\infty.$$

If we use an analogue of terminology from [32] and [48], this means that the function $\frac{\lambda_{q\infty}(t)}{t} = \left(\int_{t^n}^2 \tau^{-1}[b(\tau)]^{-q'} d\tau\right)^{1/q'}$, $t \in (0,1]$, is the *continuous envelope function* of the space $H^\sigma L_{n/(\sigma-1),q;b}(\mathbb{R}^n)$. Using also Theorem 4.5 (iii), we can see that the couple

$$\left(\left(\int_{t^n}^2 \tau^{-1}[b(\tau)]^{-q'} d\tau\right)^{1/q'}, q\right)$$

is the *continuous envelope* of the space $H^\sigma L_{n/(\sigma-1),q;b}(\mathbb{R}^n)$.

(vi) Let $r \in [q,+\infty]$. Using (4.39) and (4.38), we arrive at

$$\left\| \tau^{-1/r} \frac{\tau}{\lambda_{qr}(\tau)} \right\|_{r;(0,t)} \approx \frac{t}{\lambda_{q\infty}(t)} \quad \text{for all} \quad t \in (0,1). \tag{4.61}$$

This implies that

$$\varlimsup_{t\to 0_+} \frac{\left\|\tau^{-1/r}\frac{\tau}{\mu(\tau)}\right\|_{r;(0,t)}}{\left\|\tau^{-1/r}\frac{\tau}{\lambda_{qr}(\tau)}\right\|_{r;(0,t)}} \approx \varlimsup_{t\to 0_+} \frac{\left\|\tau^{-1/r}\frac{\tau}{\mu(\tau)}\right\|_{r;(0,t)}}{\frac{t}{\lambda_{q\infty}(t)}}. \tag{4.62}$$

Together with estimate

$$\|\tau^{-1/r}\tau/\mu(\tau)\|_{r;(0,t)} \geq \|\tau^{1-1/r}\|_{r;(0,t)}/\mu(t) \approx t/\mu(t) \quad \text{for all} \quad t \in (0,\,\delta),$$

this shows that condition (4.41) is satisfied if

$$\varlimsup_{t\to 0_+} \frac{\lambda_{q\infty}(t)}{\mu(t)} = +\infty. \tag{4.63}$$

(vii) Let $r = +\infty$ and let the function $t \mapsto t/\mu(t)$ be equivalent to a non-decreasing function on some interval $(0,\delta) \subset (0,1)$. Then

$$\|\tau^{-1/r}\tau/\mu(\tau)\|_{r;(0,t)} = \|\tau/\mu(\tau)\|_{\infty;(0,t)} \approx t/\mu(t) \quad \text{for all} \quad t \in (0,\,\delta).$$

Applying this estimate in (4.62), we can see that (4.41) is now equivalent to (4.63).

(viii) Let $r \in [q,+\infty)$. Since any function $\rho \in \mathcal{L}_r$ satisfies $\left\|\tau^{-1/r}\frac{\tau}{\rho(\tau)}\right\|_{r;(0,\delta)} < +\infty$ (cf. (2.15)), we have $\left\|\tau^{-1/r}\frac{\tau}{\rho(\tau)}\right\|_{r;(0,t)} \to 0$ as $t \to 0_+$. In particular, this holds with $\rho = \mu$ and $\rho = \lambda_{qr}$. Thus, L'Hospital's rule gives

$$\varlimsup_{t\to 0_+} \frac{\left\|\tau^{-1/r}\frac{\tau}{\mu(\tau)}\right\|_{r;(0,t)}}{\left\|\tau^{-1/r}\frac{\tau}{\lambda_{qr}(\tau)}\right\|_{r;(0,t)}} = \varlimsup_{t\to 0_+} \frac{\lambda_{qr}(t)}{\mu(t)} \tag{4.64}$$

provided that the last limit exists.

**Examples 4.9** (cf. [28, Examples 3.2]). *Let $\sigma \in (1, n+1)$, $p = n/(\sigma-1)$, $q \in (1,+\infty]$ and let $r \in [q,+\infty]$.*

**1.** *Let $\alpha, \beta \in \mathbb{R}$ and let $b \in SV(0,+\infty)$ be defined by*

$$b(t) = l_1(t)^\alpha \; l_2(t)^\beta, \quad t > 0.$$

**(i)** *If $\alpha < 1/q'$ and $\beta \in \mathbb{R}$, then*

$$H^\sigma L^{p,q}(\log L)^\alpha(\log\log L)^\beta(\mathbb{R}^n) \hookrightarrow \Lambda^{\lambda_{qq}(\cdot)}_{\infty,q}(\overline{\mathbb{R}^n}) \hookrightarrow \Lambda^{\lambda_{qr}(\cdot)}_{\infty,r}(\overline{\mathbb{R}^n}) \hookrightarrow C^{0,\lambda_{q\infty}(\cdot)}(\overline{\mathbb{R}^n}), \tag{4.65}$$

*with*

$$\begin{aligned}
\lambda_{qq}(t) &\approx t\; l_1(t)^{1-\alpha}\; l_2(t)^{-\beta}, \quad t \in (0,1];\\
\lambda_{qr}(t) &\approx t\; l_1(t)^{1/r+1/q'-\alpha}\; l_2(t)^{-\beta}, \quad t \in (0,1];\\
\lambda_{q\infty}(t) &\approx t\; l_1(t)^{1/q'-\alpha}\; l_2(t)^{-\beta}, \quad t \in (0,1].
\end{aligned}$$

**(ii)** *If $\alpha = 1/q'$ and $\beta < 1/q'$, then*

$$H^\sigma L^{p,q}(\log L)^\alpha(\log\log L)^\beta(\mathbb{R}^n) \hookrightarrow \Lambda^{\lambda_{qq}(\cdot)}_{\infty,q}(\overline{\mathbb{R}^n}) \hookrightarrow \Lambda^{\lambda_{qr}(\cdot)}_{\infty,r}(\overline{\mathbb{R}^n}) \hookrightarrow C^{0,\lambda_{q\infty}(\cdot)}(\overline{\mathbb{R}^n}),$$

*with*

$$\begin{aligned}
\lambda_{qq}(t) &\approx t\; l_1(t)^{1/q}\; l_2(t)^{1-\beta}, \quad t \in (0,1];\\
\lambda_{qr}(t) &\approx t\; l_1(t)^{1/r}\; l_2(t)^{1/r+1/q'-\beta}, \quad t \in (0,1];\\
\lambda_{q\infty}(t) &\approx t\; l_2(t)^{1/q'-\beta}, \quad t \in (0,1].
\end{aligned}$$

**(iii)** *If $\alpha = 1/q'$ and $\beta = 1/q'$, then*

$$H^\sigma L^{p,q}(\log L)^\alpha (\log\log L)^\beta(\mathbb{R}^n) \hookrightarrow \Lambda^{\lambda_{qq}(\cdot)}_{\infty,q}(\overline{\mathbb{R}^n}) \hookrightarrow \Lambda^{\lambda_{qr}(\cdot)}_{\infty,r}(\overline{\mathbb{R}^n}) \hookrightarrow C^{0,\lambda_{q\infty}(\cdot)}(\overline{\mathbb{R}^n}),$$

*with*

$$\begin{aligned} \lambda_{qq}(t) &\approx t\ l_1(t)^{1/q}\ l_2(t)^{1/q}\ l_3(t), \quad t \in (0,1]; \\ \lambda_{qr}(t) &\approx t\ l_1(t)^{1/r}\ l_2(t)^{1/r}\ l_3(t)^{1/r+1/q'}, \quad t \in (0,1]; \\ \lambda_{q\infty}(t) &\approx t\ l_3(t)^{1/q'}, \quad t \in (0,1]. \end{aligned}$$

**(iv)** *If either $\alpha > 1/q'$ or $\alpha = 1/q'$ and $\beta > 1/q'$, then*

$$H^\sigma L^{p,q}(\log L)^\alpha (\log\log L)^\beta(\mathbb{R}^n) \hookrightarrow Lip(\mathbb{R}^n).$$

**2.** *Let $\alpha \in (0,1)$, $\beta \in \mathbb{R}$ and let $b \in SV(0,+\infty)$ be defined by*

$$b(t) = l_1(t)^{-(\alpha-1)/q'} \exp(\beta\, l_1(t)^\alpha), \quad t > 0.$$

**(i)** *If $\beta < 0$, then*

$$H^\sigma L_{p,q;b}(\mathbb{R}^n) \hookrightarrow \Lambda^{\lambda_{qq}(\cdot)}_{\infty,q}(\overline{\mathbb{R}^n}) \hookrightarrow \Lambda^{\lambda_{qr}(\cdot)}_{\infty,r}(\overline{\mathbb{R}^n}) \hookrightarrow C^{0,\lambda_{q\infty}(\cdot)}(\overline{\mathbb{R}^n})$$

*with*

$$\begin{aligned} \lambda_{qq}(t) &\approx t\ l_1(t^n)^{-(\alpha-1)/q} \exp(-\beta\, l_1(t^n)^\alpha), \quad t \in (0,1]; \\ \lambda_{qr}(t) &\approx t\ l_1(t^n)^{-(\alpha-1)/r} \exp(-\beta\, l_1(t^n)^\alpha), \quad t \in (0,1]; \\ \lambda_{q\infty}(t) &\approx t\ \exp(-\beta\, l_1(t^n)^\alpha), \quad t \in (0,1]. \end{aligned}$$

**(ii)** *If $\beta > 0$, then*

$$H^\sigma L_{p,q;b}(\mathbb{R}^n) \hookrightarrow Lip(\mathbb{R}^n).$$

## 5. Embeddings into generalized Hölder spaces: the case $\sigma \in (0,1)$

In this section we present main results from [30]. First, we characterize continuous embeddings of the space $H^\sigma X(\mathbb{R}^n)$, with $\sigma \in (0,1)$ and a r. i. BFS $X = X(\mathbb{R}^n) = X(\mathbb{R}^n, \mu_n)$, into the generalized Hölder space $\Lambda^{\mu(\cdot)}_{\infty,r}(\overline{\mathbb{R}^n})$ and then we apply this result to the case when the space $X(\mathbb{R}^n)$ is the Lorentz-Karamata space $L_{p,q;b}(\mathbb{R}^n)$. Second, we describe compact embeddings of the space $H^\sigma X(\mathbb{R}^n)$, with $\sigma \in (0,1)$ and a r. i. BFS $X = X(\mathbb{R}^n) = X(\mathbb{R}^n, \mu_n)$, into the generalized Hölder space $\Lambda^{\mu(\cdot)}_{\infty,r}(\overline{\mathbb{R}^n})$ and then we make use of this result in the case when the space $X(\mathbb{R}^n)$ is the Lorentz-Karamata space $L_{p,q;b}(\mathbb{R}^n)$.

Assuming that $g_\sigma \in X'$ in the next theorem, we reduce the continuous embedding to inequality (5.2) below, which involves the Hardy-type operator (1.6).

**Theorem 5.1** ([30, Theorem 5.1]). *Let $\sigma \in (0,1)$ and let $X = X(\mathbb{R}^n) = X(\mathbb{R}^n, \mu_n)$ be a r. i. BFS such that $\|g_\sigma\|_{X'} < +\infty$. Assume that $r \in (0,+\infty]$ and $\mu \in \mathcal{L}_r$. Then*

$$H^\sigma X(\mathbb{R}^n) \hookrightarrow \Lambda^{\mu(\cdot)}_{\infty,r}(\overline{\mathbb{R}^n}) \tag{5.1}$$

*if and only if*

$$\left\| t^{-\frac{1}{r}} [\mu(t)]^{-1} \int_0^{t^n} \tau^{\frac{\sigma}{n}-1} f^*(\tau)\ d\tau \right\|_{r;(0,1)} \precsim \|f\|_X \quad \text{for all} \quad f \in X. \tag{5.2}$$

Using Theorem 5.1 and Lemma 2.3, we arrive at the following corollary.

**Corollary 5.2** ([30, Corollary 5.2]). *Let $\sigma \in (0,1)$ and let $X = X(\mathbb{R}^n)$ be a r. i. BFS. Assume that $r \in (0,+\infty]$ and $\mu \in \mathcal{L}_r$. Then* (5.1) *holds if and only if* $\|g_\sigma\|_{X'} < +\infty$ *and* (5.2) *is satisfied.*

Continuous embeddings of spaces $H^\sigma L_{p,q;b}(\mathbb{R}^n)$ with $\sigma \in (0,1)$ into generalized Hölder spaces in the superlimiting case (that is, when $p > n/\sigma$) are characterized in the next theorem. Note that part (i) of this theorem complements Corollary 4.2 (i).

**Theorem 5.3** ([30, Theorem 6.3])**.** *Let $\sigma \in (0,1)$, $p \in (\frac{n}{\sigma}, +\infty)$, $q \in [1,+\infty]$, $b \in SV(0,+\infty)$, $r \in (0,+\infty]$ and $\mu \in \mathcal{L}_r$. Let $\lambda : (0,1] \to (0,+\infty)$ be defined by*

$$\lambda(x) := x^{\sigma - \frac{n}{p}}[b(x^n)]^{-1} \quad \textit{for all} \quad x \in (0,1]. \tag{5.3}$$

*(Note that $\lambda \in \mathcal{L}_r$ for any $r \in (0,+\infty]$.)*

(i) *If $1 \le q \le r \le +\infty$, then*

$$H^\sigma L_{p,q;b}(\mathbb{R}^n) \hookrightarrow \Lambda^{\mu(\cdot)}_{\infty,r}(\overline{\mathbb{R}^n})$$

*if and only if*

$$\overline{\lim_{x\to 0_+}} \frac{\lambda(x)}{\mu(x)} < +\infty. \tag{5.4}$$

(ii) *If $0 < r < q \le +\infty$ and $q > 1$, then*

$$H^\sigma L_{p,q;b}(\mathbb{R}^n) \hookrightarrow \Lambda^{\mu(\cdot)}_{\infty,r}(\overline{\mathbb{R}^n})$$

*if and only if*

$$\int_0^1 \left(\frac{\lambda(x)}{\mu(x)}\right)^u \frac{dx}{x} < +\infty, \tag{5.5}$$

*where $\frac{1}{u} := \frac{1}{r} - \frac{1}{q}$.*

**Remark 5.4** ([30, Remark 6.4])**.** Assume that all the assumptions of Theorem 5.3 are satisfied.

(i) If $r \in [q,+\infty]$, then the embedding

$$H^\sigma L_{p,q;b}(\mathbb{R}^n) \hookrightarrow \Lambda^{\mu(\cdot)}_{\infty,r}(\overline{\mathbb{R}^n}) \tag{5.6}$$

with $\mu = \lambda$ is *sharp* with respect to the parameter $\mu$ (which means that the target space $\Lambda^{\mu(\cdot)}_{\infty,r}(\overline{\mathbb{R}^n})$ in (5.6) and the space $\Lambda^{\lambda(\cdot)}_{\infty,r}(\overline{\mathbb{R}^n})$ (that is, the target space in (5.6) with $\mu = \lambda$) satisfy $\Lambda^{\lambda(\cdot)}_{\infty,r}(\overline{\mathbb{R}^n}) \hookrightarrow \Lambda^{\mu(\cdot)}_{\infty,r}(\overline{\mathbb{R}^n})$). This follows from Theorem 5.3 (i) (condition (5.4)).

(ii) Among embeddings (5.6) that one with $\mu = \lambda$ and $r = q$ is *optimal*, that is, the target space $\Lambda^{\mu(\cdot)}_{\infty,r}(\overline{\mathbb{R}^n})$ in (5.6) and the space $\Lambda^{\lambda(\cdot)}_{\infty,q}(\overline{\mathbb{R}^n})$ (that is, the target space in (5.6) with $\mu = \lambda$ and $r = q$) satisfy

$$\Lambda^{\lambda(\cdot)}_{\infty,q}(\overline{\mathbb{R}^n}) \hookrightarrow \Lambda^{\mu(\cdot)}_{\infty,r}(\overline{\mathbb{R}^n}). \tag{5.7}$$

Indeed, if $r \in [q,+\infty]$, this follows from part (i) and the fact that $\Lambda^{\lambda(\cdot)}_{\infty,r}(\overline{\mathbb{R}^n}) \hookrightarrow \Lambda^{\lambda(\cdot)}_{\infty,s}(\overline{\mathbb{R}^n})$ if $0 < r < s \le +\infty$ (cf. [27, eq. (3.6)]). To verify it when $0 < r < q$, note that (5.7) is satisfied if

$$\left\| t^{-1/r}\frac{\omega(f,t)}{\mu(t)} \right\|_{r;(0,1)} \precsim \left\| t^{-1/q}\frac{\omega(f,t)}{\lambda(t)} \right\|_{q;(0,1)} \quad \text{for all} \quad f \in \Lambda^{\lambda(\cdot)}_{\infty,q}(\overline{\mathbb{R}^n}).$$

Since $\omega(f,\cdot)$ is non-decreasing on $(0,1)$, this inequality holds (cf. [33, Proposition 2.1 (ii)] and [47, Lemma, p. 176]) if

$$\int_0^1 \left\| t^{-1/r}[\mu(t)]^{-1} \right\|^u_{r;(x,1)} [\lambda(x)]^u \frac{dx}{x} < +\infty.$$

However, one can show that the last condition is equivalent to (5.5). The result follows from Theorem 5.3 (ii) since (5.5) is satisfied when (5.6) holds.

(iii) The embedding

$$H^\sigma L_{p,q;b}(\mathbb{R}^n) \hookrightarrow \Lambda^{\mu(\cdot)}_{\infty,r}(\overline{\mathbb{R}^n})$$

*does not hold* if $\mu = \lambda$ and $r \in (0, q)$ (this follows from Theorem 5.3 (ii)).

(iv) Using the terminology of [32] or [48], we obtain that $\left(\frac{\lambda(x)}{x}, q\right)$ is the *continuity envelope* of $H^\sigma L_{p,q;b}(\mathbb{R}^n)$ (this is a consequence of part (i) with $r = +\infty$ and part (iii)).

The following assertion is an analogue of Theorem 5.3 and concerns the limiting case when $p = n/\sigma$.

**Theorem 5.5** ([30, Theorem 6.5])**.** *Let $\sigma \in (0,1)$, $p = \frac{n}{\sigma}$, $q \in (1, +\infty]$, $r \in (0, +\infty]$, $\mu \in \mathcal{L}_r$ and let $b \in SV(0, +\infty)$ be such that $\|t^{-1/q'}[b(t)]^{-1}\|_{q';(0,1)} < +\infty$. Let $\lambda_{qr} \in \mathcal{L}_r$ be defined by*

$$\lambda_{qr}(x) := [b(x^n)]^{q'/r} \left( \int_0^{x^n} t^{-1}[b(t)]^{-q'}\, dt \right)^{1/q'+1/r}, \quad x \in (0, 1]. \tag{5.8}$$

(i) *If $1 < q \le r \le +\infty$, then*

$$H^\sigma L_{p,q;b}(\mathbb{R}^n) \hookrightarrow \Lambda^{\mu(\cdot)}_{\infty,r}(\overline{\mathbb{R}^n})$$

*if and only if*

$$\overline{\lim_{x\to 0_+}} \frac{\|t^{-1/r}[\mu(t)]^{-1}\|_{r;(x,1)}}{\|t^{-1/r}[\lambda_{qr}(t)]^{-1}\|_{r;(x,1)}} < +\infty. \tag{5.9}$$

(ii) *If $0 < r < q \le +\infty$ and $q > 1$, then*

$$H^\sigma L_{p,q;b}(\mathbb{R}^n) \hookrightarrow \Lambda^{\mu(\cdot)}_{\infty,r}(\overline{\mathbb{R}^n})$$

*if and only if*

$$\int_0^{1/2} \left( \frac{\|t^{-1/r}[\mu(t)]^{-1}\|_{r;(x,1)}}{\|t^{-1/r}[\lambda_{qr}(t)]^{-1}\|_{r;(x,1)}} \right)^u \left( \int_0^{x^n} t^{-1}[b(t)]^{-q'}\, dt \right)^{-1} [b(x^n)]^{-q'} \frac{dx}{x} < +\infty, \tag{5.10}$$

*where $\frac{1}{u} := \frac{1}{r} - \frac{1}{q}$.*

**Remark 5.6** ([30, Remark 6.6])**.** Assume that all assumptions of Theorem 5.5 are satisfied.

(i) If $r \in [q, +\infty]$, then the embedding

$$H^\sigma L_{p,q;b}(\mathbb{R}^n) \hookrightarrow \Lambda^{\mu(\cdot)}_{\infty,r}(\overline{\mathbb{R}^n}) \tag{5.11}$$

with $\mu = \lambda_{qr}$ is *sharp* with respect to the parameter $\mu$, that is, the target space $\Lambda^{\mu(\cdot)}_{\infty,r}(\overline{\mathbb{R}^n})$ in (5.11) and the space $\Lambda^{\lambda_{qr}(\cdot)}_{\infty,r}(\overline{\mathbb{R}^n})$ (i.e., the target space in (5.11) with $\mu = \lambda_{qr}$) satisfy

$$\Lambda^{\lambda_{qr}(\cdot)}_{\infty,r}(\overline{\mathbb{R}^n}) \hookrightarrow \Lambda^{\mu(\cdot)}_{\infty,r}(\overline{\mathbb{R}^n}). \tag{5.12}$$

Indeed, the last embedding is satisfied if

$$\left\| t^{-1/r} \frac{\omega(f,t)}{\mu(t)} \right\|_{r;(0,1)} \precsim \left\| t^{-1/r} \frac{\omega(f,t)}{\lambda_{qr}(t)} \right\|_{r;(0,1)} \quad \text{for all} \quad f \in \Lambda^{\lambda_{qr}(\cdot)}_{\infty,r}(\overline{\mathbb{R}^n}). \tag{5.13}$$

Since $\omega(f, \cdot)$ is non-decreasing on $(0, 1)$, this inequality holds (cf. [33, Proposition 2.1 (i)]) if

$$\sup_{x\in(0,1)} \left\| t^{-1/r}[\mu(t)]^{-1} \right\|_{r;(x,1)} \Big/ \left\| t^{-1/r}[\lambda_{qr}(t)]^{-1} \right\|_{r;(x,1)} < +\infty, \tag{5.14}$$

which is equivalent to (5.9). The result follows from Theorem 5.5 (i) since (5.9) is satisfied when (5.11) holds.

(ii) Among embeddings (5.11) that one with $\mu = \lambda_{qq}$ and $r = q$ is *optimal*, that is, the target space $\Lambda^{\mu(\cdot)}_{\infty,r}(\overline{\mathbb{R}^n})$ in (5.11) and the space $\Lambda^{\lambda_{qq}(\cdot)}_{\infty,q}(\overline{\mathbb{R}^n})$ (that is, the target space in (5.11) with $\mu = \lambda_{qq}$ and $r = q$) satisfy

$$\Lambda^{\lambda_{qq}(\cdot)}_{\infty,q}(\overline{\mathbb{R}^n}) \hookrightarrow \Lambda^{\mu(\cdot)}_{\infty,r}(\overline{\mathbb{R}^n}). \tag{5.15}$$

Indeed, if $r \in [q, +\infty]$, this follows from part (i) and the fact that $\Lambda^{\lambda_{qr}(\cdot)}_{\infty,r}(\overline{\mathbb{R}^n}) \hookrightarrow \Lambda^{\lambda_{qs}(\cdot)}_{\infty,s}(\overline{\mathbb{R}^n})$ if $q \le r \le s \le +\infty$ (the proof of this is analogous to the proof of [27, eq. (3.18)]). To verify it when $0 < r < q$, note that (5.15) is satisfied if

$$\left\| t^{-1/r} \frac{\omega(f,t)}{\mu(t)} \right\|_{r;(0,1)} \precsim \left\| t^{-1/q} \frac{\omega(f,t)}{\lambda_{qq}(t)} \right\|_{q;(0,1)} \quad \text{for all} \quad f \in \Lambda^{\lambda_{qq}(\cdot)}_{\infty,q}(\overline{\mathbb{R}^n}).$$

Since $\omega(f,\cdot)$ is non-decreasing on $(0,1)$, this inequality holds (cf. [33, Proposition 2.1 (ii)] and [47, Lemma, p. 176]) if

$$\int_0^1 \left\| t^{-1/r} [\mu(t)]^{-1} \right\|^u_{r;(x,1)} \left\| t^{-1/q} [\lambda_{qq}(t)]^{-1} \right\|^{-q\frac{u}{r}}_{q;(x,1)} [\lambda_{qq}(x)]^{-q} \frac{dx}{x} < +\infty.$$

However, one can show that the last condition is equivalent to (5.10). The result follows from Theorem 5.5 (ii) since (5.10) is satisfied when (5.11) holds.

(iii) The embedding

$$H^\sigma L_{p,q;b}(\mathbb{R}^n) \hookrightarrow \Lambda^{\mu(\cdot)}_{\infty,r}(\overline{\mathbb{R}^n})$$

*does not hold* if $\mu = \lambda_{qr}$ and $r \in (0, q)$ (this follows from Theorem 5.5 (ii)).

(iv) Using the terminology of [32] or [48], we obtain that $\left(\frac{\lambda_{q\infty}(x)}{x}, q\right)$ is the *continuity envelope* of $H^\sigma L_{p,q;b}(\mathbb{R}^n)$ (this is a consequence of part (i) with $r = +\infty$ and part (iii)).

**Examples 5.7.** *Let* $\sigma \in (0,1)$, $p = n/\sigma$, $q \in (1, +\infty]$ *and let* $r \in [q, +\infty]$.

**1.** *Let* $\alpha, \beta \in \mathbb{R}$ *and let* $b \in SV(0, +\infty)$ *be defined by*

$$b(t) = l_1(t)^\alpha \; l_2(t)^\beta, \quad t > 0.$$

**(i)** *If* $\alpha > 1/q'$ *and* $\beta \in \mathbb{R}$, *then*

$$H^\sigma L^{p,q}(\log L)^\alpha (\log\log L)^\beta(\mathbb{R}^n) \hookrightarrow \Lambda^{\lambda_{qq}(\cdot)}_{\infty,q}(\overline{\mathbb{R}^n}) \hookrightarrow \Lambda^{\lambda_{qr}(\cdot)}_{\infty,r}(\overline{\mathbb{R}^n}) \hookrightarrow C^{0,\lambda_{q\infty}(\cdot)}(\overline{\mathbb{R}^n}), \tag{5.16}$$

*with*

$$\begin{aligned} \lambda_{qq}(t) &\approx l_1(t)^{1-\alpha} \; l_2(t)^{-\beta}, \quad t \in (0,1]; \\ \lambda_{qr}(t) &\approx l_1(t)^{1/r+1/q'-\alpha} \; l_2(t)^{-\beta}, \quad t \in (0,1]; \\ \lambda_{q\infty}(t) &\approx l_1(t)^{1/q'-\alpha} \; l_2(t)^{-\beta}, \quad t \in (0,1]. \end{aligned}$$

**(ii)** *If* $\alpha = 1/q'$ *and* $\beta > 1/q'$, *then*

$$H^\sigma L^{p,q}(\log L)^\alpha (\log\log L)^\beta(\mathbb{R}^n) \hookrightarrow \Lambda^{\lambda_{qq}(\cdot)}_{\infty,q}(\overline{\mathbb{R}^n}) \hookrightarrow \Lambda^{\lambda_{qr}(\cdot)}_{\infty,r}(\overline{\mathbb{R}^n}) \hookrightarrow C^{0,\lambda_{q\infty}(\cdot)}(\overline{\mathbb{R}^n}),$$

*with*

$$\begin{aligned} \lambda_{qq}(t) &\approx l_1(t)^{1/q} \; l_2(t)^{1-\beta}, \quad t \in (0,1]; \\ \lambda_{qr}(t) &\approx l_1(t)^{1/r} \; l_2(t)^{1/r+1/q'-\beta}, \quad t \in (0,1]; \\ \lambda_{q\infty}(t) &\approx l_2(t)^{1/q'-\beta}, \quad t \in (0,1]. \end{aligned}$$

**(iii)** *If either* $\alpha < 1/q'$ *or* $\alpha = 1/q'$ *and* $\beta < 1/q'$ *or* $\alpha = 1/q'$ *and* $\beta = 1/q'$, *then we have* (3.39) *or* (3.40) *or* (3.41), *respectively.*

**2.** *Let* $\alpha \in (0,1)$, $\beta \in \mathbb{R}$ *and let* $b \in SV(0, +\infty)$ *be defined by*

$$b(t) = l_1(t)^{-(\alpha-1)/q'} \exp(\beta \, l_1(t)^\alpha), \quad t > 0.$$

**(i)** *If $\beta > 0$, then*

$$H^\sigma L_{p,q;b}(\mathbb{R}^n) \hookrightarrow \Lambda^{\lambda_{qq}(\cdot)}_{\infty,q}(\overline{\mathbb{R}^n}) \hookrightarrow \Lambda^{\lambda_{qr}(\cdot)}_{\infty,r}(\overline{\mathbb{R}^n}) \hookrightarrow C^{0,\lambda_{q\infty}(\cdot)}(\overline{\mathbb{R}^n})$$

*with*

$$\begin{aligned} \lambda_{qq}(t) &\approx l_1(t^n)^{-(\alpha-1)/q} \exp(-\beta\, l_1(t^n)^\alpha), \quad t \in (0,1]; \\ \lambda_{qr}(t) &\approx l_1(t^n)^{-(\alpha-1)/r} \exp(-\beta\, l_1(t^n)^\alpha), \quad t \in (0,1]; \\ \lambda_{q\infty}(t) &\approx \exp(-\beta\, l_1(t^n)^\alpha), \quad t \in (0,1]. \end{aligned}$$

**(ii)** *If $\beta < 0$, then* (3.42) *holds.*

**Remark 5.8.** (i) Note that the embedding

$$H^\sigma L^{p,q}(\log L)^\alpha(\log\log L)^\beta(\mathbb{R}^n) \hookrightarrow C^{0,\lambda_{q\infty}(\cdot)}(\overline{\mathbb{R}^n})$$

mentioned in part (ii) of Examples 5.7 remains true even if $\sigma \in (0,n)$ provided that $q \in (1,+\infty)$ (see [19, Theorem 4.3]).

(ii) Examples 5.7 *continue to hold even if $\sigma \in (0,n)$.* This follows from results of the next section.

Now, we make use of Theorem 2.4 to characterize compact embeddings of Bessel potential spaces $H^\sigma X(\mathbb{R}^n)$ into generalized Hölder spaces $\Lambda^{\mu(\cdot)}_{\infty,r}(\overline{\Omega})$.

**Theorem 5.9** ([30, Theorem 7.3])**.** *Let $\sigma \in (0,1)$ and let $X = X(\mathbb{R}^n) = X(\mathbb{R}^n,\mu_n)$ be a r. i. BFS such that $\|g_\sigma\|_{X'} < \infty$. Assume that $r \in (0,+\infty)$, $\mu \in \mathcal{L}_r$ and that $\Omega$ is a bounded domain in $\mathbb{R}^n$. Then*

$$H^\sigma X(\mathbb{R}^n) \hookrightarrow\hookrightarrow \Lambda^{\mu(\cdot)}_{\infty,r}(\overline{\Omega}) \quad {}^{*}) \tag{5.17}$$

*if and only if*

$$\sup_{\|f\|_X \le 1} \left\| t^{-\frac{1}{r}}[\mu(t)]^{-1} \int_0^{t^n} \tau^{\frac{\sigma}{n}-1} f^*(\tau)\, d\tau \right\|_{r;(0,\xi)} \to 0 \quad as \quad \xi \to 0_+. \tag{5.18}$$

Using Theorem 5.9 and Lemma 2.3, we arrive at the following corollary.

**Corollary 5.10** ([30, Corollary 7.4])**.** *Let $\sigma \in (0,1)$ and let $X = X(\mathbb{R}^n)$ be a r. i. BFS. Assume that $r \in (0,+\infty)$, $\mu \in \mathcal{L}_r$ and that $\Omega$ is a bounded domain in $\mathbb{R}^n$. Then* (5.17) *holds if and only if $\|g_\sigma\|_{X'} < +\infty$ and* (5.18) *is satisfied.*

Compact embeddings of spaces $H^\sigma L_{p,q;b}(\mathbb{R}^n)$ with $\sigma \in (0,1)$ into generalized Hölder spaces in the superlimiting case (that is, when $p > n/\sigma$) are characterized in the next theorem. Note that part (i) of this theorem complements Corollary 4.2 (ii).

**Theorem 5.11** ([30, Theorem 8.2])**.** *Let $\sigma \in (0,1)$, $p \in (\frac{n}{\sigma},+\infty)$, $q \in [1,+\infty]$, $b \in SV(0,+\infty)$, $r \in (0,+\infty)$ and $\mu \in \mathcal{L}_r$. Assume that $\Omega$ is a bounded domain in $\mathbb{R}^n$. Let $\lambda : (0,1] \to (0,+\infty)$ be defined by*

$$\lambda(x) := x^{\sigma-\frac{n}{p}}[b(x^n)]^{-1} \quad for\ all \quad x \in (0,1]. \tag{5.19}$$

*(Note that $\lambda \in \mathcal{L}_r$ for any $r \in (0,+\infty]$.)*

(i) *If $1 \le q \le r < +\infty$, then*

$$H^\sigma L_{p,q;b}(\mathbb{R}^n) \hookrightarrow\hookrightarrow \Lambda^{\mu(\cdot)}_{\infty,r}(\overline{\Omega})$$

*if and only if*

$$\lim_{x\to 0_+} \frac{\lambda(x)}{\mu(x)} = 0. \tag{5.20}$$

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*) Recall that this means that the mapping $u \mapsto u|_\Omega$ from $H^\sigma X(\mathbb{R}^n)$ into $\Lambda^{\mu(\cdot)}_{\infty,r}(\overline{\Omega})$ is compact.

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