THE AVERAGING INTEGRAL OPERATOR BETWEEN WEIGHTED LEBESGUE SPACES AND REVERSE HÖLDER INEQUALITIES

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Dedicated to Professor Gerard Bourdaud on the occasion of his 60th birthday

ABSTRACT. Let $1 < p \le q < +\infty$ and let v, w be weights on $(0,+\infty)$ satisfying: $v(x)x^\rho$ is equivalent to a non-decreasing function on $(0,+\infty)$ for some $\rho \ge 0$;

$$[w(x)x]^{1/q} \approx [v(x)x]^{1/p} \quad \text{for all } x \in (0,+\infty).$$

Let A be the averaging operator given by $(Af)(x) := \frac{1}{x}\int_0^x f(t)\,\mathrm{d}t$, $x \in (0,+\infty)$.

First, we prove that the operator

$$A : L^p((0,+\infty);v) \to L^p((0,+\infty);v) \text{ is bounded}$$

if and only if the operator

$$A : L^p((0,+\infty);v) \to L^q((0,+\infty);w) \text{ is bounded.}$$

Second, we show that the boundedness of the averaging operator A on the space $L^p((0,+\infty);v)$ implies that, for all $r > 0$, the weight $v^{1-p'}$ satisfies the reverse Hölder inequality over the interval $(0,r)$ with respect to the measure $\mathrm{d}t$, while the weight v satisfies the reverse Hölder inequality over the interval $(r,+\infty)$ with respect to the measure $t^{-p}\,\mathrm{d}t$. As a corollary, we obtain that the boundedness of the averaging operator A on the space $L^p((0,+\infty);v)$ is equivalent to the boundedness of the averaging operator A on the space $L^p((0,+\infty);v^{1+\delta})$ for some $\delta > 0$.

1. Introduction

Let $1 < p < +\infty$ and let v be a weight on $(0,+\infty)$, i.e., a measurable function which is positive a.e. on $(0,+\infty)$. By $L^p(v) \equiv L^p((0,+\infty);v)$ we denote the weighted Lebesgue space of all measurable functions f on $(0,+\infty)$ for which the norm

$$\|f\|_{p,v} = \left(\int_0^{+\infty} |f(x)|^p v(x)\,\mathrm{d}x\right)^{1/p}$$

is finite.

We shall consider one of the basic operators in the mathematical analysis, the averaging operator A defined by

$$(Af)(x) := \frac{1}{x}\int_0^x f(t)\,\mathrm{d}t, \quad x \in (0,+\infty).$$

It is well known (see [B] or [OK]) that if $1 < p < +\infty$ and w, v are weights on $(0,+\infty)$, then the averaging operator $A : L^p(v) \to L^q(w)$ is bounded, that is, there

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exists a constant $c > 0$ such that

$$\|Af\|_{q,w} \leq c\|f\|_{p,v} \quad \text{for all } f \in L^p(v), \tag{1}$$

if and only if

$$B := \sup_{r>0} \left(\int_r^{+\infty} w(t)t^{-q} \, \mathrm{d}t \right)^{1/q} \left(\int_0^r v(t)^{1-p'} \, \mathrm{d}t \right)^{1/p'} < +\infty, \tag{2}$$

where $p' = p/(p-1)$.

Throughout the paper we use the following convention: For two non-negative expressions (i.e. functions or functionals) F and G the symbol $F \lesssim G$ (or $F \gtrsim G$) means that $F \leq cG$ (or $cF \geq G$), where c is a positive constant independent of appropriate quantities involved in F and G. We shall write $F \approx G$ (and say that F and G are equivalent) if both relations $F \lesssim G$ and $F \gtrsim G$ hold.

Our aim is to prove the following assertions.

Theorem 1. *Let $1 < p \leq q < +\infty$ and let v, w be weights on $(0, +\infty)$ such that:*

(3) *$v(x)x^\rho$ is equivalent to a non-decreasing function on $(0, +\infty)$ for some $\rho \geq 0$;*

$$[w(x)x]^{1/q} \approx [v(x)x]^{1/p} \quad \textit{for all } x \in (0, +\infty). \tag{4}$$

Then the averaging operator

$$A : L^p(v) \to L^p(v) \quad \textit{is bounded} \tag{5}$$

if and only if the operator

$$A : L^p(v) \to L^q(w) \quad \textit{is bounded.} \tag{6}$$

Assumptions of Theorem 1 and (5) ensure that

$$\left(\int_r^{+\infty} w(t)t^{-q} \, \mathrm{d}t \right)^{1/q} \left(\int_0^r v(t)^{1-p'} \, \mathrm{d}t \right)^{1/p'} \approx 1 \quad \text{for all } r > 0,$$

which means that (w, v) is the optimal couple of weights for which (1) holds. Note also that assumption (4) is satisfied when $w = v$ and $q = p$.

In the particular case when $\rho = 0$ in (3) the statement of Theorem 1 has been communicated to us by a referee of another our paper.

It is known that the weight v satisfying both (3) with $\rho = 0$ and (5) belongs to the A_p-class of B. Muckenhoupt. Since $v \in A_p$ implies that $v^{1-p'} \in A_{p'}$, the following two reverse Hölder inequalities hold for such a weight:

$$\left(\frac{1}{r} \int_0^r [v(t)^{1-p'}]^{1+\delta} \, \mathrm{d}t \right)^{1/(1+\delta)} \lesssim \frac{1}{r} \int_0^r v(t)^{1-p'} \, \mathrm{d}t,$$

$$\left(\frac{1}{r} \int_0^r v(t)^{1+\delta} \, \mathrm{d}t \right)^{1/(1+\delta)} \lesssim \frac{1}{r} \int_0^r v(t) \, \mathrm{d}t,$$

for all $r > 0$ and some $\delta > 0$.

The next theorem shows that the former inequality remains true even when $\rho \geq 0$ in (3) while the latter inequality is then replaced by the reverse Hölder inequality for the weight v, the interval $(r, +\infty)$ and the measure $t^{-p} \, \mathrm{d}t$.

Theorem 2. *Let $1 < p < +\infty$ and let v be a weight on $(0, +\infty)$ such that (3) holds. Assume that the averaging operator*

$$A : L^p(v) \to L^p(v) \quad \textit{is bounded.} \tag{7}$$

Then there is $\delta_0 > 0$ such that

$$\left(\frac{1}{r} \int_0^r [v(t)^{1-p'}]^{1+\delta} \, \mathrm{d}t \right)^{1/(1+\delta)} \lesssim \frac{1}{r} \int_0^r v(t)^{1-p'} \, \mathrm{d}t \tag{8}$$

and

$$\left(\frac{1}{r^{1-p}}\int_r^{+\infty} v(t)^{1+\delta}t^{-p}\,\mathrm{d}t\right)^{1/(1+\delta)} \lesssim \frac{1}{r^{1-p}}\int_r^{+\infty} v(t)t^{-p}\,\mathrm{d}t \tag{9}$$

for all $r > 0$ *and* $\delta \in [0, \delta_0)$.

Corollary 1. *Let* $1 < p < +\infty$ *and let* v *be a weight on* $(0, +\infty)$ *such that* (3) *holds. Then the averaging operator*

$$A : L^p(v) \to L^p(v) \quad \textit{is bounded} \tag{10}$$

if and only if there is $\delta > 0$ *such that the operator*

$$A : L^p(v^{1+\delta}) \to L^p(v^{1+\delta}) \quad \textit{is bounded.} \tag{11}$$

Corollary 1 is a particular case of the following assertion.

Corollary 2. *Let* $1 < p \le q < +\infty$ *and let* v, w *be weights on* $(0, +\infty)$ *such that* (3) *and* (4) *hold. Then* (10) *is satisfied if and only if there is* $\delta > 0$ *such that the operator*

$$A : L^p(v(x)^{1+\delta}) \to L^q(w(x)^{1+\delta}x^{\delta(1-q/p)}) \quad \textit{is bounded.}$$

We refer to [OR] for further related results.

Remark 1. It has been said that the weight v satisfying both (3) with $\rho = 0$ and (5) belongs to the A_p-class of B. Muckenhoupt. On the other hand, there are weights which satisfy (3) and (5) but which do not belong to the A_p-class. A simple example is $v(t) = t^\beta$, $t > 0$, with $\beta \le -1$.

The paper is organized as follows. In Section 2 we prove Theorem 1 while the proof of Theorem 2 is given in Section 3. Section 4 is devoted to proofs of Corollaries 1 and 2.

2. Proof of Theorem 1

To prove Theorem 1, we shall use the following assertion. (Note that its proof is based on [N, Lemma 2] and a dual version of Nakai's result.)

Lemma 1 (see [OR, Lemma B]). *Let* $1 < p \le q < +\infty$ *and let* v, w *be weights on* $(0, +\infty)$ *such that* (3) *and* (4) *hold. Assume that the averaging operator* $A : L^p(v) \to L^q(w)$ *is bounded. Then there exists a positive constant* α_0 *such that*

$$\int_0^r [v(t)t^\alpha]^{1-p'}\,\mathrm{d}t \approx [v(r)r^{\alpha+1-p}]^{1-p'}$$

and

$$\int_r^{+\infty} w(t)t^{\alpha-q}\,\mathrm{d}t \approx w(r)r^{\alpha+1-q}$$

for all $r > 0$ *and* $\alpha \in [0, \alpha_0)$.

Proof of Theorem 1. **(i)** Assume that (6) holds. Then, by Lemma 1, there exists $\alpha_0 > 0$ such that

$$\int_0^r [v(t)t^\alpha]^{1-p'}\,\mathrm{d}t \approx [v(r)r^{\alpha+1-p}]^{1-p'} \quad \text{for all } r > 0 \text{ and } \alpha \in [0, \alpha_0). \tag{12}$$

Hence,

$$\int_0^r v(t)^{1-p'}\,\mathrm{d}t \approx v(r)^{1-p'}\,r \quad \text{for all } r > 0. \tag{13}$$

Moreover, using (12) with a fixed $\alpha \in (0, \alpha_0)$, we get

$$v(r) \approx r^{p-1-\alpha} \left(\int_0^r [v(t)t^\alpha]^{1-p'} \, \mathrm{d}t \right)^{1/(1-p')} \quad \text{for all } r > 0. \tag{14}$$

Thus, applying also the monotonicity of the function

$$t \mapsto \left(\int_0^t [v(\tau)\tau^\alpha]^{1-p'} \, \mathrm{d}\tau \right)^{1/(1-p')}, \quad t > 0, \tag{15}$$

and (12), we arrive at

$$\begin{aligned} \int_r^{+\infty} v(t)t^{-p} \, \mathrm{d}t &\approx \int_r^{+\infty} t^{p-1-\alpha} \left(\int_0^t [v(\tau)\tau^\alpha]^{1-p'} \, \mathrm{d}\tau \right)^{1/(1-p')} t^{-p} \, \mathrm{d}t \\ &\leq \left(\int_0^r [v(\tau)\tau^\alpha]^{1-p'} \, \mathrm{d}\tau \right)^{1/(1-p')} \int_r^{+\infty} t^{-1-\alpha} \, \mathrm{d}t \\ &\approx v(r)r^{1-p} \quad \text{for all } r > 0, \end{aligned}$$

which implies that

$$\left(\int_r^{+\infty} v(t)t^{-p} \, \mathrm{d}t \right)^{1/p} \lesssim v(r)^{1/p} r^{-1/p'} \quad \text{for all } r > 0. \tag{16}$$

On the other hand, by (13),

$$\left(\int_0^r v(t)^{1-p'} \, \mathrm{d}t \right)^{1/p'} \approx v(r)^{-1/p} \, r^{1/p'} \quad \text{for all } r > 0. \tag{17}$$

Estimates (16) and (17) used in (2) yield (5).

(ii) Assume now that (5) holds. By Lemma 1 (with $p = q$ and $w = v$), (12) is satisfied. (Note that (4) holds when $p = q$ and $w = v$.) Consequently, (13), (14) and (17) remain true. Thus, using also the monotonicity of the function (15), we arrive at

$$\begin{aligned} &\int_r^{+\infty} v(t)^{q/p} t^{q/p-1} t^{-q} \, \mathrm{d}t \\ &\approx \int_r^{+\infty} \left(t^{p-1-\alpha} \left(\int_0^t [v(\tau)\tau^\alpha]^{1-p'} \, \mathrm{d}\tau \right)^{1/(1-p')} \right)^{q/p} t^{q/p-1-q} \, \mathrm{d}t \\ &\leq \left(\int_0^r [v(\tau)\tau^\alpha]^{1-p'} \, \mathrm{d}\tau \right)^{q/[p(1-p')]} \int_r^{+\infty} t^{-\alpha q/p-1} \, \mathrm{d}t \\ &\approx v(r)^{q/p} r^{q/p-q} \quad \text{for all } r > 0. \end{aligned}$$

Since, by (4), $w(t) \approx v(t)^{q/p} t^{q/p-1}$ for all $t > 0$, the last estimate implies that

$$\left(\int_r^{+\infty} w(t)t^{-q} \, \mathrm{d}t \right)^{1/q} \lesssim v(r)^{1/p} r^{-1/p'} \quad \text{for all } r > 0. \tag{18}$$

Estimates (17) and (18) used in (2) yield (6). □

3. Proof of Theorem 2

Assume that (7) holds. Then, by Lemma 1 (with $q = p$ and $w = v$), there is $\alpha_0 > 0$ such that

$$\int_0^r [v(t)t^\alpha]^{1-p'} \, \mathrm{d}t \approx [v(r)r^{\alpha+1-p}]^{1-p'} \tag{19}$$

and

$$\int_r^{+\infty} v(t)t^{\alpha-p}\,\mathrm{d}t \approx v(r)r^{\alpha+1-p} \tag{20}$$

for all $r > 0$ and $\alpha \in [0, \alpha_0)$. Consequently, for all $r > 0$,

$$v(r)^{1-p'} \approx r^{-1} \int_0^r v(t)^{1-p'}\,\mathrm{d}t \tag{21}$$

and

$$v(r) \approx r^{p-1} \int_r^{+\infty} v(t)t^{-p}\,\mathrm{d}t. \tag{22}$$

Take $\delta \in (0, \delta_1)$, where $\delta_1 := \alpha_0(p'-1)$ and put $\alpha := \delta/(p'-1)$. Using (21), the monotonicity of the function

$$t \mapsto \left(\int_0^t v(\tau)^{1-p'}\,\mathrm{d}\tau\right)^\delta, \quad t > 0,$$

(19) and again (21), we arrive at

$$\begin{aligned}
\int_0^r [v(t)^{1-p'}]^{1+\delta}\,\mathrm{d}t &= \int_0^r v(t)^{1-p'}[v(t)^{1-p'}]^\delta\,\mathrm{d}t \\
&\approx \int_0^r v(t)^{1-p'}\left(t^{-1}\int_0^t v(\tau)^{1-p'}\,\mathrm{d}\tau\right)^\delta \mathrm{d}t \\
&\le \left(\int_0^r v(\tau)^{1-p'}\,\mathrm{d}\tau\right)^\delta \int_0^r [v(t)t^\alpha]^{1-p'}\,\mathrm{d}t \\
&\approx \left(\int_0^r v(\tau)^{1-p'}\,\mathrm{d}\tau\right)^\delta [v(r)r^{\alpha+1-p}]^{1-p'} \\
&= \left(\int_0^r v(\tau)^{1-p'}\,\mathrm{d}\tau\right)^\delta v(r)^{1-p'}r^{-\delta+1} \\
&\approx \left(\int_0^r v(\tau)^{1-p'}\,\mathrm{d}\tau\right)^{1+\delta} r^{-\delta}, \quad \text{for all } r > 0,
\end{aligned}$$

which implies that

$$\left(\frac{1}{r}\int_0^r [v(t)^{1-p'}]^{1+\delta}\,\mathrm{d}t\right)^{1/(1+\delta)} \lesssim \frac{1}{r}\int_0^r v(t)^{1-p'}\,\mathrm{d}t \tag{23}$$

for all $r > 0$ and $\delta \in [0, \delta_1)$.

Take $\delta \in (0, \delta_2)$, where $\delta_2 := \alpha_0/(p-1)$ and put $\alpha := \delta(p-1)$. Using (22), the monotonicity of the function

$$t \mapsto \left(\int_t^{+\infty} v(\tau)\tau^{-p}\,\mathrm{d}\tau\right)^\delta, \quad t > 0,$$

(20) and again (22), we obtain

$$\begin{aligned}\int_r^{+\infty} v(t)^{1+\delta} t^{-p}\,\mathrm{d}t &= \int_r^{+\infty} v(t)t^{-p} v(t)^{\delta}\,\mathrm{d}t \\ &\approx \int_r^{+\infty} v(t)t^{-p}\left(t^{p-1}\int_t^{+\infty} v(\tau)\tau^{-p}\,\mathrm{d}\tau\right)^{\delta}\mathrm{d}t \\ &\leq \left(\int_r^{+\infty} v(\tau)\tau^{-p}\,\mathrm{d}\tau\right)^{\delta}\int_r^{+\infty} v(t)t^{\alpha-p}\,\mathrm{d}t \\ &\approx \left(\int_r^{+\infty} v(\tau)\tau^{-p}\,\mathrm{d}\tau\right)^{\delta} v(r)r^{\alpha+1-p} \\ &= \left(\int_r^{+\infty} v(\tau)\tau^{-p}\,\mathrm{d}\tau\right)^{\delta} v(r)r^{1-p}\, r^{\delta(p-1)} \\ &\approx \left(\int_r^{+\infty} v(\tau)\tau^{-p}\,\mathrm{d}\tau\right)^{1+\delta} r^{\delta(p-1)}, \quad \text{for all } r>0,\end{aligned}$$

which implies that

$$\left(\frac{1}{r^{1-p}}\int_r^{+\infty} v(t)^{1+\delta}t^{-p}\,\mathrm{d}t\right)^{1/(1+\delta)} \lesssim \frac{1}{r^{1-p}}\int_r^{+\infty} v(t)t^{-p}\,\mathrm{d}t \tag{24}$$

for all $r>0$ and $\delta\in[0,\delta_2)$.

Putting $\delta_0 := \min\{\delta_1,\delta_2\}$, we get estimates (8) and (9) from (23) and (24). □

4. Proofs of Corollaries 1 and 2

Proof of Corollary 1. **(i)** Assume that (10) is satisfied. Then, by Theorem 2, there is $\delta_0>0$ such that reverse Hölder inequalities (8) and (9) hold. Together with (10) and (2) (used with $q=p$ and $w=v$), this implies that

$$\begin{aligned}&\left(\int_r^{+\infty} v(t)^{1+\delta}t^{-p}\,\mathrm{d}t\right)^{1/p}\left(\int_0^r [v(t)^{1+\delta}]^{1-p'}\,\mathrm{d}t\right)^{1/p'} \\ &\lesssim \left[\left(\frac{1}{r^{1-p}}\int_r^{+\infty} v(t)t^{-p}\,\mathrm{d}t\right)^{1+\delta} r^{1-p}\right]^{1/p}\left[\left(\frac{1}{r}\int_0^r v(t)^{1-p'}\,\mathrm{d}t\right)^{1+\delta} r\right]^{1/p'} \\ &= \left[\left(\int_r^{+\infty} v(t)t^{-p}\,\mathrm{d}t\right)^{1/p}\left(\int_0^r v(t)^{1-p'}\,\mathrm{d}t\right)^{1/p'}\right]^{1+\delta} \\ &\lesssim 1 \quad \text{for all } r>0 \quad \text{and} \quad \delta\in[0,\delta_0).\end{aligned}$$

Consequently, (11) holds with any $\delta\in[0,\delta_0)$.

(ii) Assume now that (11) is satisfied with some $\delta>0$. Together with the Hölder inequalities (used with the exponents $1+\delta$, $(1+\delta)/\delta$ and the measures $t^{-p}\,\mathrm{d}t$ or dt) and (2) (applied with $q = p$ and w, v replaced by $v^{1+\delta}$), this shows that

$$\begin{aligned}
&\left(\int_r^{+\infty} v(t)t^{-p}\,\mathrm{d}t\right)^{1/p}\left(\int_0^r v(t)^{1-p'}\,\mathrm{d}t\right)^{1/p'}\\
&\lesssim \left[\left(\int_r^{+\infty} v(t)^{1+\delta}t^{-p}\,\mathrm{d}t\right)^{1/(1+\delta)}\left(\int_r^{+\infty} t^{-p}\,\mathrm{d}t\right)^{\delta/(1+\delta)}\right]^{1/p}\\
&\qquad\times\left[\left(\int_0^r [v(t)^{1-p'}]^{1+\delta}\,\mathrm{d}t\right)^{1/(1+\delta)} r^{\delta/(1+\delta)}\right]^{1/p'}\\
&\approx \left[\left(\int_r^{+\infty} v(t)^{1+\delta}t^{-p}\,\mathrm{d}t\right)^{1/p}\left(\int_0^r [v(t)^{1+\delta}]^{1-p'}\,\mathrm{d}t\right)^{1/p'}\right]^{1/(1+\delta)}\\
&\lesssim 1 \quad \text{for all } r > 0.
\end{aligned}$$

Consequently, (10) holds. □

Proof of Corollary 2. By Corollary 1, (10) is equivalent to (11). Thus, putting $V(x) := v(x)^{1+\delta}$ and $W(x) := w(x)^{1+\delta}x^{\delta(1-q/p)}$, $x > 0$, we see that the result will follow from Theorem 1 provided that we show that

$V(x)x^{\overline{\rho}}$ *is equivalent to a non-decreasing function on* $(0, +\infty)$ *for some* $\overline{\rho} \geq 0$

and

$$[W(x)x]^{1/q} \approx [V(x)x]^{1/p} \quad \textit{for all } x \in (0, +\infty).$$

We can easily see that the former condition is a consequence of (3) if $\overline{\rho} \geq \rho(1+\delta)$ and that the latter one is equivalent to (4). □

References

[B] J. S. Bradley, *Hardy inequalities with mixed norms.* Canad. Math. Bull. **21** (1978), 405–408.

[OK] B. Opic and A. Kufner, *Hardy-type inequalities.* Pitman Research Notes in Mathematics Series **219**, Longman Scientific & Technical, Harlow, 1990.

[OR] B. Opic and J. Rákosník, *On weighted estimates for the averaging integral operator.* Preprint no. 186, IM AS CR, Prague, 2009.

[N] E. Nakai, *Hardy-Littlewood maximal operator, singular integral operators and the Riesz potentials on generalized Morrey spaces.* Math. Nachr. **166** (1994), 95–103.

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