Modelling with Jump Processes and Optimal Control

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Outline

Introduction

Are continuous type models satisfactory?

Empirical facts of financial time series and how Diffusion models (DM) and Models with Jumps (JM) can capture these facts

- Sudden movements, heavy tails
  - DM: extremely large volatility term need to be added
  - JM: generic property

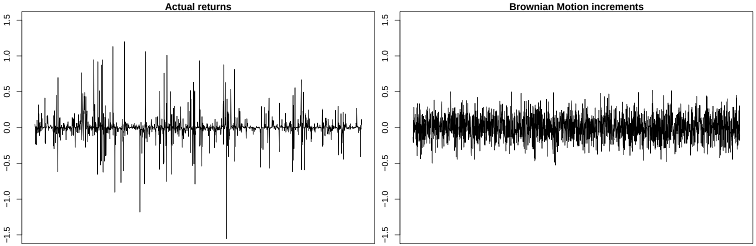

Figure: Left picture: Returns observed every 6 seconds. In the right one, Brownian Motion incements with the same mean and variance.

Glance at history

- (1900) L. Bachelier: probabilistic modelling of financial markets using Brownian Motion.
- (1st half of 20th cent.) P. Lévy: Lévy processes introduced.
- (1963) B. B. Mandelbrot: α-stable distribution to model cotton prices.
- (1973) Black and Scholes: geometric Brownian motion.
- (1976) R.C. Merton: (Poisson) Jump-Diffusion model.
- (1998) O.E. Barndorff-Nielsen: Normal Inverse Gaussian process.
- (2000 - ...) Boom in Jump processes.

Lévy Processes

What are Lévy processes

Assume a given probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathrm{P})$, with usual conditions.

Definition

We say that the process $L = (L_t, t \geq 0)$, $L_0 = 0$ is a *Lévy process* if

(i) L has *stationary increments*: $\mathcal{L}(L_t - L_s) = \mathcal{L}(L_{t-s})$, $0 \leq s < t < \infty$,

(ii) L has *independent increments*: $L_t - L_s \perp \mathcal{F}_s$, $0 \leq s < t < \infty$,

(iii) L is *continuous in probability*: $L_t \xrightarrow{P} L_s$, $t \to s$.

Examples

Poisson process $L_t \sim Po(\lambda t)$, $\lambda > 0$.

- Density

$$P(L_t = k) = \frac{(\lambda t)^k}{k!} \mathrm{e}^{-\lambda t},$$

- Characteristic function

$$\psi_{L_t}(u) = \exp\left(\lambda t \left(\mathrm{e}^{iu} - 1\right)\right).$$

Brownian motion

- Characteristic function

$$\psi_{L_t}(u) = \exp\left(\mu t u - \frac{1}{2}\sigma^2 t u^2\right).$$

Remark

L is Lévy if and only if the distribution of L_t is infinitely divisible for all $t \geq 0$.

Notation

We denote a jump size at time t

$$\Delta L(t) = L(t) - L(t-), \quad 0 \le t < \infty.$$

For $A \in \mathcal{B}(R)$ bounded below we define

$$N(t, A) = \# \left\{ 0 \le s \le t, \, \Delta L(s) \in A \right\}, \quad 0 \le t < \infty,$$

which is a Poisson process with intensity $\nu(A) = \mathrm{E}\left[N(1, A)\right]$.
We introduce a Poisson integral

$$L_t = \sum_{0 \le s \le t} \Delta L_s = \int_{[0,t]\times\mathbb{R}} z N(\mathrm{d}s, \mathrm{d}z).$$

We define a *compensated poisson random measure*

$$\tilde{N}(t, A) = N(t, A) - t\nu(A).$$

Basic theorem I.

Theorem (Lévy-Itô Decomposition)

If L is a Lévy process then there is $b \in \mathbb{R}$, $\sigma \geq 0$ and a Poisson random measure N with a Lévy measure ν satisfying

$$\int_{\mathbb{R}} (1 \wedge z^2)\nu(dz) < \infty,$$

such that

$$L_t = bt + \sigma W_t + \int_{|z|\leq 1} z\tilde{N}(t, dz) + \int_{|z|>1} zN(t, dz), \quad 0 \leq t < \infty. \tag{2.1}$$

- The small jumps part $\int_{|z|\leq 1} z\tilde{N}(t, \mathrm{d}z)$ is an L^2-martingale
- Large jumps part $\int_{|z|>1} zN(t, \mathrm{d}z)$ is of finite variation, but may have no finite moments

Basic theorem II.

Theorem (Levy-Khintchine formula)

Let L be a Lévy process, then

$$E\, e^{iuL_t} = e^{t\psi(u)},$$

$u \in \mathbb{R}$, $t \geq 0$ *and*

$$\psi(u) = ibu - \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}\setminus\{0\}} \left(e^{iuz} - 1 - iuz I_{[|z|<1]}\right) \nu(dz).$$

As an immediate result we can see that the law of a Lévy process L is uniquely determined by the law of L_1.

Pathwise properties

- Essentially driven by jumps, càdlàg paths.
- As an immediate result of Lévy-Itô decomposition we see that for every Lévy process

$$\sum_{0 \le s \le t} |\Delta L_s|^2 \mathrm{I}_{[|\Delta L_s| < 1]} < \infty, \quad \forall t \ge 0, a.s.$$

but we allow

$$\sum_{0 \le s \le t} |\Delta L_s| \mathrm{I}_{[|\Delta L_s| < 1]} = \infty, \quad \forall t \ge 0, a.s.$$

in which case L is of infinite variaton.

Modelling with Jump Processes

Outline of modelling phase

1. Making the series stationary
   - we assume that the nonstationarity is basically caused by variable intensity of trading,
   - overcome by appropriate time change.
2. Selecting a model
   - based on empirical facts (moments, variation, tail behavior).
3. Choosing a fitting procedure and get the parameters
   - if analytical density is known, MLE method is used,
   - otherwise GMM method based on characteristic function can be applied.

Variation

Remark

Let L be a Lévy process of the form (2.1), $\Delta_t^n = \{t_0, \ldots, t_n\}$ arbitrary partition of interval $[0, t]$

$$\sum_{\Delta_t^n} \left(L_{t_i} - L_{t_{i-1}}\right)^2 \xrightarrow{P} \sigma^2 t + \sum_{s \in [0,t]} \left[\Delta(L_s)\right]^2, \quad \|\Delta_t^n\| \to 0.$$

In other words, our estimator of volatility may be deformed by big jumps. Alternatives

- BiPower Variation (Barndorff 1998)

$$\frac{\pi}{2} \sum_{\Delta_t^n} |L_{t_i} - L_{t_{i-1}}||L_{t_{i-1}} - L_{t_{i-2}}|.$$

- Truncated Quadratic Variation (Hannig 2009)

$$\sum_{\Delta_t^n} \left(L_{t_i} - L_{t_{i-1}}\right)^2 \mathrm{I}_{\left[|L_{t_i} - L_{t_{i-1}}| < g(\Delta_{t_i})\right]}.$$

are both consistent estimators of $\sigma^2 t$.

Comparison of different estimates of standard deviations

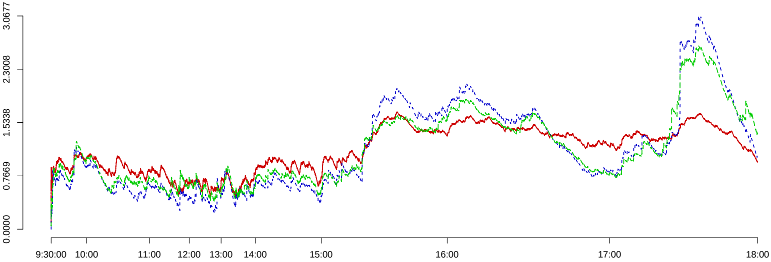

Figure: Transformed time: green line = Quadratic variation, red = truncated QV, blue = BiPower variation.

Normal Inverse Gaussian

- Process can be expressed as $L_t = B(T_t)$, where

$$T_t = \inf\left\{s > 0; W_s + \alpha s = \delta t\right\},$$

  and B_t is a Brownian motion with drift θ and volatility σ.
- Pure jump model with infinite variation.
- Exponential tail decay.
- Probability density in a closed (analytical) form (Bessel function), i.e. MLE possible.

Merton Jump-Diffusion

- Process can be expressed as

$$L_t = \alpha t + \sigma W_t + \sum_{i=1}^{N_t} Y_i, \quad t \geq 0,$$

  i.e. Brownian motion with big gaussian jumps.
- Tails a little heavier than gaussian.
- Probability density function can be expressed in a series expansion. We use first order approximation

$$f_{L_{\Delta t}}(x) = (1 - \lambda \Delta t) f_{W_{\Delta t}}(x) + \lambda \Delta t \left(f_{W_{\Delta t} + Y_1} \right)(x).$$

Estimation method

Maximum Likelihood method performed[1].

NIG model

Time scale	$\bar{\alpha}$	μ	σ	θ	
$\mathcal{T}$	0.080051	-0.00012	0.3499268	0.0001245	
$\overline{\mathcal{T}}$	0.090008	-0.00101	0.3468827	0.0010085	

Merton model

	μ	σ	γ	δ	λ
$\mathcal{T}$	-0.000201	0.087893	0.000260	0.6708204	0.316296
$\overline{\mathcal{T}}$	-0.000289	0.099935	0.000778	0.6708204	0.287545

Table: Comparison of maximum likelihood estimates.

[1] Estimation performed in software R. Quasi-Newton optimization method, which allows constraints of parameters, was used.

Graphical inference

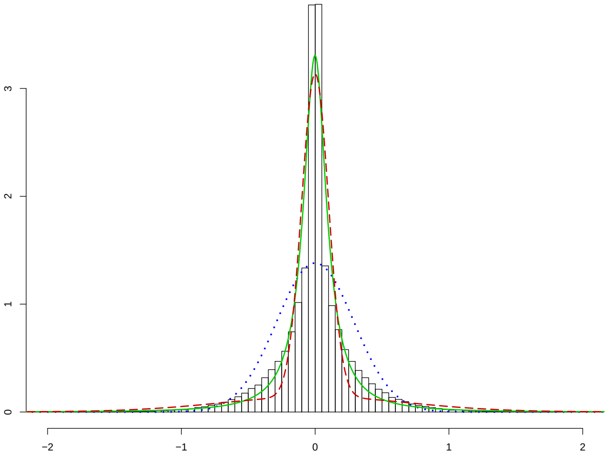

Figure: Estimated probability density function: green (solid) line = NIG, red (dashed) = Merton Jump, blue (dotted) = Gaussian.

Optimal Control

Model set-up I.

Consider an investor placing his money into two assets

- riskfree, paying interest rate r
- risky asset with dynamics

$$\mathrm{d}F_t = \alpha \mathrm{d}t + \sigma \mathrm{d}W_t + \int_{-\infty}^{\infty} z\tilde{N}(\mathrm{d}t, \mathrm{d}z). \tag{4.1}$$

An investor controls

- the number of F_t, $t \geq 0$ in his portfolio by Δ_t,
- consumption $C_t \geq 0$.

i.e. the dynamics of his portfolio is of the form

$$\mathrm{d}X_t = \Delta_t \left(\alpha \mathrm{d}t + \sigma \mathrm{d}W_t + \int_{-\infty}^{\infty} z\tilde{N}(\mathrm{d}t, \mathrm{d}z) \right) + rX_t \mathrm{d}t - C_t \mathrm{d}t \tag{4.2}$$

with $X(0) = x$, $\Delta_t \in \mathcal{F}_{t-}$ (predictable), $C_t \in \mathcal{F}_t$.

Model Set-up II.

The objective of an investor is

$$v(x) = \sup_{(\Delta_t, C_t) \in \mathcal{A}(x)} \int_0^\infty e^{-\beta t} \mathrm{E}\, U(C_t) \mathrm{d}t, \tag{4.3}$$

where $\mathcal{A}(x)$ is the set of admissible strategies, β a discount factor and U denotes a power utility function of the form

$$U(x) = \frac{x^{1-p}}{1-p}, \quad p > 1.$$

Notation

- $\theta_p(t) = \frac{\Delta_t}{X_{t-}}$ is the number of assets in the portfolio per one money unit at time t and let
- $c_t = \frac{C_t}{X_{t-}}$ denotes the proportional consumption.

Personal risk aversion

Assume geometric BM model, one needs to consider

- the maximal proportion of wealth an agent would invest. Example *toin coss*, winner takes 1.2 of a bet, agent's wealth is 1000000.

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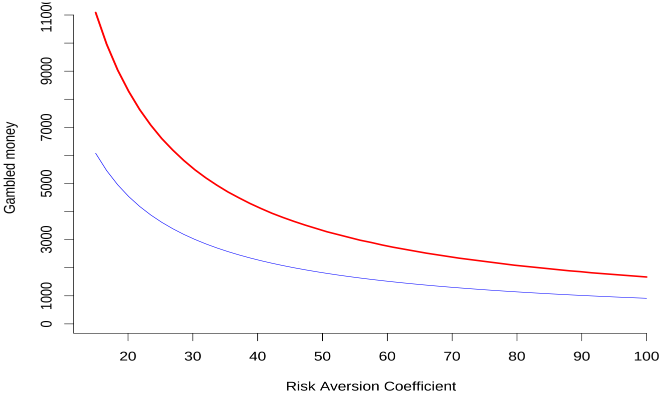

Figure: Maximal (red) and optimal (blue) invested amount of money.

Personal risk aversion

- draw-down probability $P_p(x) = x^{2p-1}$, It is the probability that the investor's discounted wealth will ever fall below fraction x of the initial wealth.

Example

- Logarithmic utility function: $P_1(x) = x$. The probability of losing $(1-x)$ percent of investment is x
- Power utility function for $p = 1/2$. An agent loses $(1-x)$ percent of investment with probability 1 for any $0 \leq x \leq 1$.

Theorem (Optimal Proportion and Consumption)

Assume the portfolio (4.2) and the objective (4.3). Let

$$\theta_p^* = argmin\, h(\theta_p) = argmin \Big\{ \alpha\theta_p(1-p) - \frac{1}{2}\sigma^2\theta_p^2 p(1-p) + \int_{-\infty}^{\infty} \left((1+\theta_p z)^{1-p} - 1 - \theta_p z(1-p)\right) \nu(dz) \Big\}.$$

Assume also that

$$\beta - r(1-p) - h(\theta_p^*) > 0. \tag{4.4}$$

Then

- θ_p^* *is the optimal proportion,*
- $c^* = (K(1-p))^{-1/p}$ *is the optimal consumption,*
- $v(z) = Kz^{1-p}$ *is the value function,*

where $K = \frac{1}{1-p}\left(\frac{\beta - r(1-p) - h(\theta_p^*)}{p}\right)^{-p}$.

A short comment on the theorem

- A similar theorem presented for geometric Lévy process with
$$\int_{\mathbb{R}} \nu(\mathrm{d}z) < \infty,$$
which is extremely restrictive, see (Framstad 1998). The authors considered power utility function with $0 < p < 1$, which describes an extremely aggressive investor.
- Assumption (4.4) grants that agent's consumption is positive and that his discounted well-being tends to zero as $t \to \infty$.

Merton proportion

Let us denote

- Merton proportion

$$\theta_p^{*M} = \frac{\alpha - r}{p\sigma^2},$$

- Merton consumption

$$c^{*M} = A(p) = \frac{\beta - r(1-p)}{p} - \frac{1}{2}\frac{(\alpha - r)^2}{\sigma^2}\frac{1-p}{p}.$$

How will the proportion and the consumption be changed after adding jumps into the model?

Optimal consumption and portfolio - preparation

An empirical study was performed.[2] Futures is a martingale with respect to the risk neutral measure. To compare optimal portfolios based on different models we:

- standardized the data, so that $\sigma \approx 30\%$,
- α is set as 7%.
- Assume that our (Futures) returns behave like stock log-returns but with different volatility and drift.

[2] Computation performed in software R. Integrals numerically evaluated, adaptive quadrature applied. Nonlinear equation solved by Newton method.

Optimal consumption and portfolio - results

	Model	Naive Merton	NIG	Merton Jump
$p = 4$	θ_p^*	0.141150	0.104904	0.086328
	c_p^*	0.042647	0.042329	0.041673
$p = 10$	θ_p^*	0.056460	0.047433	0.035010
	c_p^*	0.029270	0.029186	0.028809
$p = 40$	θ_p^*	0.014115	0.012392	0.008806
	c_p^*	0.022344	0.022328	0.022220
$p = 70$	θ_p^*	0.008066	0.007121	0.005036
	c_p^*	0.021342	0.021333	0.021270

Table: Comparison of optimal proportion and consumption for Merton and Jump models. $\beta = 10\,\%$, $r = 2\,\%$, $\alpha = 7\,\%$, $\sigma = 0.3$.

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Thank you for attention