The Numerical Solution of Compressible Flows in Time Dependent Domains

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1. ALE Formulation of the Continuous Problem
   - Continuous Problem
   - ALE Formulation

2. Discontinuous Galerkin Discretization
   - Space Semidiscretization
   - Semi-implicit Time Discretization

Let $\Omega_t \subset I\!R^2$ be a bounded domain depending on time t with boundary $\partial\Omega_t = \Gamma_I \cup \Gamma_O \cup \Gamma_{W_t}$.

Continuous Problem

Find $\mathbf{w}(\cdot,t) : \Omega_t \longrightarrow \mathbb{R}^4$ such that

$$\frac{\partial \mathbf{w}}{\partial t} + \sum_{s=1}^{2} \frac{\partial \mathbf{f}_s(\mathbf{w})}{\partial x_s} = 0$$

where

$$\mathbf{w} = (\rho, \rho v_1, \rho v_2, e)^{\mathrm{T}} \in \mathbb{R}^4,$$

$$\mathbf{f}_s(w) = (\rho v_s, \rho v_1 v_s + \delta_{1s} p, \rho v_2 v_s + \delta_{2s} p, (e+p) v_s)^{\mathrm{T}},$$

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We add the thermodynamical relation

$$p = (\gamma - 1)(e - \rho|v|^2/2).$$

and boundary conditions:

Γ_I, Γ_O : as in the FVM,

Γ_{W_t} : $\mathbf{v} \cdot \mathbf{n} = \boldsymbol{z} \cdot \mathbf{n}$, where $\boldsymbol{z}$ is the velocity of the moving wall.

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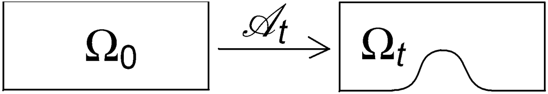


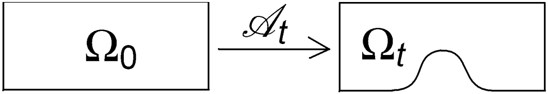
$$\mathscr{A}_t : \overline{\Omega}_0 \to \overline{\Omega}_t,$$
$$\mathscr{A}_t : \boldsymbol{X} \mapsto \boldsymbol{x} = \boldsymbol{x}(\boldsymbol{X}, t).$$

We define the ALE velocity:

$$\begin{aligned} \tilde{\boldsymbol{z}}(\boldsymbol{X}, t) &= \frac{\partial}{\partial t}\mathscr{A}_t(\boldsymbol{X}), \qquad \boldsymbol{X} \in \overline{\Omega}_0 \\ \boldsymbol{z}(\boldsymbol{x}, t) &= \tilde{\boldsymbol{z}}(\mathscr{A}_t^{-1}(\boldsymbol{x}), t), \quad \boldsymbol{x} \in \overline{\Omega}_t \end{aligned} \tag{1}$$

and the ALE derivative of a function $f = f(\boldsymbol{x}, t)$ defined in Ω_t:

$$\begin{aligned} \frac{D^A f}{Dt}(\boldsymbol{x}, t) &= \frac{\partial \tilde{f}}{\partial t}(\boldsymbol{X}, t)|_{X = \mathscr{A}_t^{-1}(\boldsymbol{x})}, \\ \tilde{f}(\boldsymbol{X}, t) &= f(\mathscr{A}_t(\boldsymbol{X}), t), \quad \boldsymbol{X} \in \Omega_0. \end{aligned} \tag{2}$$



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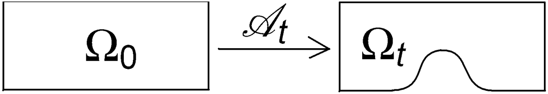

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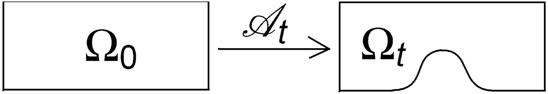

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Lemma

$$1)\ \frac{D^A f}{Dt} = \frac{\partial f}{\partial t} + \boldsymbol{z} \cdot \nabla f,$$

$$2)\ \frac{D^A f}{Dt} = \frac{\partial f}{\partial t} + \mathrm{div}(\boldsymbol{z} f) - f\,\mathrm{div}\boldsymbol{z}.$$

Using this lemma, we can reformulate the Euler equations:

Formulation 1:

$$\frac{\partial \mathbf{w}}{\partial t} + \sum_{s=1}^{2} \frac{\partial \mathbf{f}_s(\mathbf{w})}{\partial x_s} = 0 \iff \frac{D^A \mathbf{w}}{Dt} + \sum_{s=1}^{2} \frac{\partial \mathbf{f}_s(\mathbf{w})}{\partial x_s} - \boldsymbol{z} \cdot \nabla \mathbf{w} = 0.$$

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Here $\mathbf{g}_s, s = 1, 2$, are modified inviscid fluxes

$$\mathbf{g}_s(w) = \mathbf{f}_s(w) - z_s\mathbf{w}.$$

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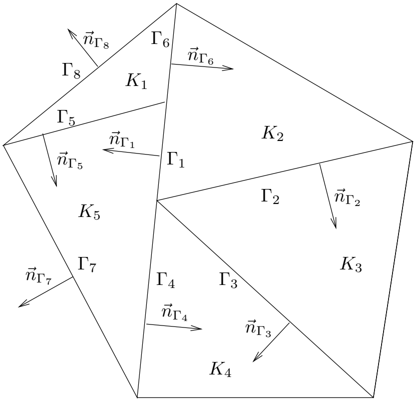
- By $\mathscr{F}_h$ we denote the set of all edges of $\mathscr{T}_h$. For a given edge $\Gamma \in \mathscr{F}_h$ we define a unit normal $\mathbf{n}_\Gamma$.



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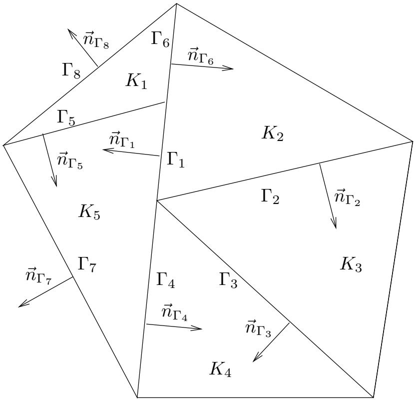

For each interior face $\Gamma \in \mathscr{F}_h$ there exist two neighbours $K_\Gamma^{(L)}, K_\Gamma^{(R)} \in \mathscr{T}_h$. We use the convention that $\mathbf{n}_\Gamma$ is the outer normal to the element $K_\Gamma^{(L)}$.

$$v^{(L)} = \text{trace of } v|_{K_\Gamma^{(L)}} \text{ on } \Gamma,$$
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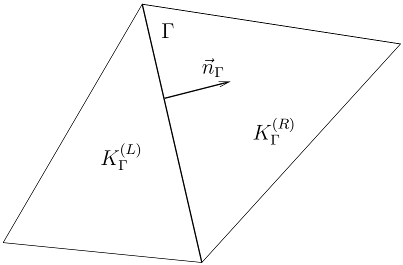

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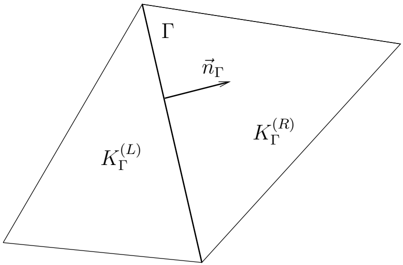

- Over $\mathscr{T}_h$ we define the *broken Sobolev space*

$$H^k(\Omega, \mathscr{T}_h) = \{v; v|_K \in H^k(K)\ \forall K \in \mathscr{T}_h\}$$

- We discretize the continuous problem in the space of discontinuous piecewise polynomial functions

$$S_h = \{v; v|_K \in P_p(K)\ \forall K \in \mathscr{T}_h\},$$

  where $P_p(K)$ is the space of all polynomials on K of degree $\leq p$.

- In order to derive a variational formulation, we multiply the Euler equations by a test function $\boldsymbol{\varphi} \in H^2(\Omega, \mathscr{T}_h)$, apply Green's theorem on individual elements and sum over all elements.

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$$\frac{D^A \mathbf{w}}{Dt} + \sum_{s=1}^{2} \frac{\partial \mathbf{f}_s(\mathbf{w})}{\partial x_s} - \boldsymbol{z} \cdot \nabla \mathbf{w} = 0$$

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- In the second term incorporate a numerical flux $\mathbf{H}$:

$$\int_\Gamma \sum_{s=1}^{2} \mathbf{f}_s(\mathbf{w}) n_\Gamma^{(s)} \cdot \boldsymbol{\varphi} \, dS \approx \int_\Gamma \mathbf{H}_f(\mathbf{w}|_\Gamma^{(L)}, \mathbf{w}|_\Gamma^{(R)}, \mathbf{n}_\Gamma) \cdot \boldsymbol{\varphi} \, dS,$$

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Formulation 1: Semi-implicit Time Discretization

$$\left(\frac{D^A \mathbf{w}_h}{Dt}, \boldsymbol{\varphi}\right) + b_h(\mathbf{w}_h, \boldsymbol{\varphi}) + c_h(\mathbf{w}_h, \boldsymbol{\varphi}) = 0$$

- A fully implicit scheme requires the solution of a nonlinear system. In the semi-implicit scheme we linearize the nonlinear terms using their specific properties.
- We solve only one linear system per time level.

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The time derivative can be approximated by the finite difference:

$$\frac{D^A \mathbf{w}_h}{Dt}(\boldsymbol{x}, t_{n+1}) = \frac{\partial \widetilde{\mathbf{w}}_h}{\partial t}(\boldsymbol{X}, t_{n+1})|_{X = \mathscr{A}^{-1}_{t_{n+1}}(\boldsymbol{x})} \approx \frac{\widetilde{\mathbf{w}}_h^{n+1}(\boldsymbol{X}) - \widetilde{\mathbf{w}}_h^{n}(\boldsymbol{X})}{\tau_n}.$$

Formulation 1: Semi-implicit Time Discretization

$$\left(\frac{D^A \mathbf{w}_h}{Dt}, \boldsymbol{\varphi}\right) + b_h^{(1)}(\mathbf{w}_h, \boldsymbol{\varphi}) + c_h^{(1)}(\mathbf{w}_h, \boldsymbol{\varphi}) = 0$$

Convective terms:

$$-\sum_{K\in\mathscr{T}_h} \int_K \sum_{s=1}^{2} \mathbf{f}_s(\mathbf{w}^{n+1}) \cdot \frac{\partial \boldsymbol{\varphi}}{\partial x_s}\, dx + \sum_{\Gamma\in\mathscr{F}_h} \int_\Gamma \mathbf{H}_f(\mathbf{w}^{n+1}|_\Gamma^{(L)}, \mathbf{w}^{n+1}|_\Gamma^{(R)}, \mathbf{n}_\Gamma) \cdot \boldsymbol{\varphi}\, dS$$

Formulation 1: Semi-implicit Time Discretization

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It holds that

$$\mathbf{f}_s(\mathbf{w}) = \mathbb{A}_s(\mathbf{w})\mathbf{w}, \text{ where } \mathbb{A}_s(\mathbf{w}) = \frac{D\mathbf{f}_s(\mathbf{w})}{D\mathbf{w}}.$$

We therefore linearize

$$\mathbf{f}_s(\mathbf{w}^{n+1}) \approx \mathbb{A}_s(\mathbf{w}^n)\mathbf{w}^{n+1}.$$

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We choose the Vijayasundaram numerical flux

$$\mathbf{H}_f(\mathbf{w}^{(L)}, \mathbf{w}^{(R)}, \mathbf{n}) = \mathbb{P}_f^{+}(\langle \mathbf{w} \rangle, \mathbf{n})\, \mathbf{w}^{(L)} + \mathbb{P}_f^{-}(\langle \mathbf{w} \rangle, \mathbf{n})\, \mathbf{w}^{(R)}$$

and linearize

$$\mathbf{H}_f(\mathbf{w}^{n+1}|_\Gamma^{(L)}, \mathbf{w}^{n+1}|_\Gamma^{(R)}, \mathbf{n}_\Gamma) \approx \mathbb{P}_f^{+}(\langle \mathbf{w}^n \rangle, \mathbf{n}_\Gamma)\, \mathbf{w}^{n+1}|_\Gamma^{(L)} + \mathbb{P}_f^{-}(\langle \mathbf{w}^n \rangle, \mathbf{n}_\Gamma)\, \mathbf{w}^{n+1}|_\Gamma^{(R)}.$$

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ALE terms are linear, we treat them implicitly:

$$-\sum_{K \in \mathscr{T}_h} \int_K \sum_{s=1}^{2} z_s \frac{\partial \mathbf{w}_h^{n+1}}{\partial x_s} \boldsymbol{\varphi} \, dx$$

Formulation 2: Semi-implicit Time Discretization

$$\left(\frac{D^A \mathbf{w}_h}{Dt}, \boldsymbol{\varphi}\right) + b_h^{(2)}(\mathbf{w}_h, \boldsymbol{\varphi}) + c_h^{(2)}(\mathbf{w}_h, \boldsymbol{\varphi}) = 0$$

Convective terms:

$$-\sum_{K \in \mathscr{T}_h} \int_K \sum_{s=1}^{2} \mathbf{g}_s(\mathbf{w}^{n+1}) \cdot \frac{\partial \boldsymbol{\varphi}}{\partial x_s} \, dx + \sum_{\Gamma \in \mathscr{F}_h} \int_\Gamma \mathbf{H}_g(\mathbf{w}^{n+1}|_\Gamma^{(L)}, \mathbf{w}^{n+1}|_\Gamma^{(R)}, \mathbf{n}_\Gamma) \cdot \boldsymbol{\varphi} \, dS$$

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$$\mathbf{g}_s(\mathbf{w}) = \mathbf{f}_s(\mathbf{w}) - z_s\mathbf{w} = \left(\mathbb{A}_s(\mathbf{w}) - z_s\mathbb{I}\right)\mathbf{w}, \text{ where } \mathbb{A}_s(\mathbf{w}) = \frac{D f_s(\mathbf{w})}{D\mathbf{w}}.$$

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We choose the Vijayasundaram numerical flux for **g**

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$$\mathbf{H}_{VS}(\mathbf{w}^{n+1}|_\Gamma^{(L)}, \mathbf{w}^{n+1}|_\Gamma^{(R)}, \mathbf{n}_\Gamma) \approx \mathbb{P}^{+}\left(\langle \mathbf{w}^n\rangle, \mathbf{n}_\Gamma\right) \mathbf{w}^{n+1}|_\Gamma^{(L)} + \mathbb{P}^{-}\left(\langle \mathbf{w}^n\rangle, \mathbf{n}_\Gamma\right) \mathbf{w}^{n+1}|_\Gamma^{(R)}.$$

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Boundary Conditions - Inlet, outlet

- BCs at Γ_I, Γ_O are imposed by choosing the "outside" boundary state $\mathbf{w}^{(R)}$ in the numerical flux.
- Appropriate coordinate system, neglecting the tangential derivatives and linearization give:

$$\frac{\partial \mathbf{q}}{\partial t} + \frac{\partial \mathbf{f}_1(\mathbf{q})}{\partial \tilde{x}_1} = 0 \Rightarrow \frac{\partial \mathbf{q}}{\partial t} + \mathbb{A}_1(\mathbf{q}_i)\frac{\partial \mathbf{q}}{\partial \tilde{x}_1} = 0,$$

- We seek q_j such that the linearized problem has sense.
- Eigenvectors of $\mathbb{A}_1(\mathbf{q}_i)$ form a basis and eigenvalues are real.

$$\mathbf{q}_i = \sum_{s=1}^{4} \alpha_s \mathbf{r}_s, \quad \mathbf{q}_j = \sum_{s=1}^{4} \beta_s \mathbf{r}_s.$$

- Substitution into the system reduces it to four independent equations that have an analytical solution.

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- Conclusion: depending on the sign of eigenvalues of $\mathbb{A}_1(\mathbf{q}_i)$ we either prescribe or extrapolate α_s, β_s
- When prescribing β_s, we evaluate from an appropriate state (e.g. far-field).
- Finally

$$\mathbf{q}_i = \mathbb{T}\boldsymbol{\alpha} \Rightarrow \boldsymbol{\alpha} = \mathbb{T}^{-1}\mathbf{q}_i,$$
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$$\mathbf{q}_j := \sum_{s=1}^{4} \gamma_s \mathbf{r}_s = \mathbb{T}\boldsymbol{\gamma}, \text{ where } \gamma_s = \begin{cases} \alpha_s, & \lambda_s \geq 0, \\ \beta_s, & \lambda_s < 0. \end{cases}$$

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Boundary Conditions - Moving walls

- On moving impermeable walls, we prescribe $\mathbf{v} \cdot \mathbf{n} = \boldsymbol{z} \cdot \mathbf{n}$
- We prescribe the numerical flux:

$$\mathbf{H}_f(\mathbf{w}^{(L)}, \mathbf{w}^{(R)}, \mathbf{n}) \approx \sum_{s=1}^{2} \mathbf{f}_s(\mathbf{w}) n_s$$

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$$\sum_{s=1}^{2} \mathbf{f}_s(\mathbf{w}) n_s = n_1 \begin{pmatrix} \rho v_1 \\ \rho v_1^2 + p \\ \rho v_1 v_2 \\ (e+p) v_1 \end{pmatrix} + n_2 \begin{pmatrix} \rho v_2 \\ \rho v_1 v_2 \\ \rho v_2^2 + p \\ (e+p) v_2 \end{pmatrix}$$

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Mesh movement

Mesh constraints

Smoothness, no crossover, efficiency, memory use.

- Velocity smoothing vs Coordinate smoothing
- Continuous vs Discrete

Poisson, Linear elasticity, spring models, entropy-based...

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Shock Capturing

In transonic and supersonic flows it is common that solutions develop discontinuities. In these cases spurious under and overshoots occur on elements near the discontinuity. Especially in the semi-implicit case, it is desirable to avoid such phenomena. We therefore locally add artificial diffusion to suppress these effects.

Shock Capturing

To the scheme we add two artificial viscosity forms. Internal diffusion:

$$\Phi_h^1(\mathbf{w}^n, \mathbf{w}^{n+1}, \boldsymbol{\varphi}) = \nu_1 \sum_{K \in \mathscr{T}_h} h_K G^n(K) \int_K \nabla \mathbf{w}^{n+1} \cdot \nabla \boldsymbol{\varphi} \, \mathrm{d}x$$

with $\nu_1 = O(1)$ a given constant. Here $G(i)$ is a discontinuity indicator which measures interelement jumps of the solution:

$$G^k(K) = \begin{cases} 1 & \text{if interelement jumps of } \mathbf{w}^n \text{ are large near } K, \\ 0 & \text{otherwise.} \end{cases}$$

Shock Capturing

Interelement diffusion:

$$\Phi_h^2(\mathbf{w}^n, \mathbf{w}^{n+1}, \boldsymbol{\varphi}) = \nu_2 \sum_{\Gamma \in \mathscr{F}_h} \langle G \rangle_\Gamma \int_\Gamma [\mathbf{w}^{n+1}] \cdot [\boldsymbol{\varphi}] \, \mathrm{d}S,$$

with $\nu_2 = O(1)$ a given constant. This term allows to strengthen the influence of neighbouring elements and improves the behavior of the method in the case, when strongly unstructured and/or anisotropic meshes are used.

Thank you for your attention