Automatic Adaptivity for Evolutionary Problems Based on the Rothe's Method

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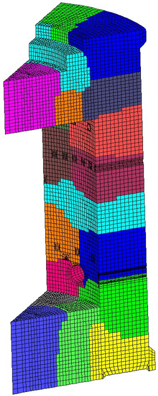
Acknowledgement

J. Kruis, P. Mayer, P. Sváček, T. Vejchodský, J. Zítko, . . .

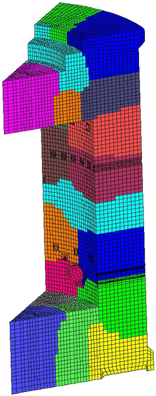
Students:
D. Andrš, J. Červený, L. Dubcová, P. Kůs, M. Lazar, M. Šimko, S. Vyvialová, M. Zítka

Multiphysics problems



- Highly nonlinear, time-depedent PDE system

Multiphysics problems



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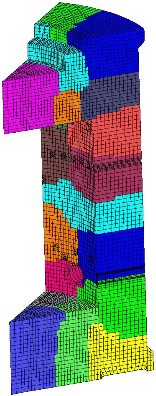
- Automatic adaptivity (*hp*-adaptivity) needed

Multiphysics problems



- Highly nonlinear, time-depedent PDE system
- Automatic adaptivity (*hp*-adaptivity) needed
- Analytical error estimates not available

hp-FEM

- *hp*-FEM = FEM with variable h and p

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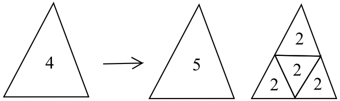
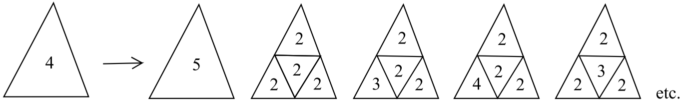
- Automatc adaptivity in *hp*-FEM differs from standard FEM
  - *"Reduced scheme"*: Analyticity of approximate solution (Melenk, ...)
    Decision between *p* and *h* (two refinement candidates)



  - *"Full scheme"*: Enrichment of FE space (Demkowicz, our group, ...)
    Many refinement candidates possible ($\approx 10^2$ in 2D, 10^3 in 3D)



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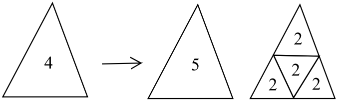

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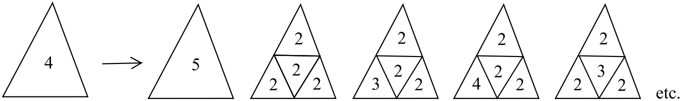
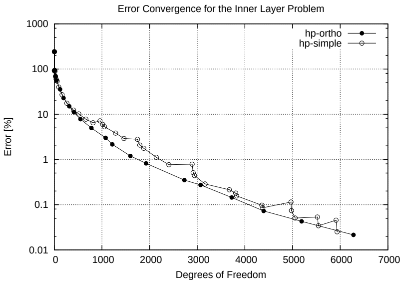
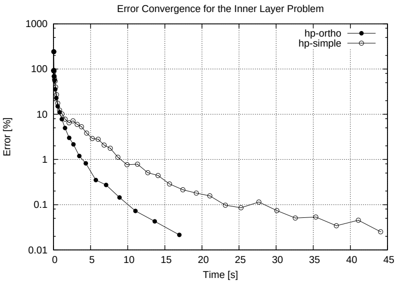


- Analytical error estimates not practical

NOTE: Full scheme refinements reproducable by multiple steps of reduced scheme





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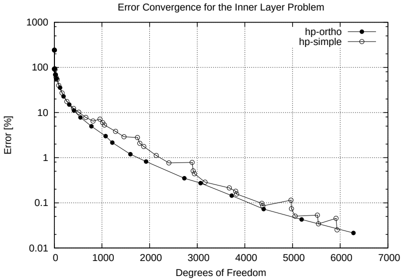
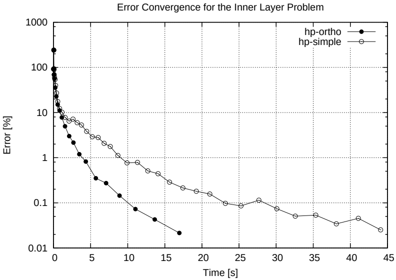




Reduced scheme: Simpler adaptivity but more refinement steps needed

Full scheme: Fewer refinement steps $\longrightarrow$ less work on matrix level

Goals and methodology

Solve a wide range of multiphysics problems with controlled accuracy in spacetime

- thermoelasticity, microwave heating, induction heating, flow of magnetorheological fluids, MHD, . . .
- need for space-time adaptive *hp*-FEM

We need error estimates that

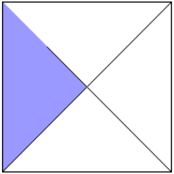
- work for a wide range of PDE problems
- are computable (free of dubious constants)
- tell the shape of the error in elements

Methodology:

- approximations with arbitrary-level hanging nodes
- PDE-independent error estimators
- adaptive multi-mesh *hp*-FEM
- space-time *hp*-FEM on dynamical meshes

Forced refinements

- Regular mesh



Forced refinements

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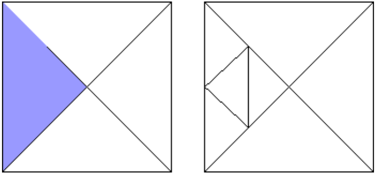

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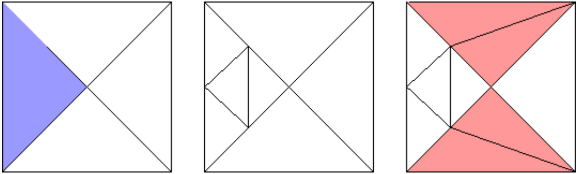

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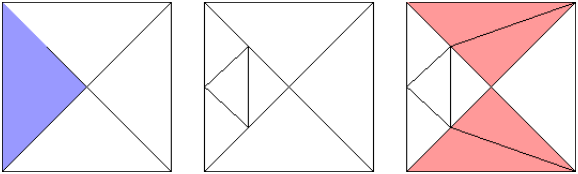
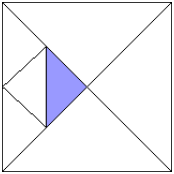


- One-level hanging nodes (1-irregular mesh)



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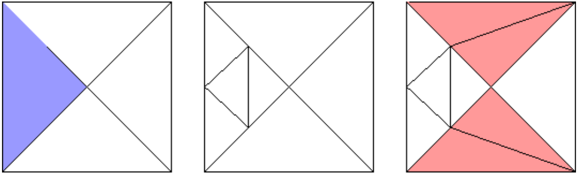

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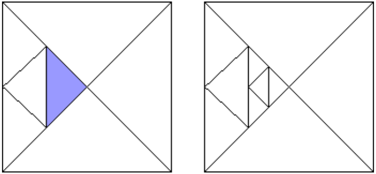

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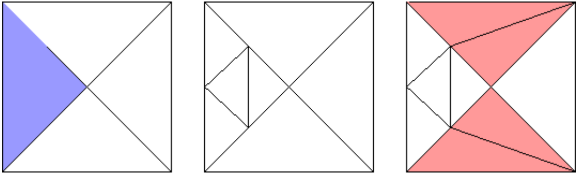

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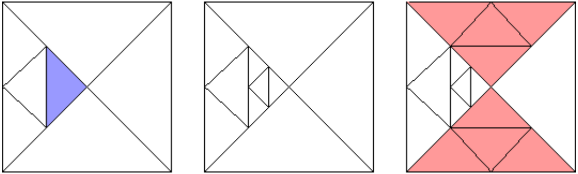
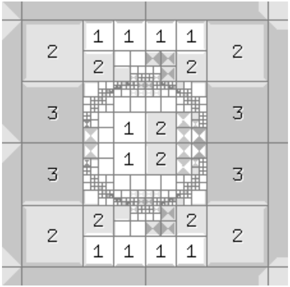


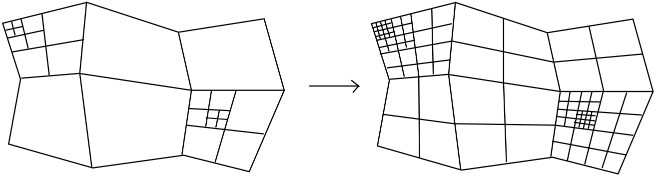
- Forced refinements
  - introduce unnecessary DOF
  - spoil element shapes
  - have recursive nature
  - cause incompatible refinements in the multi-mesh *hp*-FEM
- Arbitrary-level hanging nodes:



P.S., L. Dubcova, I. Dolezel: Adaptive hp-FEM with Arbitrary-Level Hanging Nodes for Time-Harmonic Maxwell's Equations, J. Comput. Appl. Math., submitted.
P.S., J. Cerveny, I. Dolezel: Arbitrary-Level Hanging Nodes and Automatic Adaptivity in the hp-FEM, Math. Comput. Simul. 77 (2008), 117 - 132.

PDE-independent error estimator

- Based on *approximation pairs* with different orders of accuracy
- Embedded higher-order ODE methods (Fehlberg, Hairer, Wanner et al.)



- Global OG projection on coarse mesh $\rightarrow$ shape of error
- Local OG projections on coarse mesh $\rightarrow$ optimal refinement candidates

P.S., M. Simko: PDE-Independent Adaptivity Algorithm for the hp-FEM Based on Approximation Pairs, J. Comput. Appl. Math., submitted.

Illustration: elliptic problem

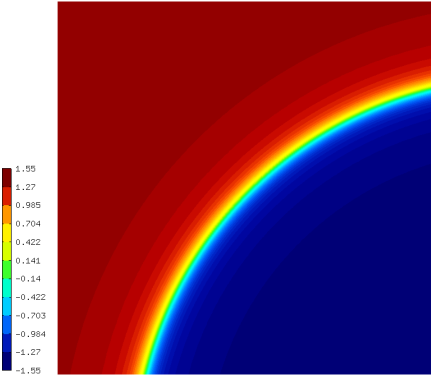
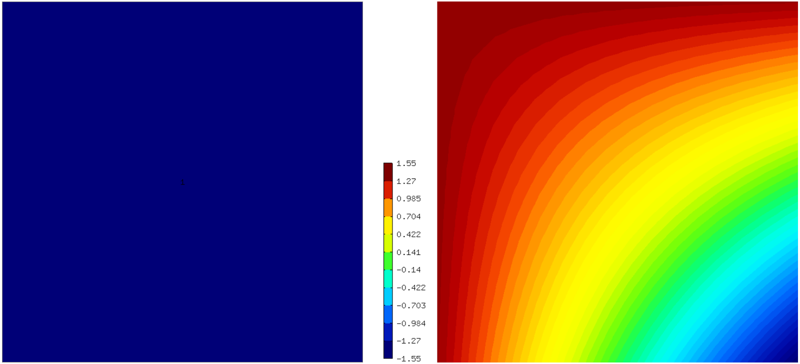


Illustration: elliptic problem (step 1)



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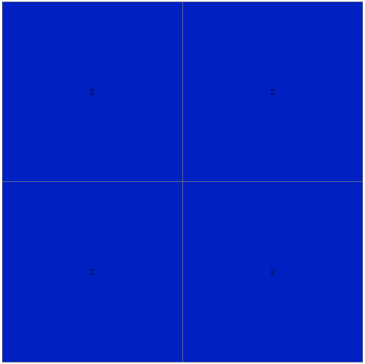

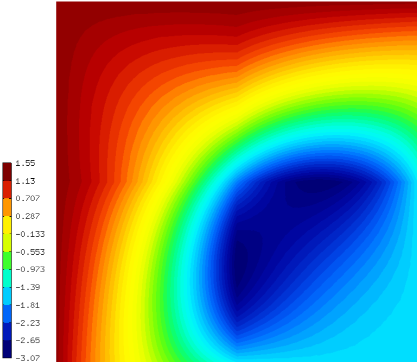

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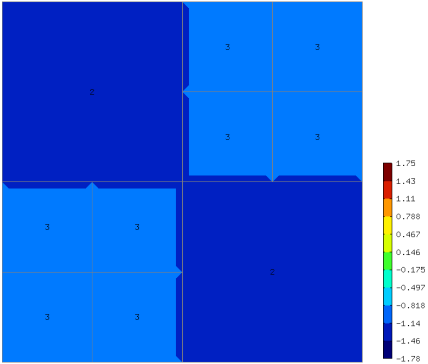


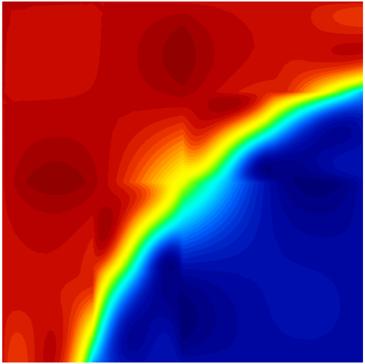


Illustration: elliptic problem (step 4)

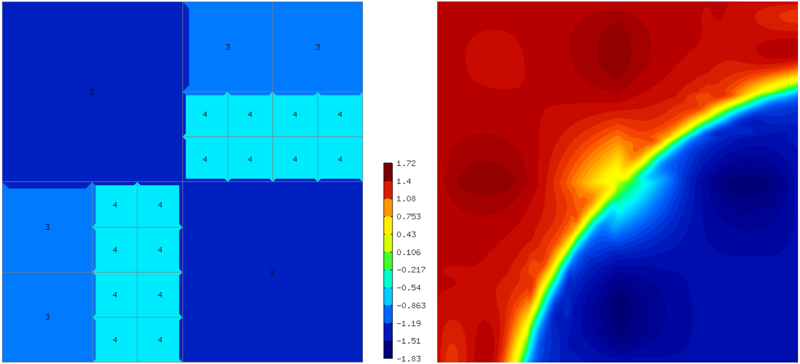
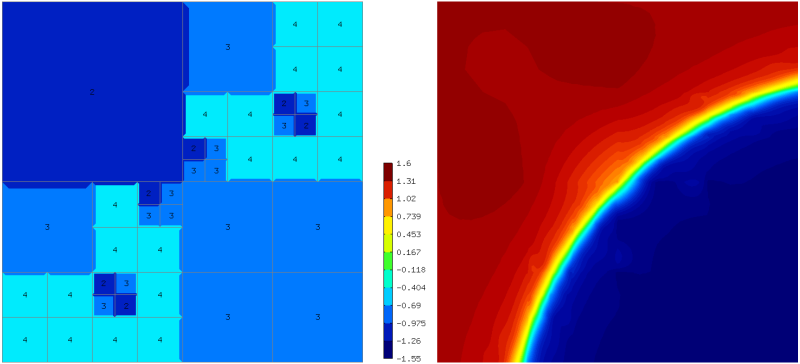


Illustration: elliptic problem (step 5)



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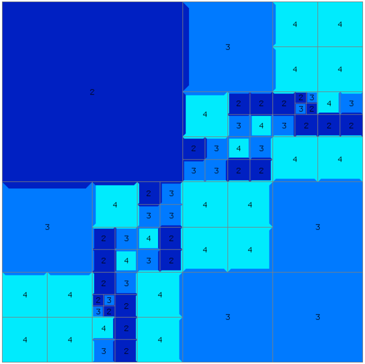

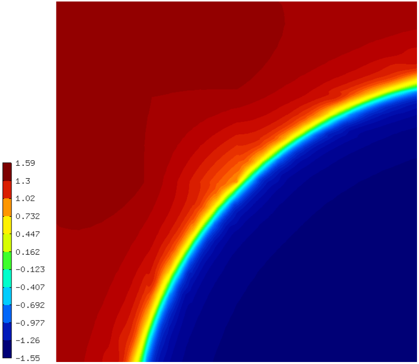

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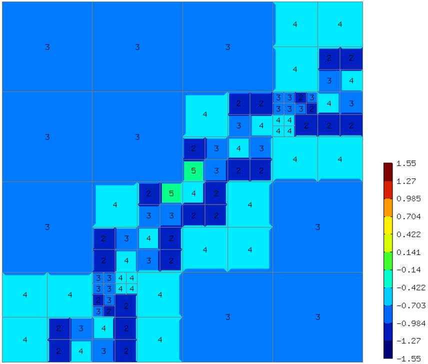

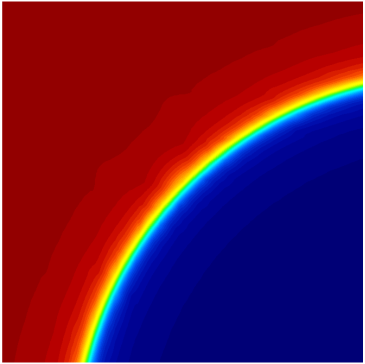

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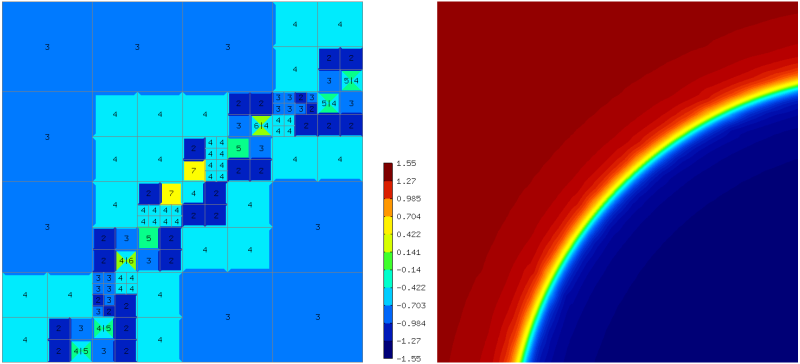
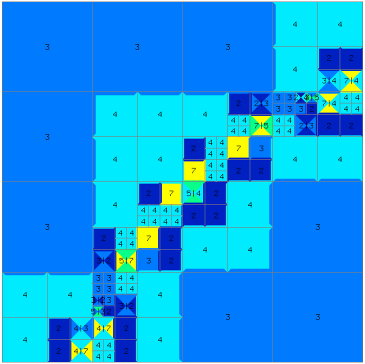




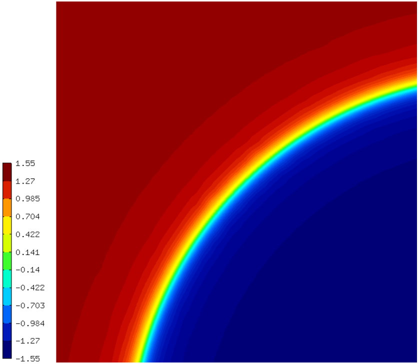
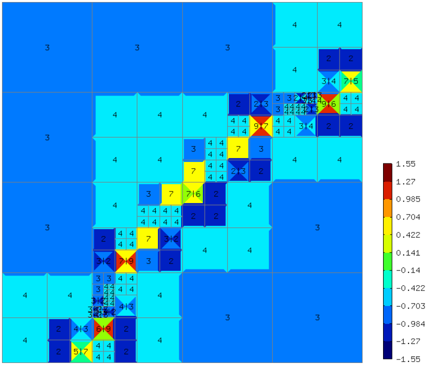

1.55
1.27
0.985
0.704
0.422
0.141
-0.14
-0.422
-0.703
-0.984
-1.27
-1.55

Illustration: elliptic problem (step 10)



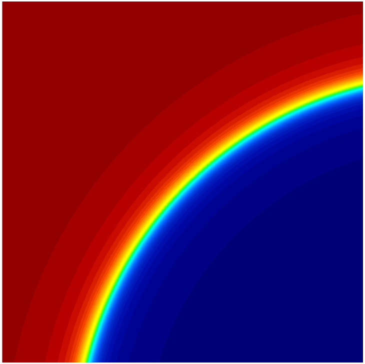
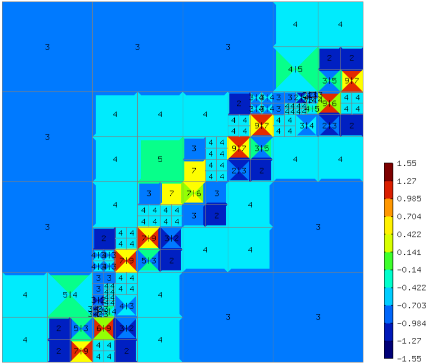


Illustration: elliptic problem (step 11)



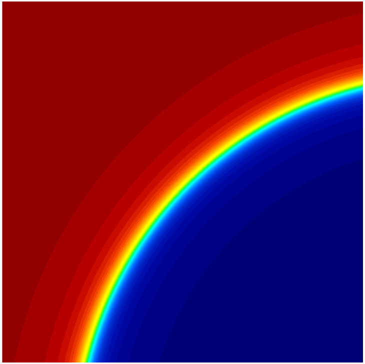
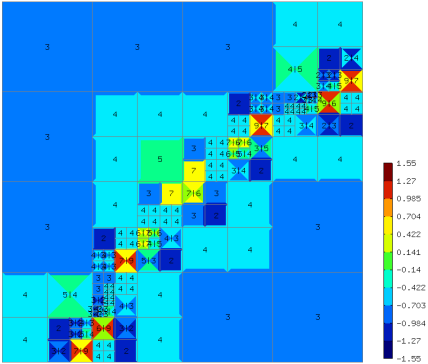


Illustration: elliptic problem (step 12)



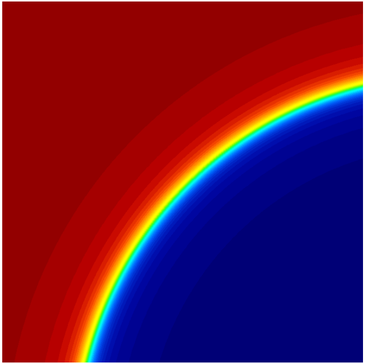
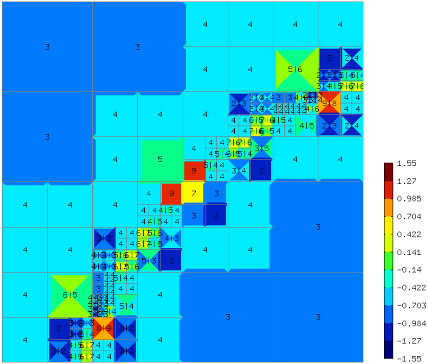
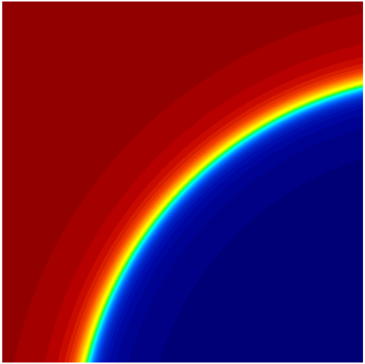


Illustration: elliptic problem (step 13)





hp-FEM vs. standard FEM

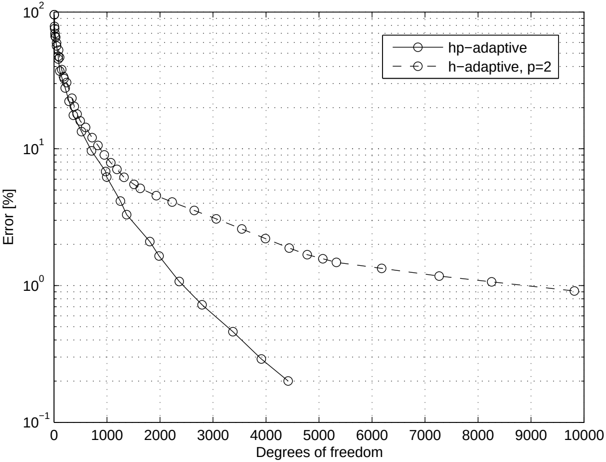
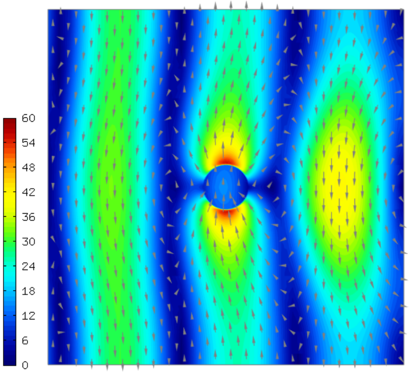


Illustration: waveguide problem



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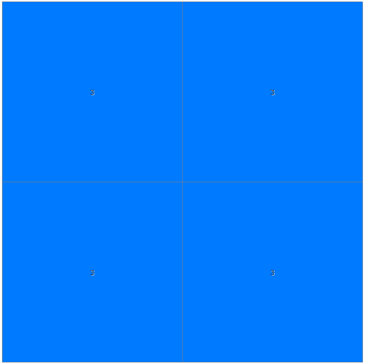


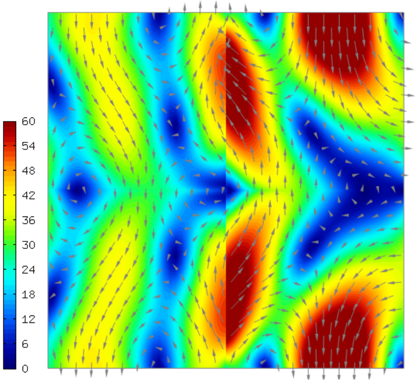
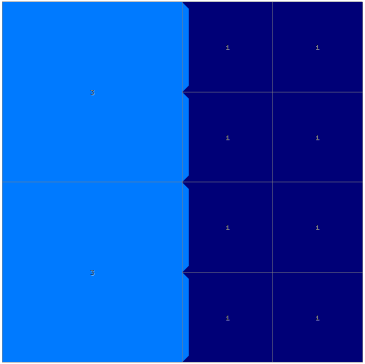


Illustration: waveguide problem (step 2)



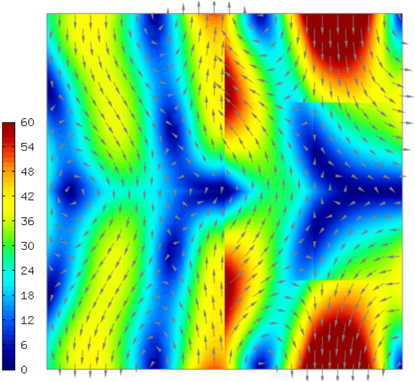
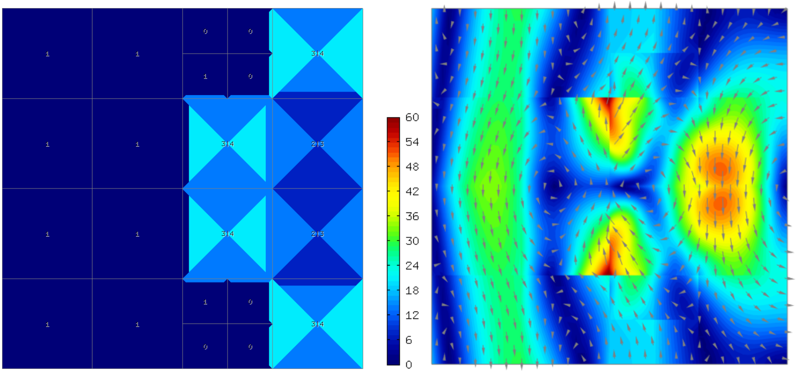


Illustration: waveguide problem (step 3)



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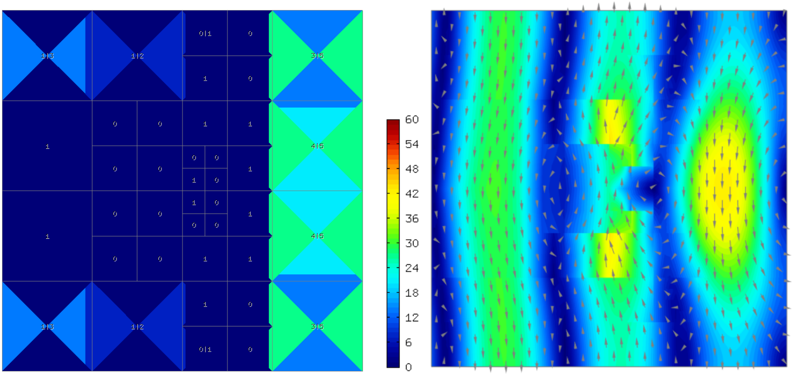

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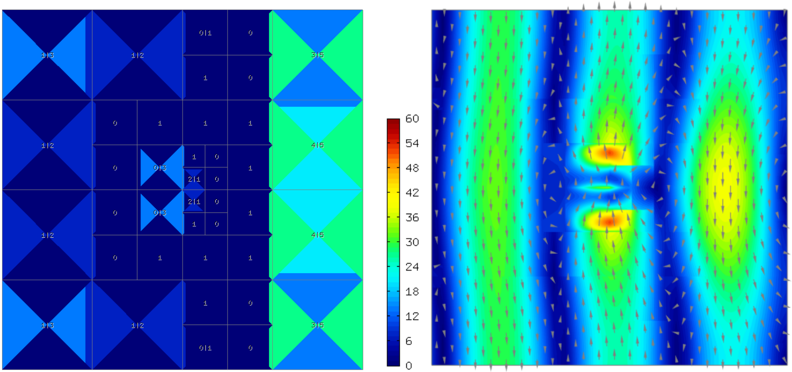
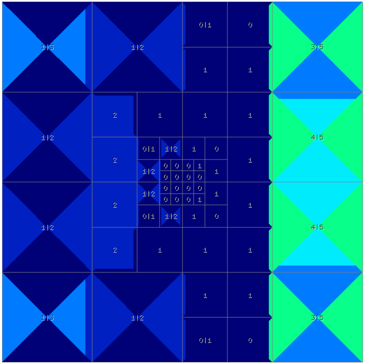


Illustration: waveguide problem (step 6)



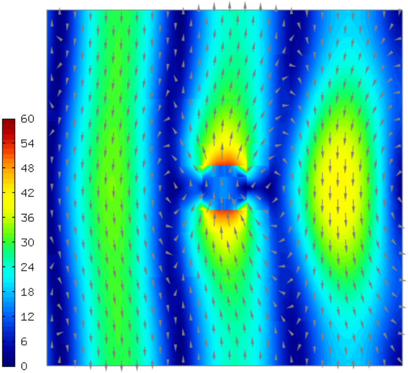
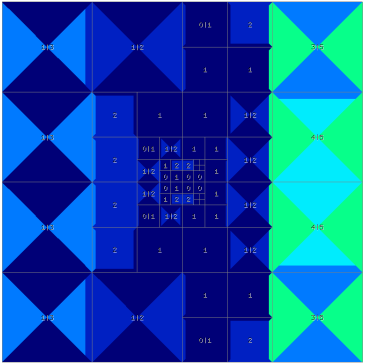


Illustration: waveguide problem (step 7)



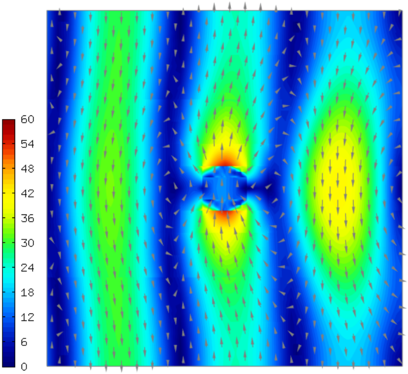


Illustration: waveguide problem (step 8)

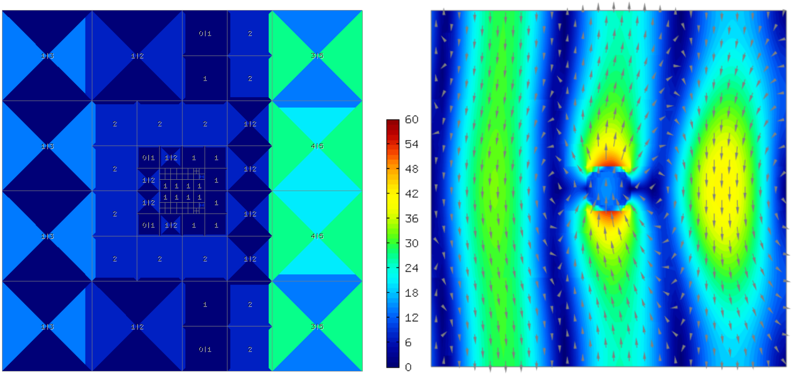
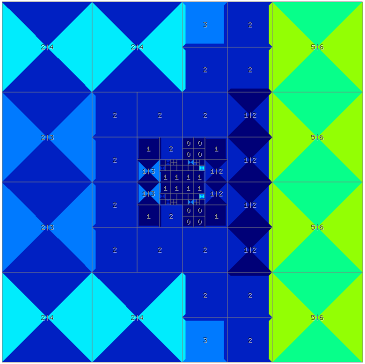


Illustration: waveguide problem (step 9)



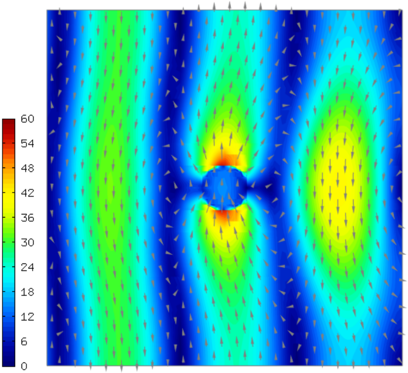
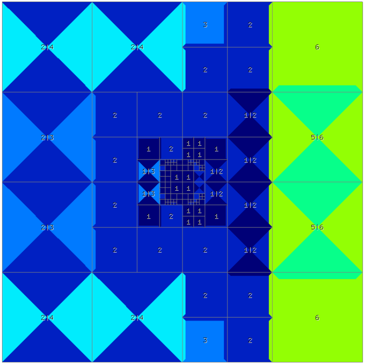


Illustration: waveguide problem (step 10)



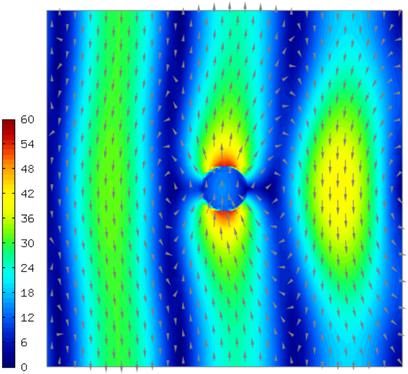


Illustration: waveguide problem (step 11)

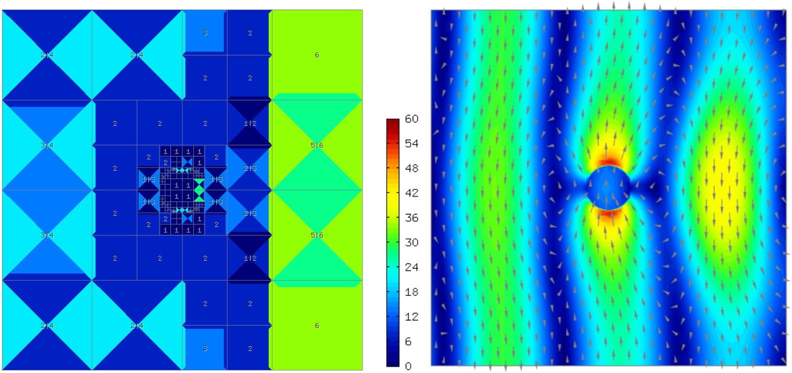


Illustration: waveguide problem (step 12)

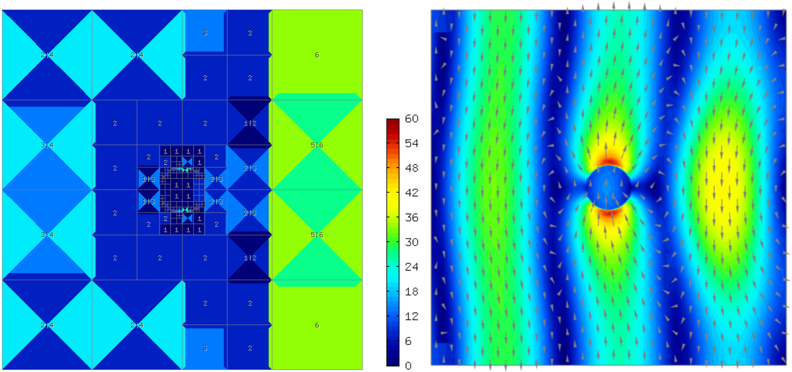


Illustration: waveguide problem (step 13)

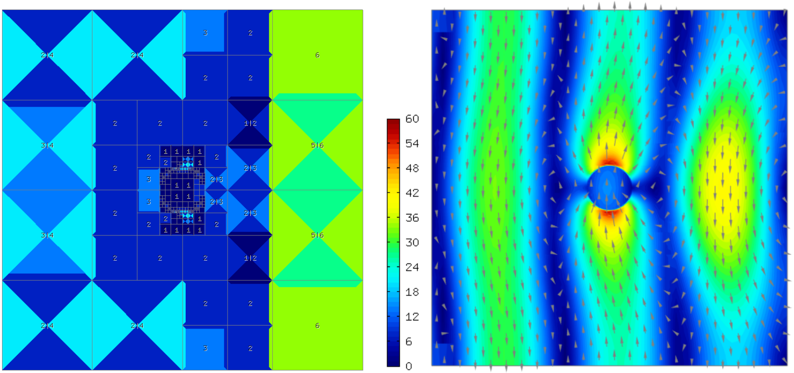


Illustration: waveguide problem (step 14)

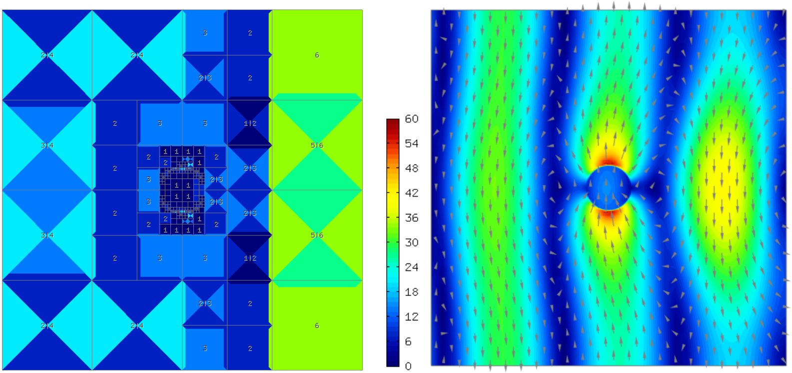


Illustration: waveguide problem (step 15)

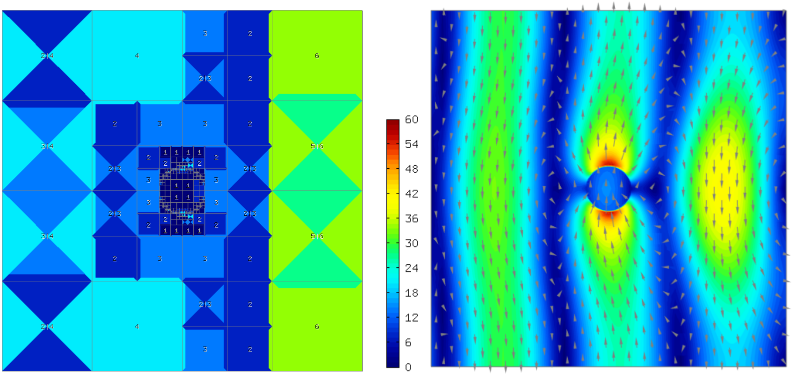
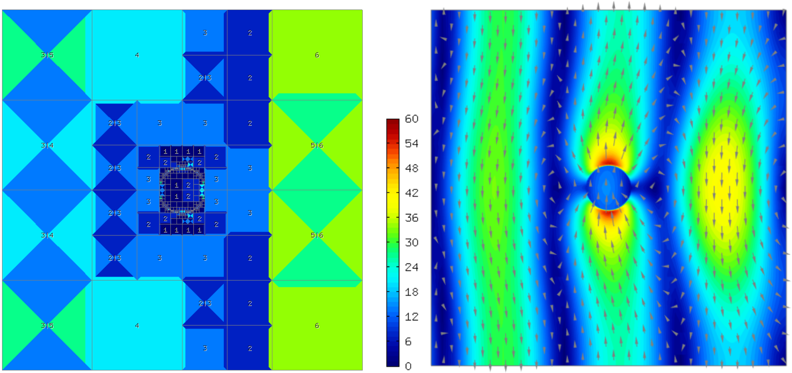
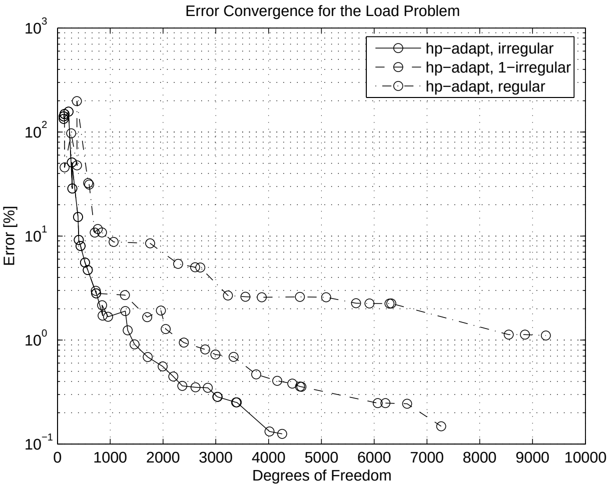


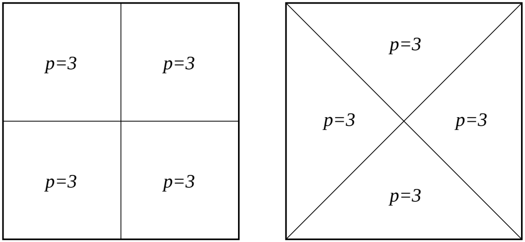
Illustration: waveguide problem (step 16)



Mesh regularity vs. convergence speed

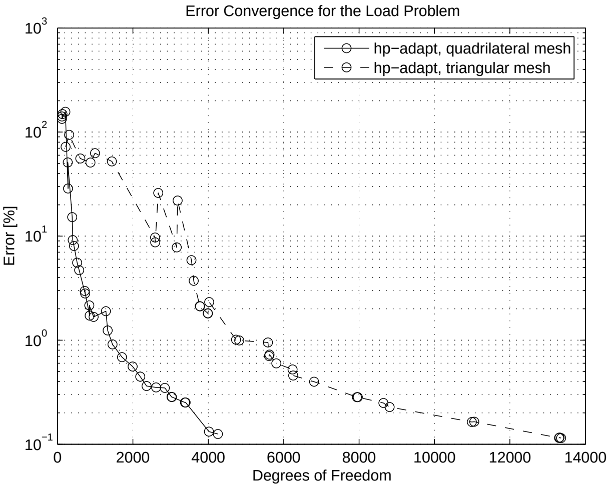


Triangles vs. quadrilaterals



Two different initial meshes for the waveguide problem.

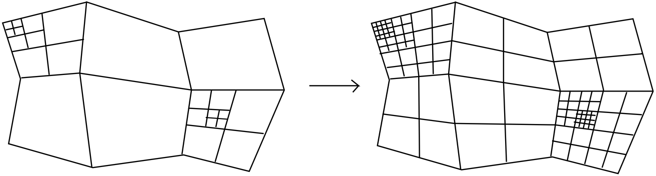
Convergence much faster on quads!



Phenomenon not understood yet.

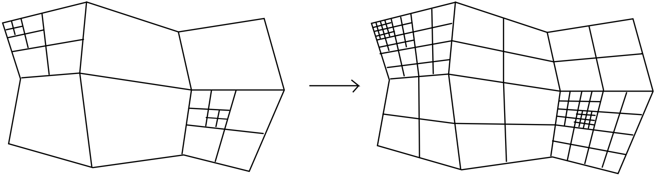
Open problems

- PDE-independent error estimation

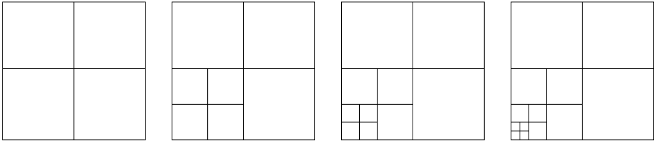


Open problems

- PDE-independent error estimation



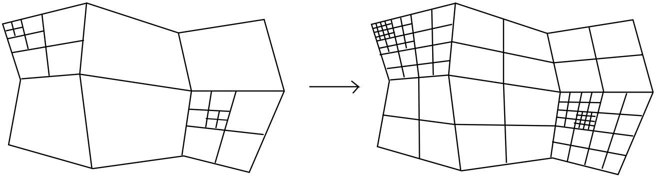
- Numerical singularity of stiffness matrices in adaptivity



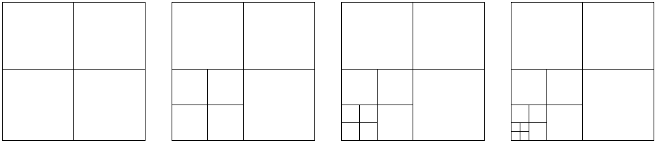
After 12 - 15 steps, max/min volume ratio $\approx 10^9$ and zero eigenvalues appear

Open problems

- PDE-independent error estimation



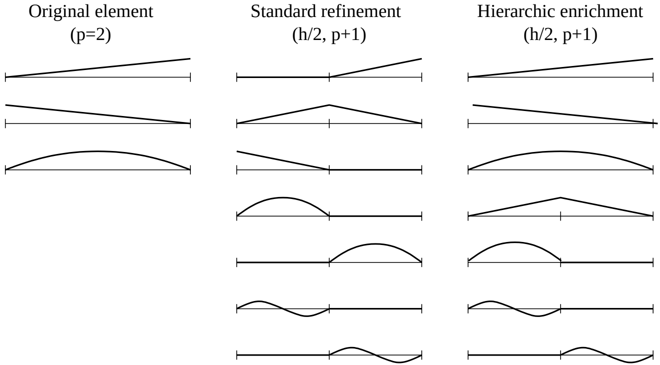
- Numerical singularity of stiffness matrices in adaptivity



After 12 - 15 steps, max/min volume ratio $\approx 10^9$ and zero eigenvalues appear

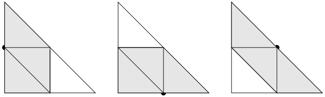
- Hierarchic basis enrichment & multilevel methods?

Hierarchic *hp*-refinement: 1D case

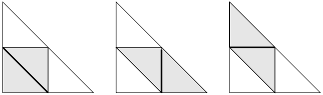


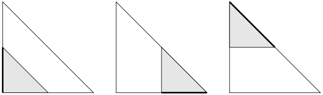
Quadrilateral elements: straightforward by product geometry

Hierarchic *hp*-refinement: triangles

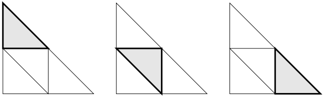


Add three vertex functions.



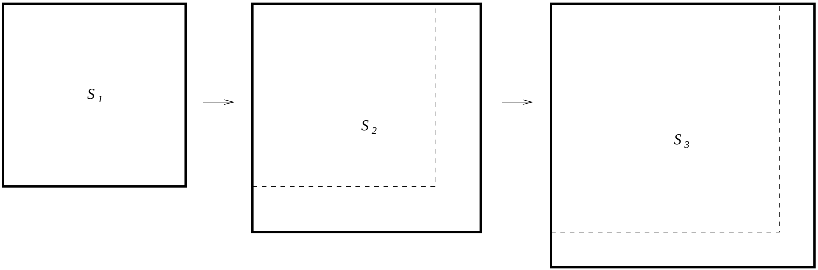


Add $p-1$ edge functions of degrees $2, 3, \ldots, p$ per highlighted edge.
Add one edge function of degree $p+1$ to each of the 9 edges.



Add bubble functions of degrees $3, 4, \ldots, p$ into highlighted subelements.
Add $p-1$ bubble functions of degree $p+1$ to each of the four subelements.

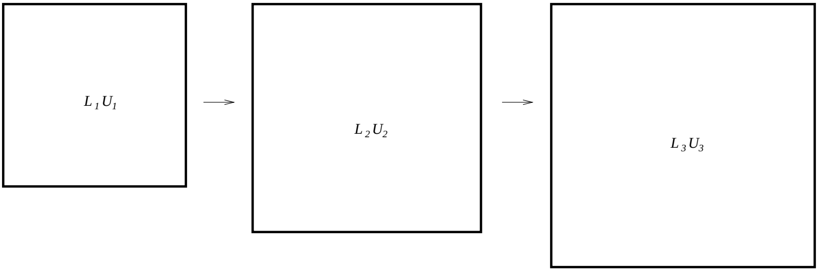
Embedded stiffness matrices



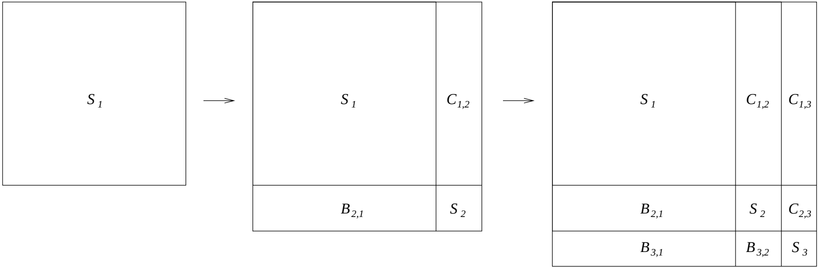
Nonsymmetric, indefinite, ill-conditioned → need for sparse direct solvers

Embedded LU decompositions

- Stiffness matrices embedded $\rightarrow$ LU decompositions embedded
- S_1 decomposed via UMFPACK – fast
- Extensions to S_2, S_3, etc. – slow!



Block Jacobi method



Global discrete problem in enriched space:

$$\begin{pmatrix} S_1 & C_{1,2} \\ B_{2,1} & S_2 \end{pmatrix} \begin{pmatrix} Y_1 + \Delta Y_1 \\ Y_2 + \Delta Y_2 \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$$

Solve $S_1 Y_1 = F_1, \; S_2 Y_2 = F_2.$

Block Jacobi method

Global discrete problem in enriched space:

$$\begin{pmatrix} S_1 & C_{1,2} \\ B_{2,1} & S_2 \end{pmatrix} \begin{pmatrix} Y_1 + \Delta Y_1 \\ Y_2 + \Delta Y_2 \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$$

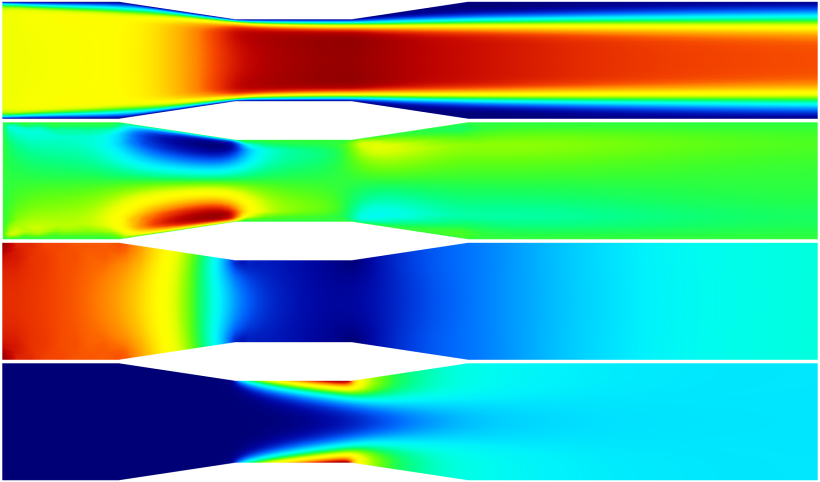
Iterative method for ΔY_1 and ΔY_2:

$$L_1 U_1 \Delta Y_1^{(k+1)} = -C_{1,2} \left(Y_2 + \Delta Y_2^{(k)} \right),$$
$$L_2 U_2 \Delta Y_2^{(k+1)} = -B_{2,1} \left(Y_1 + \Delta Y_1^{(k)} \right).$$

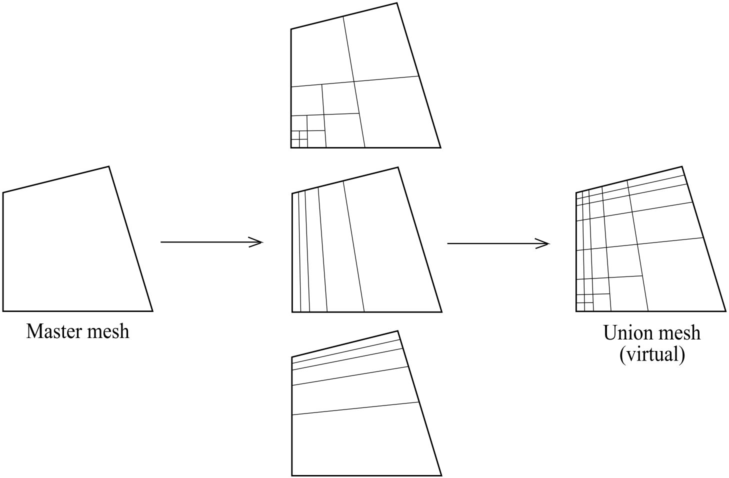
Seems to work well for a wide range of problem types.

Do you know about other methods?

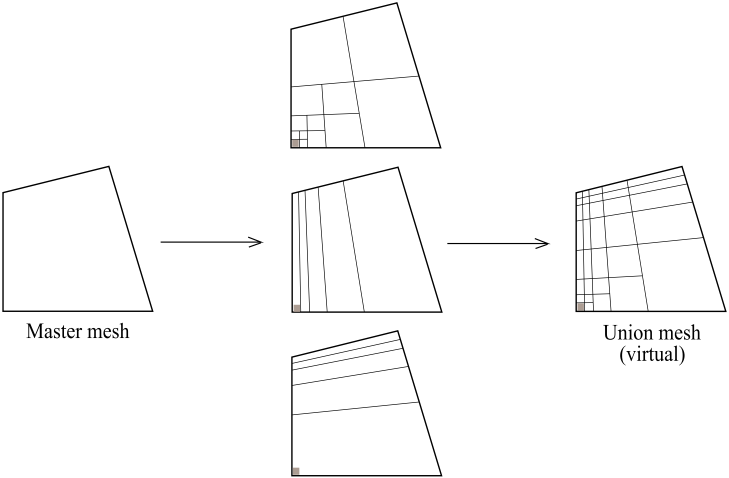
Multi-physics problems



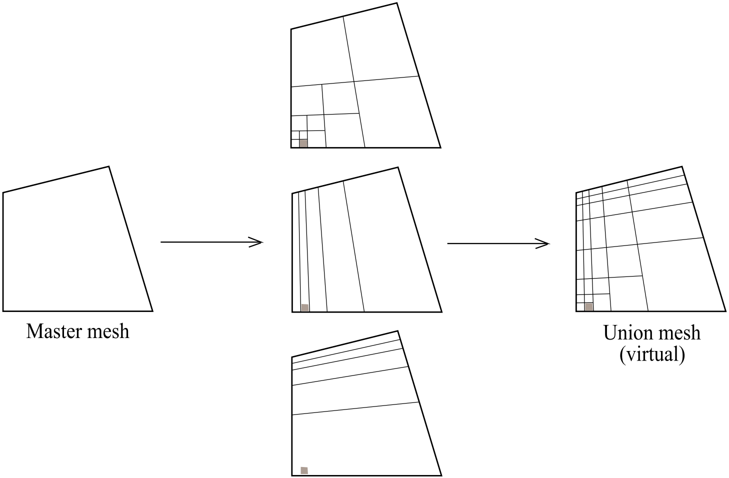
Multi-mesh *hp*-FEM



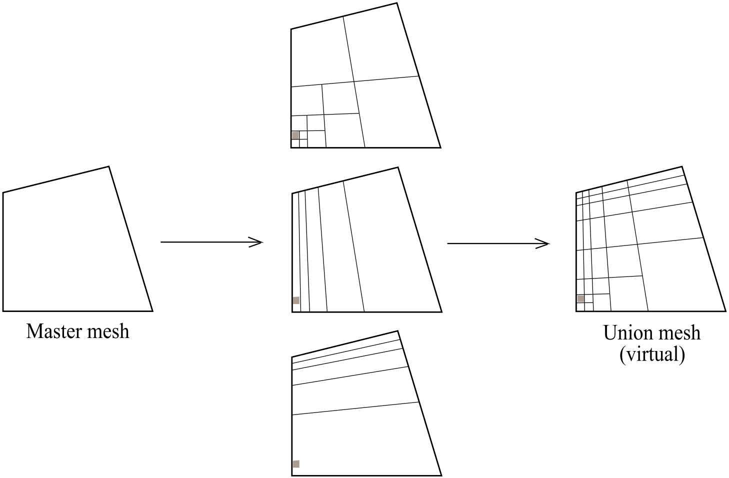
Multi-mesh *hp*-FEM



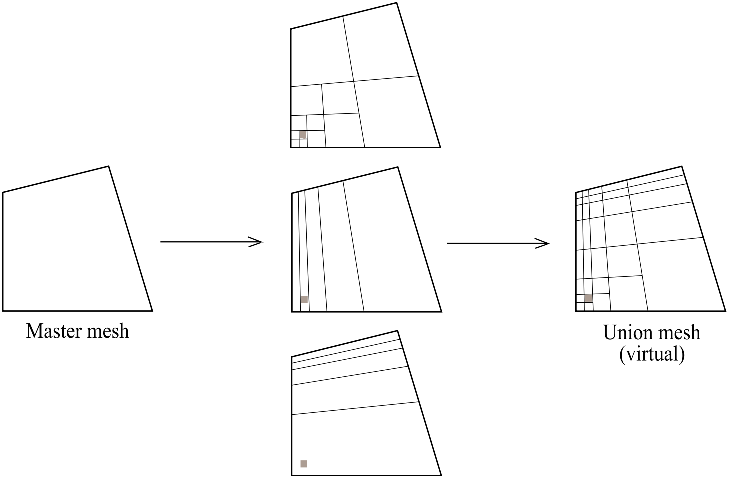
Multi-mesh *hp*-FEM



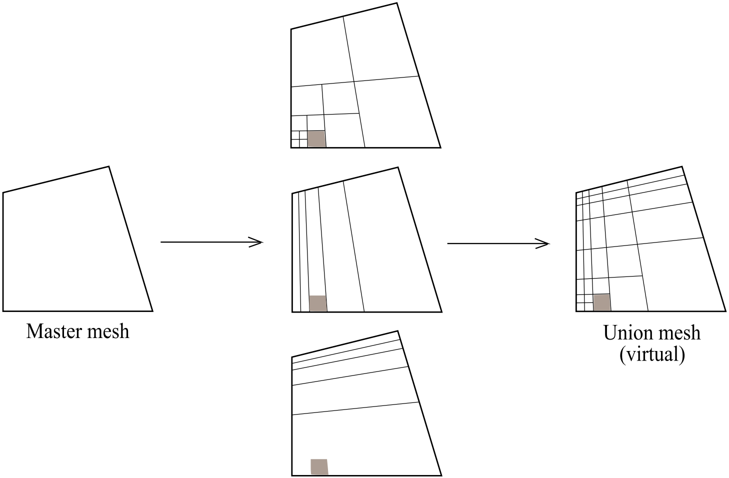
Multi-mesh *hp*-FEM



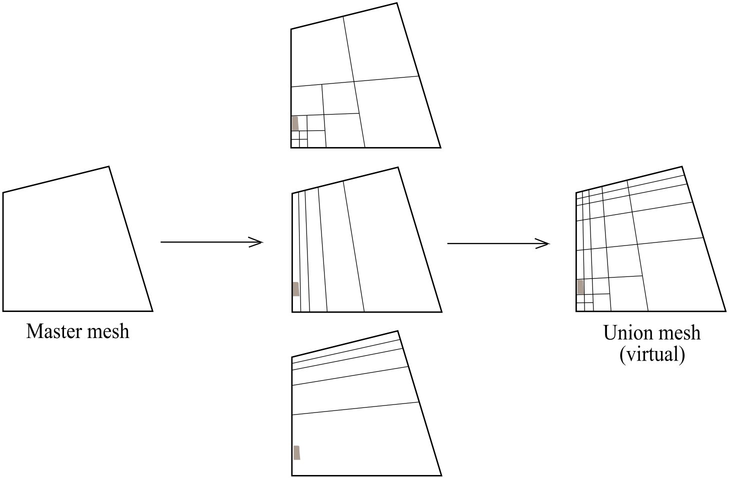
Multi-mesh *hp*-FEM



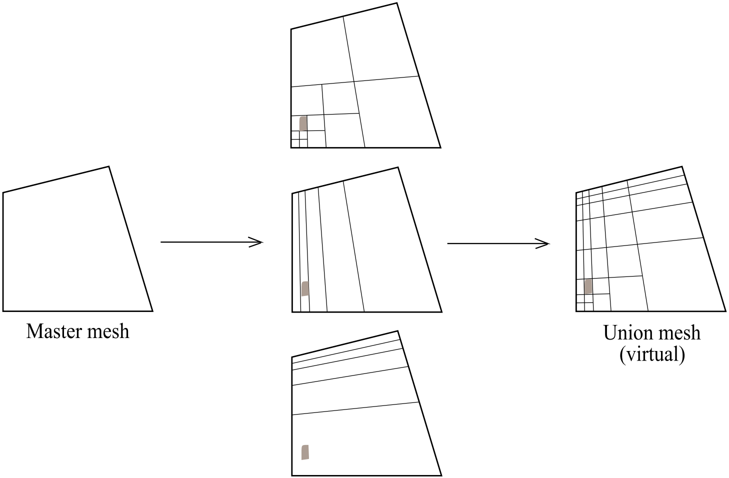
Multi-mesh *hp*-FEM



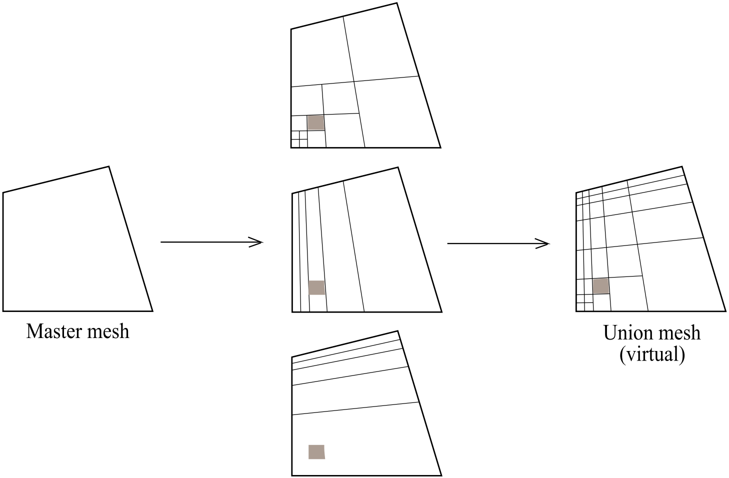
Multi-mesh *hp*-FEM



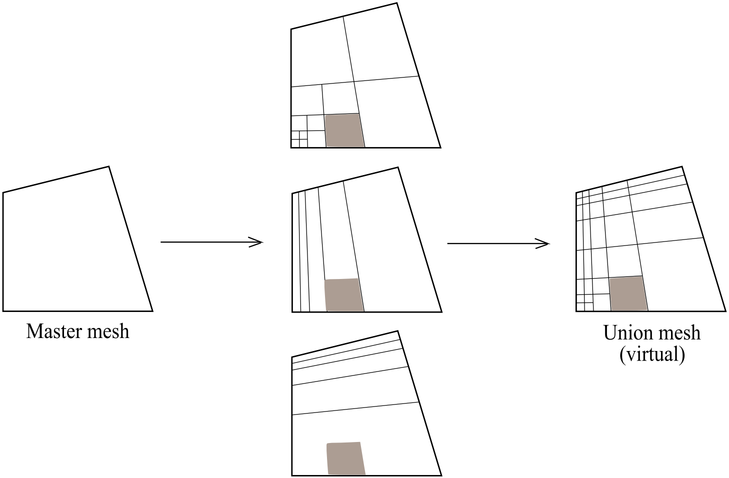
Multi-mesh *hp*-FEM



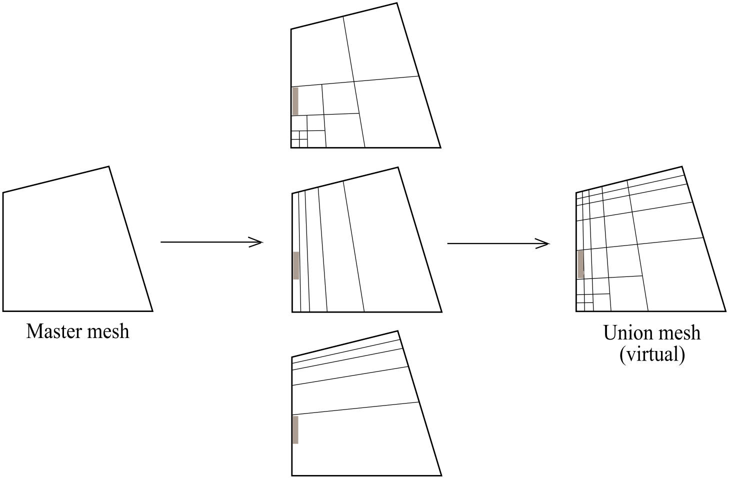
Multi-mesh *hp*-FEM



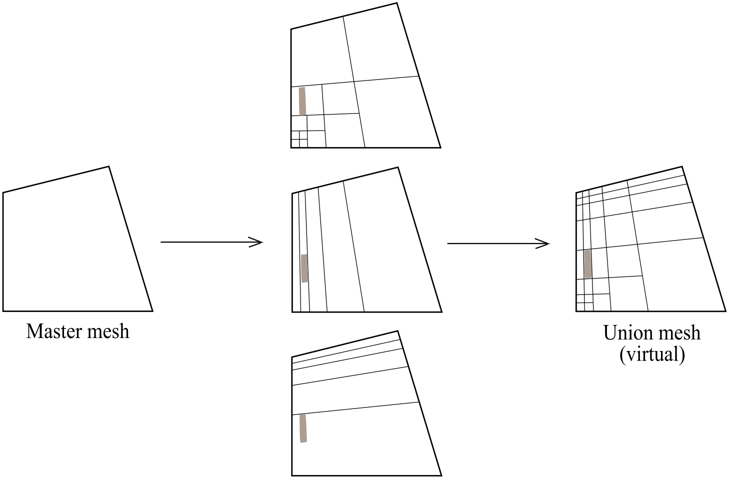
Multi-mesh *hp*-FEM



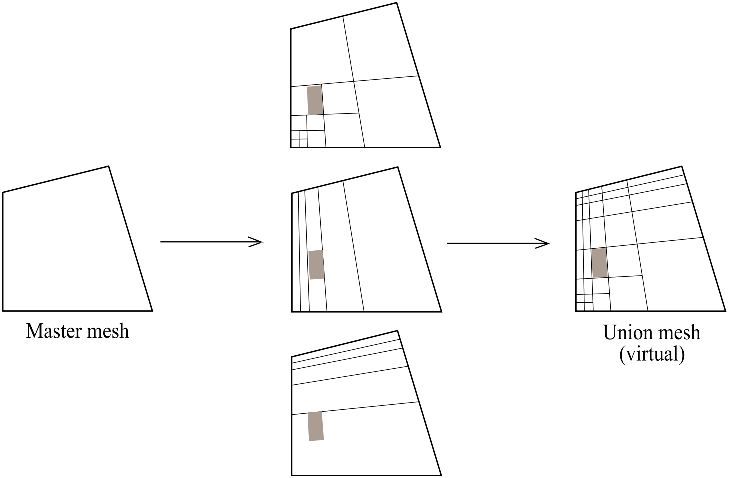
Multi-mesh *hp*-FEM



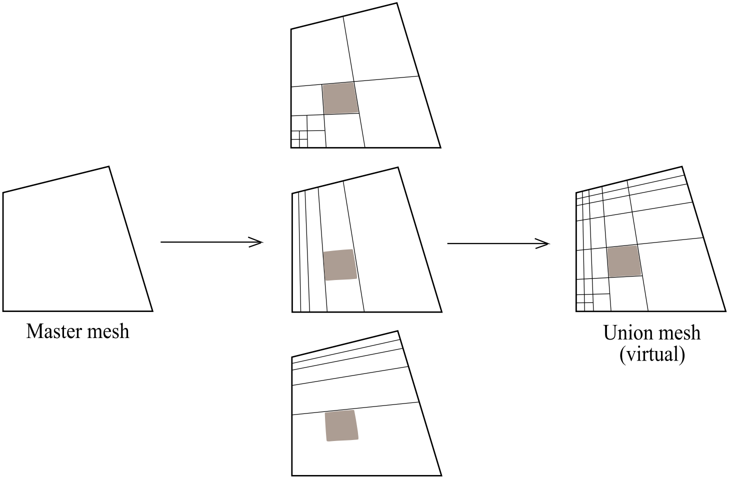
Multi-mesh *hp*-FEM



Multi-mesh *hp*-FEM



Multi-mesh *hp*-FEM



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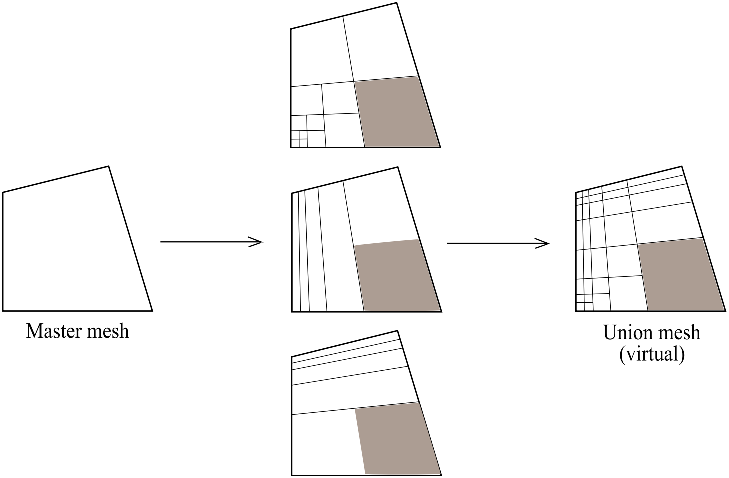

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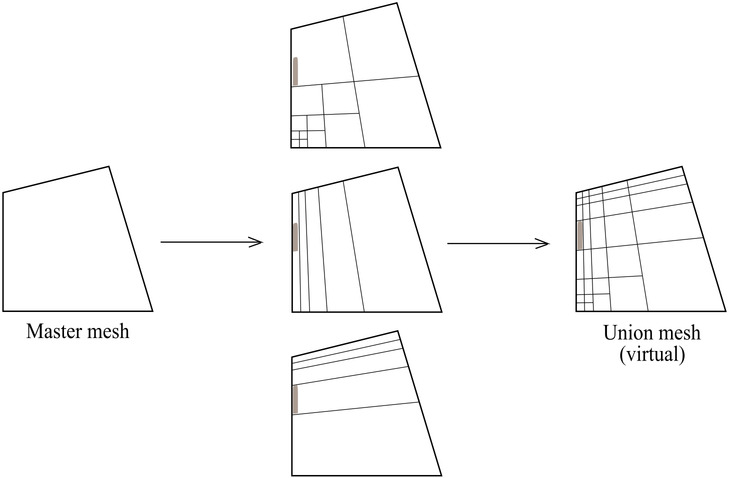
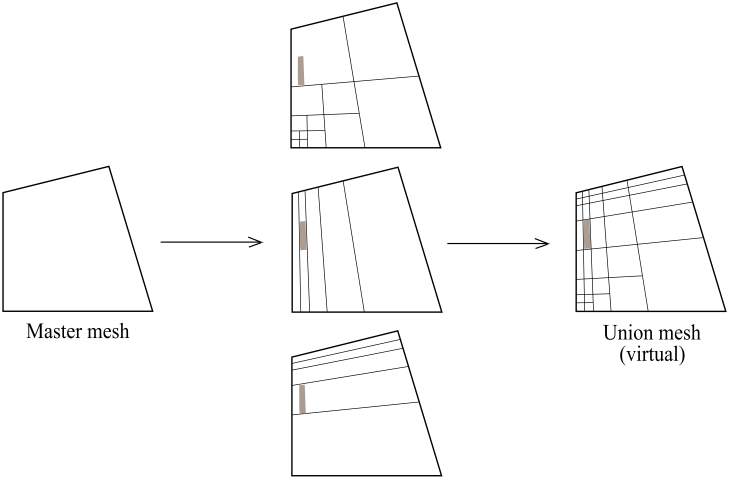
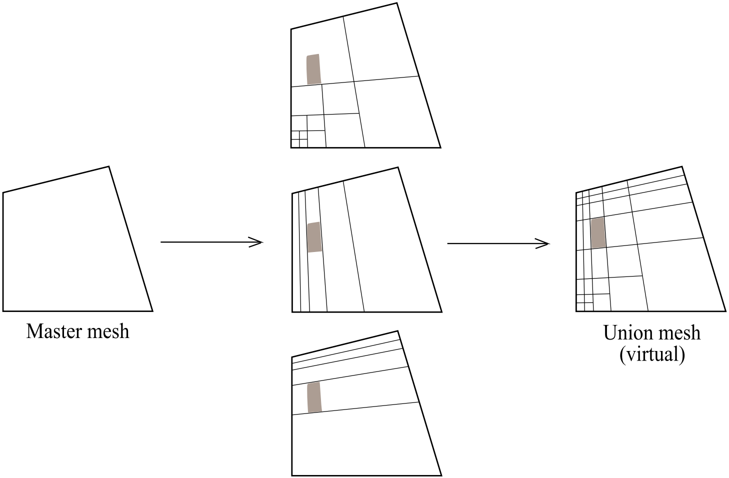


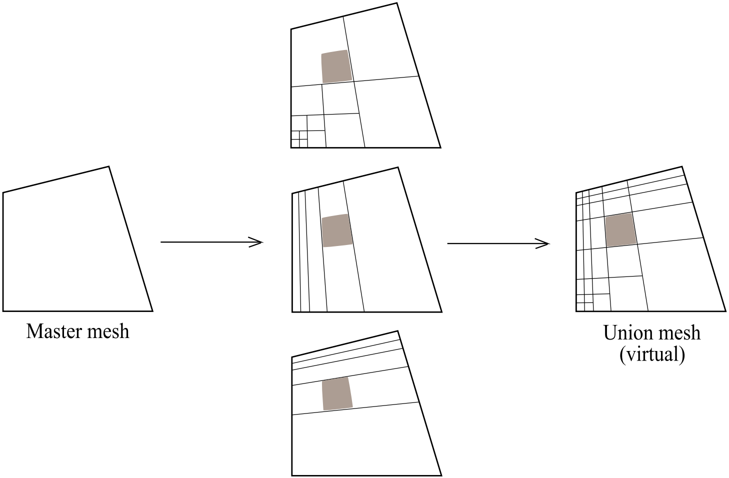
Multi-mesh *hp*-FEM



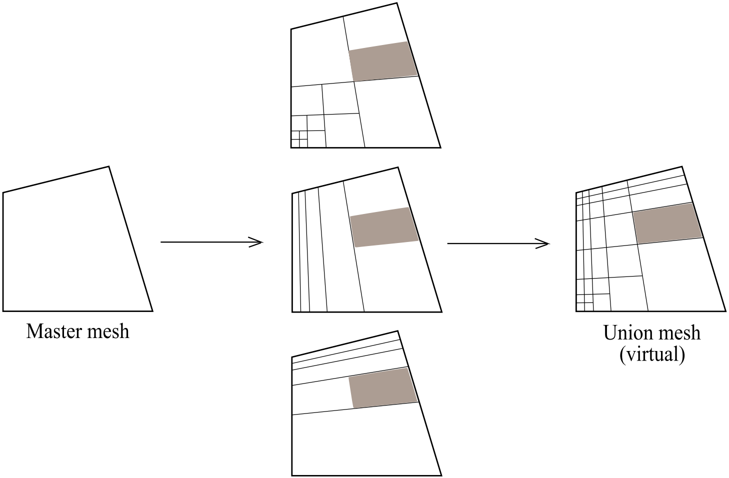
Multi-mesh *hp*-FEM



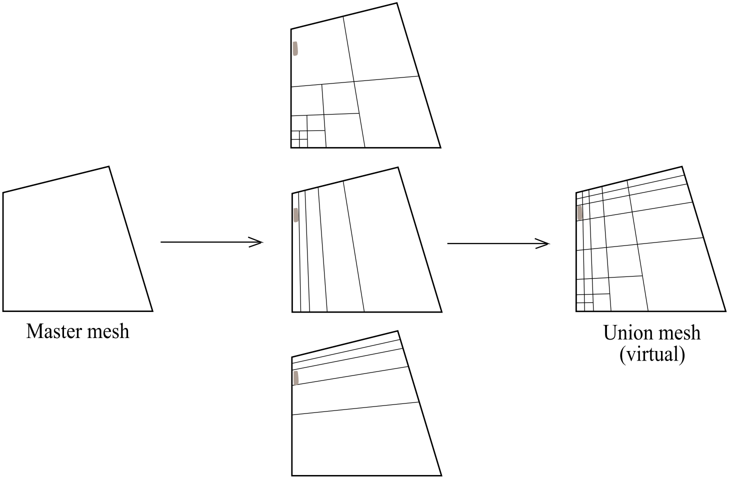
Multi-mesh *hp*-FEM



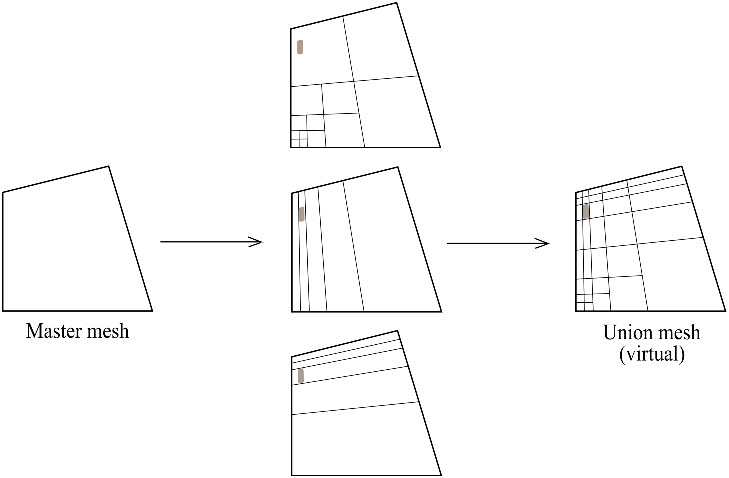
Multi-mesh *hp*-FEM



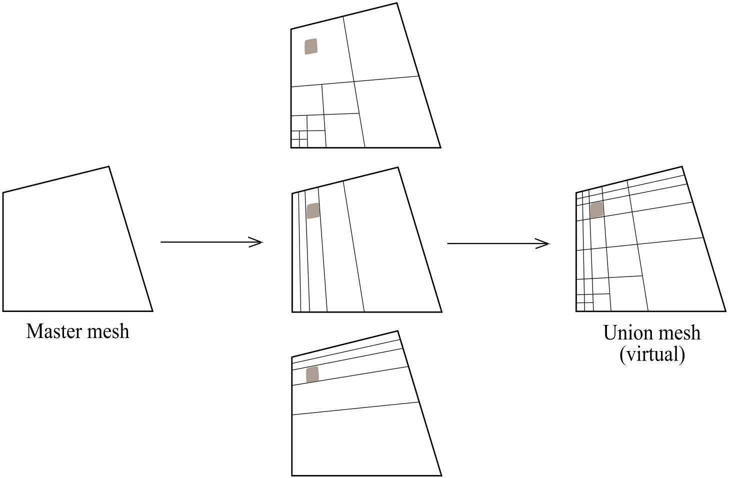
Multi-mesh *hp*-FEM



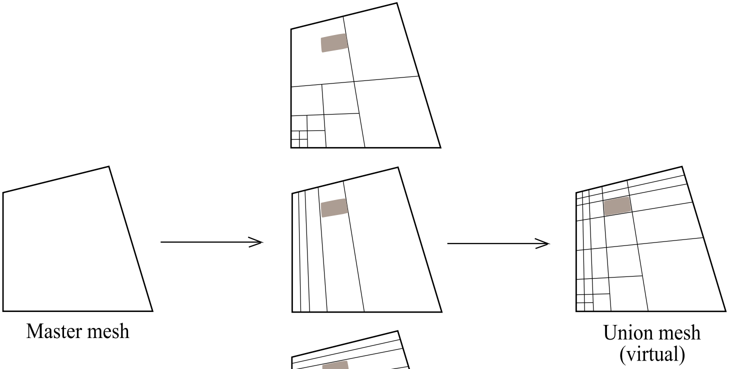
Multi-mesh *hp*-FEM



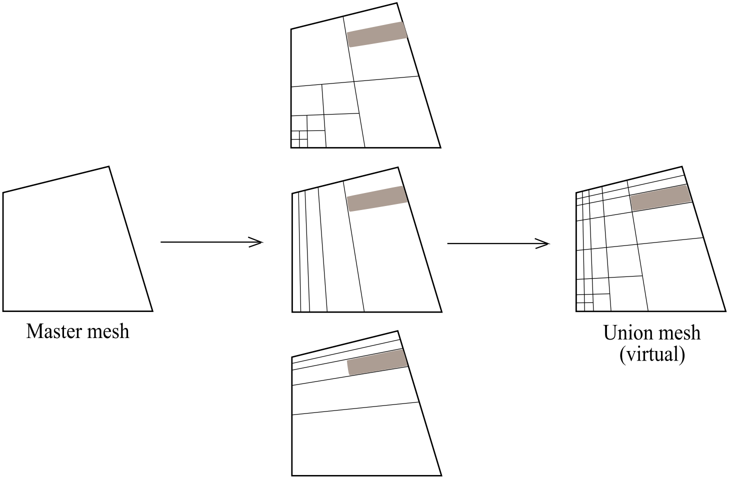
Multi-mesh *hp*-FEM



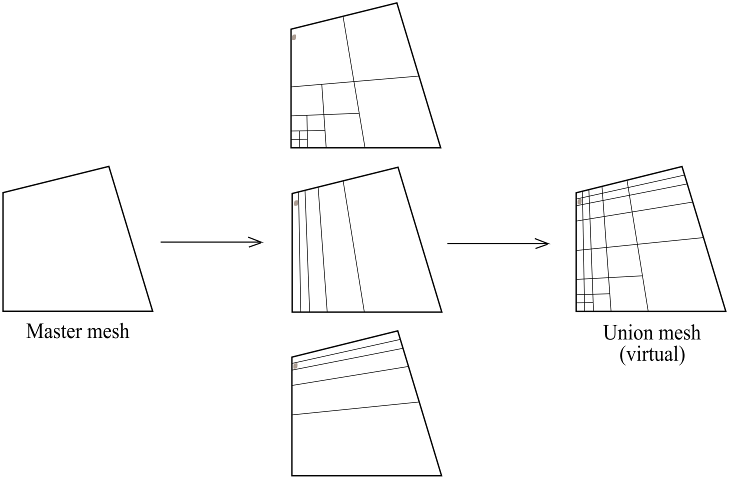
Multi-mesh *hp*-FEM



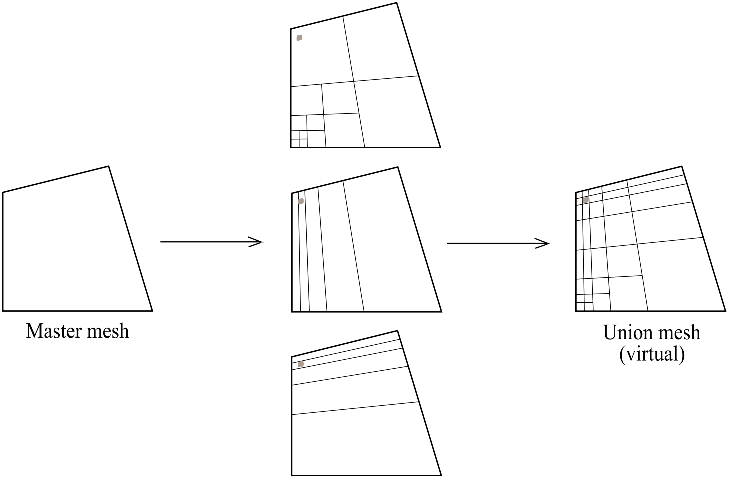
Multi-mesh *hp*-FEM



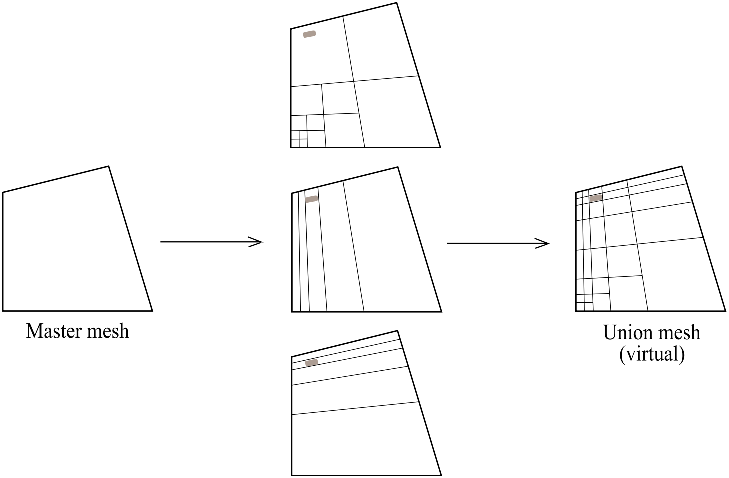
Multi-mesh *hp*-FEM



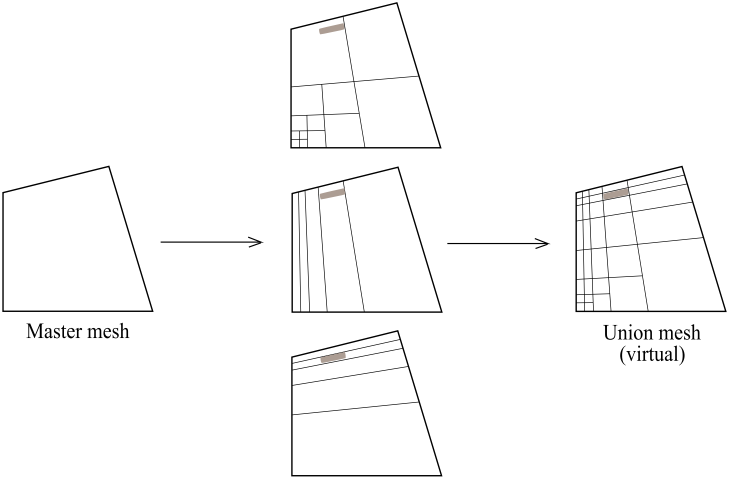
Multi-mesh *hp*-FEM



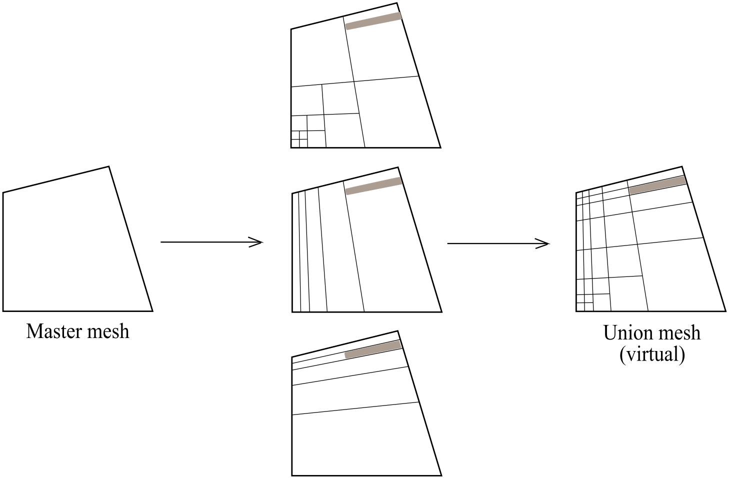
Multi-mesh *hp*-FEM



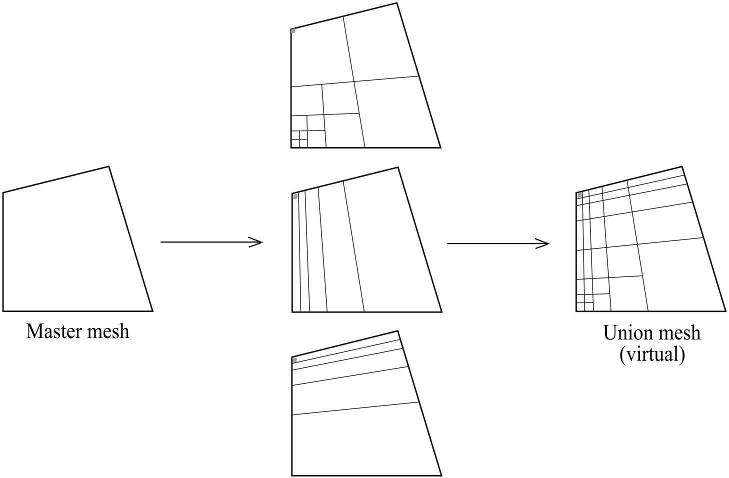
Multi-mesh *hp*-FEM



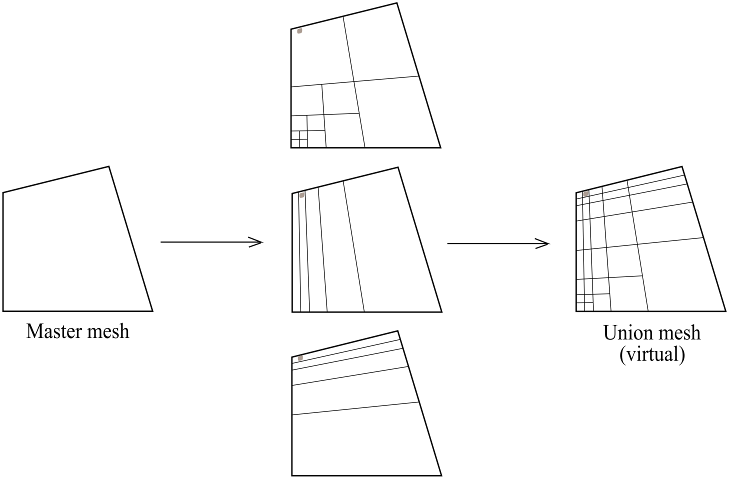
Multi-mesh *hp*-FEM



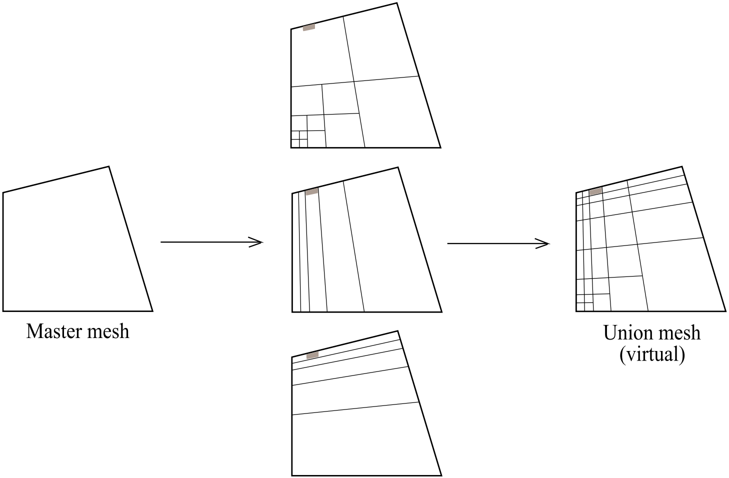
Multi-mesh *hp*-FEM



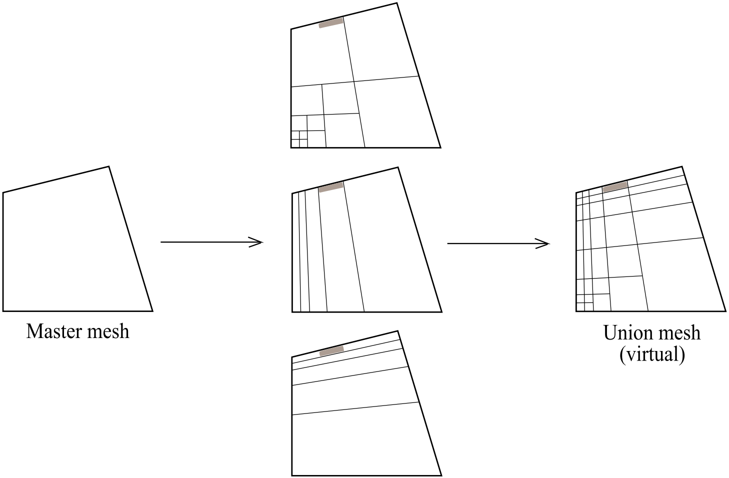
Multi-mesh *hp*-FEM



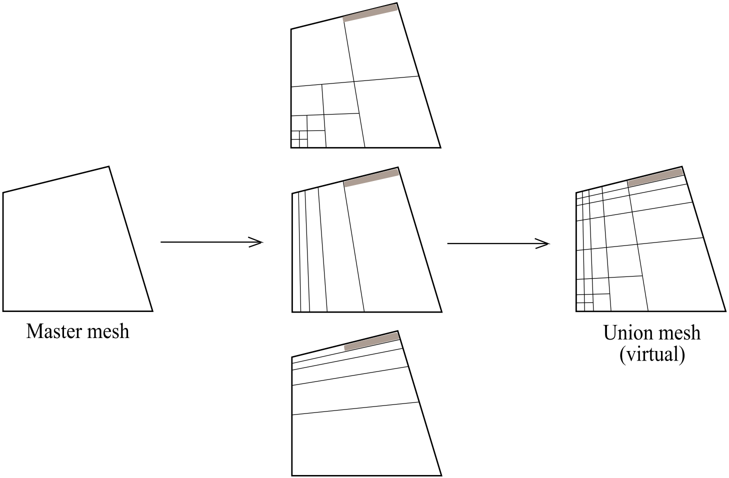
Multi-mesh *hp*-FEM



Multi-mesh *hp*-FEM



Multi-mesh *hp*-FEM



Multi-mesh *hp*-FEM

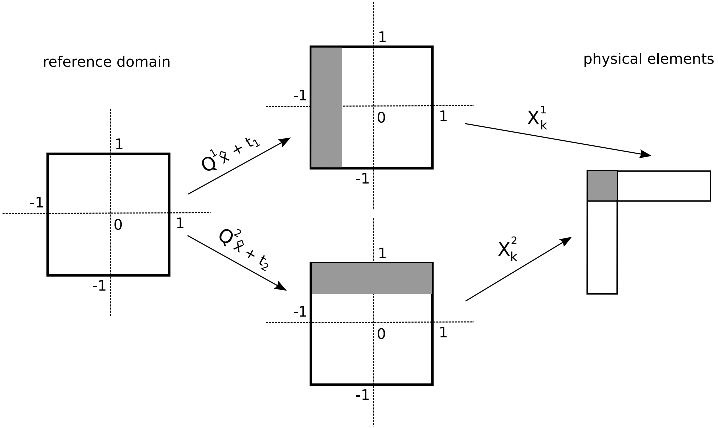
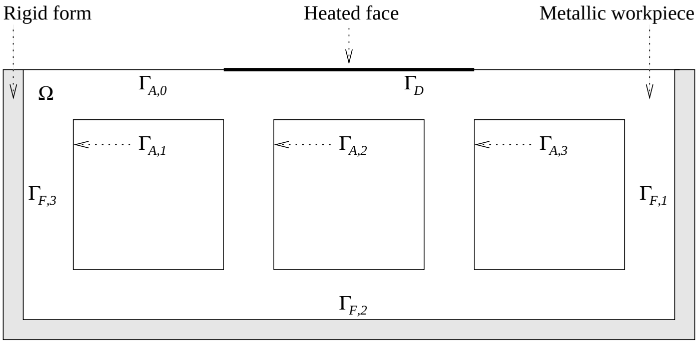


Illustration - thermoelasticity



- Solution: temperature

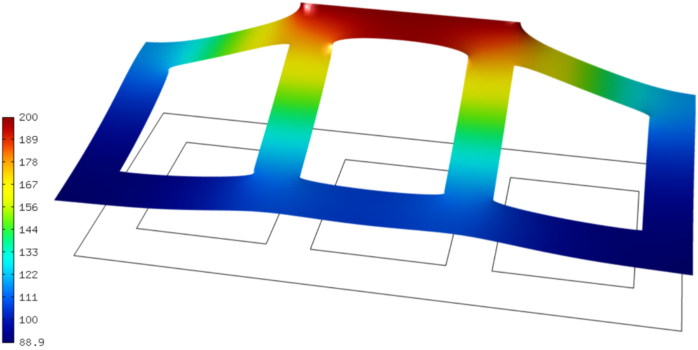


Illustration - thermoelasticity

- Solution: stress

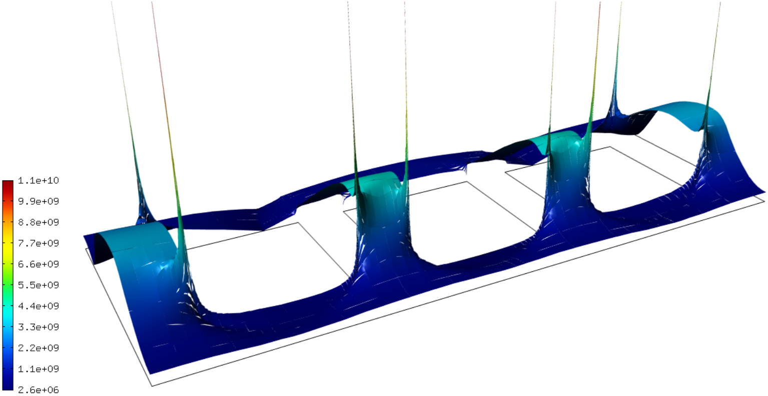
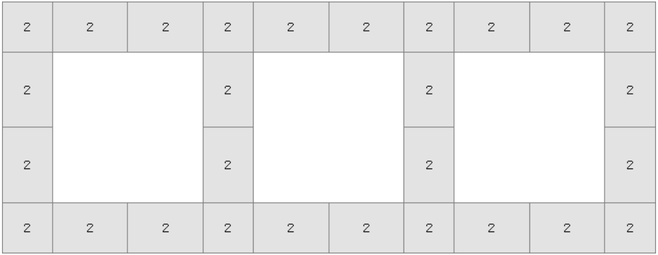


Illustration - thermoelasticity (step 1)



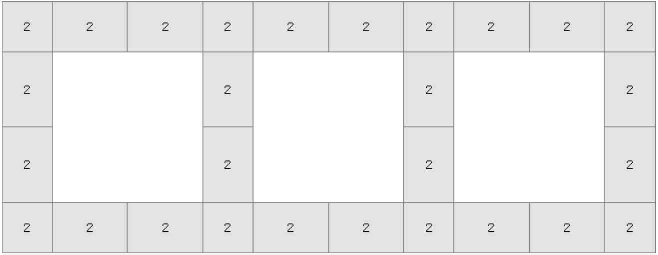
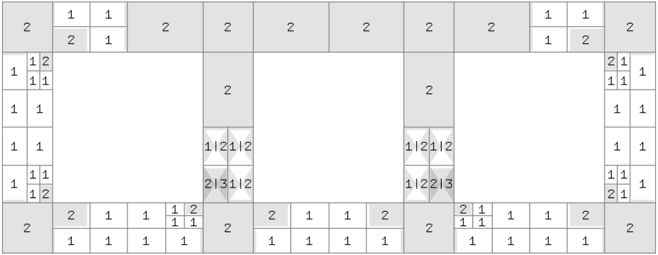


Illustration - thermoelasticity (step 2)



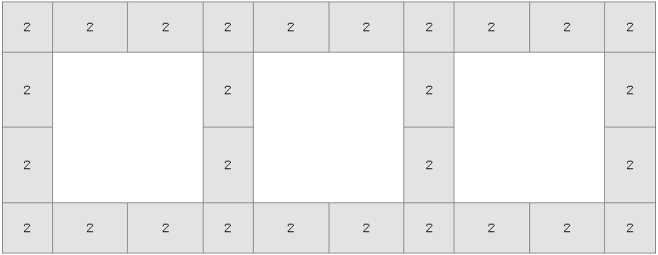


Illustration - thermoelasticity (step 3)

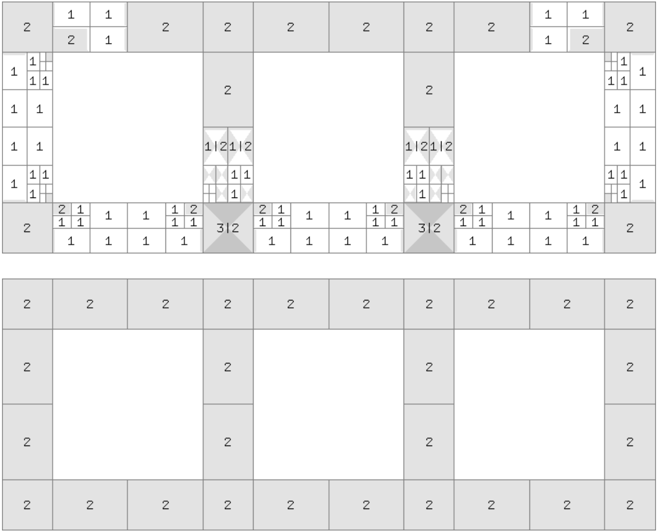


Illustration - thermoelasticity (step 4)

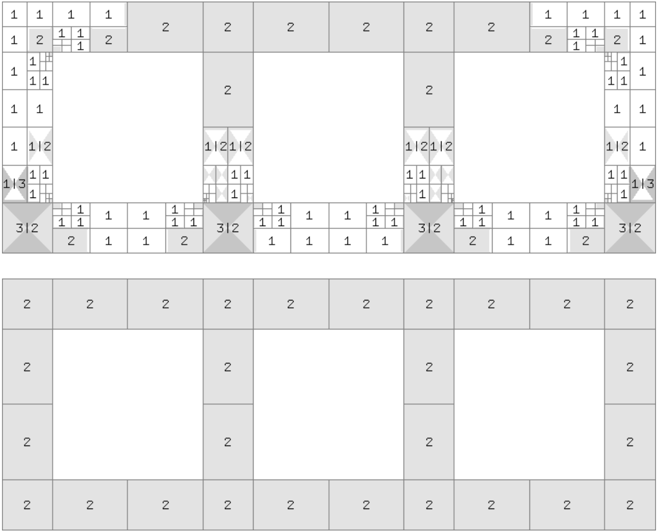


Illustration - thermoelasticity (step 5)

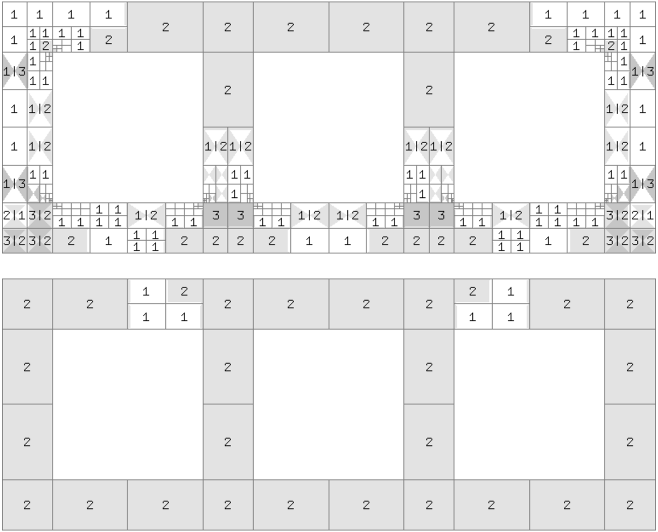
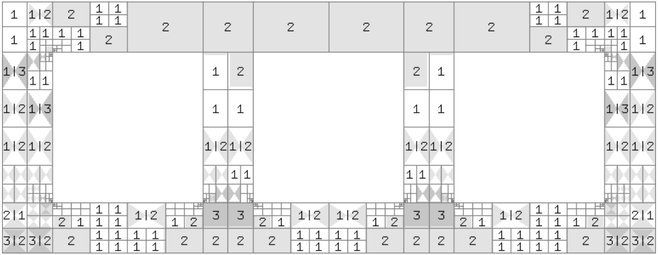


Illustration - thermoelasticity (step 6)



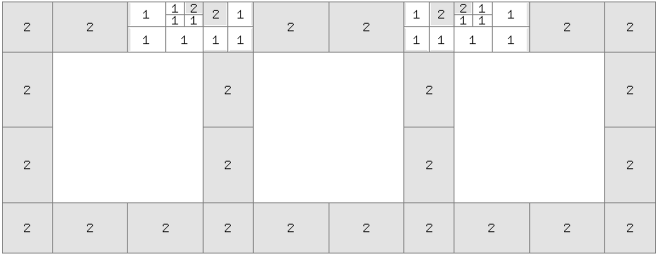


Illustration - thermoelasticity (step 7)

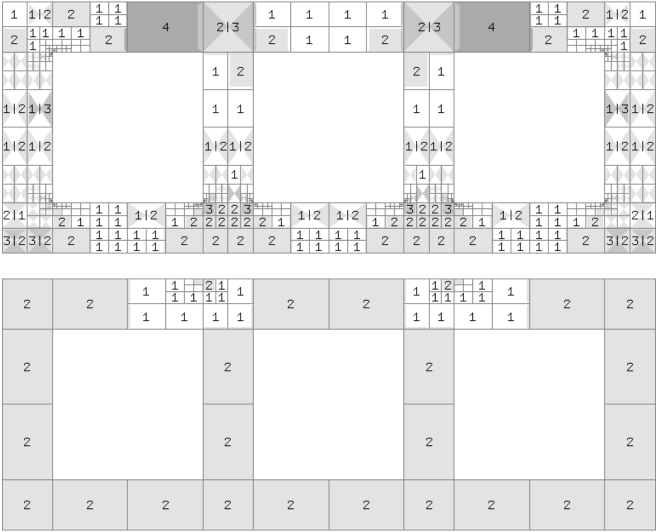


Illustration - thermoelasticity (step 8)

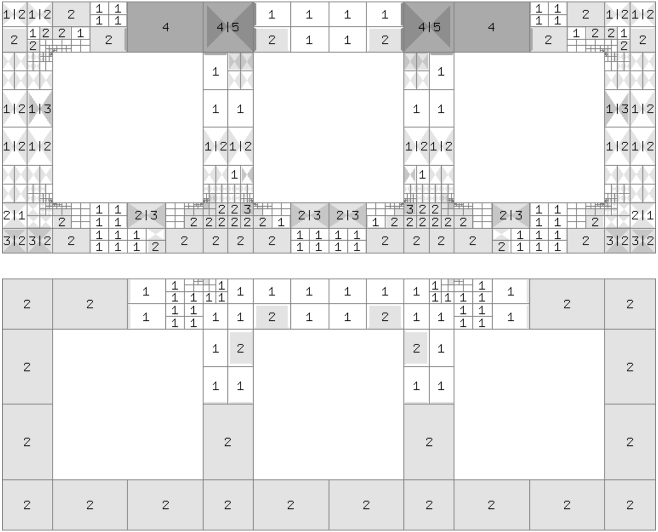


Illustration - thermoelasticity (step 9)

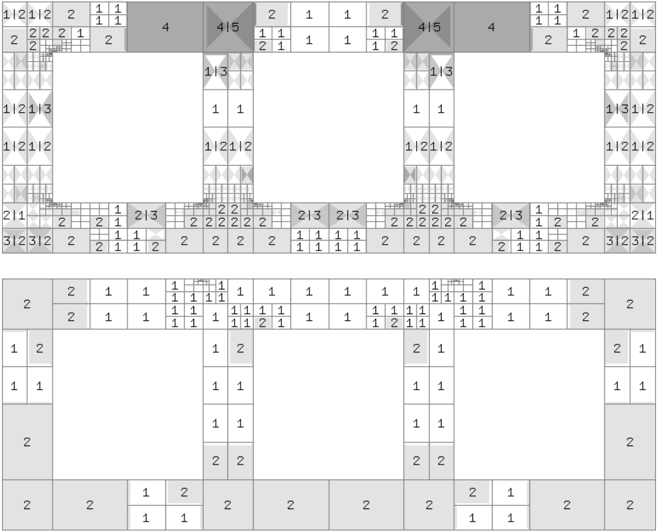


Illustration - thermoelasticity (step 10)

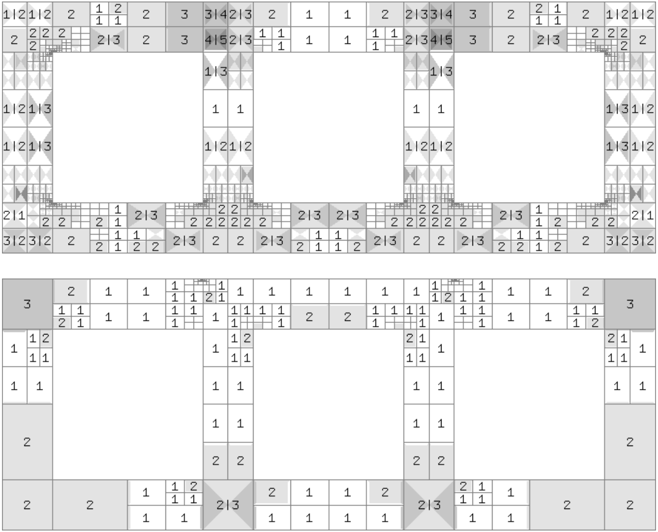


Illustration - thermoelasticity (step 11)

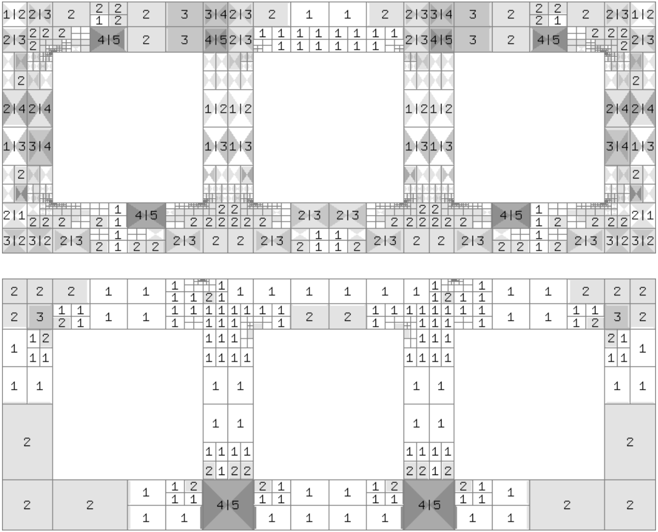


Illustration - thermoelasticity (step 12)

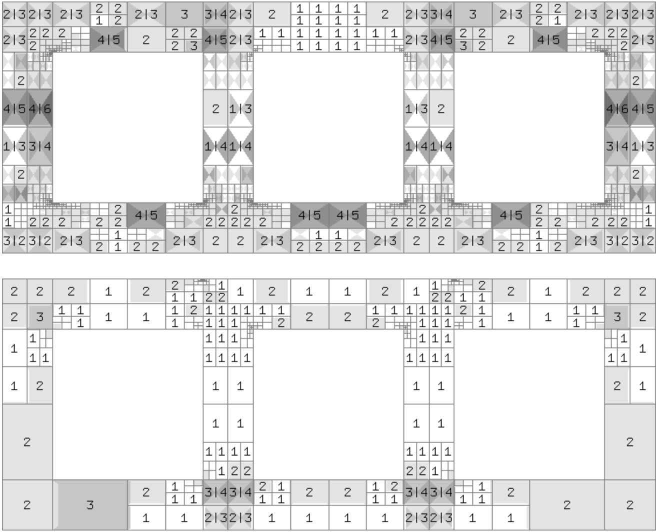


Illustration - thermoelasticity (step 13)

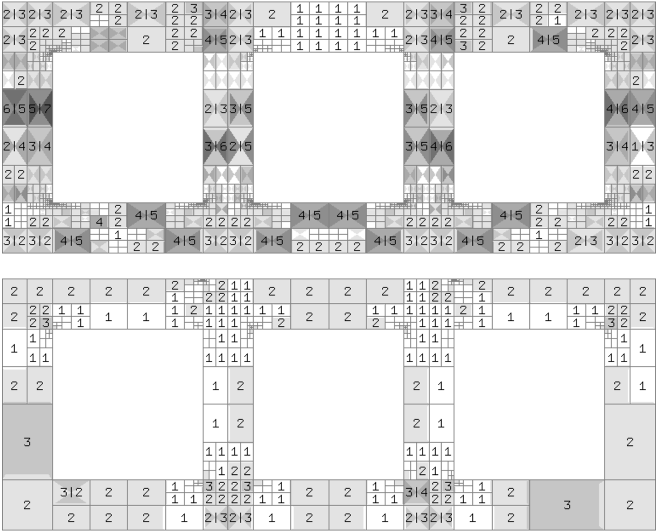
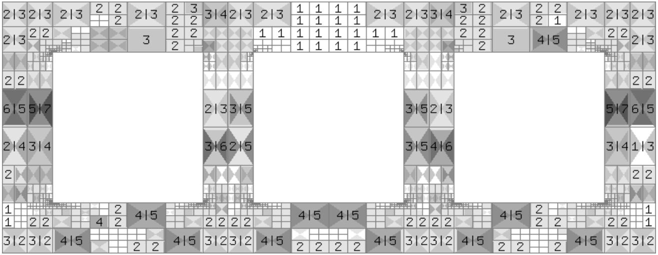
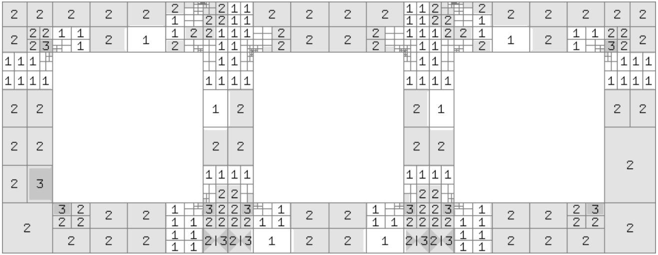
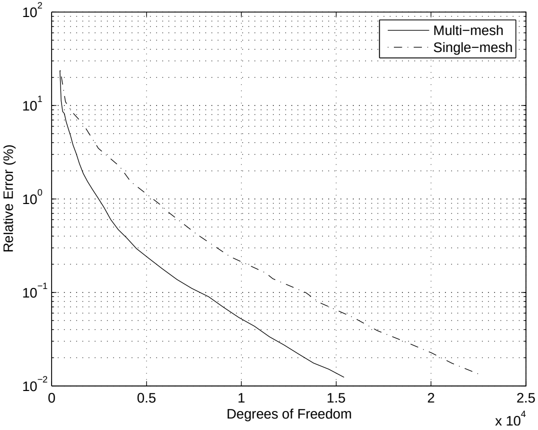


Illustration - thermoelasticity (step 14)





Convergence: multi-mesh vs. single-mesh



The Rothe's method

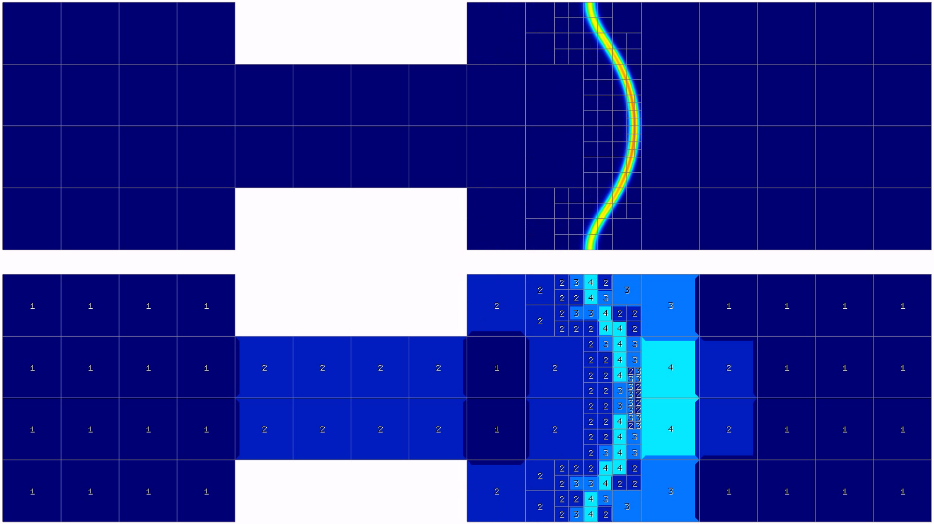
- Approximates a time-dependent PDE with system of time-independent ones.
- Illustration: $\frac{\partial u}{\partial t} - \Delta u = f$

$$\frac{\partial u}{\partial t} \approx \frac{u^{n+1} - u^n}{\Delta t} \Rightarrow -\Delta t \Delta u^{n+1} + u^{n+1} = u^n + \Delta t f^{n+1}$$

$$\frac{\partial u}{\partial t} \approx \frac{3u^{n+2} - 4u^{n+1} + u^n}{2\Delta t} \Rightarrow -2\Delta t \Delta u^{n+2} + 3u^{n+2} = 4u^{n+1} - u^n + 2\Delta t f^{n+2}$$

- Time-independent PDEs solved using spatial adaptivity
- “Simultaneous mesh refinement and coarsening”
- Adaptive control of time step as in embedded ODE methods

Illustration: flame propagation



NOTE: Algorithm limited to parabolic PDEs and linear FEM was published recently by Schmich & Vexler in SIAM J. Sci. Comput (2008).

Illustration: microwave heating

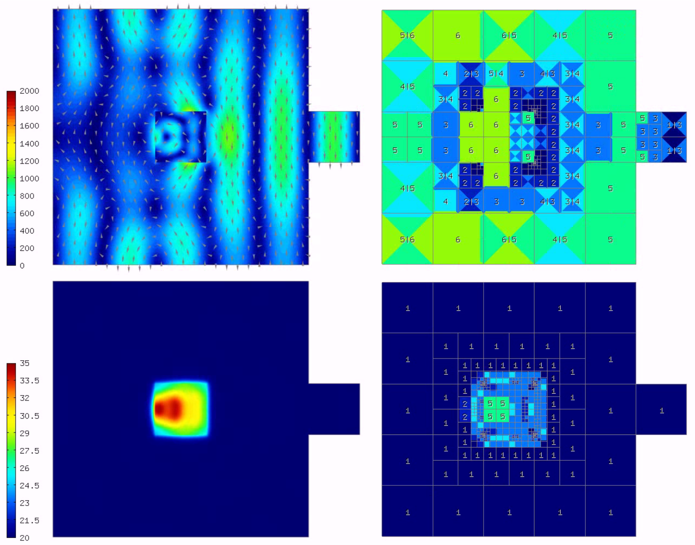


Illustration: thermally-conductive viscous flow

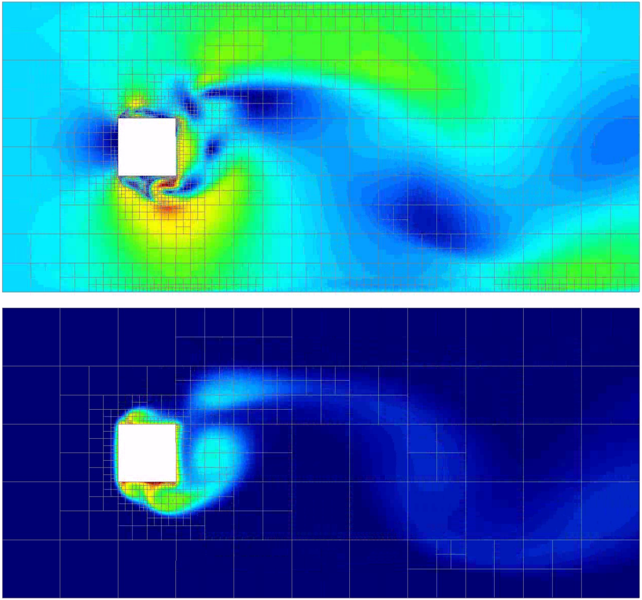


Illustration: thermally-conductive viscous flow

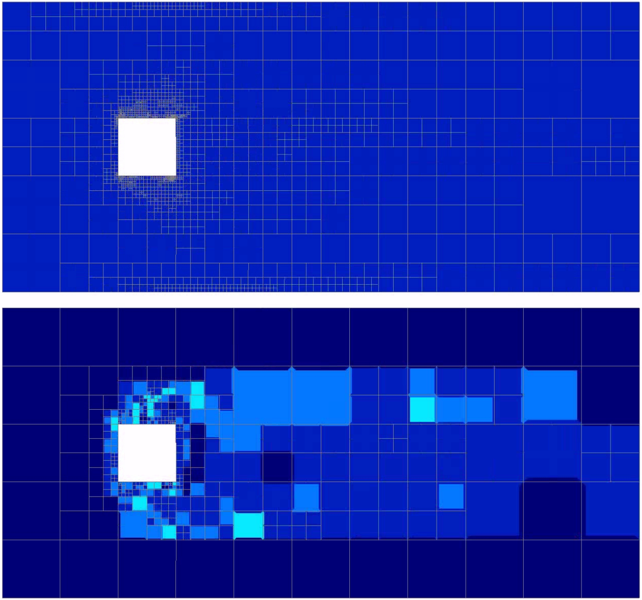
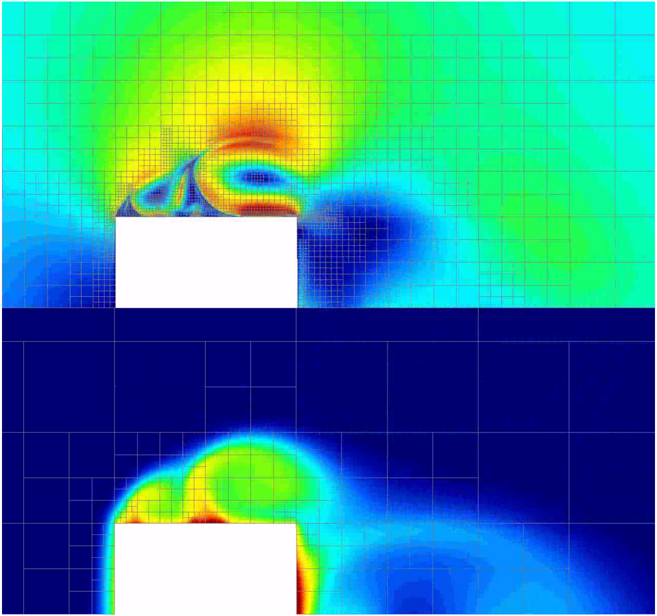


Illustration: thermally-conductive viscous flow



Outlook

- Further develop the methodology, mainly in 3D
- Solve difficult problems of practical interest
  - Interface tracking problems (with M. Shashkov, LANL)
  - MHD (with P. Bochev, Sandia NL)
  - Microfluidics in biochips (with R. Hoppe, U of Houston)
  - Radiative treatment of cancer (with J. Hesthaven, Brown U)
  - Processing of metals (with I. Dolezel, CTU)
  - Analysis of large civil structures (with J. Kruis, CTU)
- Release the software under GPL license
  - HERMES 2D during next academic year
  - HERMES 3D in couple of years
- Find good students and grow the team!

The end

Thank you for your attention!