On a traffic problem: Filippov system formulation for N cars

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PANM 14

Programy a algoritmy numerické matematiky 14

Outline

Follow-the-leader Model of the Traffic Flow I

- Consider the microscopic follow-the-leader model of the traffic flow on the circular road of length L

$$x_i' = y_i \,, \tag{1a}$$

$$y_i' = \frac{1}{\tau}\left[V(x_{i+1} - x_i) - y_i\right] \,, \quad x_{N+1} = x_1 + L \,, \tag{1b}$$

$i = 1, \ldots, N$, where N is a number of cars on the road.

- (x_i, y_i) – position and velocity of the i-th car,
- τ – reaction time of a driver,
- $V : r \mapsto V(r)$ – optimal velocity (OV) function,
    - Choice of V imposes a driving law.
    - We consider a hyperbolic OV function

$$V(r) = V^{max} \frac{\tanh(a(r-1)) + \tanh a}{1 + \tanh a} \,. \tag{2}$$

Follow-the-leader Model of the Traffic Flow II

- The difference

$$h_i = x_{i+1} - x_i \,, \quad i = 1, \ldots, N$$

is called headway of the i-th car.

- Given an initial condition $[x^0, y^0] \in \mathbb{R}^N \times \mathbb{R}^N$, the system (1) defines a flow on $\mathbb{R}^N \times \mathbb{R}^N$

$$[x^0, y^0] \mapsto [x(t), y(t)] \equiv \Phi(t, [x^0, y^0]) \,, \quad t \in \mathbb{R} \,.$$

Follow-the-leader Model of the Traffic Flow III

- Without loss of generality, we may order x^0 as

$$s \leq x_1^0 \leq x_2^0 \leq \cdots \leq x_{N-1}^0 \leq x_N^0 \leq L + s \,,$$

  where $s \in \mathbb{R}$ is an arbitrary phase shift.
- We also assume that all initial velocity components are positive,

$$y^0 = (y_1^0, \ldots, y_N^0)\,, \quad y_i^0 > 0\,, \quad i = 1, \ldots, N\,.$$

Non Physical Solutions I

- It is well known that solutions of the follow-the-leader model may become non physical.
- The model breaks down at the time instant when two cars collide.

Collision

The collision occurs at the time t_E for which there exists $k \in \{1, 2, \ldots, N\}$ such that,

$$h_k(t_E) = x_{k+1}(t_E) - x_k(t_E) = 0\,, \quad y_k(t_E) > y_{k+1}(t_E)\,.$$

By continuity argument, $h_k(t)$ becomes negative for $t_E < t < T$.

Remark

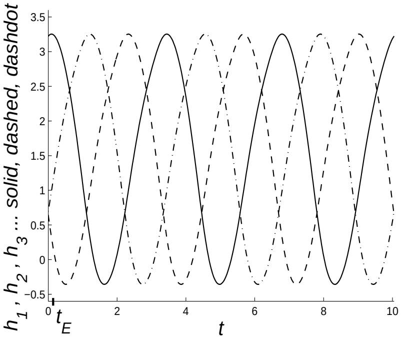
The inequality $y_k(t_E) > y_{k+1}(t_E)$ holds generically.

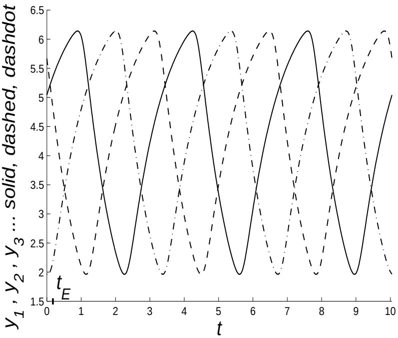
Non Physical Solutions II

Example (Periodic non physical solution)

Consider $N = 3$, $L = 4.56281$, $V^{max} = 7$, $a = 2$, $\tau = 1$ and

$$x^0 = [0, 3.2189, 3.8664]\,, \quad y^0 = [5.0324, 5.6519, 1.9646]\,.$$





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One can observe that at $t = t_E = 0.2074$,

$$h_2(t_E) = 0\,, \quad y_2(t_E) > y_3(t_E)\,.$$

The natural interpretation is that at $t = t_E$ the car No 2 is about to overtake the car No 3.

Overtaking Model

Overtaking Model

1. Given an initial condition, system (1) is solved numerically.
2. Locate the least time t_E for which there is $k \in \{1, \dots, N\}$ such that $h_k(t_E) = 0$.
3. Create new initial condition by swapping $[x_k(t_E), y_k(t_E)]$ and $[x_{k+1}(t_E), y_{k+1}(t_E)]$.
4. Recursively repeating steps 1–3 we get event sequence $\{t_E(j)\}$.
5. Reconstructing car permutations, the trajectory of each car is assembled from pieces defined on intervals $[t_E(j-1), t_E(j))$.

- The resulting trajectory is piecewise smooth.

L. B., Vladimír Janovský, *On pattern formation in a class of traffic models*, Physica D 237 (2008), 28–49.

Event Map

- Our main interest is to analyze long-time behavior of Overtaking Model.
- For given event sequence $\{t_E(j)\}_{j=1}^{Z}$ we construct a sequence of symbols called event map

$$G_E = \{[i_s \to j_s]\}_{s=1}^{Z}.$$

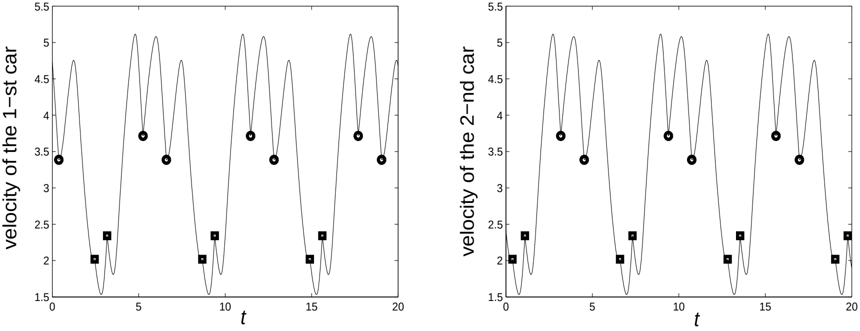
- Car No i_k overtakes the car No j_k at $t = t_E(k)$, $k = 1, \dots, Z$.
- Event map proves to be useful to identify and classify invariant objects of Overtaking Model, in particular oscillatory patterns.
- For $N = 3$, we managed to identify five types of oscillatory patterns.

Rotating Wave of Class 1

Example (Rotating wave of class 1)

Consider $N = 3$, $L = 3.093725$, $V^{max} = 7$, $a = 2$, $\tau = 1$ and

$$x^0 = [0, 0.7440, 0.9102]\,, \quad y^0 = [4.7421, 2.4211, 3.6912]\,.$$



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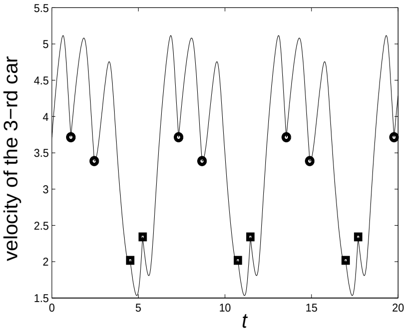

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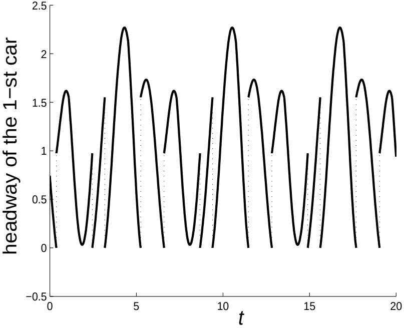

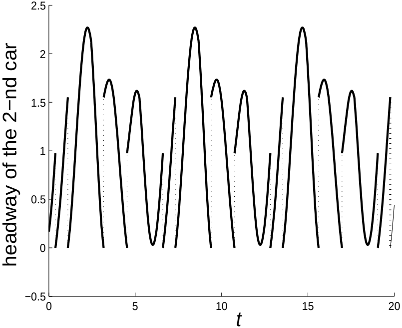

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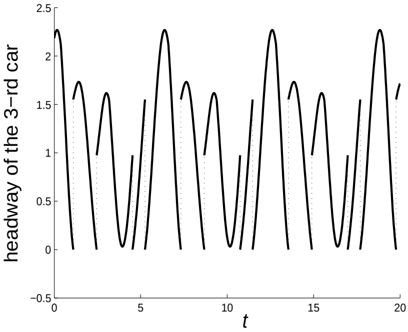

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$$x^0 = [0, 0.7440, 0.9102]\,, \quad y^0 = [4.7421, 2.4211, 3.6912]\,.$$

- The trajectory is periodic with period $T = 6.2226$.
- The event map reads

$$G_E = \{[1 \to 2], [3 \to 2], [3 \to 1], [2 \to 1], [2 \to 3], [1 \to 3], \ldots\}\,.$$

- The event map G_E is periodic with period $p_E = 6$.

Filippov Systems

Given a partition $\{S_\alpha\}$ of the state space $\mathbb{R}^n$, the Filippov system of differential equations is generally defined by

$$x' = f^{(\alpha)}(x)\,, \quad x \in S_\alpha\,. \tag{3}$$

Hence, system (3) has different right-hand sides in different subsets of the phase space. The right-hand side of (3) may be discontinuous in x. In our model, the subsets S_α are determined in an adaptive way by the current car ordering on the circuit.

A. F. Filippov, *Differential Equations with Discontinuous Righthand Sides*, Kluwer, Dordrecht, 1988.

M. Di Bernardo, C. J. Budd, A. R. Champneys, P. Kowalczyk, *Piecewise-smooth Dynamical Systems. Theory and Applications*, Springer, London, 2008.

Equivalent Formulation of Follow-the-leader Model

- Follow-the-leader model in headway–velocity coordinates

$$h_i' = y_{i+1} - y_i \,, \quad y_{N+1} = y_1 \tag{4a}$$

$$y_i' = \frac{1}{\tau} \left[V(h_i) - y_i \right] \,, \tag{4b}$$

$i = 1, \ldots, N$

- System (4) is equivalent to (1) up to a shift in x variable.
- To handle the overtaking without creating new initial conditions at $t = t_E(j)$ we need to
  - generalize the notion of headway,
  - modify the OV function.

Overtaking Model as a Filippov System I

Gap variable

Definition (Gap variable)

The quantity $h_{i,j}$, $i \neq j$, defined as

$$h_{i,j} = x_j - x_i \,, \quad i < j \,, \quad h_{i,j} = L - h_{j,i} \,, \quad i > j$$

is called gap between car No i and car No j.

Proposition

$$h_{i,j} = \sum_{k=i}^{j-1} h_{k,k+1} \,, \quad i < j \,, \quad h_{i,j} = L - \sum_{k=j}^{i-1} h_{k,k+1} \,, \quad i > j \tag{5}$$

Remark

It is essential that we allow $h_{i,j} < 0$ and $h_{i,j} > L$.

Overtaking Model as a Filippov System II

L-periodic discontinuous OV function

- The driving law imposed by the OV function depends on the current headway of each car, it does not depend on how many laps the leader is ahead / behind.
- Therefore, to make the gap variables acceptable as the OV function arguments we need to modify the OV function to be:
  - L-periodic,
  - and consequently discontinuous.

Definition (L-periodic discontinuous OV function)

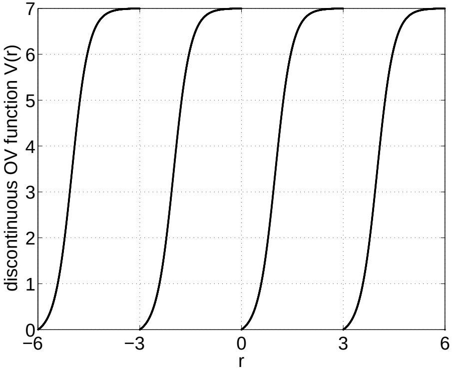
L-periodic discontinuous OV function $\widetilde{V} : r \mapsto \widetilde{V}(r)$ is defined via the formula (2) on the interval $[0, L)$ and extended periodically on the whole $\mathbb{R}$.

Overtaking Model as a Filippov System III

L-periodic discontinuous OV function

Example

Let $L = 3$, $a = 2$, $V^{max} = 7$.



Overtaking Model as a Filippov System IV

Differential equations

The Overtaking Model in gap–velocity coordinates reads

$$h'_{i,i+1} = y_{i+1} - y_i \,, \quad i = 1, \ldots, N-1 \,, \tag{6a}$$

$$y'_i = \frac{1}{\tau}\left[\widetilde{V}\left(h_{i,v(i)}\right) - y_i\right] \,, \quad i = 1, \ldots, N \,. \tag{6b}$$

Index $v(i)$ is a number of the car which is the current leader of the i-th car. Note that $h_{i,v(i)}$ is computed by relation (5).

Overtaking Model as a Filippov System V

Detection of overtaking

- Overtaking occur at the time t_E if there exist $i, j \in \{1, 2, \ldots, N\}$, $i \neq j$, such that $h_{i,j}(t_E) = kL$, $k \in \mathbb{Z}$.
  - If $h'_{i,j}(t_E) < 0$ then the car No i overtakes the car No j.
  - If $h'_{i,j}(t_E) > 0$ then the car No i is overtaken by the car No j.
- Since any car can overtake only its leader, it is sufficient to look for roots of equations

$$f_i(t) = h_{i,v(i)}(t) - \left\lfloor \frac{h_{i,v(i)}(t_E(j))}{L} \right\rfloor L = 0\,, \quad i = 1, \ldots, N\,,$$

where

$$\lfloor x \rfloor = \max_{n \in \mathbb{Z}} \{n < x\}\,,$$

$t_E(j)$ stands for the last instant at which the overtaking occured.

Overtaking Model as a Filippov System VI

Update of v(i)

- Let the cars are running ordered

$$\ldots, p, i, j, q, \ldots$$

  Then

$$v(p) = i\,, \quad v(i) = j\,, \quad v(j) = q\,.$$

- Let $[i \rightarrow j]$ at the time t_E.
- For $t > t_E$, the running order is

$$\ldots, p, j, i, q, \ldots$$

- Therefore we need to update $v(p)$, $v(i)$ and $v(j)$ as follows

$$v^{new}(p) = j = v(i)\,, \quad v^{new}(i) = q = v(j)\,, \quad v^{new}(j) = i = v(p)$$

Simulation Procedure I

Initialization

1. Set initial condition

$$h_{i,i+1}(t_0) = h^0_{i,i+1} \in \mathbb{R}\,, \quad i = 1, \ldots, N-1\,,$$

$$y_i(t_0) = y_i^0 > 0\,, \quad i = 1, \ldots, N\,.$$

   Set $t_E(0) = t_0$, $j = 0$.

2. Set $x_1 = 0$ and

$$x_i = \sum_{k=1}^{i-1} h_{k,k+1}(t_0)\,, \quad i = 2, \ldots, N\,.$$

3. Compute initial leaders via formula

$$v(i) = \arg\min_{j \neq i} \mathrm{mod}\left(x_j - x_i,\, L\right)\,.$$

Simulation Procedure II

Simulation

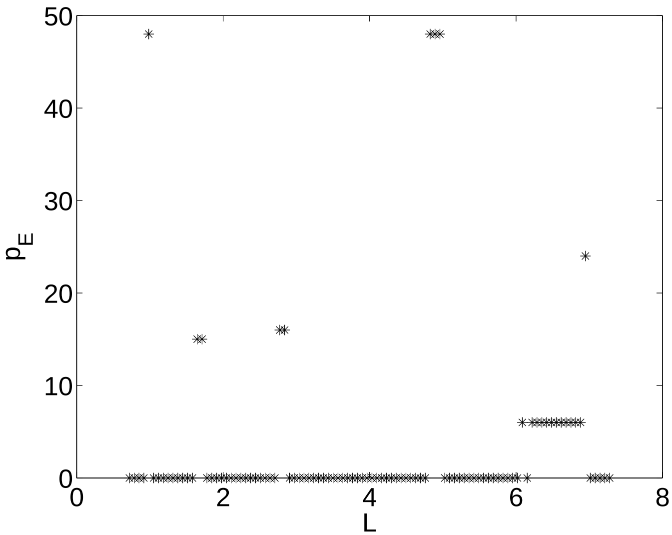
1. Integrate system (6) and monitor test functions

$$f_i(t) = h_{i,v(i)}(t) - \left\lfloor \frac{h_{i,v(i)}(t_E(j))}{L} \right\rfloor L\,, \quad i = 1, \ldots, N\,.$$

2. Locate least t_E for which there is $k \in \{1, 2, \ldots, N\}$ such that $f_k(t_E) = 0$.
3. $j = j + 1$, $t_E(j) = t_E$, $G_E(j) = [k \to v(k)]$, update $v(i)$.
4. Repeat steps 1-3.

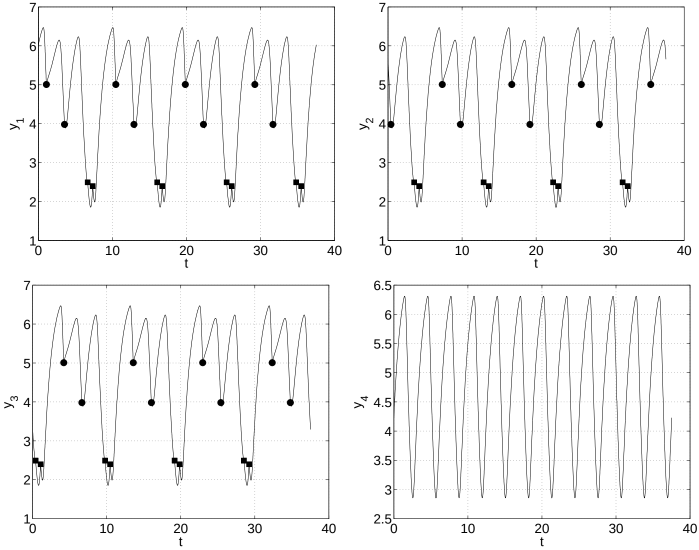
“Bifurcation Diagram"

$N = 4$, $V^{max} = 7$, $a = 2$, $\tau = 1$, $L \in [0.72, 7.28]$



$L = 6.4186,\ T = 9.3842,\ p_E = 6$

$N = 4,\ V^{max} = 7,\ a = 2,\ \tau = 1$



$L = 6.4186$, $T = 9.3842$, $p_E = 6$

$N = 4$, $V^{max} = 7$, $a = 2$, $\tau = 1$

- Event map:

$$G_E = \{[2 \to 3], [1 \to 3], [1 \to 2], [3 \to 2], [3 \to 1], [2 \to 1], \ldots\}$$

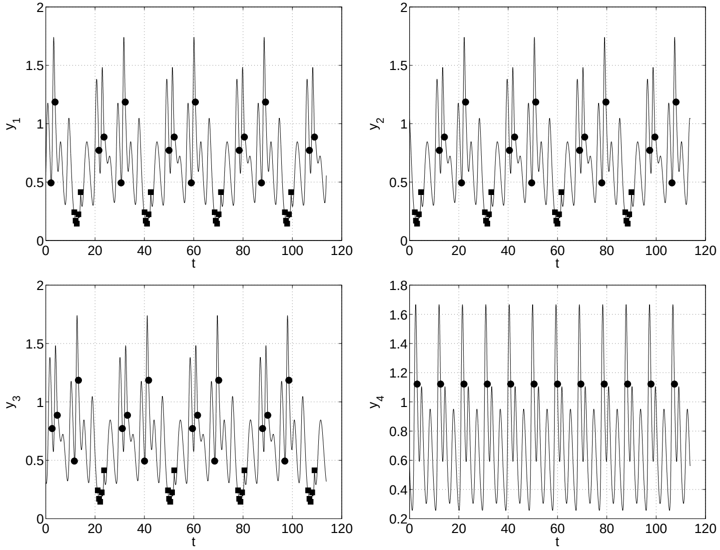
- Car No 4 neither overtakes nor is being overtaken.
- G_E is identical (up to a shift) to the Event Map corresponding to rotating wave of class 1 for $N = 3$.
- The pattern satisfies

$$y_i(t + T) = y_i(t)\,, \quad i = 1, 2, 3, 4\,,$$

$$h_{i,j}(t + T) = h_{i,j}(t)\,, \quad i, j = 1, 2, 3, 4,\ i \neq j\,,$$

$L = 1.6477$, $T = 28.4403$, $p_E = 15$

$N = 4$, $V^{max} = 7$, $a = 2$, $\tau = 1$



$L = 1.6477$, $T = 28.4403$, $p_E = 15$

$N = 4$, $V^{max} = 7$, $a = 2$, $\tau = 1$

- Event map:

$$\begin{aligned} G_E = \{ & [1 \to 2], [3 \to 2], [4 \to 2], [1 \to 2], [3 \to 2], \\ & [3 \to 1], [2 \to 1], [4 \to 1], [3 \to 1], [2 \to 1], \\ & [2 \to 3], [1 \to 3], [4 \to 3], [2 \to 3], [1 \to 3], \ldots \} \end{aligned}$$

- Car No 4 is never overtaken.
- The pattern satisfies

$$\begin{aligned} y_i(t+T) &= y_i(t)\,, \quad i = 1,2,3,4\,, \\ h_{i,j}(t+T) &= h_{i,j}(t)\,, \quad i,j = 1,2,3,\ i \neq j\,, \\ h_{4,j}(t+T) &= h_{4,j}(t) - L\,, \quad j = 1,2,3\,, \\ h_{i,4}(t+T) &= h_{i,4}(t) + L\,, \quad i = 1,2,3\,. \end{aligned}$$

Conclusion and Outlook

- Formulation of the Overtaking Model as a Filippov system was proposed.
- Preliminary simulation results:
  - several oscillatory patterns identified,
  - construction of the “bifurcation diagram".
- The main advantage of the Filipov system formulation is the possibility of using standard software (AUTO97, MATCONT, etc.) to continue these patterns with respect to a parameter.