Computational homogenization for multiscale modeling of heterogeneous materials

Eduard Rohan   **Robert Cimrman**   **Vladimír Lukeš**

Department of Mechanics
University of West Bohemia, Plzeň, Czech Republic

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Homogenization of problems with scale-dependent parameters

scale dependent material parameters (s.d.m.p.)

- existence of underlying structures (micro-level)
- upscaling by "$\varepsilon \to 0$" ... meso $\to$ macro

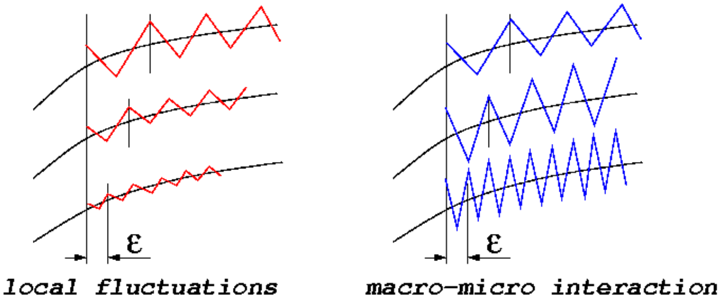
scale:	micro	meso	macro
description:	s.d.m.p.	heterogeneous description, of *micro*model	homogenized model

- fluid saturated *double-porous* media (f.s.dp.m.) – s.d. permeability
  $\Rightarrow$ **solids with microflow**
- phononic crystals – s.d. elasticity
  $\Rightarrow$ **materials with negative mass** (gaps in wave propagation)

layered media – scale dependent thickness

- prefusion in structured transversally periodic f.s.dp.m.
- acoustic transmission on perforated interfaces

Coefficient scaling – interactions & fluctuations



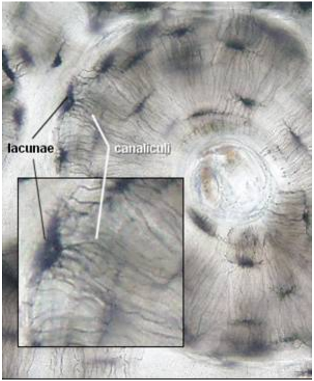
$$|\text{grad}| \lesssim C \qquad\qquad |\text{grad}| \lesssim \frac{C}{\varepsilon}$$

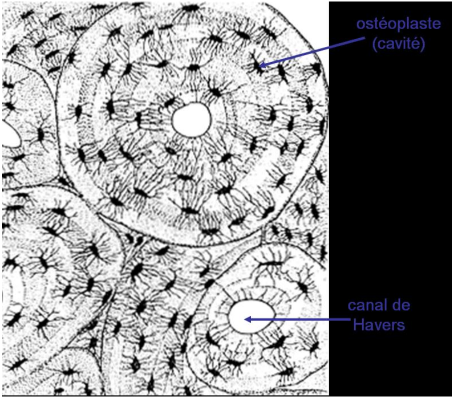
…strong heterogeneity

Cortical bone — double porous medium?

3 scales — distinguishable porosities

- Macro-scale: a piece of compact bone (10 mm)
- Meso-scale – level 1 – Haversian and Volkmann channels (100 μm)
- Micro-scale – level 2 – canaliculi, lacunae in "solid matrix" (1 μm)

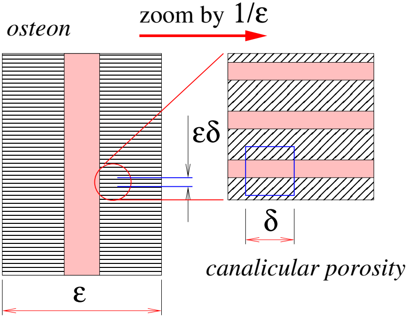




3 scales – Dual porosity – a model of compact bone

Meso-scale → *micro*-scale

scale	zoom to:	char. length	
macro-scale	piece of bone	L	≈ 10 mm
meso-scale	osteon – Haversian porosity	εL	$\approx 100\ \mu$m
micro-scale	canalicular prosity	$\varepsilon\delta L$	$\approx 1\ \mu$m



⇒ one may deduce:

$$\varepsilon \approx \delta \approx 1/100$$

Seepage velocity in the matrix:

$$\mathbf{w}^{\varepsilon,\delta} = \phi\bar{\mathbf{v}}^{\varepsilon,\delta} = -\delta^2 \mathbf{K}_v^0 \nabla p^\varepsilon \ ,$$

$$\delta = \delta(\varepsilon) \approx \varepsilon \quad \text{. . . proportional}$$

⇒ "absolute scale" of canaliculi:

$$\varepsilon\delta \approx \varepsilon^2$$

Biot model of porous fluid saturated solid

$$-\partial_j D_{ijkl} e_{kl}(\mathbf{u}^s) + \partial_j(\alpha_{ij} p) = f_i$$
$$\alpha_{ij} e_{ij}(\frac{\mathrm{d}}{\mathrm{d}\,t}\mathbf{u}^s) + \mathrm{div}\mathbf{w} + \frac{1}{\mu}\frac{\mathrm{d}}{\mathrm{d}\,t} p = 0$$
$$\mathbf{K}\nabla p + \mathbf{w} = 0\ ,$$

elasticity	D_{ijkl}
permeability	$(K_{ij}) = \mathbf{K}$
Biot coefficients	α_{ij}
Biot modulus	$\frac{1}{\mu} = \frac{1}{\mathcal{N}} + \frac{\phi_0}{k_f}$

Boundary value problem – two filed formulation

Find $\mathbf{u}^s$ and p such that

$$-\partial_j D_{ijkl} e_{kl}(\mathbf{u}^s) + \partial_j(\alpha_{ij} p) = f_i \quad \text{in } \Omega,$$
$$\alpha_{ij} e_{ij}(\frac{\mathrm{d}}{\mathrm{d}\,t}\mathbf{u}^s) - \partial_i K_{ij} \partial_j p + \frac{1}{\mu}\frac{\mathrm{d}}{\mathrm{d}\,t} p = 0 \quad \text{in } \Omega,$$

and

$$\mathbf{u}^s(t,\cdot) = \bar{\mathbf{u}}^s(t,\cdot) \quad \text{on } \partial\Omega,\ \text{for } t \in]0, T[\ ,$$
$$\mathbf{w}(t,\cdot) = 0 \quad \text{on } \partial\Omega,\ \text{for } t \in]0, T[\ ,$$
$$\mathbf{u}^s(0,\cdot) = 0 \quad \text{in } \Omega,$$
$$p(0,\cdot) = 0 \quad \text{in } \Omega.$$

Geometry – domain decomposition

$$\Omega = \Omega_m \cup \Omega_c \cup \Gamma_{mc}, \quad \text{with} \quad \Omega_m \cap \Omega_c = \emptyset \, .$$

Representative periodic cell $Y = \Pi_{i=1}^{3}]0, \bar{y}_i[$

$$Y_m = Y \setminus \overline{Y_c} \, ,$$
$$\partial_m Y_c = \partial_c Y_m = \overline{Y_c} \cap \overline{Y_m} \, ,$$
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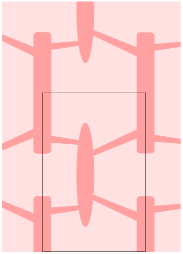

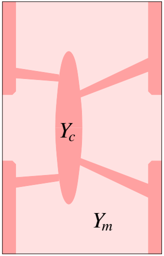

Periodic unfolding — macro-micro decomposition

$$x = \varepsilon \left[\frac{x}{\varepsilon} \right]_Y + \varepsilon \left\{ \frac{x}{\varepsilon} \right\}_Y \qquad y = \left\{ \frac{x}{\varepsilon} \right\}_Y \in Y \, .$$

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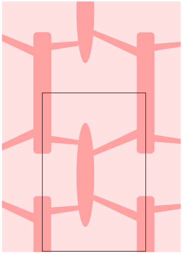
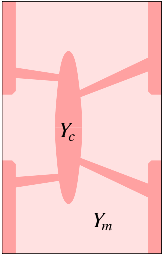
Representative periodic cell
$Y = \Pi_{i=1}^{3}]0, \bar{y}_i[$

$$\begin{aligned} Y_m &= Y \setminus \overline{Y_c} \,, \\ \partial_m Y_c = \partial_c Y_m &= \overline{Y_c} \cap \overline{Y_m} \,, \\ \partial_c Y_c &= \overline{Y_c} \cap \partial Y \,, \\ \partial_m Y_m &= \overline{Y_m} \cap \partial Y \,, \end{aligned}$$





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Biot model with oscillating coefficients – dual porosity

Material coefficients

$$D^{\varepsilon}_{ijkl}(x) = D^{c}_{ijkl}(\{\frac{x}{\varepsilon}\})\chi^{\varepsilon}_{c}(x) + D^{m}_{ijkl}(\{\frac{x}{\varepsilon}\})\chi^{\varepsilon}_{m}(x)\;,$$
$$\alpha^{\varepsilon}_{ij}(x) = \alpha^{c}_{ij}(\{\frac{x}{\varepsilon}\})\chi^{\varepsilon}_{c}(x) + \alpha^{m}_{ij}(\{\frac{x}{\varepsilon}\})\chi^{\varepsilon}_{m}(x)\;,$$
$$K^{\varepsilon}_{ij}(x) = K^{c}_{ij}(\{\frac{x}{\varepsilon}\})\chi^{\varepsilon}_{c}(x) + \varepsilon^2 K^{m}_{ij}(\{\frac{x}{\varepsilon}\})\chi^{\varepsilon}_{m}(x)\;,$$
$$\mu^{\varepsilon}(x) = \mu^{c}(\{\frac{x}{\varepsilon}\})\chi^{\varepsilon}_{c}(x) + \mu^{m}(\{\frac{x}{\varepsilon}\})\chi^{\varepsilon}_{m}(x)\;.$$

Weak formulation — time-integrated pressure P

$$\int_\Omega D^{\varepsilon}_{ijkl} e_{kl}(\mathbf{u}^\varepsilon) e_{ij}(\mathbf{v}) - \int_\Omega \frac{\mathrm{d}\,P^\varepsilon}{\mathrm{d}\,t}\,\alpha^{\varepsilon}_{ij} e_{ij}(\mathbf{v}) = \int_\Omega \mathbf{f}\cdot\mathbf{v}\;,\quad \forall \mathbf{v}\in V_0\;,$$
$$\int_\Omega q\,\alpha^{\varepsilon}_{ij} e_{ij}(\mathbf{u}^\varepsilon) + \int_\Omega K^{\varepsilon}_{ij}\partial_j P^\varepsilon\,\partial_i q + \int_\Omega \frac{1}{\mu^\varepsilon}\frac{\mathrm{d}\,P^\varepsilon}{\mathrm{d}\,t}\,q = 0\;,\quad \forall q\in H^1(\Omega)\;,$$
$$\text{where } P^\varepsilon(t,x) = \int_0^t p^\varepsilon(t,x)\,dt\;,$$

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D^{\varepsilon}_{ijkl}(x) &= D^{c}_{ijkl}(\{\frac{x}{\varepsilon}\})\chi^{\varepsilon}_{c}(x) + D^{m}_{ijkl}(\{\frac{x}{\varepsilon}\})\chi^{\varepsilon}_{m}(x) \;, \\
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K^{\varepsilon}_{ij}(x) &= K^{c}_{ij}(\{\frac{x}{\varepsilon}\})\chi^{\varepsilon}_{c}(x) + \varepsilon^{2} K^{m}_{ij}(\{\frac{x}{\varepsilon}\})\chi^{\varepsilon}_{m}(x) \;, \\
\mu^{\varepsilon}(x) &= \mu^{c}(\{\frac{x}{\varepsilon}\})\chi^{\varepsilon}_{c}(x) + \mu^{m}(\{\frac{x}{\varepsilon}\})\chi^{\varepsilon}_{m}(x) \;.
\end{aligned}
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Weak formulation — time-integrated pressure P

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\int_\Omega D^{\varepsilon}_{ijkl} e_{kl}(\mathbf{u}^{\varepsilon}) e_{ij}(\mathbf{v}) - \int_\Omega \frac{\mathrm{d}\, P^{\varepsilon}}{\mathrm{d}\, t}\, \alpha^{\varepsilon}_{ij} e_{ij}(\mathbf{v}) = \int_\Omega \mathbf{f}\cdot\mathbf{v} \;, \quad \forall \mathbf{v} \in V_0 \;,
$$

$$
\int_\Omega q\, \alpha^{\varepsilon}_{ij} e_{ij}(\mathbf{u}^{\varepsilon}) + \int_\Omega K^{\varepsilon}_{ij} \partial_j P^{\varepsilon}\, \partial_i q + \int_\Omega \frac{1}{\mu^{\varepsilon}} \frac{\mathrm{d}\, P^{\varepsilon}}{\mathrm{d}\, t}\, q = 0 \;, \quad \forall q \in H^1(\Omega) \;,
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Homogenization $\varepsilon \to 0$, limit model

Periodic unfolding *(Cioranescu, Damlamian, Griso ... 2002)*

oscillating fuction: $f^\varepsilon(x) \longrightarrow \mathcal{T}_\varepsilon(f^\varepsilon)(x, y)$

Basic steps of the periodic unfolding method

1. weak formulation (WF)
2. a priori estimation of unknown functions
3. limit functions (gradients)
4. definition of test functions
5. unfolding the integrals in WF, passing to the limit
6. scale decoupling — microproblems,
   corrector basis functions, global functions
7. solving the microproblems
8. evaluation of homogenized coefficients (HC)
9. solving the macroscopic problem — defined in terms of HC
10. interpretation for $\varepsilon > 0$ — given finite scale

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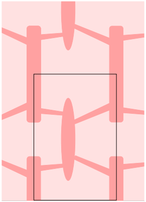
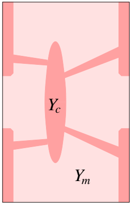
Limit model — uncoupled scales

Phys. field	Function	Scale	Domain
displac.	$\mathbf{u}(x)$	Macro	Ω
displac.	$\mathbf{u}^1(x,y)$	micro	$\Omega \times Y$
press.	$P(x)$	Macro	Ω
press.	$P^1(x,y)$	micro	$\Omega \times Y_c$
press.	$\widehat{P}^0(x,y)$	micro	$\Omega \times Y_m$





Global equations:

to be satisfied for all $\mathbf{w} \in V_0 = \mathbf{H}^1_0(\Omega)$ and $q^0 \in H^1(\Omega)$

$$\fint_{\Omega\times Y} D_{ijkl}[e^x_{kl}(\mathbf{u}) + e^y_{kl}(\mathbf{u}^1)]\, e^x_{ij}(\mathbf{v}^0) - \fint_{\Omega\times Y} \alpha_{ij} e^x_{ij}(\mathbf{v}^0) \left(\frac{\mathrm{d}\,P}{\mathrm{d}\,t} + \chi_m \frac{\mathrm{d}\,\widehat{P}^0}{\mathrm{d}\,t}\right) = \int_\Omega \mathbf{f}\cdot\mathbf{v}^0 \,,$$

$$\fint_{\Omega\times Y_c} K^c_{ij}\,[\partial^x_j P + \partial^y_j P^1]\,\partial^x_i q^0 + \fint_{\Omega\times Y} \alpha_{ij}[e^x_{ij}(\mathbf{u}) + e^y_{ij}(\mathbf{u}^1)]\, q^0$$

$$+ \fint_{\Omega\times Y_c} \frac{1}{\mu^c}\frac{\mathrm{d}\,P}{\mathrm{d}\,t} q^0 + \fint_{\Omega\times Y_m} \frac{1}{\mu^m}\left(\frac{\mathrm{d}\,P}{\mathrm{d}\,t} + \frac{\mathrm{d}\,\widehat{P}^0}{\mathrm{d}\,t}\right) q^0 = 0 \,,$$

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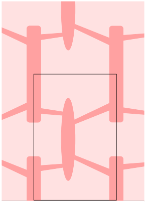

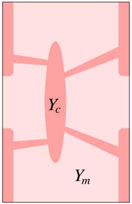

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$$\int\!\!\!\!\!-_{\Omega\times Y_c} K^c_{ij}\, [\partial^x_j P + \partial^y_j P^1]\, \partial^x_i q^0 + \int\!\!\!\!\!-_{\Omega\times Y} \alpha_{ij}[e^x_{ij}(\mathbf{u}) + e^y_{ij}(\mathbf{u}^1)]\, q^0$$
$$+ \int\!\!\!\!\!-_{\Omega\times Y_c} \frac{1}{\mu^c} \frac{\mathrm{d}\, P}{\mathrm{d}\, t} q^0 + \int\!\!\!\!\!-_{\Omega\times Y_m} \frac{1}{\mu^m} \left(\frac{\mathrm{d}\, P}{\mathrm{d}\, t} + \frac{\mathrm{d}\, \widehat{P}^0}{\mathrm{d}\, t} \right) q^0 = 0 \,,$$

Limit model — uncoupled scales

Phys. field	Function	Scale	Domain	space
displac.	$\mathbf{u}(x)$	Macro	Ω	$\mathbf{L}^2(0,T;\mathbf{H}^1_0(\Omega))$
displac.	$\mathbf{u}^1(x,y)$	micro	$\Omega\times Y$	$\mathbf{L}^2(0,T;\mathbf{L}^2(\Omega;\mathbf{H}^1_\#(Y)))$
press.	$P(x)$	Macro	Ω	$L^\infty(0,T;L^2(\Omega))$
press.	$P^1(x,y)$	micro	$\Omega\times Y_c$	$L^\infty(0,T;L^2(\Omega;H^1_\#(Y_c)))$
press.	$\widehat{P}^0(x,y)$	micro	$\Omega\times Y_m$	$L^\infty(0,T;L^2(\Omega;H^1(Y_m)))$

Local equation — diffusion in the channels

$$\fint_{Y_c} K^c_{ij}\,\partial^y_j P^1\,\partial^y_i\psi = -\partial^x_j P(x)\fint_{Y_c} K^c_{ij}\,\partial^y_i\psi \qquad \forall\psi\in H^1_\#(Y_c)$$

Local equation — diffusion-deformation in the marix

$$\fint_Y D_{ijkl}[e^x_{kl}(\mathbf{u})+e^y_{kl}(\mathbf{u}^1)]\,e^y_{ij}(\mathbf{w}) - \fint_Y \alpha_{ij}e^y_{ij}(\mathbf{w})\frac{\mathrm{d}\,P}{\mathrm{d}\,t} - \fint_{Y_m}\alpha^m_{ij}e^y_{ij}(\mathbf{w})\frac{\mathrm{d}\,\widehat{P}^0}{\mathrm{d}\,t} = 0\,,$$

$$\fint_{Y_m} K^m_{ij}\,\partial^y_j\widehat{P}^0\,\partial^y_i\phi + \fint_{Y_m}\alpha^m_{ij}[e^x_{ij}(\mathbf{u})+e^y_{ij}(\mathbf{u}^1)]\,\phi + \fint_{Y_m}\frac{1}{\mu^m}\left(\frac{\mathrm{d}\,P}{\mathrm{d}\,t}+\frac{\mathrm{d}\,\widehat{P}^0}{\mathrm{d}\,t}\right)\phi = 0\,,$$

for all $\mathbf{w}\in\mathbf{H}^1_\#(Y)$ and $\phi\in H^1_{\#0}(Y_m)$.

Scale decoupling — Time dependent problem

Local fields $\mathbf{u}^1(t,x,y), \widehat{P}^0(t,x,y)$ — convolution form

Introduce split using corrector basis functions:

$$\mathbf{u}^1(t,x,y) = \int_0^t \boldsymbol{\omega}^{rs}(t-\tau)\frac{\mathrm{d}}{\mathrm{d}\,\tau} e^x_{rs}(\mathbf{u}(\tau))\,d\tau + \int_0^t \boldsymbol{\omega}^{P}(t-\tau)\frac{\mathrm{d}}{\mathrm{d}\,\tau} P(\tau)\,d\tau\ ,$$

$$\widehat{P}^0(t,x,y) = \int_0^t \pi^{rs}(t-\tau,y)\frac{\mathrm{d}}{\mathrm{d}\,\tau} e^x_{rs}(\mathbf{u}(\tau))\,d\tau + \int_0^t \pi^{P}(t-\tau,y)\frac{\mathrm{d}}{\mathrm{d}\,\tau} P(\tau)\,d\tau\ .$$

Local problems – notation

$$a_Y(\mathbf{u},\,\mathbf{v}) = \fint_Y D_{ijkl}(y) e^y_{kl}(\mathbf{u})\, e^y_{ij}(\mathbf{v})\ ,$$

$$b_{Y_m}(\varphi,\,\mathbf{v}) = \fint_{Y_m} \varphi\,\alpha^m_{ij}(y) e^y_{ij}(\mathbf{v})\ ,$$

$$c_{Y_m}(\varphi,\,\psi) = \fint_{Y_m} K^m_{ij}(y)\partial^y_j\varphi\partial^y_i\psi\ ,$$

$$d_{Y_m}(\varphi,\,\psi) = \fint_{Y_m} (\mu^m)^{-1}\psi\,\varphi\ .$$

Microscopic problems — correctors w.r.t. $e^x_{rs}(\mathbf{u})$

The correctors $(\boldsymbol{\omega}^{rs}, \pi^{rs})$ can be expressed in the form

$$\boldsymbol{\omega}^{rs}(t) = [\tilde{\boldsymbol{\omega}}^{rs}(t) + \bar{\boldsymbol{\omega}}^{rs}]\, H_+(t)\;,$$
$$\pi^{rs}(t) = [\tilde{\pi}^{rs}(t) + \bar{\pi}^{rs}]\, H_+(t)\;.$$

Steady problem correctors – compute $(\bar{\boldsymbol{\omega}}^{rs}, \bar{\pi}^{rs})$

$$a_Y\left(\bar{\boldsymbol{\omega}}^{rs},\, \mathbf{v}\right) = a_Y\left(\boldsymbol{\Pi}^{rs},\, \mathbf{v}\right) \quad \forall \mathbf{v} \in \mathbf{H}^1_{\#}(Y)\;,$$
$$c_{Y_m}\left(\bar{\pi}^{rs},\, q\right) = -b_{Y_m}\left(q,\, \bar{\boldsymbol{\omega}}^{rs} + \boldsymbol{\Pi}^{rs}\right) \quad \forall q \in H^1_{\#0}(Y_m)\;,$$

where $\boldsymbol{\Pi}^{rs} = (\Pi^{rs}_i)$ is defined as $\Pi^{rs}_i = y_s \delta_{ir}$.

Evolutionary problem correctors – compute $(\tilde{\boldsymbol{\omega}}^{rs}, \tilde{\pi}^{rs})$

$$a_Y\left(\tilde{\boldsymbol{\omega}}^{rs},\, \mathbf{v}\right) - b_{Y_m}\left(\frac{\mathrm{d}}{\mathrm{d}\,t}\tilde{\pi}^{rs},\, \mathbf{v}\right) = 0 \quad \forall \mathbf{v} \in \mathbf{H}^1_{\#}(Y)\;,$$
$$b_{Y_m}\left(q,\, \tilde{\boldsymbol{\omega}}^{rs}\right) + c_{Y_m}\left(\tilde{\pi}^{rs},\, q\right) + d_{Y_m}\left(\frac{\mathrm{d}}{\mathrm{d}\,t}\tilde{\pi}^{rs},\, q\right) = 0 \quad \forall q \in H^1_{\#0}(Y_m)\;,$$

where $\tilde{\pi}^{rs}(0) = -\bar{\pi}^{rs}$.

Microscopic problems — correctors w.r.t. P

The correctors $(\boldsymbol{\omega}^{rs}, \pi^{rs})$ can be expressed in the form

$$\begin{aligned}\boldsymbol{\omega}^P(t) &= \tilde{\boldsymbol{\omega}}^P(t)\, H_+(t) + \boldsymbol{\omega}^{*P}\, \delta_+(t)\,,\\ \pi^P(t) &= \tilde{\pi}^P(t)\, H_+(t)\,,\end{aligned}$$

Steady problem correctors – compute $(\boldsymbol{\omega}^{*,P}, \tilde{\pi}^P(0))$

$$\begin{aligned} a_Y\left(\boldsymbol{\omega}^{*,P},\, \mathbf{v}\right) - b_{Y_m}\left(\tilde{\pi}^P(0_+),\, \mathbf{v}\right) &= b_Y\left(1,\, \mathbf{v}\right) \quad \forall \mathbf{v} \in \mathbf{H}^1_\#(Y)\,,\\ b_{Y_m}\left(q,\, \boldsymbol{\omega}^{*,P}\right) + d_{Y_m}\left(\tilde{\pi}^P(0_+),\, q\right) &= -d_{Y_m}\left(1,\, q\right) \quad \forall q \in H^1_{\#0}(Y_m)\,. \end{aligned}$$

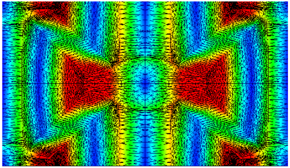
Evolutionary problem correctors – compute $(\tilde{\boldsymbol{\omega}}^P, \tilde{\pi}^P)$

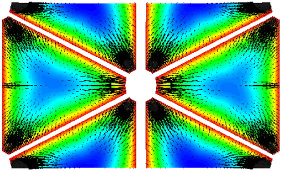
$$\begin{aligned} a_Y\left(\tilde{\boldsymbol{\omega}}^P,\, \mathbf{v}\right) - b_{Y_m}\left(\frac{\mathrm{d}}{\mathrm{d}\,t}\tilde{\pi}^P,\, \mathbf{v}\right) &= 0 \quad \forall \mathbf{v} \in \mathbf{H}^1_\#(Y)\,,\\ b_{Y_m}\left(q,\, \boldsymbol{\omega}^{*,P}\right) + c_{Y_m}\left(\tilde{\pi}^P,\, q\right) + d_{Y_m}\left(\frac{\mathrm{d}}{\mathrm{d}\,t}\tilde{\pi}^P,\, q\right) &= 0 \quad \forall q \in H^1_{\#0}(Y_m)\,. \end{aligned}$$

where $\tilde{\pi}^P(0)$ is given by the "steady problem".

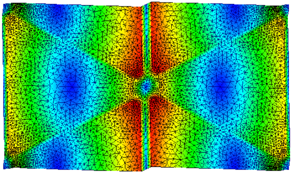
corrector shape functions

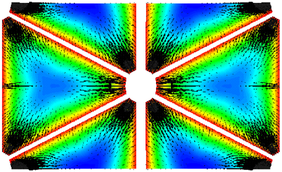
- steady correctors:
  - left: $\bar{\omega}^{11}$, right: $\bar{p}^{11} \in [-0.673, 0]$ + perfusion velocities





  - left: $\bar{\omega}^{21}$ (scaled 0.1x), right: $\bar{p}^{21} \in [-0.542, 0]$ + perfusion velocities





Homogenized coefficients

Defined in terms of the corrector basis functions

- the homogenized elastic tensor

$$\mathcal{E}_{ijkl} = a_Y\left(\boldsymbol{\Pi}^{kl} + \bar{\boldsymbol{\omega}}^{kl},\, \boldsymbol{\Pi}^{ij} + \bar{\boldsymbol{\omega}}^{ij}\right)\,,$$

- the homogenized viscosity tensor of the fading memory

$$\mathcal{H}_{ijkl}(t) = c_{Y_m}\left(\frac{\mathrm{d}}{\mathrm{d}\,t}\tilde{\pi}^{kl},\, \bar{\pi}^{ij}\right)\,.$$

…*effects of microcirculation in the matrix*

- Instantaneous homogenized Biot coefficients

$$\mathcal{B}_{ij} = \frac{1}{|Y|}\int_Y \alpha_{ij} + b_Y\left(1,\, \bar{\boldsymbol{\omega}}^{ij}\right)\,,$$

- transition ("fading memory") homogenized Biot coefficients

$$\mathcal{F}_{ij}(t) = b_{Y_m}\left(\tilde{\pi}^P - \tilde{\pi}^P(0),\, \tilde{\boldsymbol{\omega}}^{ij}\right) + c_{Y_m}\left(\tilde{\pi}^{ij},\, \tilde{\pi}^P\right)\,.$$

- homogenized *reciprocal Biot modulus* – instantaneous response

$$\mathcal{M} = \fint_Y \frac{1}{\mu} + d_{Y_m}\left(\tilde{\pi}^P(0_+),\, 1\right) + b_Y\left(1,\, \boldsymbol{\omega}^{*,P}\right)\ ,$$

- fading memory part of the homogenized *reciprocal Biot modulus*

$$\mathcal{G}(t) = d_{Y_m}\left(\frac{\mathrm{d}}{\mathrm{d}\,t}\tilde{\pi}^P,\, 1\right) + b_Y\left(1,\, \tilde{\boldsymbol{\omega}}^P\right)\ .$$

Effective permeability $\mathcal{C}_{ij}$ — autonomous problem

correctors $\eta \in H^1_\#(Y_c)$ in channels

$$\mathcal{C}_{kl} = \fint_{Y_c} K^c_{ij}\partial^y_j(\eta^l + y_l)\,\partial^y_i(\eta^k + y_k)\ ,$$

$$\text{where} \quad 0 = \fint_{Y_c} K^c_{ij}\partial^y_j(\eta^k + y_k)\,\partial^y_i\psi \quad \forall\psi \in H^1_\#(Y)$$

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Macroscopic model — upscaled double porous medium

For a.a. $t \in]0, T[$ find $\mathbf{u} \in V$ and $P \in Q$ (with $P(0) = 0$) such that

$$\int_\Omega \mathcal{E}_{ijkl} e_{kl}(\mathbf{u}) e_{ij}(\mathbf{v}) + \int_\Omega \int_0^t \mathcal{H}_{ijkl}(t-\tau) e_{kl}(\frac{\mathrm{d}}{\mathrm{d}\,t}\mathbf{u}(\tau))\, d\tau\, e_{ij}(\mathbf{v})$$

$$- \int_\Omega (\mathcal{B}_{ij} + \mathcal{F}_{ij}(0_+)) \frac{\mathrm{d}}{\mathrm{d}\,t} P\, e_{ij}(\mathbf{v}) - \int_\Omega \int_0^t \mathcal{F}_{ij}(t-\tau) \frac{\mathrm{d}}{\mathrm{d}\,\tau} P(\tau)\, d\tau\, e_{ij}(\mathbf{v}) = \int_\Omega \mathbf{f} \cdot \mathbf{v}\ ,$$

$$\int_\Omega \mathcal{B}_{ij} e_{ij}(\mathbf{u})\, q + \int_\Omega \int_0^t \mathcal{F}_{ij}(t-\tau) e_{ij}(\frac{\mathrm{d}}{\mathrm{d}\,\tau}\mathbf{u}(\tau))\, d\tau\, q + \int_\Omega \mathcal{C}_{ij} \partial_j P \partial_i q$$

$$+ \int_\Omega \mathcal{M} \frac{\mathrm{d}}{\mathrm{d}\,t} P\, q + \int_\Omega \int_0^t \mathcal{G}(t-\tau) \frac{\mathrm{d}}{\mathrm{d}\,\tau} P(\tau)\, d\tau\, q = 0\ ,$$

for all $\mathbf{v} \in V_0$ and $q \in Q_0$.

Postprocessing the microflow in double-porous structure

Macro-level — overall flow

Due to interconnected network of Haversian and Volkmann canals

$$\mathbf{w}^M = -\mathbf{K}^M \nabla p^M \quad \ldots \quad \text{from the homogenized macro-model}$$

Micro-level 1 — Haversian porosity flow

$$\begin{aligned}\mathbf{w}^{\mu 1} &= -\mathbf{K}^c (\nabla p)^{\mu 1, corr} \\ &= -\mathbf{K}^c \left(\nabla_x p^M(x) + \nabla_y p^1\right)\end{aligned}$$

Micro-level 2 — canalicular porosity flow

$$\mathbf{w}^{\mu 2, ref} = -\mathbf{K}^m \nabla_y \hat{p} \quad \ldots \quad \hat{p} = \hat{p}(e^x_{rs}(\mathbf{u}^M), p^M) \text{ by the convolution}$$

Let ε_0 be the scale: $\varepsilon_0 = \mu 1/M = \mu 2/\mu 1 \approx 1/100$
an ϕ be the canalicular porosity (vol. frac.), then

$$\mathbf{w}^{\mu 2, real} = {\varepsilon_0}^2 \mathbf{w}^{\mu 2, ref}$$

$$\bar{\mathbf{v}}^{\mu 2, real} = \phi^{-1} \mathbf{w}^{\mu 2, real} \quad \ldots \textit{mean velocity of the Poiseuille flow}$$

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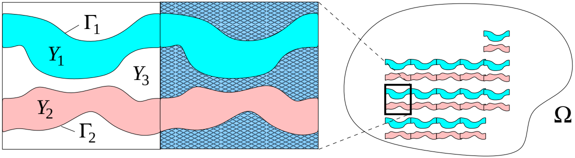
Perfusion in deforming tissue – parallel flows

We consider *two systems of channels* separated by the matrix interface.



Biot continuum:

- incompressible medium
- Double porous matrix $\rightarrow K_{ij}^{\varepsilon} \approx \varepsilon^2$ in the *matrix*.

Find $\mathbf{u}^{\varepsilon}(t) \in V$ and $p^{\varepsilon}(t) \in H^1(\Omega)$ such that

$$\int_{\Omega} D_{ijkl}^{\varepsilon} e_{kl}(\mathbf{u}^{\varepsilon}) e_{ij}(\mathbf{v}) - \int_{\Omega} p^{\varepsilon} \, \mathrm{div} \mathbf{v} = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, , \quad \forall \mathbf{v} \in V_0 \, ,$$

$$\int_{\Omega} q \, \mathrm{div} \frac{\mathrm{d}}{\mathrm{d}\, t} \mathbf{u}^{\varepsilon} + \int_{\Omega} K_{ij}^{\varepsilon} \partial_j p^{\varepsilon} \, \partial_i q = 0 \, , \quad \forall q \in H^1(\Omega) \, ,$$

where $\mathbf{u}(0,x) = 0$ and $p(0,x) = 0$.

Homogenized model

- macroscopic displacements $\mathbf{u}(t)$
- two macroscopic pressures $p_1(t), p_2(t)$

equilibrium of forces (virtual work):

$$\int_\Omega \left[\mathcal{E}_{ijkl} e^x_{kl}(\mathbf{u}(t)) + \int_0^t \mathcal{H}_{ijkl}(t-\tau) \frac{\mathrm{d}}{\mathrm{d}\,\tau} e^x_{kl}(\mathbf{u}(\tau))\, d\tau \right] e^x_{ij}(\mathbf{v})$$
$$- \int_\Omega e^x_{ij}(\mathbf{v}) \int_0^t \tilde{\mathcal{R}}^1_{ij}(t-\tau)[p_1(\tau) - p_2(\tau)]\, d\tau$$
$$- \sum_{\alpha=1,2} \int_\Omega \left[\frac{|Y_\alpha|}{|Y|}\delta_{ij} + \bar{\mathcal{P}}^\alpha_{ij} \right] p_\alpha(t)\, e^x_{ij}(\mathbf{v}) = L(\mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V}_0 \,,$$

two balance-of-mass equations for $\alpha, \beta = 1, 2,\ \beta \neq \alpha$

$$\int_\Omega \mathcal{C}^\alpha_{ij}\, \partial^x_j p_\alpha(t)\, \partial^x_i q$$
$$+ \int_\Omega q\, \mathcal{G}^* \frac{\mathrm{d}}{\mathrm{d}\,t} (p_\alpha(t) - p_\beta(t)) + \int_\Omega q \int_0^t \tilde{\mathcal{G}}_+(t-\tau) \frac{\mathrm{d}}{\mathrm{d}\,\tau} (p_\alpha(\tau) - p_\beta(\tau))\, d\tau$$
$$+ \int_\Omega q \int_0^t \tilde{\mathcal{R}}^\alpha_{ij}(t-\tau) \frac{\mathrm{d}}{\mathrm{d}\,\tau} e^x_{ij}(\mathbf{u}(\tau))\, d\tau + \int_\Omega q \left[\frac{|Y_\alpha|}{|Y|}\delta_{ij} + \bar{\mathcal{P}}^\alpha_{ij} \right] \frac{\mathrm{d}}{\mathrm{d}\,t} e^x_{ij}(\mathbf{u}(t)) = 0 \,, \quad \forall q \in$$

. . . *fluid flows in the two channels and its redistribution between them.*

Local problems for t-variant correctors

Find $(\tilde{\boldsymbol{\omega}}^{rs}, \tilde{\pi}^{rs}) \in \mathbf{H}^1_\#(Y) \times H^1_{\#0}(Y_3)$

such that $\tilde{\pi}^{rs}(0) = -\bar{\pi}^{rs}$ and for $t > 0$

$$a_Y\left(\tilde{\boldsymbol{\omega}}^{rs}(t), \mathbf{v}\right) - \left(\frac{\mathrm{d}}{\mathrm{d}\,t}\tilde{\pi}^{rs}(t), \mathrm{div}_y \mathbf{v}\right)_{Y_3} = 0 \quad \forall \mathbf{v} \in \mathbf{H}^1_\#(Y)$$

$$\left(\psi, \mathrm{div}_y \tilde{\boldsymbol{\omega}}^{rs}(t)\right)_{Y_3} + c_{Y_3}\left(\tilde{\pi}^{rs}(t), \psi\right)_{Y_3} = 0 \quad \forall \psi \in H^1_{\#0}(Y_3)\ ,$$

Find $(\tilde{\boldsymbol{\omega}}^{\alpha}, \tilde{\pi}^{\alpha}) \in \mathbf{H}^1_\#(Y) \times H^1_{\#}(Y_3)$

such that $\tilde{\pi}^{\alpha}(0)$ "is given" (a Lagrange multiplier) and for $t > 0$

$$a_Y\left(\tilde{\boldsymbol{\omega}}^{\alpha}(t), \mathbf{v}\right) - \left(\frac{\mathrm{d}}{\mathrm{d}\,t}\tilde{\pi}^{\alpha}(t), \mathrm{div}_y \mathbf{v}\right)_{Y_3} = 0 \quad \forall \mathbf{v} \in \mathbf{H}^1_\#(Y)$$

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where $\tilde{\pi}^{\alpha}(t) = \delta_{\alpha\beta}$ on $\Gamma_\beta\ ,\ \beta = 1, 2\ .$

Corrector *t*-variant basis functions – Local problems

$\boldsymbol{\omega}^{rs}(t,y)$	displacement in Y	effect of macro-strain $e^x_{rs}(\mathbf{u})$
$\pi^{rs}(t,y)$	pressure in $Y_3 \subset Y$	effect of macro-strain $e^x_{rs}(\mathbf{u})$
$\boldsymbol{\omega}^{\alpha}(t,y)$	displacement in Y	effect of macro-pressure P_α
$\pi^{\alpha}(t,y)$	pressure in $Y_3 \subset Y$	effect of macro-pressure P_α

Microscopic Corrector Problems – generic form

FE approximation

$$\mathbf{A}\tilde{\mathbf{u}}^{\spadesuit}(t) - \mathbf{B}^T \frac{\mathrm{d}}{\mathrm{d}\,t}\tilde{\mathbf{p}}^{\spadesuit}(t) = 0$$

$$\mathbf{B}\tilde{\mathbf{u}}^{\spadesuit}(t) + \mathbf{C}\tilde{\mathbf{p}}^{\spadesuit}(t) = \mathbf{g}^{\spadesuit}$$

initial condition: $\tilde{\mathbf{p}}^{\spadesuit}(t=0)$

Matrix operators:

- **A** stiffness in Y
- **C** permeability in $Y_3 \subset Y$
- **B** $\mathrm{div}(\cdot)$ op. in $Y_3 \subset Y$

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Efficient computing of microscopic correctors ???

Schur complement method: *eigenvalue problem:*

$$\mathbf{C}\mathbf{q}^k = \nu^k \mathbf{G}\mathbf{q}^k \;, \quad k = 1,\dots,N_\mathrm{p} \;, \quad \text{where} \quad \mathbf{G} = \mathbf{B}\mathbf{A}^{-1}\mathbf{B}^T$$

$$\mathbf{E} := \mathrm{diag}(\{\nu^k\}) \qquad \mathbf{Q} := \{\mathbf{q}^k\}$$

Microscopic correctors ... [small/large] eigenvalues needed?

$$\tilde{\mathbf{p}}^\alpha(t) = \mathbf{Q}\left[\exp\{-\mathbf{E}t\}\tilde{\zeta}^\alpha(0) + \mathbf{E}^{-1}(\mathbf{I} - \exp\{-\mathbf{E}t\})\mathbf{Q}^T\tilde{\mathbf{g}}^\alpha\right] \;,$$

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\tilde{\mathbf{p}}^\alpha(t) = \mathbf{Q}\left[-\underbrace{\mathbf{E}\exp\{-\mathbf{E}t\}}_{\text{at } t\approx 0???}\tilde{\zeta}^\alpha(0) + \exp\{-\mathbf{E}t\}\mathbf{Q}^T\tilde{\mathbf{g}}^\alpha\right]$$

$\Rightarrow$ Initial singularity
$t \approx 0$ ... large eigenvalues dominate
$t > 0$ ... small eigenvalues are relevant $\Rightarrow$ approximation
$\Rightarrow$ approximate computing of convolution kernels

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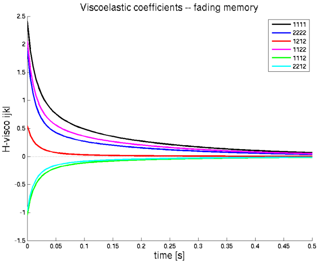
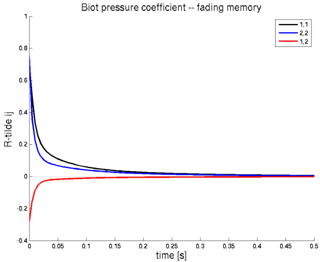
Fading Memory Kernels – *synchronous decay ???*

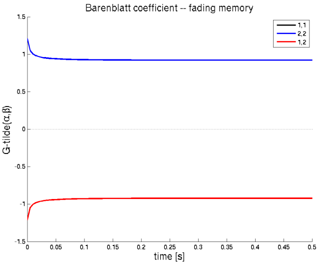
$\mathcal{H}_{ijkl}$ **2D**



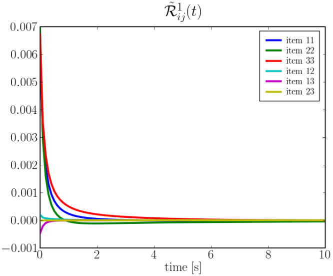
$\tilde{\mathcal{R}}^1_{ij}$ **2D**



$\tilde{\mathcal{G}}$ **2D**



$\tilde{\mathcal{R}}^1_{ij}$ **3D**



Homogenized coefficients — Convolutions

Discrete convolution kernels

$$\int_0^{t_j} \mathcal{F}(t-s) f(s)\, ds \approx \sum_{k=1}^{j} F^{(j-k)} \overline{f}^{(k)}$$

Approximation of the initial singularity

$$\begin{aligned} \mathcal{F}(t) &\approx \mathcal{F}^* \delta_+(t) + \tilde{\mathcal{F}}(t) \\ \mathcal{F}^* &= \int_0^{t_\epsilon} \mathcal{A}_{\max \nu}(\mathcal{F}(t))\, dt \\ \tilde{\mathcal{F}}(t) &= \mathcal{A}_{\min \nu}(\mathcal{F}(t)) \end{aligned}$$

where

$\mathcal{A}_{\max \nu}(\cdot)$ approximation by large eigenalues

$\mathcal{A}_{\min \nu}(\cdot)$ approximation by small eigenalues

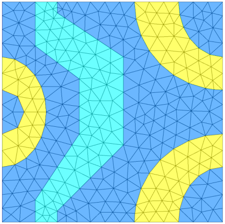
Reduced computation of eigenvalues – Approximation

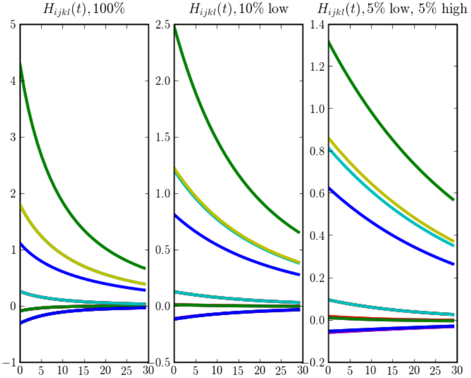
Strategies

compute ??? % smallest and/or largest eigenvalues,
given *fine/coars* mesh

Viscoelasticity – fading memory kernels $\mathcal{H}_{ijkl}(t)$

Rough mesh
584 elements





42% error 70% error

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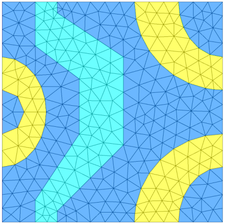

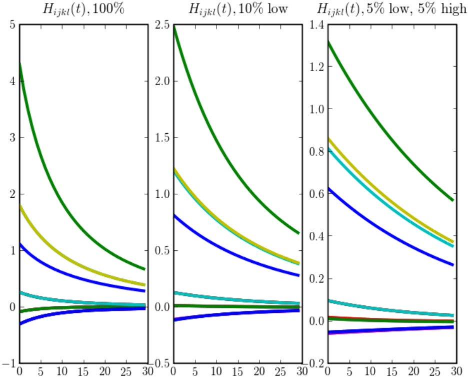

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Reduced computation of eigenvalues – Approximation

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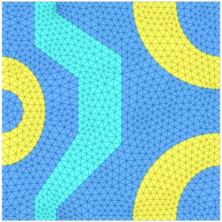
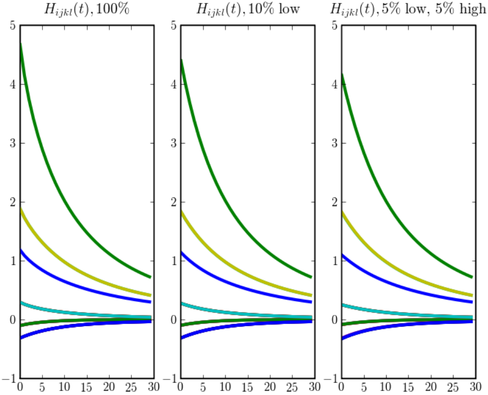
Fine mesh
2336 elements





4% error   11% error

$\Rightarrow$ observation

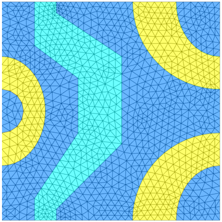
- use **finer mesh**
- **small eigenvalues** more important than the large ones

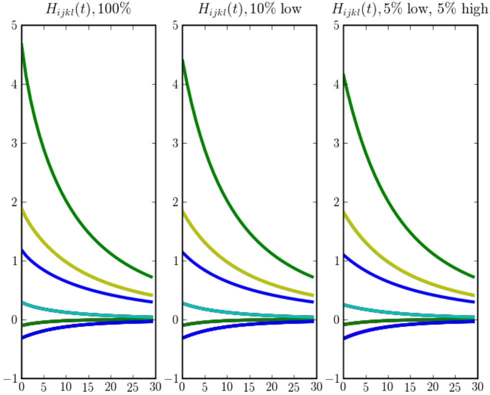
Reduced computation of eigenvalues – Approximation

Viscoelasticity – fading memory kernels $\mathcal{H}_{ijkl}(t)$

Fine mesh

2336 elements





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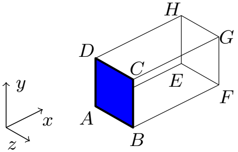
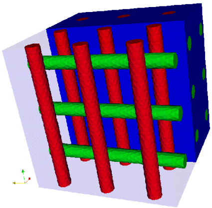
3D Parallel Flows - Problem Setting

in orthogonal straight channels

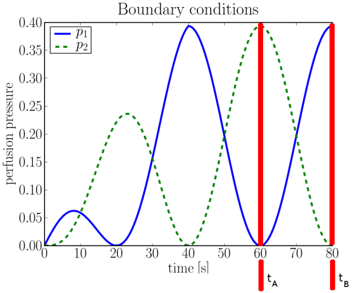
- macroscopic domain:



- BC on face:
  - ABCD: fixed ($\mathbf{u} = 0$)
  - BFGC: given $p_1(t)$
  - CGHD: given $p_2(t)$
- microstructure:



- prescribed perfusion pressures:

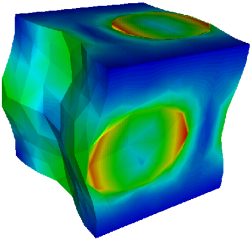
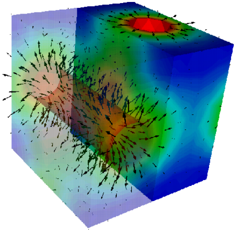


- we will show:
  - some micro-corrector shape functions
  - some fading memory coefficients
  - macro solution snapshots for $t = t_A$, $t = t_B$

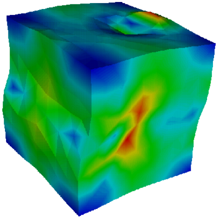
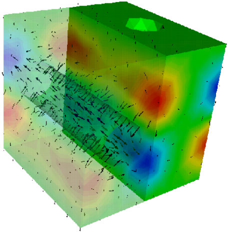
Microscopic Solution

corrector shape functions

- steady correctors:
  - left: $\bar{\omega}^{11}$, right: $\bar{p}^{11} \in [-53, 0]$ + perfusion velocities

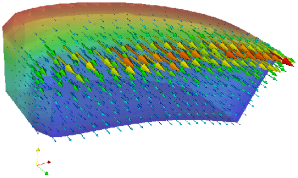
  - left: $\bar{\omega}^{21}$, right: $\bar{p}^{21} \in [-0.41, 0.43]$ + perfusion velocities

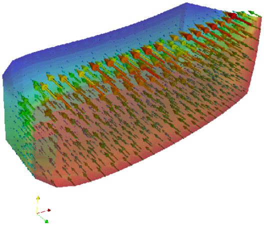
Macroscopic Solution

- perfused block: color = pressures, arrows = perfusion velocities
- deformation enlarged for visualization (10x)

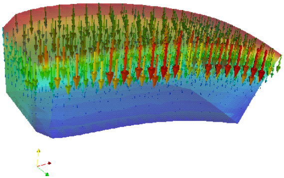
$t = t_A$: p_1, $\mathbf{w}_1$



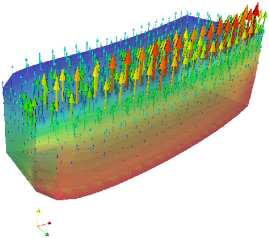
$t = t_B$: p_1, $\mathbf{w}_1$



$t = t_A$: p_2, $\mathbf{w}_2$



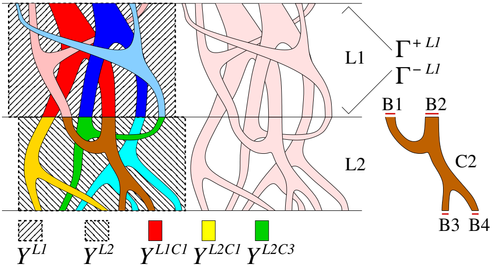
$t = t_B$: p_2, $\mathbf{w}_2$



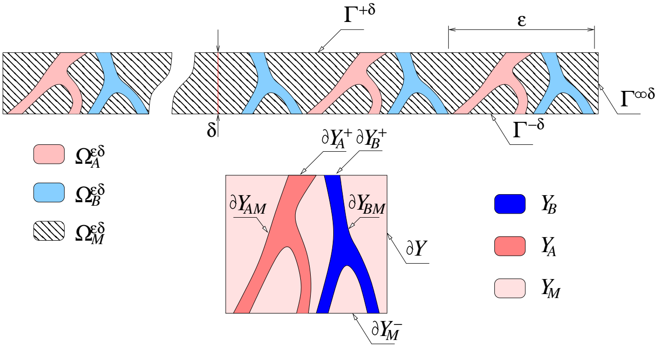
Perfusion of layered structure — model of *brain perfusion*

Idea:

- brain represented by "spherical" shells
- each shell has a periodic structures
- representation of disconnected arterial and venous trees



Geometry – Layered structure



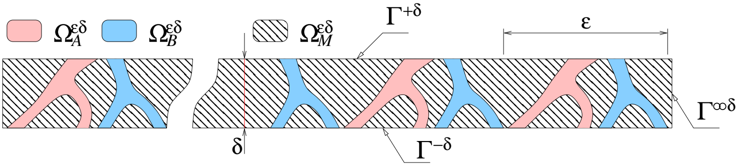
channels $\Omega_A^{\varepsilon\delta}, \Omega_B^{\varepsilon\delta}$
& matrix $\Omega_M^{\varepsilon\delta}$

$$\Omega^\delta = \Omega_M^{\varepsilon\delta} \cup \Omega_A^{\varepsilon\delta} \cup \Omega_B^{\varepsilon\delta}$$
$$\emptyset = \Omega_A^{\varepsilon\delta} \cap \Omega_B^{\varepsilon\delta}$$

double-porous medium:
strongly heterogeneous permeability

$$\kappa_{ij}^\varepsilon(x) = \delta_{ij} \times \begin{cases} \kappa(\{\frac{x}{\varepsilon}\}) & x \in \Omega_A^{\varepsilon\delta} \cup \Omega_B^{\varepsilon\delta}, \\ \varepsilon^2 \bar{\kappa}(\{\frac{x}{\varepsilon}\}) & x \in \Omega_M^{\varepsilon\delta}, \end{cases}$$

Boundary value problem in Ω^δ



$$\nabla \cdot \kappa^\varepsilon \nabla p^\varepsilon = 0 \quad \text{in } \Omega^\delta ,$$
$$\mathbf{n} \cdot \nabla p^\varepsilon = 0 \quad \text{on } \Gamma^{\infty\delta} ,$$
$$\mathbf{n} \cdot \nabla p^\varepsilon = g^{\varepsilon\pm} \text{ on } \Gamma^{\delta+} \cup \Gamma^{\delta-} .$$

solvability: $\int_{\Gamma^{\pm\varepsilon\delta}} g^{\varepsilon\pm} \, dS = 0$,

uniqueness: $\int_{\Omega_A^{\varepsilon\delta} \cup \Omega_B^{\varepsilon\delta}} p^\varepsilon = 0$.

Weak formulation $\forall q \in H^1(\Omega^\delta)/\mathbb{R}$

$$\int_{\Omega_A^{\varepsilon\delta} \cup \Omega_B^{\varepsilon\delta}} \kappa \nabla p^\varepsilon \cdot \nabla q + \int_{\Omega_M^{\varepsilon\delta}} \varepsilon^2 \bar{\kappa} \nabla p^\varepsilon \cdot \nabla q = \int_{\Gamma^{\delta+} \cup \Gamma^{\delta-}} g^{\varepsilon\pm} q \, dS ,$$

Acoustic waves in ducts with perforated interfaces

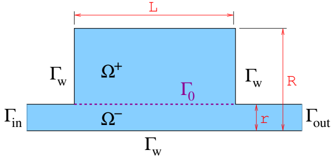
Model

- Acoustic medium – air
- Perforated interface Γ_0

$$\Omega^G = \Omega^+ \cup \Omega^- \cup \Gamma_0$$

- Problem: Compute acoustic pressure p^+, p^- in Ω^+, Ω^-

$$\begin{aligned} c^2\nabla^2 p^+ + \omega^2 p^+ &= 0 \quad \text{in } \Omega^+ \,, \\ c^2\nabla^2 p^- + \omega^2 p^- &= 0 \quad \text{in } \Omega^- \,, \\ + \text{ boundary conditions} & \quad \text{on } \partial\Omega^G \,, \end{aligned}$$



Transmission condition on Γ_0 ???

semi-empirical formulae: impedance Z (complex number)

$$\frac{\partial p^+}{\partial n^+} = -\mathrm{i}\frac{\omega\rho}{Z}(p^+ - p^-), \quad \frac{\partial p^-}{\partial n^-} = -\mathrm{i}\frac{\omega\rho}{Z}(p^- - p^+) \,,$$

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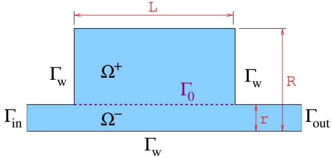

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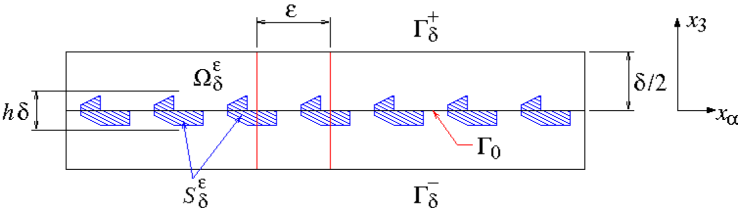
Treatment of transmission conditions

Existing approaches

- Quasi-empirical approach: $\frac{\partial p^{\pm}}{\partial n^{\pm}} = \mp \mathrm{i} \frac{\omega \rho}{Z} (p^{+} - p^{-})$
- Homogenization – inner and outer expansions
  [Sanchez-Hubert, Sanchez-Palencia,1982], [Bonnet-Ben-Dhia etal., 2007]
- only "flat perforated thin sieve"

Homogenization of acoustic waves in *thin perforated layer*

- Arbitrary geometry in the layer
- thickness ≈ period of perforation ≈ scale of holes
- *Goal:* to replace the perforated layer by a "homogenized surface"



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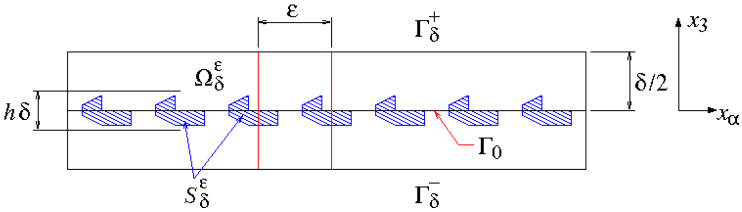
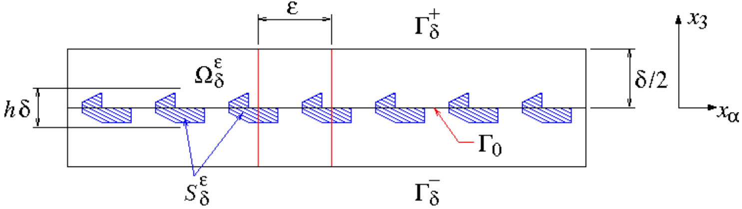


Boundary value problem in the transmission layer



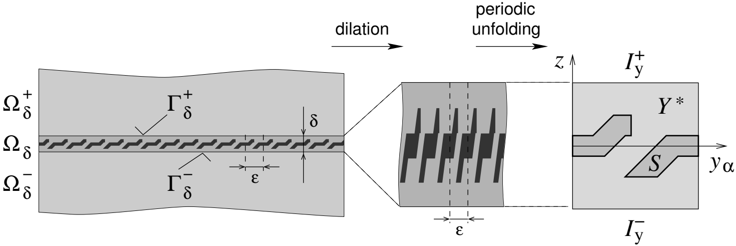
- thickness proportional to the perforation size: $\delta = \varkappa\varepsilon$
- low frequencies (ω independent of ε)
- Neumann problem:

$$c^2\nabla^2 p^{\varepsilon\delta} + \omega^2 p^{\varepsilon\delta} = 0 \quad \text{in } \Omega_\delta^\varepsilon \,,$$

$$c^2\frac{\partial p^{\varepsilon\delta}}{\partial n^\delta} = -\mathrm{i}\omega g^{\varepsilon\delta\pm} \quad \text{on } \Gamma_\delta^\pm \qquad \ldots\text{transversal velocity}$$

$$\frac{\partial p^{\varepsilon\delta}}{\partial n^\delta} = 0 \quad \text{on } \partial S_\delta^\varepsilon \cup \partial\Omega_\delta^\infty \qquad \ldots\text{solid obstacle,}$$

Dilatation and Periodic unfolding



fluid representation: $\Omega_\delta^\varepsilon \xrightarrow{\text{dilatation}} \Omega^\varepsilon \xrightarrow{\text{unfolding}} Y^*$

Dilated weak formulation

Find $p^\varepsilon \in H^1(\Omega^\varepsilon)$ such that

$$c^2 \int_{\Omega^\varepsilon} \left(\partial_\alpha p^\varepsilon \partial_\alpha q + \frac{1}{\delta^2} \partial_z p^\varepsilon \partial_z q \right) - \omega^2 \int_{\Omega^\varepsilon} p^\varepsilon q =$$

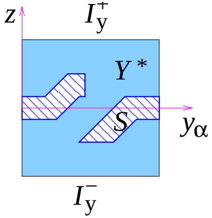
$$-\mathrm{i}\omega \frac{1}{\delta} \left(\int_{\Gamma^+} g^{\varepsilon\delta+} q\, dS + \int_{\Gamma^-} g^{\varepsilon\delta-} q\, dS \right)$$

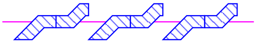
for all $q \in H^1(\Omega^\varepsilon)$.

Geometry of the perforated interface layer

Microscopic scale — periodic perforation

- representative cell
  $Y = I_y \times]-1/2, +1/2[$
  where
  $I_y = \{(y_\alpha, 0) : y_\alpha \in]0,1[,\ \alpha = 1,2\}$
- solid (rigid) part: S
- air in $Y^* = Y \setminus S$
- functions periodic in y_α, $\alpha = 1, 2$





Limit interface fluxes ($\approx$ velocities)

r.h.s. terms

$$\cdots = -\mathrm{i}\omega \frac{1}{\delta}\left(\int_{\Gamma^+} g^{\varepsilon\delta+} q \, dS + \int_{\Gamma^-} g^{\varepsilon\delta-} q \, dS\right)$$

$g^{\varepsilon\delta\pm}$ must be specified:

- assume existence of $g^{0-}, g^{0+} \in L^2(\Gamma_0)$ such that

$$\mathcal{T}_\varepsilon^b\left(g^{\varepsilon+}\right) \rightharpoonup g^{0+} \text{ and } \mathcal{T}_\varepsilon^b\left(g^{\varepsilon-}\right) \rightharpoonup g^{0-} \text{ weakly in } L^2(\Gamma_0 \times I_y^\pm) \,.$$

- transversal acoustic velocity does not change when passing the perforated layer, thus, we require

$$\frac{1}{\varepsilon}\left(\int_{\Gamma^+} \phi g^{\varepsilon+} + \int_{\Gamma^-} \phi g^{\varepsilon-}\right) \to 0 \quad \forall \phi \in \mathcal{D}(\Gamma_0) \,.$$

- *... therefore* $g^{0\pm} \equiv g^{0+} = -g^{0-}$.

Limit equation – tangent acoustic wave in plane Γ_0

$$c^2 \int_{\Gamma_0 \times Y^*} \left(\partial_\alpha^x p^0 + \partial_\alpha^y p^1\right)\left(\partial_\alpha^x q^0 + \partial_\alpha^y q^1\right) + c^2 \frac{1}{\varkappa^2} \int_{\Gamma_0 \times Y^*} \partial_z p^1 \partial_z q^1$$
$$- \omega^2 \int_{\Gamma_0 \times Y^*} p^0\, q^0 = -\frac{\mathrm{i}\omega}{\varkappa} \int_{\Gamma_0} g^{0\pm} \left[\int_{I_y^+} q^1\, dS_y - \int_{I_y^-} q^1\, dS_y\right]$$

- local (microscopic) problem: $q^0 \equiv 0$,
  $q^1(x_\alpha, y) = \theta(x_\alpha)\phi(y)$
- global (macroscopic) problem: $q^1 \equiv 0$,
  $q^0 = q^0(x_\alpha)$

Acoustic pressure jump on Γ_0 – acoustic impedance X

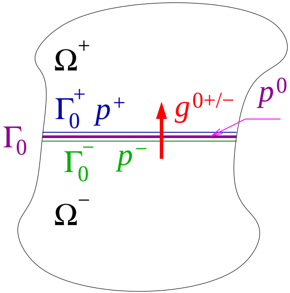
Coupling the limit interface acoustic pressures $p^+, p^- \in L^2(\Gamma_0)$ and the "transversal velocity" such that for any $\phi \in \mathcal{D}(\Gamma_0)$

$$\frac{1}{\varepsilon_0}\int_{\Gamma_0} \phi(p^+ - p^-) \approx \int_{\Gamma_0} \phi \frac{1}{|I_y|}\left[\int_{I_y^+} p^1 \, d\Gamma_y - \int_{I_y^-} p^1 \, d\Gamma_y\right] .$$

Acoustic impedance (implicit)

$$p^+ - p^- = X g^{0\pm}$$

- pressure jump $p^+ - p^-$
- transversal velocity $g^{0\pm}$



Local microscopic problems — pressure correctors

- Scale decoupling — corrector functions:

$$p^1(x_\alpha, y) = \pi^\beta(y)\partial_\beta p^0(x_\alpha) + \mathrm{i}\omega\xi^\pm(y)g^{0\pm}(x_\alpha) \ ,$$

- Corrector of the tangent interface velocity $v_t \approx \partial_\alpha p^0$
find $\pi^\beta \in H^1_{\#(1,2)}(Y)$, $\beta = 1,2$, such that

$$\int_{Y^*} \left[\partial^y_\alpha \pi^\beta \, \partial^y_\alpha q + \frac{1}{\varkappa^2}\partial_z \pi^\beta \partial_z q\right] = -\int_{Y^*} \partial^y_\beta q \qquad \forall q \in H^1_{\#(1,2)}(Y)$$

- Corrector of the normal interface velocity $v_n \approx g^{0\pm}$
find $\xi^\pm \in H^1_{\#(1,2)}(Y)/\mathbb{R}$, such that

$$\int_{Y^*} \left[\partial^y_\alpha \xi^\pm \, \partial^y_\alpha q + \frac{1}{\varkappa^2}\partial_z \xi^\pm \partial_z q\right] = -\frac{|Y|}{c^2\varkappa}\left(\int_{I^+_y} q \, dS_y - \int_{I^-_y} q \, dS_y\right) \ ,$$

for all $q \in H^1_{\#(1,2)}(Y)/\mathbb{R}$

Homogenized interface conditions

Homogenized coefficients

- Tangent acoustic diffusion coefficients

$$A_{\alpha\beta} = \frac{c^2}{|Y|} \int_{Y^*} \partial_\gamma^y (y^\beta + \pi^\beta)\, \partial_\gamma^y (y^\alpha + \pi^\alpha) + \frac{c^2}{\varkappa^2 |Y|} \int_{Y^*} \partial_z \pi^\beta \partial_z \pi^\alpha \,.$$

- Coefficients of transversal-to-tangent coupling of velocity

$$B_\alpha = \frac{c^2}{|Y|} \int_{Y^*} \partial_\alpha^y \xi^\pm \,,$$

$$\frac{\varkappa}{|I_y|} B_\alpha = D_\alpha = \frac{1}{|I_y|} \left(\int_{I_y^+} \pi^\alpha \, dS_y - \int_{I_y^-} \pi^\alpha \, dS_y \right) \,,$$

- Local transversal impedance

$$F = \frac{1}{|I_y|} \left(\int_{I_y^+} \xi^\pm \, dS_y - \int_{I_y^-} \xi^\pm \, dS_y \right)$$

Homogenized interface conditions

Acoustic transmission

- Given pressure jump $[p^+ - p^-]$
- Find $p^0 \in H^1(\Gamma_0)$ and $g^{0\pm} \in L^2(\Gamma_0)$ such that

$$\int_{\Gamma_0} A_{\alpha\beta} \partial^x_\beta p^0 \partial^x_\alpha q - \frac{|Y^*|}{|Y|}\omega^2 \int_{\Gamma_0} p^0 q + \mathrm{i}\omega \int_{\Gamma_0} g^{0\pm} B_\alpha \partial^x_\alpha q = 0$$

$$-\mathrm{i}\omega \int_{\Gamma_0} \psi D_\beta \partial^x_\beta p^0 + \omega^2 \int_{\Gamma_0} F g^{0\pm} \psi = -\frac{\mathrm{i}\omega}{\varepsilon_0} \int_{\Gamma_0} (p^+ - p^-)\psi$$

for all $q \in H^1(\Gamma_0)$ and $\psi \in L^2(\Gamma_0)$

Explanation

- transversal pressure jump induces transversal and in-plane fluxes (velocities)
- in-plane waves – $A_{\alpha\beta} \approx c^2$ anisotropic velocity^2 of propagation
  in-plane resonance: eigenpairs $(\hat{\omega}^2, \hat{p})$ satisfy

$$-\partial_\alpha A_{\alpha\beta} \partial_\beta \hat{p} = \frac{|Y^*|}{|Y|} \hat{\omega}^2 \hat{p} \quad \text{in } \Gamma_0$$

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- Given pressure jump $[p^+ - p^-]$
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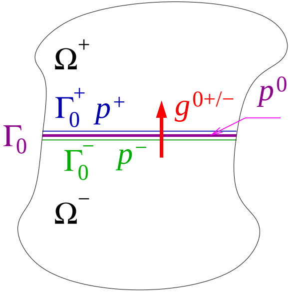
Acoustic problem with homogenized perforation

Acoustic behaviour in $\Omega^+ \cup \Omega^-$

$$c^2\nabla^2 p^+ + \omega^2 p^+ = 0 \quad \text{in } \Omega^+ ,$$
$$c^2\nabla^2 p^- + \omega^2 p^- = 0 \quad \text{in } \Omega^- ,$$
$$+ \text{ boundary conditions} \quad \text{on } \partial\Omega^G ,$$

$\Omega^G = \Omega^+ \cup \Omega^- \cup \Gamma_0$



Transmission condition — in terms of p^0 and $g^{0\pm}$

$$c^2 \frac{\partial p^+}{\partial n^+} = \mathrm{i}\omega g^{0\pm} \text{ on } \Gamma_0 ,$$
$$c^2 \frac{\partial p^-}{\partial n^-} = -\mathrm{i}\omega g^{0\pm} \text{ on } \Gamma_0 ,$$

Interface problem for p^0 and $g^{0\pm}$ be satisfied

Acoustic problem with homogenized perforation

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$$
\begin{aligned}
c^2\nabla^2 p^+ + \omega^2 p^+ &= 0 \quad \text{in } \Omega^+ \,, \\
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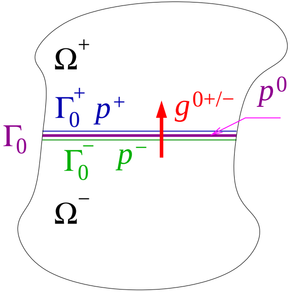
+ \text{ boundary conditions} & \quad \text{on } \partial\Omega^G \,,
\end{aligned}
$$

$\Omega^G = \Omega^+ \cup \Omega^- \cup \Gamma_0$



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\end{aligned}
$$

Interface problem for p^0 and $g^{0\pm}$ be satisfied

Discretized interface problem

$$\mathbf{A}\mathbf{p}^0 - \phi^*\omega^2\mathbf{M}\mathbf{p}^0 + \mathrm{i}\omega\mathbf{B}^T\mathbf{g}^0 = 0$$
$$-\mathrm{i}\omega\mathbf{D}\mathbf{p}^0 + \omega^2\mathbf{F}\mathbf{g}^0 = -\mathrm{i}\omega\mathbf{M}(\mathbf{p}^+ - \mathbf{p}^-)\frac{1}{\varepsilon_0}$$

- Schur complement (for ω out-of-resonance)

$$\mathbf{p}^0 = -\mathrm{i}\omega(\mathbf{A} - \phi^*\omega^2\mathbf{M})^{-1}\mathbf{B}^T\mathbf{g}^0 \,,$$
$$\omega^2[\mathbf{F} - \mathbf{D}(\mathbf{A} - \phi^*\omega^2\mathbf{M})^{-1}\mathbf{B}^T]\,\mathbf{g}^0 = -\mathrm{i}\omega\mathbf{M}(\mathbf{p}^+ - \mathbf{p}^-)\frac{1}{\varepsilon_0}$$

- Coupled impedance $\mathbf{X}(\omega^2) = \omega^2[\mathbf{F} - \mathbf{D}(\mathbf{A} - \phi^*\omega^2\mathbf{M})^{-1}\mathbf{B}^T]$

$$\varepsilon_0\mathbf{X}(\omega^2)\mathbf{g}^0 = -\mathrm{i}\omega\mathbf{M}(\mathbf{p}^+ - \mathbf{p}^-)$$

- ... resembles the structure of the standard conditions, since $\mathbf{g}^0 \approx \partial p^+/\partial n^+ = -\partial p^-/\partial n^-$.

Global problem – FEM discretization

notation		explanation
$\mathbf{p}$	...	pressure in $\Omega^+ \cup \Omega^- \cup \partial\Omega$
$\mathbf{p}^{+/-}$	...	pressure on $\Gamma_0^+ \cup \Gamma_0^-$
$\mathbf{C}(\omega), \mathbf{Q}^+(\omega), \mathbf{Q}^-(\omega)$	...	matrices assoc. with $c^2\nabla^2 p + \omega^2 p$ and with B.C. in $\bar{\Omega} \setminus \Gamma_0^{+/-}$
$\bar{\mathbf{C}}^{+/-}(\omega)$	...	matrix assoc. with $c^2\nabla^2 p + \omega^2 p$ on $\Gamma_0^+ \cup \Gamma_0^-$
$\bar{\mathbf{h}}$	...	r.h.s. (boundary conditions)

$$\begin{bmatrix} \mathbf{C}(\omega), & (\mathbf{Q}^+)^H(\omega), & (\mathbf{Q}^-)^H(\omega), & 0 \\ \mathbf{Q}^+(\omega), & \bar{\mathbf{C}}^+(\omega) & 0, & -\mathrm{i}\omega\mathbf{M} \\ \mathbf{Q}^-(\omega), & 0, & \bar{\mathbf{C}}^-(\omega), & +\mathrm{i}\omega\mathbf{M} \\ 0, & +\mathrm{i}\omega\mathbf{M}, & -\mathrm{i}\omega\mathbf{M}, & \varepsilon\mathbf{X}(\omega^2) \end{bmatrix} \cdot \begin{bmatrix} \mathbf{p} \\ \mathbf{p}^+ \\ \mathbf{p}^- \\ \mathbf{g}^0 \end{bmatrix} = \mathrm{i}\omega \begin{bmatrix} \bar{\mathbf{h}} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\varepsilon\mathbf{X}(\omega^2)$ ... coupled impedance for finite scale of the perforation

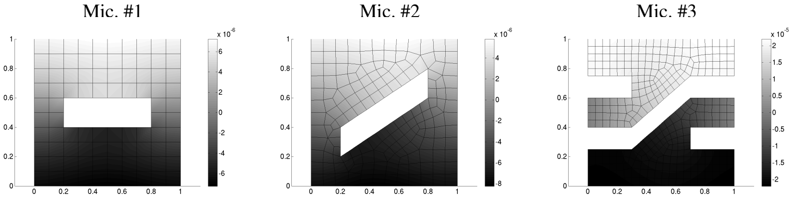
$$\varepsilon\varkappa = \delta = \text{ layer thickness [m]} \qquad \varkappa \approx 1$$

Influence of the perforation geometry

TEST:

- 3 microstructures (2D acoustics problem)
- Variation of homogenized transmission parameters

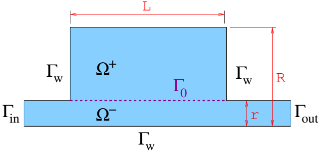
Mic.	$A[(\mathrm{m/s})^2]$	$B[\mathrm{m}]$	$F[\mathrm{s}^2]$
#1	$1.155 \cdot 10^5$	0	$1.391 \cdot 10^{-5}$
#2	$1.704 \cdot 10^5$	-0.251	$1.324 \cdot 10^{-5}$
#3	$2.186 \cdot 10^5$	-0.897	$4.265 \cdot 10^{-5}$

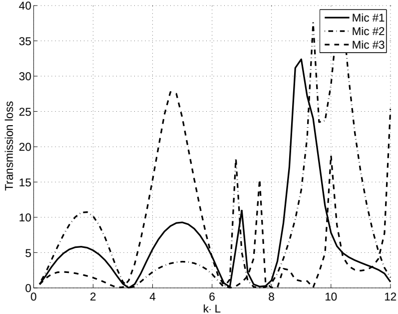


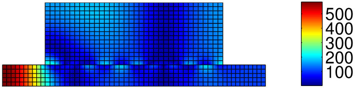
Distribution of $\xi^{\pm}$ in Y^*.

Transmission loss TL – Global response – 3 microstructures

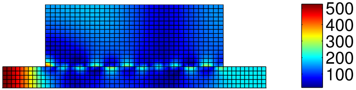
- $TL = 20\,log\left(|\bar{p}_{|\Gamma_{\rm in}}| / |p_{|\Gamma_{\rm out}}|\right)$
- test wave guide (2D):
  Γ_0 . . . thickness $0.007 \cdot L$



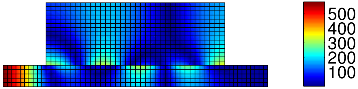




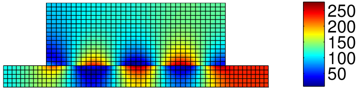
Mic. #1; $k \cdot L = 5$



Mic. #2; $k \cdot L = 5$



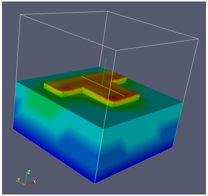
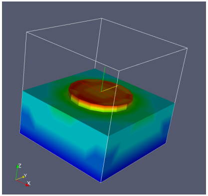
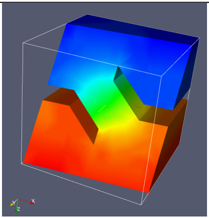
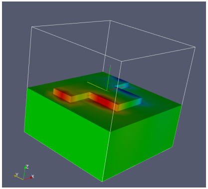
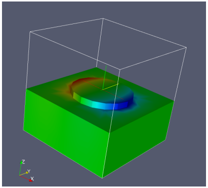
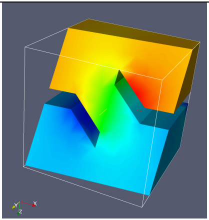
Mic. #3; $k \cdot L = 5$



Mic. #3; $k \cdot L = 1$

Modulus of the acoustic pressure in Ω^G for $k \cdot L = 5$ (1 in the last picture)

3D micro-problems – periodic microstructures

	isotropic "flat"	anisotropic "flat"	anisotropic "relief"
$\xi^{\pm}$			
π^1			
coupl.:	$B_\alpha = 0$	$B_\alpha = 0$	$B_\alpha = [0.303, -0.011]$
imped.:	$F = 1.32 \cdot 10^{-5}$	$F = 1.42 \cdot 10^{-5}$	$F = 2.56 \cdot 10^{-5}$
sound velo.:	$A = 10^5 \begin{bmatrix} 1.19, & 0.0 \\ 0.0, & 1.2 \end{bmatrix}$	$A = 10^5 \begin{bmatrix} 1.19, & 0.0 \\ 0.0, & 1.19 \end{bmatrix}$	$A = 10^5 \begin{bmatrix} 1.44, & -0.01 \\ -0.01, & 1.80 \end{bmatrix}$

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