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SHARP EMBEDDINGS OF BESOV SPACES INVOLVING ONLY LOGARITHMIC SMOOTHNESS

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Abstract

We use Kolyada's inequality and its converse form to prove sharp embeddings of Besov spaces $B^{0,\beta}_{p,r}$ (involving the zero classical smoothness and a logarithmic smoothness with the exponent β) into Lorentz-Zygmund spaces. We also determine growth envelopes of spaces $B^{0,\beta}_{p,r}$. In distinction to the case when the classical smoothness is positive, we show that we cannot describe all embeddings in question in terms of growth envelopes.

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key-words: Besov spaces with generalized smoothness, Lorentz-Zygmund spaces, sharp embeddings, growth envelopes.

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1 Introduction

In this paper we study sharp embeddings of Besov spaces $B^{0,\beta}_{p,r} = B^{0,\beta}_{p,r}(\mathbb{R}^n)$, $1 \le p < \infty$, $1 \le r \le \infty$ and $\beta + 1/r > 0$, into Lorentz-Zygmund spaces $L^{loc}_{p,q;\gamma} = L^{loc}_{p,q;\gamma}(\mathbb{R}^n)$, $1 \le p < \infty$, $1 \le r \le q \le \infty$ and $\gamma \in \mathbb{R}$. The Besov spaces $B^{0,\beta}_{p,r}$ are defined by means of the modulus of continuity and they involve the zero classical smoothness and a logarithmic smoothness with the exponent β — cf. Definition 2.1 in Section 2. By the Lorentz-Zygmund space $L^{loc}_{p,q;\gamma}$ we mean the set of all measurable functions on $\mathbb{R}^n$ with the finite quasi-norm

$$\left(\int_0^1 t^{q/p} (1 + |\ln t|)^{\gamma q} f^*(t)^q \, \frac{dt}{t} \right)^{1/q} \tag{1}$$

(with the usual modification when $q = \infty$).

First, Theorem 3.1 mentioned below states that the (continuous) embedding

$$B^{0,\beta}_{p,r} \hookrightarrow L^{loc}_{p,q;\gamma} \tag{2}$$

with

$$\gamma = \beta + 1/r + 1/\max\{p,q\} - 1/q \tag{3}$$

holds if and only if $q \geq r$. Consequently, when $q \geq r$, (2) holds with any γ satisfying

$$\gamma \leq \beta + 1/r + 1/\max\{p,q\} - 1/q.$$

Second, if $q \geq r$, then, by Theorem 3.2 mentioned below, embedding (2) cannot hold with $\gamma > \beta + 1/r + 1/\max\{p,q\} - 1/q$. This means that embedding (2) with γ given by (3) is sharp. Actually, Theorem 3.2 states even more. For example, it shows that we cannot make the target space in (2) (with γ from (3)) smaller by writing some powers of iterated logarithms inside the quasi-norm (1) of the space $L^{loc}_{p,q;\gamma}$.

There are two main ingredients of our proofs of these results. The first one is Kolyada's inequality recalled in Proposition 4.7. This inequality gives an estimate from below of the modulus of continuity of a function $f \in L_p = L_p(\mathbb{R}^n)$, $1 \leq p < \infty$, in terms of its non-increasing rearrangement. The second one is the "inverse Kolyada inequality" which is formulated in Proposition 3.5 and proved in this paper. Using these inequalities, we can reduce embedding (2) to a reverse Hardy inequality restricted to the cone of non-increasing functions — cf. Proposition 3.6.

Embeddings of Besov spaces into rearrangement invariant spaces were considered by Goldman [7], Goldman and Kerman [8], and Netrusov [14]. These authors used different methods and considered a more general setting. However, as mentioned in [7], the characterization of embedding (2) can be obtained from [14] only when $q = r$. Furthermore, the methods used in [7] also do not allow to consider the full range of parameters. Indeed, after a careful checking, one can see that the restriction $1 < p \leq r$ appears in the relevant theorem (cf. Theorem 3 of [7]).

Our results and techniques enable us to determine the (local) growth envelope (cf. Definition 2.2) of the Besov space $B^{0,\beta}_{p,r}$. Recall that the concept of the (local) growth envelope was introduced in [12] and [16], where also growth envelopes of some fundamental function spaces were calculated. In particular, it was shown that the growth envelope of the (classical) Besov space $B^s_{p,r}(\mathbb{R}^n)$, $0 < s < n/p$, $1 \leq p < \infty$ and $1 \leq r \leq \infty$, is the pair $(t^{s/n-1/p}, r)$. (Here we report only results from [12] and [16] with $p, r \geq 1$, when the Besov space in question is a Banach space.) The limiting case s/p was treated there as well: the growth envelope of the Besov space $B^{n/p}_{p,r}(\mathbb{R}^n)$, $1 \leq p < \infty$, $1 < r \leq \infty$, is the pair $((1+|\ln t|)^{1/r'}, r)$, where r' stands for the conjugate exponent of r. We should also mention that in [12] and [16] the (equivalent) Fourier analytical definition of Besov spaces was used. With this definition, the notion of the growth envelope is meaningful even when $s = 0$, $1 \le p \le \infty$ and $1 \le r \le \min\{p, 2\}$ (a so-called borderline case). The best what is known in such a case — cf. [12] — is that the growth envelope function is $t^{-1/p}$ (as expected), and that the fine index should be between r and p.

Growth envelopes have been also studied for Besov spaces $B^{(s,\Psi)}_{p,r}$ in [4], [3] and [9], where Ψ stands for a function of log-type and $s \in (0, n/p]$. We refer to [2], [10] and [1] for results on growth envelopes of more general Besov (and also Triebel-Lizorkin) spaces of generalized smoothness. While in [4], [3] and [2] the Fourier analytical definition of spaces was used, in [9] and [10] an equivalent definition based on the modulus of smoothness was employed.

On the other hand, no information has been obtained for the borderline case mentioned above when $s = 0$ and when all the known techniques do not work.

In this paper we determine the growth envelope of the Besov space $B^{0,\beta}_{p,r}$ (that is when $s = 0$) defined by means of the modulus of continuity. If $1 \le p < \infty$, $1 \le r \le \infty$ and $\beta > -1/r$, then the growth envelope of the space $B^{0,\beta}_{p,r}$ is the pair $(t^{-\frac{1}{p}}(1 + |\ln t|)^{-\beta-\frac{1}{r}}, \max\{p, r\})$ — cf. Theorem 3.3. There are some interesting features of this result. In distinction to results on growth envelopes of Besov spaces $B^s_{p,r}$ with $s \in (0, n/p]$, the first index p *plays a new role* here: it is involved in the fine index, which is not r now but $\max\{p, r\}$. Furthermore, another *new phenomenon appears* here. Namely, the embedding of the Besov space $B^{0,\beta}_{p,r}$ given by Theorem 3.1 cannot be described in terms of the growth envelope of the space $B^{0,\beta}_{p,r}$ when $1 \le r \le q < p < \infty$ — cf. Remark 3.4.

The paper is organized as follows. In Section 2 we give notation and basic definitions. Main results are presented in Section 3. Section 4 is devoted to auxiliary assertions. In subsequent sections (Sections 5-9) main results are proved.

2 Notation and basic definitions

For two non-negative expressions $\mathcal{A}$ and $\mathcal{B}$, the symbol $\mathcal{A} \lesssim \mathcal{B}$ means that $\mathcal{A} \le c\mathcal{B}$ for some positive constant c independent of the variables in the expressions $\mathcal{A}$ and $\mathcal{B}$. (To avoid misunderstandings, we will make clear in every instance on which variables the constant is independent using an expression like "for all".) If $\mathcal{A} \lesssim \mathcal{B}$ and $\mathcal{B} \lesssim \mathcal{A}$, we write $\mathcal{A} \approx \mathcal{B}$ and say that $\mathcal{A}$ and $\mathcal{B}$ are *equivalent*.

Given a set A, its characteristic function is denoted by χ_A. Given two sets A and B, we write $A \Delta B$ for their symmetric difference. For $a \in \mathbb{R}^n$ and $r \ge 0$, the notation $B(a, r)$ stands for the closed ball in $\mathbb{R}^n$ centered at a with radius r. The volume of $B(0, 1)$ in $\mathbb{R}^n$ is denoted by V_n though, in general, we use the notation $|\cdot|_n$ for Lebesgue measure in $\mathbb{R}^n$.

Let Ω be a Borel subset of $\mathbb{R}^n$. The symbol $\mathcal{M}_0(\Omega)$ is used to denote the family of all complex-valued or extended real-valued (Lebesgue-)measurable functions defined and finite a.e. on Ω. By $\mathcal{M}_0^+(\Omega)$ we mean the subset of $\mathcal{M}_0(\Omega)$ consisting of those functions which are non-negative a.e. on Ω. If $\Omega = (a,b) \subset \mathbb{R}$, we write simply $\mathcal{M}_0(a,b)$ and $\mathcal{M}_0^+(a,b)$ instead of $\mathcal{M}_0((a,b))$ and $\mathcal{M}_0^+((a,b))$, respectively. By $\mathcal{M}_0^+(a,b;\downarrow)$ or $\mathcal{M}_0^+(a,b;\uparrow)$ we mean the collection of all $f \in \mathcal{M}_0^+(a,b)$ which are non-increasing or non-decreasing on (a,b), respectively. Finally, by $AC(a,b)$ we denote the family of all real-valued functions which are locally absolutely continuous on (a,b) (that is, absolutely continuous on any closed subinterval of (a,b)).

For $f \in \mathcal{M}_0(\mathbb{R}^n)$, we define the *non-increasing rearrangement* f^* by

$$f^*(t) := \inf\{\lambda \geq 0 : |\{x \in \mathbb{R}^n : |f(x)| > \lambda\}|_n \leq t\}, \quad t \geq 0.$$

The corresponding maximal function f^{**} is given by

$$f^{**}(t) := \frac{1}{t}\int_0^t f^*(s)\,ds$$

and is also non-increasing on the interval $(0,\infty)$.

Given a Borel subset Ω of $\mathbb{R}^n$ and $0 < r \leq \infty$, $L_r(\Omega)$ is the usual space of measurable functions for which the quasi-norm

$$\|f\|_{r,\Omega} := \begin{cases} (\int_\Omega |f(t)|^r\,dt)^{1/r} & \text{if } 0 < r < \infty \\ \operatorname{ess\,sup}_{t\in\Omega}|f(t)| & \text{if } r = \infty \end{cases}$$

is finite. When $\Omega = \mathbb{R}^n$, we simplify $L_r(\Omega)$ to L_r and $\|\cdot\|_{r,\Omega}$ to $\|\cdot\|_r$.

Given $f \in L_p$, $1 \leq p < \infty$, the first difference operator Δ_h of step $h \in \mathbb{R}^n$ transforms f in $\Delta_h f$ defined by

$$(\Delta_h f)(x) := f(x+h) - f(x), \quad x \in \mathbb{R}^n,$$

whereas the modulus of continuity of f is given by

$$\omega_1(f,t)_p := \sup_{\substack{h\in\mathbb{R}^n \\ |h|\leq t}} \|\Delta_h f\|_p, \quad t > 0.$$

Now we introduce the Besov function spaces with the zero classical smoothness which we shall consider. Our smoothness will be controlled by some power of $\ell(t)$, where $\ell(t) := 1 + |\ln t|$, $t > 0$.

Definition 2.1 *Given $1 \leq p < \infty$, $1 \leq r \leq \infty$ and $\beta \in \mathbb{R}$,*

$$B^{0,\beta}_{p,r} := \{f \in L_p : \|f\|_{B^{0,\beta}_{p,r}} := \|f\|_p + \|t^{-1/r}\ell^\beta(t)\,\omega_1(f,t)_p\|_{r,(0,1)} < \infty\}.$$

Note that, since $\omega_1(f,t)_p \lesssim \|f\|_p$, only the case $\|t^{-1/r}\ell^\beta(t)\|_{r,(0,1)} = \infty$ (or, equivalently, $\beta r + 1 \geq 0$ if r is finite and $\beta > 0$ if r is infinity) is of interest; otherwise $B^{0,\beta}_{p,r} = L_p$.

We shall occasionally need the notion of Borel measure μ associated with a non-decreasing function $g : (a,b) \to \mathbb{R}$, where $-\infty \leq a < b \leq \infty$. By this we mean the unique (non-negative) measure μ on the Borel subsets of (a,b) such that $\mu([c,d]) = g(d+) - g(c-)$ for all $[c,d] \subset (a,b)$.

We finish this section by recalling the notion of growth envelope of the function space A (we refer to [12] for details).

Definition 2.2 *Let $(A, \|\cdot\|_A) \subset \mathcal{M}_0(\mathbb{R}^n)$ be a quasi-normed space such that $A \not\hookrightarrow L_\infty$. A positive, non-increasing, continuous function h defined on some interval $(0,\varepsilon]$, $\varepsilon \in (0,1)$, is called the* (local) growth envelope function *of the space A provided that*

$$h(t) \approx \sup_{\|f\|_A \leq 1} f^*(t) \quad \text{for all } \ t \in (0,\varepsilon].$$

Given a growth envelope function h of the space A (determined up to equivalence near zero) and a number $u \in (0,\infty]$, we call the pair (h,u) the (local) growth envelope of the space A when the inequality

$$\Big(\int_{(0,\varepsilon)} \Big(\frac{f^*(t)}{h(t)} \Big)^q \, d\mu_H(t) \Big)^{1/q} \lesssim \|f\|_A$$

(with the usual modification when $q = \infty$) holds for all $f \in A$ if and only if the positive exponent q satisfies $q \geq u$. Here μ_H is the Borel measure associated with the non-decreasing function $H(t) := -\ln h(t)$, $t \in (0,\varepsilon)$. The component u in the growth envelope pair is called the fine index.

3 Main Results

Theorem 3.1 *If $1 \leq p < \infty$, $1 \leq r \leq \infty$, $\beta > -1/r$ and $0 < q \leq \infty$, then the inequality*

$$\|t^{1/p-1/q}\ell^{\beta+1/r+1/\max\{p,q\}-1/q}(t) f^*(t)\|_{q,(0,1)} \lesssim \|f\|_{B^{0,\beta}_{p,r}} \tag{4}$$

holds for all $f \in B^{0,\beta}_{p,r}$ if and only if $q \geq r$.

Theorem 3.2 *Let $1 \leq p < \infty$, $1 \leq r \leq q \leq \infty$, $\beta > -1/r$ and let $\kappa \in \mathcal{M}^+_0(0,1;\downarrow)$. Then the inequality*

$$\|t^{1/p-1/q}\ell^{\beta+1/r+1/\max\{p,q\}-1/q}(t)\, \kappa(t) f^*(t)\|_{q,(0,1)} \lesssim \|f\|_{B^{0,\beta}_{p,r}} \tag{5}$$

holds for all $f \in B^{0,\beta}_{p,r}$ if and only if κ is bounded.

Theorem 3.3 *If* $1 \le p < \infty$, $1 \le r \le \infty$ *and* $\beta > -1/r$, *then the growth envelope of* $B^{0,\beta}_{p,r}$ *is the pair*

$$(t^{-\frac{1}{p}}\ell^{-\beta-\frac{1}{r}}(t), \max\{p,r\}).$$

Remark 3.4 *Put* $h(t) := t^{-1/p}\ell^{-\beta-1/r}(t)$ *and* $H(t) := -\ln h(t)$ *for* $t \in (0,\varepsilon)$, *where* $\varepsilon \in (0,1)$ *is small enough. Since* $H'(t) \approx \frac{1}{t}$ *for all* $t \in (0,\varepsilon)$, *the measure* μ_H *associated with the function* H *satisfies* $d\mu_H(t) \approx \frac{dt}{t}$. *Thus, by Definition 2.2 and Theorem 3.3,*

$$\Big\|t^{-1/q}\frac{f^*(t)}{h(t)}\Big\|_{q,(0,\varepsilon)} \lesssim \|f\|_{B^{0,\beta}_{p,r}} \quad \textit{for all } f \in B^{0,\beta}_{p,r} \tag{6}$$

provided that

$$q \ge \max\{p,r\}. \tag{7}$$

Hence, if (7) *holds, then inequality* (6) *gives the same result as inequality* (4) *of Theorem 3.1. However, if* $r \le q < p$, *inequality* (6) *does not hold (cf. Theorem 3.2), while inequality* (4) *does. This means that the embeddings of Besov spaces* $B^{0,\beta}_{p,r}$ *given by Theorem 3.1 cannot be described in terms of growth envelopes when* $1 \le r \le q < p < \infty$.

Two of the main ingredients in the proofs of Theorems 3.1, 3.2 and 3.3 are Proposition 4.7 (Kolyada's inequality) and Proposition 3.5 (which we call the "inverse" Kolyada inequality) mentioned below.

Proposition 3.5 **(i)** *Let* $f \in L_1$ *and let* $F(x) := f^*(V_n|x|^n)$, $x \in \mathbb{R}^n$. *Then*

$$\begin{aligned}\omega_1(F,t)_1 &\lesssim n\int_0^{t^n} f^*(s)\,ds + (n-1)\,t\int_{t^n}^{\infty} f^*(s)s^{-1/n}\,ds \\ &= t\Big(\int_{t^n}^{\infty} s^{-1/n}\int_0^s (f^*(u)-f^*(s))\,du\,\frac{ds}{s}\Big)\end{aligned} \tag{8}$$

for all $t > 0$ *and* $f \in L_1$.

(ii) *Let* $1 < p < \infty$, $f \in L_p$ *and let* $F(x) = f^{**}(V_n|x|^n)$, $x \in \mathbb{R}^n$. *Then*

$$\omega_1(F,t)_p \lesssim t\Big(\int_{t^n}^{\infty} s^{-p/n}\int_0^s (f^*(u)-f^*(s))^p\,du\,\frac{ds}{s}\Big)^{1/p}$$

for all $t > 0$ *and* $f \in L_p$.

In fact, Propositions 4.7 and 3.5 enable us to reduce the embedding in question to the following assertion:

Proposition 3.6 *Let* $1 \le p < \infty$, $1 \le r \le \infty$, $0 < q \le \infty$, $\beta \in \mathbb{R}$ *and let* ω *be a measurable function on* $(0,1)$. *Then*

$$\|\omega(t) f^*(t)\|_{q,(0,1)} \lesssim \|f\|_{B^{0,\beta}_{p,r}} \tag{9}$$

for all $f \in B^{0,\beta}_{p,r}$ *if and only if*

$$\|\omega(t) f^*(t)\|_{q,(0,1)} \lesssim \Big\| t^{1-1/r} \ell^{\beta}(t) \Big(\int_{t^n}^{2} s^{-p/n} \int_0^s (f^*(u) - f^*(s))^p \, du \, \frac{ds}{s} \Big)^{1/p} \Big\|_{r,(0,1)} \tag{10}$$

for all $f \in \mathcal{M}_0(\mathbb{R}^n)$ *such that* $|\mathrm{supp} f|_n \le 1$.

4 Preliminaries

The following easy estimates are quite useful and will be used without further notice whenever convenient: if $\varepsilon > 0$, $r \in (0, \infty]$ and $b \in \mathbb{R}$, then

$$\|t^{\varepsilon - 1/r} \ell(t)^b\|_{r,(0,T)} \approx T^{\varepsilon} \ell(T)^b \quad \text{and} \quad \|t^{-\varepsilon - 1/r} \ell(t)^b\|_{r,(T,\infty)} \approx T^{-\varepsilon} \ell(T)^b,$$

for all $T \in (0, \infty)$.

We shall also need the following geometric estimate:

Proposition 4.1 *For all* $a, b \in \mathbb{R}^n$ *and* $r \ge 0$,

$$|B(a,r) \Delta B(b,r)|_n \lesssim |b-a| \, r^{n-1}. \tag{11}$$

Proof. Since the cases $a = b$ or $r = 0$ are obvious, we assume that $a \ne b$ and $r > 0$.

If $|b-a| > r/2$, then $|B(a,r) \Delta B(b,r)|_n \lesssim r^n < 2\,|b-a|\, r^{n-1}$ and (11) follows.

If $|b-a| \le r/2$, then the inclusion $B(a, r - |b-a|) \subset B(b,r)$ and its symmetric counterpart $B(b, r-|b-a|) \subset B(a,r)$ imply that

$$B(a,r) \Delta B(b,r) \subset \big(B(a,r) \setminus B(a, r-|b-a|)\big) \cup \big(B(b,r) \setminus B(b, r-|b-a|)\big).$$

Consequently,

$$|B(a,r) \Delta B(b,r)|_n \lesssim r^n - (r - |b-a|)^n,$$

which gives (11) when $n = 1$. Assuming that $n \geq 2$, we obtain from the last estimate that

$$\begin{aligned}
&|B(a,r)\Delta B(b,r)|_n \\
&\lesssim \; n\,|b-a|\,r^{n-1} - \sum_{j=2}^{n} \tbinom{n}{j}\,(-1)^j |b-a|^j r^{n-j} \\
&= \; |b-a|\,r^{n-1}\big(n - \sum_{j=2}^{n} \tbinom{n}{j}\,(-1)^j (|b-a|^{j-1} r^{-j+1})\big) \\
&\leq \; |b-a|\,r^{n-1}\big(n + \sum_{j=2}^{n} \tbinom{n}{j}\,2^{-(j-1)}\big) \\
&\approx \; |b-a|\,r^{n-1}.
\end{aligned}$$

□

Next we present two monotonicity results, which will be often used:

Proposition 4.2 *Given $p > 0$ and a non-increasing function $g : (0,\infty) \to \mathbb{R}$, the function*

$$t \mapsto \int_0^t (g(s) - g(t))^p\, ds \tag{12}$$

is non-decreasing on $(0,\infty)$. In particular, if $f \in \mathcal{M}_0(\mathbb{R}^n)$, then the functions

$$t \mapsto \int_0^t (f^*(s) - f^*(t))^p\, ds \tag{13}$$

and

$$t \mapsto t(f^{**}(t) - f^*(t)) \tag{14}$$

are non-decreasing on $(0,\infty)$.

Proof. Given $0 < t_1 < t_2 < \infty$,

$$\int_0^{t_1} (g(s) - g(t_1))^p\, ds \leq \int_0^{t_1} (g(s) - g(t_2))^p\, ds \leq \int_0^{t_2} (g(s) - g(t_2))^p\, ds.$$

□

Proposition 4.3 *Let μ be a (non-negative) measure on $(0,\infty)$ such that $\mu[t,\infty) \in (0,\infty)$ for all $t \in (0,\infty)$. Let $g \in \mathcal{M}_0^+(0,\infty;\uparrow)$. Then the function*

$$t \mapsto \mu[t,\infty)^{-1} \int_{[t,\infty)} g\, d\mu$$

is also non-decreasing on $(0,\infty)$.

Proof. First note that the conclusion is plain if $\int_{[t,\infty)} g\,d\mu$ is infinite for all t. On the other hand, if it is finite for some t, it is finite for all t (due to the hypotheses of the proposition). Therefore, for $0 < t_1 < t_2 < \infty$,

$$\begin{aligned}
&\frac{1}{\mu[t_1,\infty)} \int_{[t_1,\infty)} g\,d\mu \\
&= \frac{\int_{[t_1,t_2)} g\,d\mu + \int_{[t_2,\infty)} g\,d\mu}{\mu[t_1,\infty)} \\
&\le \frac{\mu[t_1,t_2)\,g(t_2) + \int_{[t_2,\infty)} g\,d\mu}{\mu[t_1,\infty)} \\
&= \frac{\mu[t_1,t_2)\,\mu[t_2,\infty)^{-1}\,\mu[t_2,\infty)\,g(t_2) + \int_{[t_2,\infty)} g\,d\mu}{\mu[t_1,\infty)} \\
&\le \frac{\mu[t_1,t_2)\,\mu[t_2,\infty)^{-1}\,\int_{[t_2,\infty)} g\,d\mu + \int_{[t_2,\infty)} g\,d\mu}{\mu[t_1,\infty)} \\
&= \frac{\mu[t_1,t_2) + \mu[t_2,\infty)}{\mu[t_1,\infty)\,\mu[t_2,\infty)} \int_{[t_2,\infty)} g\,d\mu \\
&= \frac{1}{\mu[t_2,\infty)} \int_{[t_2,\infty)} g\,d\mu.
\end{aligned}$$

□

Now we proceed by recalling some properties of the maximal functions f^{**} of elements $f \in L_p$, $1 \le p \le \infty$. Such functions f are locally integrable in $\mathbb{R}^n$ and so the function $t \mapsto \int_0^t f^*(s)\,ds$ belongs to $AC(0,\infty)$ and

$$\frac{d}{dt}\int_0^t f^*(s)\,ds = f^*(t) \quad a.e. \text{ in } (0,\infty).$$

Consequently,

$$(f^{**})'(t) = -\frac{1}{t}(f^{**}(t) - f^*(t)) \quad a.e. \text{ in } (0,\infty). \tag{15}$$

On the other hand, since the function $t \mapsto 1/t$ also belongs to $AC(0,\infty)$, the same can be said about f^{**} and we can write, for any $0 < t_1 \le t_2 < \infty$,

$$f^{**}(t_2) - f^{**}(t_1) = \int_{t_1}^{t_2} (f^{**})'(s)\,ds = \int_{t_2}^{t_1} \frac{1}{s}(f^{**}(s) - f^*(s))\,ds. \tag{16}$$

In order to prove our next proposition involving f^* and f^{**}, we need classical Hardy's inequalities (see, for example, [11, pp. 240, 244]):

Given $1 < p < \infty$ and a non-negative, measurable function f on $(0,\infty)$,

$$\int_0^\infty \Big(\frac{1}{x}\int_0^x f(t)\,dt\Big)^p dx \le \big(\frac{p}{p-1}\big)^p \int_0^\infty f(x)^p\,dx \tag{17}$$

and

$$\int_0^\infty \Big(\int_x^\infty f(t)\,dt\Big)^p dx \le p^p \int_0^\infty (xf(x))^p\,dx. \tag{18}$$

Remark 4.4 *Inserting the function $f\chi_{(0,y)}$, $y>0$, instead of f in* (17) *and* (18)*, we see that inequalities* (17) *and* (18) *remain true with ∞ replaced by $y>0$.*

Proposition 4.5 *If $1<p<\infty$, then*

$$\int_0^t (f^{**}(s)-f^*(s))^p\,ds \lesssim \int_0^t (f^*(s)-f^*(t))^p\,ds \lesssim \int_0^{2t} (f^{**}(s)-f^*(s))^p\,ds$$

for all $t>0$ and $f\in L_p$.

Proof. Using classical Hardy inequality (17) and Remark 4.4,

$$\begin{aligned}
&\int_0^t (f^{**}(s)-f^*(s))^p\,ds\\
\lesssim\ &\int_0^t (f^{**}(s)-f^*(t))^p\,ds + \int_0^t (f^*(s)-f^*(t))^p\,ds\\
=\ &\int_0^t \Big(\frac{1}{s}\int_0^s (f^*(\tau)-f^*(t))\,d\tau\Big)^p ds + \int_0^t (f^*(s)-f^*(t))^p\,ds\\
\lesssim\ &\int_0^t (f^*(s)-f^*(t))^p\,ds.
\end{aligned}$$

On the other hand, using (16), classical Hardy inequality (18), Remark 4.4 and Proposition 4.2, we get

$$\begin{aligned}
&\int_0^t (f^*(s)-f^*(t))^p\,ds\\
\le\ &\int_0^t (f^{**}(s)-f^*(t))^p\,ds\\
\lesssim\ &\int_0^t (f^{**}(s)-f^{**}(t))^p\,ds + \int_0^t (f^{**}(t)-f^*(t))^p\,ds\\
=\ &\int_0^t \Big(\int_s^t \frac{f^{**}(\tau)-f^*(\tau)}{\tau}\,d\tau\Big)^p ds + t\,(f^{**}(t)-f^*(t))^p\\
\lesssim\ &\int_0^t (f^{**}(s)-f^*(s))^p\,ds + \Big(\int_t^{2t} s^{-p}\,ds\Big)\big(t\,(f^{**}(t)-f^*(t))\big)^p\\
\le\ &\int_0^{2t} (f^{**}(s)-f^*(s))^p\,ds.
\end{aligned}$$

□

We shall also need the following Hardy-type inequalities (consequences of [15, Thms. 5.9 and 6.2]):

Proposition 4.6 *Let* $1 \le P \le \infty$, $\nu \in \mathbb{R} \setminus \{0\}$ *and* $b \in \mathbb{R}$.

(i) *The inequality*

$$\|t^{\nu-1/P}\ell(t)^b \int_0^t g(\tau)\, d\tau\|_{P,(0,1)} \lesssim \|t^{\nu+1-1/P}\ell(t)^b g(t)\|_{P,(0,1)}$$

holds for all $g \in \mathcal{M}_0^+(0,1)$ *if and only if* $\nu < 0$.

(ii) *The inequality*

$$\|t^{\nu-1/P}\ell(t)^b \int_t^1 g(\tau)\, d\tau\|_{P,(0,1)} \lesssim \|t^{\nu+1-1/P}\ell(t)^b g(t)\|_{P,(0,1)}$$

holds for all $g \in \mathcal{M}_0^+(0,1)$ *if and only if* $\nu > 0$.

One of the basic ingredients in the proofs of our main results, presented in Section 3, is the following inequality of Kolyada, giving an estimate from below of the modulus of continuity in terms of non-increasing rearrangements of functions:

Proposition 4.7 ([13]) *If* $1 \le p < \infty$, *then*

$$t\Big(\int_{t^n}^{\infty} s^{-p/n} \int_0^s (f^*(u) - f^*(s))^p\, du\, \frac{ds}{s}\Big)^{1/p} \lesssim \omega_1(f,t)_p$$

for all $t > 0$ and $f \in L_p$.

We shall also make use of the next two assertions which are consequences of more general results of Gogatishvili and Pick [5, Thm. 4.2 (ii)], [6, Thm. 1.8 (i)]:

Proposition 4.8 *Let* $1 \le Q < P < \infty$ *and* $R = PQ/(P-Q)$. *Let* v, w *be non-negative functions on* $[0,\infty)$ *such that* $V(t) := \int_0^t v(s)\, ds$ *and* $W(t) := \int_0^t w(s)\, ds$ *are finite for all* $t > 0$. *Assume that*

$$\int_{[0,1]} \frac{v(s)}{s^P}\, ds \;=\; \int_{[1,\infty)} v(s)\, ds \;=\; \infty$$

and that

$$\int_{[0,\infty)} \frac{v(s)}{s^P + t^P}\, ds < \infty$$

for all $t \in (0,\infty)$*. Then the inequality*

$$\Big(\int_0^\infty w(t) f^*(t)^Q\,dt\Big)^{1/Q} \lesssim \Big(\int_0^\infty v(t) f^{**}(t)^P\,dt\Big)^{1/P} \tag{19}$$

holds for all measurable f *on* $\mathbb{R}^n$ *if and only if*

$$\int_0^\infty \frac{t^R \sup_{y\in[t,\infty)} y^{-R} W(y)^{R/Q}}{(V(t) + t^P \int_t^\infty s^{-P} v(s)\,ds)^{R/P+2}}\, V(t) \int_t^\infty s^{-P} v(s)\,ds\, t^{P-1}\,dt < \infty. \tag{20}$$

Proposition 4.9 *Let* $1 \le Q < \infty$*, let* v, w *be non-negative, locally integrable functions on* $(0,\infty)$ *and put* $W(t) := \int_0^t w(s)\,ds$*,* $t > 0$*. Consider the function*

$$\varphi(t) := \operatorname{ess\,sup}_{s\in(0,t)} s \operatorname{ess\,sup}_{\tau\in(s,\infty)} \frac{v(\tau)}{\tau}, \quad t \in (0,\infty). \tag{21}$$

This function is quasi-concave (that is, φ *is equivalent to a function in* $\mathcal{M}_0^+(0,\infty;\uparrow)$ *while* $\varphi(t)/t$ *is equivalent to a function in* $\mathcal{M}_0^+(0,\infty;\downarrow)$*). Assume that* φ *is non-degenerate, that is,*

$$\lim_{t\to 0+} \varphi(t) = \lim_{t\to\infty} \frac{1}{\varphi(t)} = \lim_{t\to\infty} \frac{\varphi(t)}{t} = \lim_{t\to 0+} \frac{t}{\varphi(t)} = 0. \tag{22}$$

Let ν *be a non-negative Borel measure on* $[0,\infty)$ *such that*

$$\frac{1}{\varphi(t)^Q} \approx \int_{[0,\infty)} \frac{d\nu(s)}{s^Q + t^Q} \quad \text{for all } t \in (0,\infty).$$

Then the inequality

$$\Big(\int_0^\infty w(t) f^*(t)^Q\,dt\Big)^{1/Q} \lesssim \operatorname{ess\,sup}_{t\in(0,\infty)} v(t) f^{**}(t) \tag{23}$$

holds for all measurable f *on* $\mathbb{R}^n$ *if and only if*

$$\int_{[0,\infty)} \sup_{s\in(t,\infty)} \frac{W(s)}{s^Q}\, d\nu(t) < \infty. \tag{24}$$

5 Proof of Proposition 3.5

First we prove the following auxiliary result:

Lemma 5.1 *Let* $g \in \mathcal{M}_0^+(0,\infty;\downarrow)$ *and let* $F(x) := g(V_n|x|^n)$*,* $x \in \mathbb{R}^n$*. Then*

$$\|\Delta_h F\|_1 \lesssim n \int_0^{V_n|h|^n} g(s)\,ds + (n-1) V_n^{1/n} |h| \int_{V_n|h|^n}^\infty g(s) s^{-1/n}\,ds \tag{25}$$

for all $h \in \mathbb{R}^n \setminus \{0\}$ *and* $g \in \mathcal{M}_0^+(0,\infty;\downarrow)$.

Moreover, if $g \in AC(0,\infty)$ *and* $1 \le p < \infty$, *then*

$$\begin{aligned}\|\Delta_h F\|_p \quad &\lesssim \quad \Big(\int_0^{V_n 3^n |h|^n} (g(s) - g(V_n 3^n |h|^n))^p \, ds\Big)^{1/p} \\ &\quad + |h| \Big(\int_{V_n 2^n |h|^n}^{\infty} s^{(1-1/n)p} \operatorname{ess\,sup}_{s/2^n \le u \le 3^n s/2^n} |g'(u)|^p \, ds\Big)^{1/p} \quad (26)\end{aligned}$$

for all $h \in \mathbb{R}^n \setminus \{0\}$ *and* $g \in \mathcal{M}_0^+(0,\infty;\downarrow) \cap AC(0,\infty)$.

Proof.

Step 1.

Assume that $g \in \mathcal{M}_0^+(0,\infty;\downarrow)$. Then

$$\|\Delta_h F\|_1 = \int_{|x|<2|h|} |F(x+h)-F(x)| \, dx + \int_{|x|>2|h|} |F(x+h)-F(x)| \, dx =: I + II.$$

Using polar coordinates, the definition of F and a further change of variables, we obtain

$$\begin{aligned} I \quad &\le \quad \int_{|x|<2|h|} F(x+h) \, dx + \int_{|x|<2|h|} F(x) \, dx \\ &= \quad \int_{|y-h|<2|h|} F(y) \, dy + \int_{|x|<2|h|} F(x) \, dx \\ &\lesssim \quad \int_{|x|<3|h|} F(x) \, dx \approx \int_0^{3|h|} g(V_n t^n) t^{n-1} \, dt \\ &\approx \quad \int_0^{V_n 3^n |h|^n} g(s) \, ds \lesssim \int_0^{V_n |h|^n} g(s) \, ds. \qquad (27) \end{aligned}$$

Denoting by μ_{-g} the Borel measure associated with $-g$ on $(0,\infty)$, using Fubini's Theorem and Proposition 4.1, we arrive at

$$\begin{aligned} II \quad &= \quad \int_{|x|>2|h|} |g(V_n |x+h|^n) - g(V_n |x|^n)| \, dx \\ &\le \quad \int_{|x|>2|h|} \int_0^\infty \chi_{[V_n \min\{|x|^n, |x+h|^n\}, V_n \max\{|x|^n, |x+h|^n\}]}(s) \, d\mu_{-g}(s) \, dx \\ &= \quad \int_{\mathbb{R}^n} \int_0^\infty \chi_{]2|h|,\infty[}(|x|) \chi_{B(0, V_n^{-1/n} s^{1/n}) \Delta B(-h, V_n^{-1/n} s^{1/n})}(x) \, d\mu_{-g}(s) \, dx \\ &\le \quad \int_0^\infty \chi_{]|h|,\infty[}(V_n^{-1/n} s^{1/n}) |B(0, V_n^{-1/n} s^{1/n}) \Delta B(-h, V_n^{-1/n} s^{1/n})|_n \, d\mu_{-g}(s) \\ &\lesssim \quad \int_{]V_n |h|^n, \infty[} |h| \, s^{1-1/n} \, d\mu_{-g}(s). \end{aligned}$$

If $n = 1$,

$$\int_{]V_1|h|,\infty[} |h|\, d\mu_{-g}(s) \leq |h|\, g(V_1|h|) \approx \int_0^{V_1|h|} g(V_1|h|)\, ds \ \leq \ \int_0^{V_1|h|} g(s)\, ds. \tag{28}$$

If $n > 1$, integration by parts (for the Riemann-Stieltjes integral), gives

$$\begin{aligned} II &\lesssim |h| \lim_{M\to\infty} \int_{V_n|h|^n}^{M} s^{1-1/n}\, d(-g_{\big|[V_n|h|^n,M]})(s) \\ &\lesssim |h|^n g(V_n|h|^n) + |h| \int_{V_n|h|^n}^{\infty} g(s)\, s^{-1/n}\, ds \\ &\lesssim \int_0^{V_n|h|^n} g(s)\, ds + |h| \int_{V_n|h|^n}^{\infty} g(s)\, s^{-1/n}\, ds. \end{aligned}$$

Now, (25) is a consequence of the last estimate, (28) and (27).

Step 2.
Assume that $g \in \mathcal{M}_0^+(0,\infty;\downarrow) \cap AC(0,\infty)$. Given any $1 \leq p < \infty$,

$$\begin{aligned} \|\Delta_h F\|_p &\leq \Big(\int_{|x|<2|h|} |F(x+h) - F(x)|^p\, dx\Big)^{1/p} \\ &\quad + \Big(\int_{|x|>2|h|} |F(x+h) - F(x)|^p\, dx\Big)^{1/p} \\ &=: I + II. \end{aligned} \tag{29}$$

Furthermore,

$$\begin{aligned} I &\leq \Big(\int_{|x|<2|h|} |F(x+h) - F(3h)|^p\, dx\Big)^{1/p} + \Big(\int_{|x|<2|h|} |F(x) - F(3h)|^p\, dx\Big)^{1/p} \\ &= \Big(\int_{|y-h|<2|h|} |F(y) - F(3h)|^p\, dy\Big)^{1/p} + \Big(\int_{|x|<2|h|} |F(x) - F(3h)|^p\, dx\Big)^{1/p} \\ &\lesssim \Big(\int_{|x|<3|h|} |F(x) - F(3h)|^p\, dx\Big)^{1/p} \\ &\approx \Big(\int_0^{3|h|} (g(V_n t^n) - g(V_n 3^n |h|^n))^p\, t^{n-1}\, dt\Big)^{1/p} \\ &\approx \Big(\int_0^{V_n 3^n |h|^n} (g(s) - g(V_n 3^n |h|^n))^p\, dt\Big)^{1/p} \end{aligned} \tag{30}$$

and

$$
\begin{aligned}
II &= \Big(\int_{|x|>2|h|} |g(V_n|x+h|^n) - g(V_n|x|^n)|^p\,dx\Big)^{1/p} \\
&= \Big(\int_{|x|>2|h|} \Big(\int_{V_n \min\{|x|^n,|x+h|^n\}}^{V_n \max\{|x|^n,|x+h|^n\}} |g'(u)|\,du\Big)^p\,dx\Big)^{1/p} \\
&\lesssim \Big(\int_{|x|>2|h|} ||x+h|^n - |x|^n|^p \operatorname{ess\,sup}_{V_n|x|^n/2^n \le u \le V_n 3^n|x|^n/2^n} |g'(u)|^p\,dx\Big)^{1/p} \\
&\lesssim \Big(\int_{|x|>2|h|} |h|^p\,|x|^{(n-1)p} \operatorname{ess\,sup}_{V_n|x|^n/2^n \le u \le V_n 3^n|x|^n/2^n} |g'(u)|^p\,dx\Big)^{1/p} \\
&\approx |h|\Big(\int_{2|h|}^\infty t^{(n-1)p} \operatorname{ess\,sup}_{V_n t^n/2^n \le u \le V_n 3^n t^n/2^n} |g'(u)|^p\,t^{n-1}\,dt\Big)^{1/p} \\
&\approx |h|\Big(\int_{V_n 2^n|h|^n}^\infty s^{(1-1/n)p} \operatorname{ess\,sup}_{s/2^n \le u \le 3^n s/2^n} |g'(u)|^p\,ds\Big)^{1/p}.
\end{aligned}
$$

Together with (30) and (29), this yields (26). □

Proof of Proposition 3.5.

Step 1.

To prove (i), take $f \in L_1$ and $g = f^*$ in Lemma 5.1. Consequently,

$$\|\Delta_h F\|_1 \lesssim n\int_0^{V_n|h|^n} f^*(s)\,ds + (n-1)V_n^{1/n}|h|\int_{V_n|h|^n}^\infty f^*(s)s^{-1/n}\,ds. \quad (31)$$

Aplying Fubini's theorem and the fact that f^* is integrable on $(0,\infty)$, we can rewrite the last expression as

$$
\begin{aligned}
& V_n^{1/n}|h|\Big(\int_{V_n|h|^n}^\infty s^{-1/n}\int_0^s (f^*(u) - f^*(s))\,du\,\frac{ds}{s}\Big) \\
&= V_n^{1/n}|h|\Big(\int_{V_n|h|^n}^\infty s^{-1/n-1}\int_0^s f^*(u)\,du\,ds\Big) \\
&\quad -V_n^{1/n}|h|\Big(\int_{V_n|h|^n}^\infty s^{-1/n-1}\int_0^s f^*(s)\,du\,ds\Big) \\
&= n\int_0^{V_n|h|^n} f^*(u)\,du + nV_n^{1/n}|h|\int_{V_n|h|^n}^\infty f^*(u)\,u^{-1/n}\,du \\
&\quad -V_n^{1/n}|h|\int_{V_n|h|^n}^\infty f^*(s)\,s^{-1/n}\,ds \\
&= n\int_0^{V_n|h|^n} f^*(u)\,du + (n-1)V_n^{1/n}|h|\int_{V_n|h|^n}^\infty f^*(u)\,u^{-1/n}\,du. \quad (32)
\end{aligned}
$$

When $n = 1$, it is plain that the right-hand side of (31) is non-decreasing in $|h|$. When $n > 1$, it is also non-decreasing in $|h|$, which can be seen from the equivalent expression given in (32) and from Proposition 4.3 (with $d\mu(s) = s^{-1/n-1}ds$ and $g(s) = n\int_0^s (f^*(u) - f^*(s))\,du$; the fact that $g \in \mathcal{M}_0^+(0,\infty;\uparrow)$ follows from Proposition 4.2).

Now, (31) and (32) imply that

$$\begin{aligned}\omega_1(F,t)_1 &\lesssim n\int_0^{V_n t^n} f^*(s)\,ds + (n-1)V_n^{1/n}t\int_{V_n t^n}^\infty f^*(s)s^{-1/n}\,ds \\ &= V_n^{1/n}t\Big(\int_{V_n t^n}^\infty s^{-1/n}\int_0^s (f^*(u)-f^*(s))\,du\,\frac{ds}{s}\Big).\end{aligned}$$

In order to complete the proof of Proposition 3.5(i), note that the factors V_n and $V_n^{1/n}$ can be omitted in the preceding formulæ (this follows again by arguments used in (32) and the discussion following it).

Step 2.

To prove part (ii), take $f \in L_p$, $1 < p < \infty$, and $g = f^{**}$ in Lemma 5.1. Consequently,

$$\begin{aligned}\|\Delta_h F\|_p &\lesssim \Big(\int_0^{V_n 3^n |h|^n} (f^{**}(s) - f^{**}(V_n 3^n |h|^n))^p\,ds\Big)^{1/p} \\ &+ |h|\Big(\int_{V_n 2^n |h|^n}^\infty s^{(1-1/n)p}\operatorname{ess\,sup}_{s/2^n \le u \le 3^n s/2^n} |(f^{**})'(u)|^p\,ds\Big)^{1/p}. \qquad (33)\end{aligned}$$

Since $f \in L_p$, (15) yields

$$(f^{**})'(u) = -\frac{1}{u}(f^{**}(u) - f^*(u)) = -\frac{1}{u^2}\int_0^u (f^*(\tau) - f^*(u))\,d\tau$$

a.e. in $(0,\infty)$. Therefore, a change of variables and Hölder's inequality show that the last term in (33) can be estimated from above (up to multiplicative positive constants) by

$$\begin{aligned}&|h|\left(\int_{V_n 2^n |h|^n}^\infty s^{(1-1/n)p}\,s^{-2p}\Big(\int_0^{3^n s/2^n} (f^*(\tau) - f^*(3^n s/2^n))\,d\tau\Big)^p\,ds\right)^{1/p} \\ &\approx |h|\left(\int_{V_n 3^n |h|^n}^\infty u^{-p-p/n}\Big(\int_0^u (f^*(\tau) - f^*(u))\,d\tau\Big)^p\,du\right)^{1/p} \\ &\le |h|\left(\int_{V_n 3^n |h|^n}^\infty u^{-1-p/n}\int_0^u (f^*(\tau) - f^*(u))^p\,d\tau\,du\right)^{1/p}. \qquad (34)\end{aligned}$$

Similar facts as those used in the discussion following (32) imply that the last expression in (34) is a non-decreasing function of $|h|$. On the other hand, Proposition 4.2 shows that the first term on the righ-hand side of (33) is also a non-decreasing function of $|h|$. Therefore,

$$\omega_1(F,t)_p \lesssim \Big(\int_0^{V_n 3^n t^n} (f^{**}(s) - f^{**}(V_n 3^n t^n))^p \, ds\Big)^{1/p} + t\Big(\int_{V_n 3^n t^n}^{\infty} s^{-1-p/n} \int_0^s (f^*(u) - f^*(s))^p \, du \, ds\Big)^{1/p}. \quad (35)$$

We claim that the latter sum is dominated by its last term. Indeed, we obtain by means of Remark 4.4 (recall that $1 < p < \infty$) and Proposition 4.2 that, for all $t > 0$,

$$\begin{aligned}
&\Big(\int_0^{V_n 3^n t^n} (f^{**}(u) - f^{**}(V_n 3^n t^n))^p \, du\Big)^{1/p} \\
&\le \Big(\int_0^{V_n 3^n t^n} (f^{**}(u) - f^{*}(V_n 3^n t^n))^p \, du\Big)^{1/p} \\
&= \Big(\int_0^{V_n 3^n t^n} \Big(\frac{1}{u}\int_0^u f^*(s) - f^*(V_n 3^n t^n)\, ds\Big)^p du\Big)^{1/p} \\
&\lesssim \Big(\int_0^{V_n 3^n t^n} (f^{*}(u) - f^{*}(V_n 3^n t^n))^p \, du\Big)^{1/p} \\
&\approx t\Big(\int_{V_n 3^n t^n}^{\infty} s^{-1-p/n}\, ds\Big)^{1/p}\Big(\int_0^{V_n 3^n t^n} (f^{*}(u) - f^{*}(V_n 3^n t^n))^p \, du\Big)^{1/p} \\
&= t\Big(\int_{V_n 3^n t^n}^{\infty} s^{-1-p/n} \int_0^{V_n 3^n t^n} (f^{*}(u) - f^{*}(V_n 3^n t^n))^p \, du\, ds\Big)^{1/p} \\
&\le t\Big(\int_{V_n 3^n t^n}^{\infty} s^{-1-p/n} \int_0^{s} (f^{*}(u) - f^{*}(s))^p \, du\, ds\Big)^{1/p}.
\end{aligned}$$

To complete our proof, note that the factor $V_n 3^n$ can be omitted from the last term in (35) (as follows by arguments used in the discussion following (32)). □

6 Proof of Proposition 3.6

We shall start with the following result:

Lemma 6.1 *Let $1 \le p < \infty$, $1 \le r \le \infty$ and $\beta \in \mathbb{R}$. Let $f \in \mathcal{M}_0(\mathbb{R}^n)$ satisfy $|\mathrm{supp} f|_n \le 1$ and*

$$\big\| t^{1-1/r} \ell^{\beta}(t) \Big(\int_{t^n}^{2} s^{-p/n} \int_0^s (f^*(u) - f^*(s))^p \, du \, \frac{ds}{s}\Big)^{1/p} \big\|_{r,(0,1)} < \infty. \quad (36)$$

Then $f \in L_p$ and the function F defined by

$$F(x) = f^*(V_n|x|^n) \ \text{if } p = 1 \ \text{ or } \ F(x) = f^{**}(V_n|x|^n) \ \text{if } 1 < p < \infty \quad (37)$$

belongs to $B^{0,\beta}_{p,r}$. *Moreover,*

$$\|F\|_{B^{0,\beta}_{p,r}} \lesssim \Big\|t^{1-1/r}\ell^{\beta}(t)\Big(\int_{t^n}^{2} s^{-p/n}\int_0^s (f^*(u)-f^*(s))^p\,du\,\frac{ds}{s}\Big)^{1/p}\Big\|_{r,(0,1)} \quad (38)$$

for all f *mentioned above.*

Proof. Take $f \in \mathcal{M}_0(\mathbb{R}^n)$ with $|\mathrm{supp} f|_n \leq 1$. Then $f^*(t) = 0$ for $t \geq 1$. Therefore, when $s \in (1,\infty)$, $\int_0^s (f^*(u)-f^*(s))^p\,du = \int_0^s f^*(u)^p\,du = \int_0^1 f^*(u)^p\,du$. Hence,

$$\begin{aligned}
\|f\|_p &= \Big(\int_0^1 f^*(u)^p\,du\Big)^{1/p}\\
&\approx \Big(\int_1^2 s^{-p/n-1}\,ds\int_0^1 f^*(u)^p\,du\Big)^{1/p}\\
&\approx \|t^{1-1/r}\ell^{\beta}(t)\|_{r,(0,1)}\Big(\int_1^2 s^{-p/n-1}\int_0^s (f^*(u)-f^*(s))^p\,du\,ds\Big)^{1/p}\\
&\leq \Big\|t^{1-1/r}\ell^{\beta}(t)\Big(\int_{t^n}^2 s^{-p/n-1}\int_0^s (f^*(u)-f^*(s))^p\,du\,ds\Big)^{1/p}\Big\|_{r,(0,1)}. \quad (39)
\end{aligned}$$

Together with (36), this shows that $f \in L_p$.

On the other hand, using (39),

$$\begin{aligned}
&\Big\|t^{1-1/r}\ell^{\beta}(t)\Big(\int_{t^n}^{\infty} s^{-p/n}\int_0^s (f^*(u)-f^*(s))^p\,du\,\frac{ds}{s}\Big)^{1/p}\Big\|_{r,(0,1)}\\
&\leq \Big\|t^{1-1/r}\ell^{\beta}(t)\Big(\int_{t^n}^{1} s^{-p/n}\int_0^s (f^*(u)-f^*(s))^p\,du\,\frac{ds}{s}\Big)^{1/p}\Big\|_{r,(0,1)}\\
&\quad + \Big\|t^{1-1/r}\ell^{\beta}(t)\Big(\int_{1}^{\infty} s^{-p/n}\int_0^1 f^*(u)^p\,du\,\frac{ds}{s}\Big)^{1/p}\Big\|_{r,(0,1)}\\
&\lesssim \Big\|t^{1-1/r}\ell^{\beta}(t)\Big(\int_{t^n}^{2} s^{-p/n}\int_0^s (f^*(u)-f^*(s))^p\,du\,\frac{ds}{s}\Big)^{1/p}\Big\|_{r,(0,1)}. \quad (40)
\end{aligned}$$

Now, since $\|F\|_p \lesssim \|f\|_p$, (38) follows from Proposition 3.5 and estimates (39) and (40). □

Proof of Proposition 3.6.

Step 1. Assume that (9) holds. Take $f \in \mathcal{M}_0(\mathbb{R}^n)$ with $|\mathrm{supp} f|_n \leq 1$. Then either the right-hand side of (10) is finite or infinite. If it is infinite, (10) is clear. So assume that the right-hand side of (10) is finite. In such a case, we apply Lemma 6.1 to get that the function F given by (37) satisfies (38). Using hypothesis (9) with F instead of f and the estimate $F^*(t) \geq f^*(t)$, inequality (10) follows.

Step 2. Assume now that (10) holds. Take $f \in B^{0,\beta}_{p,r}$ with $|\mathrm{supp} f|_n \leq 1$. Since $f \in L_p$, Proposition 4.7 and (10) yield (9).

Consider now a general $f \in B^{0,\beta}_{p,r}$ and put $g(x) := f^*(V_n|x|^n)\chi_{[0,1)}(V_n|x|^n)$, $x \in \mathbb{R}^n$. Clearly, $|\operatorname{supp} g|_n \le 1$ and $g^*(t) = f^*(t)\chi_{[0,1)}(t)$, $t \ge 0$. In particular, $g \in L_p$. Applying our hypothesis (10) to g instead of f and using Proposition 4.7, we arrive at

$$\begin{aligned}
&\|\omega(t)f^*(t)\chi_{[0,1)}(t)\|_{q,(0,1)} \\
&\lesssim \Big\| t^{1-1/r}\ell^\beta(t)\Big(\int_{t^n}^1 s^{-p/n}\int_0^s (f^*(u)-f^*(s))^p\,du\,\frac{ds}{s}\Big)^{1/p}\Big\|_{r,(0,1)} \\
&\qquad + \Big\| t^{1-1/r}\ell^\beta(t)\Big(\int_1^2 s^{-p/n}\int_0^1 f^*(u)^p\,du\,\frac{ds}{s}\Big)^{1/p}\Big\|_{r,(0,1)} \\
&\lesssim \|t^{-1/r}\ell^\beta(t)\,\omega_1(f,t)_p\|_{r,(0,1)} + \|f\|_p \\
&= \|f\|_{B^{0,\beta}_{p,r}}
\end{aligned} \tag{41}$$

and (9) follows. □

7 Proof of Theorem 3.1

To prove Theorem 3.1, we shall need a variant of Lemma 6.1. This is why we start with the following:

Remark 7.1 *Lemma 6.1 continues to hold if we assume additionally that $p \le r$ and if expression* (36) *is replaced by*

$$\Big\| t^{-1/r}\ell^\beta(t)\Big(\int_0^t f^*(u)^p\,du\Big)^{1/p}\Big\|_{r,(0,1)}.$$

Indeed, using the triangle inequality, Fubini's Theorem, a change of variables, the assumption on the support of f and Proposition 4.6(ii), we can see that the right-hand side of (38) is dominated by

$$\begin{aligned}
&\Big\| t^{1-1/r}\ell^\beta(t)\Big(\int_{t^n}^2 s^{-p/n}\int_0^{t^n} f^*(u)^p\,du\,\frac{ds}{s}\Big)^{1/p}\Big\|_{r,(0,1)} \\
&\qquad + \Big\| t^{1-1/r}\ell^\beta(t)\Big(\int_{t^n}^2 s^{-p/n}\int_{t^n}^s f^*(u)^p\,du\,\frac{ds}{s}\Big)^{1/p}\Big\|_{r,(0,1)} \\
&\lesssim \Big\| t^{-1/r}\ell^\beta(t)\Big(\int_0^{t^n} f^*(u)^p\,du\Big)^{1/p}\Big\|_{r,(0,1)} \\
&\qquad + \Big\| t^{1-1/r}\ell^\beta(t)\Big(\int_{t^n}^1 f^*(u)^p\int_u^2 s^{-p/n-1}\,ds\,du\Big)^{1/p}\Big\|_{r,(0,1)}
\end{aligned}$$

$$\approx \Big\| t^{-1/r}\ell^{\beta}(t)\Big(\int_0^t f^*(u)^p\,du\Big)^{1/p}\Big\|_{r,(0,1)}$$
$$+ \Big\| t^{p/n-p/r}\ell^{\beta p}(t)\int_t^1 u^{-p/n} f^*(u)^p\,du\Big\|_{r/p,(0,1)}^{1/p}$$
$$\lesssim \Big\| t^{-1/r}\ell^{\beta}(t)\Big(\int_0^t f^*(u)^p\,du\Big)^{1/p}\Big\|_{r,(0,1)}$$
$$+ \| t^{1-p/r}\ell^{\beta p}(t) f^*(t)^p\|_{r/p,(0,1)}^{1/p}$$
$$\lesssim \Big\| t^{-1/r}\ell^{\beta}(t)\Big(\int_0^t f^*(u)^p\,du\Big)^{1/p}\Big\|_{r,(0,1)}.$$

So, the conclusion follows immediately from Lemma 6.1.

Proof of Theorem 3.1.

Step 1. Here we prove the sufficiency of the condition $q \geq r$ under the additional assumption $q \geq p$.

Due to Proposition 4.7, it is enough to show that, for all $f \in B^{0,\beta}_{p,r}$,

$$\|t^{1/p-1/q}\ell^{\beta+1/r}(t) f^*(t)\|_{q,(0,1)}$$
$$\lesssim \|f\|_p + \Big\| t^{-1/r}\ell^{\beta}(t)\, t\Big(\int_{t^n}^{\infty} s^{-p/n}\int_0^s (f^*(u)-f^*(s))^p\,du\,\frac{ds}{s}\Big)^{1/p}\Big\|_{r,(0,1)}. \quad (42)$$

(***i***) First consider the case $p = 1$.

Since $\int_0^{t^n} f^*(u)\,du \lesssim t\int_{t^n}^{\infty} s^{-1/n}\int_0^s (f^*(u)-f^*(s))\,du\,\frac{ds}{s}$ (cf. (32) with $V_n^{1/n}|h|$ replaced by $t \in (0,1)$), we see that it is sufficient to prove that, for all $f \in B^{0,\beta}_{p,r}$,

$$\|t^{1-1/q}\ell^{\beta+1/r}(t) f^*(t)\|_{q,(0,1)} \lesssim \|f\|_1 + \Big\| t^{-1/r}\ell^{\beta}(t)\int_0^{t^n} f^*(s)\,ds\Big\|_{r,(0,1)}. \quad (43)$$

If $r = \infty$, then, by our assumption, also $q = \infty$ and (43) is trivial. Thus, we suppose that $1 \leq r < \infty$. For simplicity, we consider only the case when $q < \infty$ (the case $q = \infty$ can be handled similarly). Using the fact that singularities of functions of the form $t \mapsto t^{\alpha}\ell^{\delta}(t)$, $t \in (0,1)$, $\alpha, \delta \in \mathbb{R}$, may be only at the origin, and the monotonicity of functions in question, we obtain

$$\begin{aligned}
&\|t^{1-1/q}\ell^{\beta+1/r}(t)f^*(t)\|_{q,(0,1)}\\
&\quad\lesssim \|f\|_1 + \|t^{1-1/q}\ell^{\beta+1/r}(t)f^*(t)\|_{q,(0,1/4)}\\
&\quad= \|f\|_1 + \Big(\sum_{k=1}^{\infty}\int_{2^{-2^{k+1}}}^{2^{-2^k}} t^{q-1}\ell^{\beta q+q/r}(t)f^*(t)^q\,dt\Big)^{1/q}\\
&\quad\le \|f\|_1 + \Big(\sum_{k=1}^{\infty}\ell^{\beta q+q/r}(2^{-2^{k+1}})\sum_{i=0}^{2^k-1}\int_{2^{-2^{k+1}}2^i}^{2^{-2^{k+1}}2^{i+1}} t^{q-1}f^*(t)^q\,dt\Big)^{1/q}\\
&\quad\lesssim \|f\|_1 + \Big(\sum_{k=1}^{\infty}\ell^{\beta q+q/r}(2^{-2^{k+1}})\sum_{i=0}^{2^k-1}(2^{-2^{k+1}}2^i)^q f^*(2^{-2^{k+1}}2^i)^q\Big)^{1/q}.
\end{aligned}$$

Write $1/q=(r/q)(1/r)$, take the exponent r/q inside the outer sum and afterwards take the factor $1/q$ of this exponent inside the inner sum (all this is possible because we are assuming $q \ge r \ge 1$), to get

$$\begin{aligned}
&\|t^{1-1/q}\ell^{\beta+1/r}(t)f^*(t)\|_{q,(0,1)}\\
&\quad\lesssim \|f\|_1 + \bigg(\sum_{k=1}^{\infty}\ell^{\beta r+1}(2^{-2^k})\Big(\sum_{i=0}^{2^k-1}2^{-2^{k+1}}2^i f^*(2^{-2^{k+1}}2^i)\Big)^r\bigg)^{1/r}\\
&\quad\lesssim \|f\|_1 + \bigg(\sum_{k=1}^{\infty}\ell^{\beta r+1}(2^{-2^k})\Big(\sum_{i=0}^{2^k-1}\int_{2^{-2^{k+1}}2^{i-1}}^{2^{-2^{k+1}}2^i} f^*(t)\,dt\Big)^r\bigg)^{1/r}\\
&\quad\le \|f\|_1 + \bigg(\sum_{k=1}^{\infty}\ell^{\beta r+1}(2^{-2^k})\Big(\int_0^{2^{-2^k}} f^*(t)\,dt\Big)^r\bigg)^{1/r}\\
&\quad\approx \|f\|_1 + \bigg(\sum_{k=0}^{\infty}(\ell^{\beta r+1}(2^{-2^{k+1}})-\ell^{\beta r+1}(2^{-2^k}))\Big(\int_0^{2^{-2^{k+1}}} f^*(s)\,ds\Big)^r\bigg)^{1/r}\\
&\quad\lesssim \|f\|_1 + \bigg(\sum_{k=0}^{\infty}\int_{2^{-2^{k+1}}}^{2^{-2^k}}\ell^{\beta r}(t)\Big(\int_0^t f^*(s)\,ds\Big)^r\frac{dt}{t}\bigg)^{1/r}\\
&\quad\le \|f\|_1 + \Big\|t^{-1/r}\ell^{\beta}(t)\int_0^t f^*(s)\,ds\Big\|_{r,(0,1)},
\end{aligned}$$

which, after a change of variables, proves (43).

(***ii***) Now consider the case $1<p<\infty$. Using the monotonicity of function (13), we obtain, for all $t>0$,

$$\Big(\int_0^{t^n}(f^*(s)-f^*(t^n))^p\,ds\Big)^{1/p} \lesssim t\Big(\int_{t^n}^{\infty}s^{-p/n}\int_0^s(f^*(u)-f^*(s))^p\,du\,\frac{ds}{s}\Big)^{1/p}.$$

Thus, in view of (42), it is enough to prove that

$$\|t^{1/p-1/q}\ell^{\beta+1/r}(t)f^*(t)\|_{q,(0,1)} \lesssim \|f\|_p + \Big\|t^{-1/r}\ell^{\beta}(t)\Big(\int_0^{t^n}(f^*(s)-f^*(t^n))^p\,ds\Big)^{1/p}\Big\|_{r,(0,1)}. \quad (44)$$

Applying the estimate $f^* \le f^{**}$, (16) and the Hardy-type inequality from Proposition 4.6(ii), we arrive at

$$\begin{aligned}
&\|t^{1/p-1/q}\ell^{\beta+1/r}(t)f^*(t)\|_{q,(0,1)}\\
&\le \|t^{1/p-1/q}\ell^{\beta+1/r}(t)(f^{**}(1)+(f^{**}(t)-f^{**}(1)))\|_{q,(0,1)}\\
&\lesssim \|f\|_p + \|t^{1/p-1/q}\ell^{\beta+1/r}(t)(f^{**}(t)-f^{**}(1))\|_{q,(0,1)}\\
&= \|f\|_p + \Big\|t^{1/p-1/q}\ell^{\beta+1/r}(t)\Big(\int_t^1 \frac{f^{**}(s)-f^*(s)}{s}\,ds\Big)\Big\|_{q,(0,1)}\\
&\lesssim \|f\|_p + \|t^{1/p-1/q}\ell^{\beta+1/r}(t)(f^{**}(t)-f^*(t))\|_{q,(0,1)}. \quad (45)
\end{aligned}$$

If $r=\infty$, we use Hölder's inequality to get

$$f^{**}(t)-f^*(t) \le t^{-1/p}\Big(\int_0^t (f^*(u)-f^*(t))^p\,du\Big)^{1/p}.$$

Consequently,

$$\|t^{1/p-1/q}\ell^{\beta+1/r}(t)f^*(t)\|_{q,(0,1)} \lesssim \|f\|_p + \|t^{-1/q}\ell^{\beta+1/r}(t)\Big(\int_0^t (f^*(u)-f^*(t))^p\,du\Big)^{1/p}\|_{q,(0,1)},$$

and (44) follows immediately since our assumption $r \le q$ implies that also $q=\infty$.

If $1 \le r < \infty$, then (45), the obvious estimate

$$\|t^{1/p-1/q}\ell^{\beta+1/r}(t)(f^{**}(t)-f^*(t))\|_{q,(1/4,1)} \le \|f\|_p,$$

(44) and Proposition 4.5 show that it is enough to prove that

$$\|t^{1/p-1/q}\ell^{\beta+1/r}(t)(f^{**}(t)-f^*(t))\|_{q,(0,1/4)} \lesssim \Big\|t^{-1/r}\ell^{\beta}(t)\Big(\int_0^{t^n}(f^{**}(s)-f^*(s))^p\,ds\Big)^{1/p}\Big\|_{r,(0,1)}. \quad (46)$$

For simplicity, we consider only the case when $q<\infty$ (the case $q=\infty$ can be handled similarly). Having the monotonicity of function (14) in mind, we obtain

$$
\begin{aligned}
&\|t^{1/p-1/q}\ell^{\beta+1/r}(t)(f^{**}(t)-f^{*}(t))\|_{q,(0,1/4)}\\
&= \Big(\sum_{k=1}^{\infty}\int_{2^{-2^{k+1}}}^{2^{-2^{k}}} t^{q/p-1}\ell^{\beta q+q/r}(t)(f^{**}(t)-f^{*}(t))^{q}\,dt\Big)^{1/q}\\
&\le \Big(\sum_{k=1}^{\infty}\ell^{\beta q+q/r}(2^{-2^{k+1}})\sum_{i=0}^{2^k-1}\int_{2^{-2^{k+1}}2^{i}}^{2^{-2^{k+1}}2^{i+1}} t^{q/p-1}(f^{**}(t)-f^{*}(t))^{q}\,dt\Big)^{1/q}\\
&\lesssim \Big(\sum_{k=1}^{\infty}\ell^{\beta q+q/r}(2^{-2^{k}})\\
&\qquad\times\sum_{i=0}^{2^k-1}(2^{-2^{k+1}}2^{i})^{q/p}(f^{**}(2^{-2^{k+1}}2^{i+1})-f^{*}(2^{-2^{k+1}}2^{i+1}))^{q}\Big)^{1/q} \qquad (47)
\end{aligned}
$$

Write $1/q=(r/q)(1/r)$ and take the exponent r/q inside the outer sum (since $r/q\le 1$). Then the inner sum will have the exponent r/q, which we write as $(p/q)(r/p)$ and then take its factor p/q inside the inner sum (since $p/q\le 1$). This leads to an upper estimate by

$$
\begin{aligned}
&\Big(\sum_{k=1}^{\infty}\ell^{\beta r+1}(2^{-2^{k}})\\
&\qquad\times\Big(\sum_{i=0}^{2^k-1}(2^{-2^{k+1}}2^{i})(f^{**}(2^{-2^{k+1}}2^{i+1})-f^{*}(2^{-2^{k+1}}2^{i+1}))^{p}\Big)^{r/p}\Big)^{1/r}. \quad (48)
\end{aligned}
$$

The estimate $a=a^pa^{1-p}\approx a^p\int_{2a}^{4a}t^{-p}\,dt$, for all $a:=2^{-2^{k+1}}2^{i}$, and the monotonicity of functions (14) and (13) allow to dominate the last expression (up to a multiplicative positive constant) by

$$
\begin{aligned}
&\Big(\sum_{k=1}^{\infty}\ell^{\beta r+1}(2^{-2^{k}})\Big(\sum_{i=0}^{2^k-1}\int_{2^{-2^{k+1}}2^{i+1}}^{2^{-2^{k+1}}2^{i+2}}(f^{**}(t)-f^{*}(t))^{p}\,dt\Big)^{r/p}\Big)^{1/r}\\
&\le \Big(\sum_{k=0}^{\infty}\ell^{\beta r+1}(2^{-2^{k+1}})\Big(\int_0^{2^{-2^{k+1}+1}}(f^{**}(s)-f^{*}(s))^{p}\,ds\Big)^{r/p}\Big)^{1/r}\\
&\approx \Big(\sum_{k=0}^{\infty}(\ell^{\beta r+1}(2^{-2^{k+1}+1})-\ell^{\beta r+1}(2^{-2^{k}+1}))\\
&\qquad\times\Big(\int_0^{2^{-2^{k+1}+1}}(f^{**}(s)-f^{*}(s))^{p}\,ds\Big)^{r/p}\Big)^{1/r}\\
&\lesssim \Big(\sum_{k=0}^{\infty}\int_{2^{-2^{k+1}+1}}^{2^{-2^{k}+1}}\ell^{\beta r}(t)\,t^{-1}\Big(\int_0^{t}(f^{**}(s)-f^{*}(s))^{p}\,ds\Big)^{r/p}dt\Big)^{1/r}\\
&= \Big\|t^{-1/r}\ell^{\beta}(t)\Big(\int_0^{t}(f^{**}(s)-f^{*}(s))^{p}\,ds\Big)^{1/p}\Big\|_{r,(0,1)},
\end{aligned}
$$

which, after a change of variables, together with the estimates obtained above, gives (46).

Step 2. Now, we prove the sufficiency of the condition $q \geq r$ even when $q < p$. Thus, assume that $r \leq q < p$. In particular, $p > 1$.

It is enough to prove (42) (for all $q \in [r,p)$) but with

$$\|t^{1/p-1/q}\ell^{\beta+1/r+1/p-1/q}(t)f^*(t)\|_{q,(0,1)}$$

on its left-hand side.

Essentially, we can follow part (ii) of Step 1. The only modifications are that the case $r = \infty$ does not occur and also the way used to estimate the expression corresponding to the last term in (47) by (48) is a different one. First we apply Hölder's inequality with the exponent p/q in the inner sum (taking one of the factors to be 1), then we write $1/q = (r/q)(1/r)$ and take the exponent r/q inside the outer sum.

Step 3.

We prove the necessity of the condition $q \geq r$ when $q < p$.

Take $\omega \in (0,1]$ in such a way that the function $t \mapsto t^{-1/p}\ell^{-\beta-1/r-1/p}(t)$ is non-increasing in $[0,\omega)$. For any given $y \in (0,\omega/2)$, put

$$\begin{aligned} f_y(x) \quad := \quad & y^{-1/p}\ell^{-\beta-1/r-1/p}(y)\chi_{[0,y]}(V_n|x|^n) \\ & + (V_n|x|^n)^{-1/p}\ell^{-\beta-1/r-1/p}(V_n|x|^n)\chi_{(y,\omega)}(V_n|x|^n), \quad x \in \mathbb{R}^n. \end{aligned}$$

Then

$$f_y^*(t) = y^{-1/p}\ell^{-\beta-1/r-1/p}(y)\chi_{[0,y]}(t) + t^{-1/p}\ell^{-\beta-1/r-1/p}(t)\chi_{(y,\omega)}(t), \quad t > 0.$$

(***i***) Case $1 < p < \infty$.

Defining $F_y(x) = f_y^{**}(V_n|x|^n)$, $x \in \mathbb{R}^n$, we get $\|F_y\|_p = \|F_y^*\|_p = \|f_y^{**}\|_p \lesssim \|f_y^*\|_p \lesssim \ell^{-\beta-1/r}(\omega) \approx 1$ for all $y \in (0,\omega/2)$. Moreover, Proposition 3.5(ii), a change of variables, the triangle inequality and the fact that f_y^* is constant in $(0,y)$ imply that

$$\begin{aligned} & \|t^{-1/r}\ell^{\beta}(t)\,\omega_1(F_y,t)_p\|_{r,(0,1)} \\ \lesssim \quad & \|t^{-1/r}\ell^{\beta}(t)\,t^{1/n}\Big(\int_t^\infty s^{-p/n-1}\int_0^s (f_y^*(u)-f_y^*(s))^p\,du\,ds\Big)^{1/p}\|_{r,(0,1)} \\ \lesssim \quad & \|t^{-1/r}\ell^{\beta}(t)\,t^{1/n}\|_{r,(0,y)}\Big(\int_y^\infty s^{-p/n-1}\int_0^s (f_y^*(u)-f_y^*(s))^p\,du\,ds\Big)^{1/p} \\ & + \|t^{-1/r}\ell^{\beta}(t)\,t^{1/n}\Big(\int_t^\infty s^{-p/n-1}\int_0^s (f_y^*(u)-f_y^*(s))^p\,du\,ds\Big)^{1/p}\|_{r,(y,1)} \\ =: \quad & A + B. \end{aligned} \tag{49}$$

Furthermore, since $f_y^*(u) - f_y^*(s) \le f_y^*(u)$,

$$\begin{aligned} A &\lesssim y^{1/n}\ell^\beta(y)\Big(\int_y^\infty s^{-p/n-1}\int_y^s f_y^*(u)^p\,du\,ds + \int_y^\infty s^{-p/n-1}\int_0^y f_y^*(u)^p\,du\,ds\Big)^{1/p} \\ &\lesssim y^{1/n}\ell^\beta(y)\Big(y^{-p/n}\ell^{-\beta p-p/r}(y) + y^{-p/n}\ell^{-\beta p-p/r-1}(y)\Big)^{1/p} \\ &\lesssim \ell^{-1/r}(y) \lesssim 1 \end{aligned} \tag{50}$$

and

$$\begin{aligned} B &\lesssim \|t^{-1/r}\ell^\beta(t)\,t^{1/n}\Big(\int_t^\infty s^{-p/n-1}\int_y^s f_y^*(u)^p\,du\,ds\Big)^{1/p}\|_{r,(y,1)} \\ &\qquad + \|t^{-1/r}\ell^\beta(t)\,t^{1/n}\Big(\int_t^\infty s^{-p/n-1}\int_0^y f_y^*(u)^p\,du\,ds\Big)^{1/p}\|_{r,(y,1)} \\ &\lesssim \|t^{-1/r}\ell^{-1/r}(t)\|_{r,(y,1)} + \|t^{-1/r}\ell^\beta(t)\|_{r,(y,1)}\,\ell^{-\beta-1/r-1/p}(y) \\ &\lesssim (\ln\ell(y))^{1/r}, \end{aligned} \tag{51}$$

for all $y \in (0,\omega/2)$. Therefore $F_y \in B^{0,\beta}_{p,r}$ and $\|F_y\|_{B^{0,\beta}_{p,r}} \lesssim (\ln\ell(y))^{1/r}$ for all $y \in (0,\omega/2)$. This estimate, (4), the inequality $f_y^* \le f_y^{**} = F_y^*$ and the assumption $q < p$ imply that, for all $y \in (0,\omega/2)$,

$$\|t^{1/p-1/q}\ell^{\beta+1/r+1/p-1/q}(t)f^*(t)\|_{q,(0,1)} \lesssim (\ln\ell(y))^{1/r}. \tag{52}$$

Since the left-hand side of (52) can be estimated from below by

$$\Big(\int_y^\omega t^{-1}\ell^{-1}(t)\,dt\Big)^{1/q} \approx (\ln\ell(y))^{1/q} \quad \text{for all } y \in (0,\omega/2),$$

we conclude that it must be $q \ge r$.

(***ii***) Case $p = 1$.

We slightly modify the approach of part (i). Now, we put $F_y(x) := f_y^*(V_n|x|^n)$, $x \in \mathbb{R}^n$, we apply Proposition 3.5(i) (with the expression on the second line of (8)) instead of Proposition 3.5(ii) and make use of the equality $F_y^* = f_y^*$.

Step 4.

Now we prove the necessity of the condition $q \ge r$ when $q \ge p$.

On the contrary, suppose that $q < r$. Hence, $1 \le p \le q < r \le \infty$.

Since (4) is assumed to hold for all functions from $B^{0,\beta}_{p,r}$, Proposition 3.6 and Remark 7.1 imply that

$$\|t^{1/p-1/q}\ell^{\beta+1/r}(t)f^*(t)\|_{q,(0,1)} \lesssim \Big\|t^{-1/r}\ell^\beta(t)\,\Big(\int_0^t f^*(u)^p\,du\Big)^{1/p}\Big\|_{r,(0,1)} \tag{53}$$

for all $f \in \mathcal{M}_0(\mathbb{R}^n)$ with $|\mathrm{supp} f|_n \le 1$. One can see that (53) remains true if we omit the assumption $|\mathrm{supp} f|_n \le 1$. (Indeed, if $f \in \mathcal{M}_0(\mathbb{R}^n)$, take

$f_1 := f^*(V_n|\cdot|^n)\,\chi_{[0,1)}(V_n|\cdot|^n)$. Consequently, $f_1^*(t) = f^*(t)$ for all $t \in (0,1)$, and $|\mathrm{supp} f_1|_n \le 1$. Thus, applying (53) to f_1, we obtain the result.) Let $g \in \mathcal{M}_0(\mathbb{R}^n)$ and $f := |g|^{1/p}$. Then (53) yields

$$\|t^{1-p/q}\ell^{\beta p+p/r}(t)\, g^*(t)\|_{q/p,(0,1)} \lesssim \|t^{1-p/r}\ell^{\beta p}(t)\, g^{**}(t)\|_{r/p,(0,1)} \tag{54}$$

for all $g \in \mathcal{M}_0(\mathbb{R}^n)$ (or even for any measurable function g on $\mathbb{R}^n$).

Assume first that $1 \le r < \infty$. Then (54) implies that the inequality

$$\Big(\int_0^\infty w(t) g^*(t)^{q/p}\, dt\Big)^{p/q} \lesssim \Big(\int_0^\infty v(t) g^{**}(t)^{r/p}\, dt\Big)^{p/r} \tag{55}$$

holds for all measurable g on $\mathbb{R}^n$, where, for all $t \in (0,\infty)$,

$$w(t) = t^{q/p-1}\ell^{\beta q+q/r}(t)\chi_{(0,1)}(t)$$

and

$$v(t) = t^{r/p-1}\ell^{\beta r}(t)\chi_{(0,1)}(t) + \chi_{[1,\infty)}(t).$$

By Proposition 4.8 (with $Q = q/p$ and $P = r/p$), inequality (55) holds only if

$$\infty > \int_0^1 \frac{t^{\frac{rq}{(r-q)p}} \sup_{y\in[t,1)} y^{-\frac{rq}{(r-q)p}}\big(y^{\frac{rq}{(r-q)p}}\ell^{\beta\frac{rq}{r-q}+\frac{q}{r-q}}(y)\big)}{\Big(t^{\frac{r}{p}}\ell^{\beta r}(t) + t^{\frac{r}{p}}\Big(\int_t^1 s^{-\frac{r}{p}}(s^{\frac{r}{p}-1}\ell^{\beta r}(s))\, ds + \int_1^\infty s^{-\frac{r}{p}}\, ds\Big)\Big)^{\frac{q}{r-q}+2}}$$
$$\times\, t^{\frac{r}{p}}\ell^{\beta r}(t)\int_t^1 s^{-\frac{r}{p}}(s^{\frac{r}{p}-1}\ell^{\beta r}(s))\, ds\, t^{\frac{r}{p}-1}\, dt \;=: I.$$

However,

$$I \gtrsim \int_0^{1/2} \frac{\ell^{\beta\frac{rq}{r-q}+\frac{q}{r-q}}(t)\;\ell^{\beta r}(t)\ell^{\beta r+1}(t)\; t^{-1}}{(\ell^{\beta r}(t) + (\ell^{\beta r+1}(t) + \frac{p}{r-p}))^{\frac{q}{r-q}+2}}\, dt \;\approx\; \int_0^{1/2} t^{-1}\ell^{-1}(t)\, dt \;=\; \infty,$$

which is a contradiction. Consequently, $q \ge r$.

Assume now that $r = \infty$. Therefore, $\beta > 0$. Inequality (54) implies that

$$\Big(\int_0^\infty w(t) g^*(t)^{q/p}\, dt\Big)^{p/q} \lesssim \operatorname{ess\,sup}_{t\in(0,\infty)} v(t) g^{**}(t) \tag{56}$$

for all measurable g in $\mathbb{R}^n$, where, for all $t \in (0,\infty)$,

$$w(t) = t^{q/p-1}\ell^{\beta q}(t)\chi_{(0,1)}(t)$$

and

$$v(t) = t\,\ell^{\beta p}(t)\chi_{(0,1)}(t) + \ell(t)\chi_{[1,\infty)}(t).$$

Let ν be the measure on $[0,\infty)$ which is absolutely continuous with respect to the Lebesgue measure on $[0,\infty)$ and satisfies

$$d\nu(t) = \begin{cases} t^{-1}\ell^{-\beta q-1}(t)\, dt & \text{if } 0 < t \le 1 \\ t^{q/p-1}\ell^{-q/p-1}(t)\, dt & \text{if } t > 1 \end{cases}.$$

By Proposition 4.9 (with $Q = q/p$), inequality (56) implies that

$$\infty > \int_0^\infty \sup_{s\in(t,\infty)} \frac{\int_0^s \tau^{\frac{q}{p}-1}\ell^{\beta q}(\tau)\chi_{(0,1)}(\tau)\,d\tau}{s^{\frac{q}{p}}}\,d\nu(t) =: I.$$

However,

$$I \gtrsim \int_0^1 \big(\sup_{s\in(t,1)} \ell^{\beta q}(s)\big)\, t^{-1}\ell^{-\beta q-1}(t)\,dt \approx \int_0^1 t^{-1}\ell^{-1}(t)\,dt = \infty,$$

which is a contradiction. Consequently, $q \geq r$. □

8 Proof of Theorem 3.2

In view of Theorem 3.1, the sufficiency of the condition that κ is bounded is obvious. Thus, we prove that this condition is also necessary.

Step 1.

Assume $q \geq p$. Take $y \in (0, 1/2)$ and $f_y \in L_p(\mathbb{R}^n)$ with $f_y^* = \chi_{[0,y)}$. It is easy to see that

$$t\Big(\int_{t^n}^\infty s^{-p/n}\int_0^s (f_y^*(u) - f_y^*(s))^p\,du\,\frac{ds}{s}\Big)^{1/p} \approx \min\{y^{1/p}, t\,y^{1/p-1/n}\} \quad (57)$$

for all $t > 0$ and $y \in (0, 1/2)$.

(***i***) Case $1 < p < \infty$.

Defining $F_y(x) = f_y^{**}(V_n|x|^n)$, $x \in \mathbb{R}^n$, we get $\|F_y\|_p = \|F_y^*\|_p = \|f_y^{**}\|_p \approx \|f_y^*\|_p = y^{1/p}$ for all $y \in (0, 1/2)$. Moreover, Proposition 3.5(ii) and (57) imply that $\omega_1(F_y, t)_p \lesssim \min\{y^{1/p}, t\,y^{1/p-1/n}\}$ for all $y \in (0, 1/2)$ and $t > 0$. Hence,

$$\begin{aligned}
&\|t^{-1/r}\ell^\beta(t)\,\omega_1(F_y,t)_p\|_{r,(0,1)}\\
&\lesssim y^{1/p-1/n}\,\|t^{1-1/r}\ell^\beta(t)\|_{r,(0,y^{1/n})} + y^{1/p}\,\|t^{-1/r}\ell^\beta(t)\|_{r,(y^{1/n},1)}\\
&\approx y^{1/p}\,\ell^{\beta+1/r}(y)
\end{aligned}$$

for all $y \in (0, 1/2)$. Therefore, $F_y \in B^{0,\beta}_{p,r}$ and

$$\|F_y\|_{B^{0,\beta}_{p,r}} \lesssim y^{1/p}\,\ell^{\beta+1/r}(y) \quad \text{for all } y \in (0,1/2). \quad (58)$$

This estimate, (5), the inequality $f_y^* \leq f_y^{**} = F_y^*$ and the assumption $q \geq p$ imply that

$$\|t^{1/p-1/q}\ell^{\beta+1/r}(t)\kappa(t)\|_{q,(0,y)} \lesssim y^{1/p}\,\ell^{\beta+1/r}(y).$$

Thus,

$$\kappa(y)y^{1/p}\,\ell^{\beta+1/r}(y) \lesssim y^{1/p}\,\ell^{\beta+1/r}(y) \quad \text{for all } y \in (0,1/2).$$

Hence, κ must be bounded.

(***ii***) Case $p = 1$.

Defining $F_y(x) = f_y^*(V_n|x|^n)$, $x \in \mathbb{R}^n$, we get $\|F_y\|_1 = \|F_y^*\|_1 = \|f_y^*\|_1 = y$. Moreover, Proposition 3.5(i) and (57) yield $\omega_1(F_y, t)_1 \lesssim \min\{y, t\, y^{1-1/n}\}$ for all $y \in (0, 1/2)$ and $t > 0$. The rest follows essentially as in part (i) (now with $p = 1$ and $F_y^* = f_y^*$).

Step 2. Assume now that $1 \le r \le q < p < \infty$. In particular, $p > 1$.

For any given $y \in (0, 1/2)$, put $f_y(x) := y^{-1/p}\ell^{1/q-1/p}(y)\chi_{[0,y]}(V_n|x|^n) + (V_n|x|^n)^{-1/p}\ell^{1/q-1/p}(V_n|x|^n)\chi_{(y,1)}(V_n|x|^n)$, $x \in \mathbb{R}^n$. Then

$$f_y^*(t) = y^{-1/p}\ell^{1/q-1/p}(y)\chi_{(0,y]}(t) + t^{-1/p}\ell^{1/q-1/p}(t)\chi_{(y,1)}(t), \quad t > 0.$$

We proceed as in part (i) of Step 3 of the proof of Theorem 3.1. Defining $F_y(x) = f_y^{**}(V_n|x|^n)$, $x \in \mathbb{R}^n$, we see that $\|F_y\|_p \lesssim \ell^{1/q}(y)$ for all $y \in (0, 1/2)$. Moreover, we obtain (49), where now

$$A \lesssim \ell^{\beta+1/q}(y) \quad \text{and} \quad B \lesssim \ell^{\beta+1/r+1/q}(y)$$

for all $y \in (0, 1/2)$. Therefore, $F_y \in B^{0,\beta}_{p,r}$ and $\|F_y\|_{B^{0,\beta}_{p,r}} \lesssim \ell^{\beta+1/r+1/q}(y)$ for all $y \in (0, 1/2)$. This estimate, (5), the inequality $f_y^* \le f_y^{**} = F_y^*$ and the assumption $q < p$ imply that

$$\|t^{1/p-1/q}\ell^{\beta+1/r+1/p-1/q}(t)\, \kappa(t) f_y^*(t)\|_{q,(y,\sqrt{y})} \lesssim \ell^{\beta+1/r+1/q}(y)$$

for all $y \in (0, 1/2)$. Since the left-hand side of the last expression can be estimated from below by

$$\kappa(\sqrt{y})\|t^{-1/q}\, \ell^{\beta+1/r}(t)\|_{q,(y,\sqrt{y})} \approx \kappa(\sqrt{y})\, \ell^{\beta+1/r+1/q}(y) \quad \text{for all } y \in (0, 1/2),$$

we conclude that κ must be bounded. $\square$

9 Proof of Theorem 3.3

We refer only to the case $1 < p < \infty$; the case $p = 1$ can be easily adapted.

Put $A := B^{0,\beta}_{p,r}$. By Theorem 3.1 with $q = \infty$,

$$t^{1/p}\ell^{\beta+1/r}(t) f^*(t) \lesssim 1$$

for all $t \in (0, 1)$ and $f \in A$ with $\|f\|_A \le 1$. Therefore,

$$\sup_{\|f\|_A \le 1} f^*(t) \lesssim t^{-1/p}\ell^{-\beta-1/r}(t) \quad \text{for all } t \in (0, 1). \tag{59}$$

On the other hand, consider the functions F_y, $y \in (0, 1/2)$, from Step 1 of the proof of Theorem 3.2. By (58), there exists $c > 0$ such that

$$\|F_y\|_A \le c\, y^{1/p}\ell^{\beta+1/r}(y) \quad \text{for all } y \in (0, 1/2).$$

Together with the inequality $F_y^* \equiv f_y^{**} \geq f_y^* \equiv \chi_{[0,y)}$, this implies that

$$\sup_{\|f\|_A \leq 1} f^*(t) \geq c^{-1}\, y^{-1/p} \ell^{-\beta-1/r}(y) \chi_{[0,y)}(t) \tag{60}$$

for all $t > 0$ and $y \in (0, 1/2)$. Thus, taking $y = 2t$ for every $t \in (0, 1/4)$, we obtain from (60) that

$$\begin{aligned} \sup_{\|f\|_A \leq 1} f^*(t) &\geq c^{-1}(2t)^{-1/p} \ell^{-\beta-1/r}(2t)\chi_{[0,2t)}(t) \\ &\approx t^{-1/p}\ell^{-\beta-1/r}(t) \end{aligned}$$

for all $t \in (0, 1/4)$. Together with (59), this gives

$$\sup_{\|f\|_A \leq 1} f^*(t) \approx t^{-1/p}\ell^{-\beta-1/r}(t) =: h(t) \quad \text{for all small } t > 0.$$

Since the function h is positive, continuous and non-increasing on some $(0, \varepsilon]$, $\varepsilon \in (0, 1/2)$, and $\lim_{t\to 0+} h(t) = \infty$, this function h is a growth envelope function of the space $A = B^{0,\beta}_{p,r}$.

As to the fine index, notice that $H(t) := -\ln h(t)$ satisfies $H'(t) \approx \frac{1}{t}$ on some small interval $(0, \varepsilon)$. Therefore, $d\mu_H(t) \approx \frac{1}{t}\, dt$ and Theorem 3.1 implies that

$$\Big(\int_{(0,\varepsilon)} \Big(\frac{f^*(t)}{h(t)}\Big)^q d\mu_H(t)\Big)^{1/q} \lesssim \|f\|_A \quad \text{for all } f \in A \tag{61}$$

(with the usual modification in the case $q = \infty$) whenever $q \in [\max\{p, r\}, \infty]$. On the other hand, it is also possible to prove that this cannot hold for $q \in (0, \max\{p, r\})$.

In order to see this, we shall show first that if (61) holds then it must be $q \geq p$. We follow the same construction as in the proof of Step 3 of Theorem 3.1, now with $\omega \in (0, \varepsilon]$. Since we use (61) instead of (4), now the counterpart of (52) reads as

$$\|t^{1/p-1/q}\ell^{\beta+1/r}(t) f^*(t)\|_{q,(0,\varepsilon)} \lesssim (\ln \ell(y))^{1/r} \quad \text{for all } y \in (0, \omega/2). \tag{62}$$

If we assumed that $q < p$, then the left-hand side of (62) could be estimated from below by

$$\Big(\int_y^\omega t^{-1}\ell^{-q/p}(t)\, dt\Big)^{1/q} \approx \ell^{1/q-1/p}(y) \quad \text{for all } y \in (0, \omega/2),$$

and we would get a contradiction.

So, we have just shown that (61) implies $q \geq p$. Consequently, $1/\max\{p, q\} - 1/q = 0$ and we can now use Theorem 3.1 to show that $q \geq r$.

Therefore, (61) holds if and only if $q \in [\max\{p, r\}, \infty]$. □

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