Academy of Sciences of the Czech Republic
Mathematical Institute

Preprint Series
No.148/2002

Periodic Boundary Value Problems for Nonlinear Second Order Differential Equations with Impulses - Part I

Irena Rachůnková* and Milan Tvrdý†

September 24, 2002

Summary. This paper deals with the nonlinear impulsive periodic boundary value problem

$$u'' = f(t,u,u'), \tag{1.1}$$

$$u(t_i+) = \mathrm{J}_i(u(t_i)), \quad u'(t_i+) = \mathrm{M}_i(u'(t_i)), \quad i = 1,2,\dots,m, \tag{1.2}$$

$$u(0) = u(T), \quad u'(0) = u'(T). \tag{1.3}$$

We establish the existence results which rely on the presence of a well ordered pair (σ_1, σ_2) of lower/upper functions $(\sigma_1 \le \sigma_2$ on $[0,T])$ associated with the problem. In contrast to previous papers investigating such problems, the monotonicity of the impulse functions J_i, M_i is not required here.

Mathematics Subject Classification 2000. 34B37, 34B15, 34C25

Keywords. Second order nonlinear ordinary differential equation with impulses, periodic solutions, lower and upper functions, Leray-Schauder topological degree, a priori estimates.

0. Introduction

In recent years, the theory of impulsive differential equations has become a well respected branch of mathematics. This is because of its characteristic features

*Supported by the grant No. 201/01/1451 of the Grant Agency of the Czech Republic and by the Council of Czech Government J14/98:153100011

†Supported by the grant No. 201/01/1199 of the Grant Agency of the Czech Republic

which provide many interesting problems that cannot be solved by applying standard methods from the theory of ordinary differential equations. It can also give a natural description of many real models from applied sciences (see the examples mentioned in [1], [2]).

In particular, starting with [7], periodic boundary value problems for nonlinear second order impulsive differential equations of the form (1.1) - (1.3) have received considerable attention; see e.g. [1], [3] [5], [6], [8], [9] and [14], where the existence results in terms of lower and upper functions can also be found. However, all impose certain monotonicity requirements on the impulse functions. In contrast to these papers, we provide existence results using weaker conditions (1.12) - (1.13) instead of monotonicity.

Throughout the paper we keep the following notation and conventions:

For a real valued function u defined a.e. on $[0,T]$, we put

$$\|u\|_\infty = \sup_{t\in[0,T]} \text{ess}\, |u(t)| \quad \text{and} \quad \|u\|_1 = \int_0^T |u(s)|\,\mathrm{d}s.$$

For for a given interval $J \subset \mathbb{R}$, let $\mathbb{C}(J)$ denote the set of real valued functions which are continuous on J. Furthermore, let $\mathbb{C}^1(J)$ be the set of functions having continuous first derivatives on J and $\mathbb{L}(J)$ is the set of functions which are Lebesgue integrable on J.

Let $m \in \mathbb{N}$ and

$$0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = T$$

be a division of the interval $[0,T]$. We denote

$$\mathrm{D} = \{t_1, t_2, \ldots, t_m\}$$

and define $\mathbb{C}^1_\mathrm{D}[0,T]$ as the set of functions $u : [0,T] \mapsto \mathbb{R}$,

$$u(t) = \begin{cases} u_{[0]}(t) & \text{if } t \in [0,t_1], \\ u_{[1]}(t) & \text{if } t \in (t_1,t_2], \\ \cdots & \cdots \\ u_{[m]}(t) & \text{if } t \in (t_m,T], \end{cases}$$

where $u_{[i]} \in \mathbb{C}^1[t_i,t_{i+1}]$ for $i = 0,1,\ldots,m$. Moreover, $\mathbb{AC}^1_\mathrm{D}[0,T]$ stands for the set of functions $u \in \mathbb{C}^1_\mathrm{D}[0,T]$ having first derivatives absolutely continuous on each subinterval (t_i,t_{i+1}), $i = 0,1,\ldots,m$. For $u \in \mathbb{C}^1_\mathrm{D}[0,T]$ and $i = 1,2,\ldots,m+1$ we write

$$u'(t_i) = u'(t_i-) = \lim_{t\to t_i-} u'(t), \quad u'(0) = u'(0+) = \lim_{t\to 0+} u'(t) \tag{0.1}$$

and

$$\|u\|_{\mathrm{D}} = \|u\|_\infty + \|u'\|_\infty .$$

Note that the set $\mathbb{C}^1_{\mathrm{D}}[0,T]$ becomes a Banach space when equipped with the norm $\|.\|_{\mathrm{D}}$ and with the usual algebraic operations.

We say that $f : [0,T] \times \mathbb{R}^2 \mapsto \mathbb{R}$ satisfies the *Carathéodory conditions* on $[0,T] \times \mathbb{R}^2$ if

(i) for each $x \in \mathbb{R}$ and $y \in \mathbb{R}$ the function $f(.,x,y)$ is measurable on $[0,T]$;

(ii) for almost every $t \in [0,T]$ the function $f(t,.,.)$ is continuous on $\mathbb{R}^2$;

(iii) for each compact set $K \subset \mathbb{R}^2$ there is a function $m_K(t) \in \mathbb{L}[0,T]$ such that $|f(t,x,y)| \le m_K(t)$ holds for a.e. $t \in [0,T]$ and all $(x,y) \in K$.

The set of functions satisfying the Carathéodory conditions on $[0,T] \times \mathbb{R}^2$ will be denoted by $\mathrm{Car}([0,T] \times \mathbb{R}^2)$.

Given a Banach space $\mathbb{X}$ and its subset M, let $\mathrm{cl}(M)$ and ∂M denote the closure and the boundary of M, respectively.

Let Ω be an open bounded subset of $\mathbb{X}$. Assume that the operator $\mathrm{F} : \mathrm{cl}(\Omega) \mapsto \mathbb{X}$ is completely continuous and $\mathrm{F}\,u \ne u$ for all $u \in \partial\,\Omega$. Then $\deg(\mathrm{I} - \mathrm{F}, \Omega)$ denotes the *Leray-Schauder topological degree* of $\mathrm{I} - \mathrm{F}$ with respect to Ω, where I is the identity operator on $\mathbb{X}$. For a definition and properties of the degree see e.g. [4] or [10].

1. Formulation of the problem and main assumptions

Here we study the existence of solutions to the following problem

$$u'' = f(t,u,u'), \tag{1.1}$$

$$u(t_i+) = \mathrm{J}_i(u(t_i)), \quad u'(t_i+) = \mathrm{M}_i(u'(t_i)), \quad i = 1,2,\ldots,m, \tag{1.2}$$

$$u(0) = u(T), \quad u'(0) = u'(T), \tag{1.3}$$

where $u'(t_i)$ are understood in the sense of (0.1), $f \in \mathrm{Car}([0,T] \times \mathbb{R}^2)$, $\mathrm{J}_i \in \mathbb{C}(\mathbb{R})$ and $\mathrm{M}_i \in \mathbb{C}(\mathbb{R})$.

1.1 Definition. A *solution of the problem* (1.1) - (1.3) is a function $u \in \mathbb{AC}^1_{\mathrm{D}}[0,T]$ which satisfies the impulsive conditions (1.2), the periodic conditions (1.3) and for a.e. $t \in [0,T]$ fulfils the equation $u''(t) = f(t,u(t),u'(t))$.

1.2 Definition. A function $\sigma_1 \in \mathbb{AC}^1_{\mathrm{D}}[0,T]$ is called a *lower function of the problem* (1.1) - (1.3) if

$$\sigma_1''(t) \ge f(t,\sigma_1(t),\sigma_1'(t)) \qquad \text{for a.e. } t \in [0,T], \tag{1.4}$$

$$\sigma_1(t_i+) = \mathrm{J}_i(\sigma_1(t_i)), \quad \sigma_1'(t_i+) \ge \mathrm{M}_i(\sigma_1'(t_i)), \quad i = 1,2,\dots,m, \tag{1.5}$$

$$\sigma_1(0) = \sigma_1(T), \quad \sigma_1'(0) \ge \sigma_1'(T). \tag{1.6}$$

Similarly, a function $\sigma_2 \in \mathbb{AC}^1_{\mathrm{D}}[0,T]$ is an *upper function of the problem* (1.1) - (1.3) if

$$\sigma_2''(t) \le f(t,\sigma_2(t),\sigma_2'(t)) \qquad \text{for a.e. } t \in [0,T], \tag{1.7}$$

$$\sigma_2(t_i+) = \mathrm{J}_i(\sigma_2(t_i)), \quad \sigma_2'(t_i+) \le \mathrm{M}_i(\sigma_2'(t_i)), \quad i = 1,2,\dots,m, \tag{1.8}$$

$$\sigma_2(0) = \sigma_2(T), \quad \sigma_2'(0) \le \sigma_2'(T). \tag{1.9}$$

Throughout the paper we assume:

$$\begin{cases} 0 = t_0 < t_1 < t_2 < \dots < t_m < t_{m+1} = T < \infty, \\ \mathrm{D} = \{t_1, t_2, \dots, t_m\}, \\ f \in \mathrm{Car}([0,T] \times \mathbb{R}^2), \ \mathrm{J}_i \in \mathbb{C}(\mathbb{R}), \ \mathrm{M}_i \in \mathbb{C}(\mathbb{R}), \ i = 1,2,\dots,m; \end{cases} \tag{1.10}$$

$$\begin{cases} \sigma_1 \text{ and } \sigma_2 \text{ are respectively lower and upper functions of (1.1)-(1.3)} \\ \text{and} \quad \sigma_1 \le \sigma_2 \quad \text{on} \ [0,T]; \end{cases} \tag{1.11}$$

$$\begin{cases} \sigma_1(t_i) < x < \sigma_2(t_i) \implies \mathrm{J}_i(\sigma_1(t_i)) < \mathrm{J}_i(x) < \mathrm{J}_i(\sigma_2(t_i)), \\ i = 1,2,\dots,m; \end{cases} \tag{1.12}$$

$$\begin{cases} y \le \sigma_1'(t_i) \implies \mathrm{M}_i(y) \le \mathrm{M}_i(\sigma_1'(t_i)), \\ y \ge \sigma_2'(t_i) \implies \mathrm{M}_i(y) \ge \mathrm{M}_i(\sigma_2'(t_i)), \quad i = 1,2,\dots,m. \end{cases} \tag{1.13}$$

1.3 Remark. If $\mathrm{M}_i(0) = 0$ for $i = 1,2,\dots,m$ and $r_1 \in \mathbb{R}$ is such that $\mathrm{J}_i(r_1) = r_1$ for $i = 1,2,\dots,m$ and

$$f(t,r_1,0) \le 0 \ \text{ for a.e. } t \in [0,T],$$

then $\sigma_1(t) \equiv r_1$ on $[0,T]$ is a lower function of the problem (1.1) - (1.3). Similarly, if $r_2 \in \mathbb{R}$ is such that $\mathrm{J}_i(r_2) = r_2$ for all $i = 1,2,\dots,m$ and

$$f(t,r_2,0) \ge 0 \ \text{ for a.e. } t \in [0,T],$$

then $\sigma_2(t) \equiv r_2$ is an upper function of the problem (1.1) - (1.3).

2. A priori estimates

At the beginning of this section we introduce a class of auxiliary problems and prove uniform a priori estimates for their solutions.

Take $d \in \mathbb{R}$, $\widetilde{f} \in \mathrm{Car}([0,T] \times \mathbb{R}^2)$, $\widetilde{\mathrm{J}}_i \in \mathbb{C}(\mathbb{R})$ and $\widetilde{\mathrm{M}}_i \in \mathbb{C}(\mathbb{R})$, $i = 1, 2, \dots, m$, such that

$$(2.1) \quad \begin{cases} \widetilde{f}(t,x,y) < f(t,\sigma_1(t),\sigma_1'(t)) \ \text{ for a.e. } t \in [0,T], \text{ all } x \in (-\infty, \sigma_1(t)) \\ \qquad \text{and all } y \in \mathbb{R} \text{ such that } |y - \sigma_1'(t)| \le \dfrac{\sigma_1(t) - x}{\sigma_1(t) - x + 1}, \\ \widetilde{f}(t,x,y) > f(t,\sigma_2(t),\sigma_2'(t)) \ \text{ for a.e. } t \in [0,T], \text{ all } x \in (\sigma_2(t), \infty) \\ \qquad \text{and all } y \in \mathbb{R} \text{ such that } |y - \sigma_2'(t)| \le \dfrac{x - \sigma_2(t)}{x - \sigma_2(t) + 1}, \end{cases}$$

$$(2.2) \quad \begin{cases} \widetilde{\mathrm{J}}_i(x) < \mathrm{J}_i(\sigma_1(t_i)) & \text{if } x < \sigma_1(t_i) \\ \widetilde{\mathrm{J}}_i(x) = \mathrm{J}_i(x) & \text{if } x \in [\sigma_1(t_i), \sigma_2(t_i)] \\ \widetilde{\mathrm{J}}_i(x) > \mathrm{J}_i(\sigma_2(t_i)) & \text{if } x > \sigma_2(t_i), \qquad i = 1, 2, \dots, m, \end{cases}$$

$$(2.3) \quad \begin{cases} \widetilde{\mathrm{M}}_i(y) \le \mathrm{M}_i(\sigma_1'(t_i)) & \text{if } y \le \sigma_1'(t_i) \\ \widetilde{\mathrm{M}}_i(y) \ge \mathrm{M}_i(\sigma_2'(t_i)) & \text{if } y \ge \sigma_2'(t_i), \qquad i = 1, 2, \dots, m, \end{cases}$$

$$(2.4) \quad \sigma_1(0) \le d \le \sigma_2(0)$$

and consider an auxiliary Dirichlet problem

$$(2.5) \quad u'' = \widetilde{f}(t, u, u'),$$

$$(2.6) \quad u(t_i+) = \widetilde{\mathrm{J}}_i(u(t_i)), \quad u'(t_i+) = \widetilde{\mathrm{M}}_i(u'(t_i)), \quad i = 1, 2, \dots, m,$$

$$(2.7) \quad u(0) = u(T) = d.$$

2.1 Lemma. *Let* (1.10) – (1.12) *and* (2.1) - (2.4) *hold. Then every solution* u *of* (2.5) – (2.7) *satisfies*

$$(2.8) \quad \sigma_1 \le u \le \sigma_2 \ \text{ on } [0,T].$$

Proof. Let u be a solution of (2.5) - (2.7). Put $v(t) = u(t) - \sigma_2(t)$ for $t \in [0, T]$. Then, by (2.4), we have

$$v(0) = v(T) \leq 0. \tag{2.9}$$

So, it remains to prove that $v \leq 0$ on $(0, T)$.

• Part (i). First, we show that v does not have a positive local maximum at any point of $(0, T) \setminus \mathrm{D}$. Assume, on the contrary, that there is $\alpha \in (0, T) \setminus \mathrm{D}$ such that v has a positive local maximum at α; i.e.,

$$v(\alpha) > 0 \quad \text{and} \quad v'(\alpha) = 0. \tag{2.10}$$

This guarantees the existence of β such that $[\alpha, \beta] \subset (0, T) \setminus \mathrm{D}$ and

$$v(t) > 0 \quad \text{and} \quad |v'(t)| < \frac{v(t)}{v(t) + 1} < 1 \tag{2.11}$$

for $t \in [\alpha, \beta]$. Using (1.7), (2.1) and (2.11), we get

$$v''(t) = u''(t) - \sigma_2''(t) = \widetilde{f}(t, u(t), u'(t)) - \sigma_2''(t) > f(t, \sigma_2(t), \sigma_2'(t)) - \sigma_2''(t) \geq 0$$

for a.e. $t \in [\alpha, \beta]$. Hence,

$$0 < \int_\alpha^t v''(s)\,\mathrm{d}s = v'(t)$$

for all $t \in (\alpha, \beta]$. This contradicts that v has a local maximum at α.

• Part (ii). Now, assume that there is $t_j \in \mathrm{D}$ such that

$$\max_{t \in (t_{j-1}, t_j]} v(t) = v(t_j) > 0.$$

Then $v'(t_j) \geq 0$. By (2.2) and (2.3), we get

$$\widetilde{\mathrm{J}}_j(u(t_j)) > \mathrm{J}_j(\sigma_2(t_j)) \quad \text{and} \quad \widetilde{\mathrm{M}}_j(u'(t_j)) \geq \mathrm{M}_j(\sigma_2'(t_j));$$

by (2.6) and (1.8), the relations

$$v(t_j+) > 0 \quad \text{and} \quad v'(t_j+) \geq 0 \tag{2.12}$$

follow. If $v'(t_j+) > 0$, then there is $\beta \in (t_j, t_{j+1})$ such that

$$v'(t) > 0 \ \text{ on } \ (t_j, \beta]. \tag{2.13}$$

If $v'(t_j+) = 0$, then we can find β such that $(t_j, \beta] \subset (0,T) \setminus \mathrm{D}$ and (2.11) is satisfied on $(t_j, \beta]$. Consequently, (2.13) is valid in this case, as well. As by Part (i) v' cannot change its sign on (t_j, t_{j+1}), in both these cases we have

$$v'(t) \geq 0 \quad \text{on} \quad (t_j, t_{j+1}). \tag{2.14}$$

Now, by (2.12) - (2.14) we get

$$\max_{t \in (t_j, t_{j+1}]} v(t) = v(t_{j+1}) > 0.$$

Continuing inductively we get $v(T) > 0$, contrary to (2.9).

- Part (iii). Finally, assume that

$$\sup_{t \in (t_j, t_{j+1}]} v(t) = v(t_j+) > 0 \tag{2.15}$$

for some $t_j \in \mathrm{D}$. In view of (2.2), this is possible only if

$$\widetilde{\mathrm{J}}_j(u(t_j)) > \mathrm{J}_j(\sigma_2(t_j)). \tag{2.16}$$

If $u(t_j) \in [\sigma_1(t_j), \sigma_2(t_j)]$, then by (2.2) and (1.12) we have

$$\widetilde{\mathrm{J}}_j(u(t_j)) = \mathrm{J}_j(u(t_j)) \leq \mathrm{J}_j(\sigma_2(t_j)),$$

contrary to (2.16). If $u(t_j) < \sigma_1(t_j)$, then by (2.2), (1.11) and (1.12) we get

$$\widetilde{\mathrm{J}}_j(u(t_j)) < \mathrm{J}_j(\sigma_1(t_j)) \leq \mathrm{J}_j(\sigma_2(t_j)),$$

again a contradiction to (2.16). Therefore $u(t_j) > \sigma_2(t_j)$, i.e. $v(t_j) > 0$. Further, (2.15) gives $v'(t_j+) \leq 0$. If $v'(t_j+) = 0$, then, as in Part (ii), we get (2.13), which contradicts (2.15). Therefore, $v'(t_j+) < 0$ which yields, with (2.3), that $v'(t_j) < 0$. Thus, in view of Part (i), we deduce that $v' \leq 0$ on (t_{j-1}, t_j); i.e., $\sup_{t \in (t_{j-1}, t_j]} v(t) = v(t_{j-1}+) > 0$. Continuing inductively we get $v(0) > 0$, contradicting (2.9).

To summarize: we have proved that $v \leq 0$ on $[0,T]$ which means that $u \leq \sigma_2$ on $[0,T]$.

If we put $v = \sigma_1 - u$ on $[0,T]$ and use the properties of σ_1 instead of σ_2, we can prove $\sigma_1 \leq u$ on $[0,T]$ by an analogous argument. $\square$

In the proof of Theorem 3.1 we need a priori estimates for derivatives of solutions. To this aim we prove the following lemma.

2.2 Lemma. *Assume that* $r \in (0,\infty)$ *and that*

$$h \in \mathbb{L}[0,T] \quad \text{is nonnegative a.e. on} \quad [0,T], \tag{2.17}$$

$$\omega \in \mathbb{C}([1,\infty)) \quad \text{is positive on} \quad [1,\infty) \text{ and } \int_1^\infty \frac{ds}{\omega(s)} = \infty. \tag{2.18}$$

Then there exists $r^* \in (1,\infty)$ *such that the estimate*

$$\|u'\|_\infty \le r^* \tag{2.19}$$

holds for each function $u \in \mathbb{AC}^1_{\mathrm{D}}[0,T]$ *satisfying* $\|u\|_\infty \le r$ *and*

$$|u''(t)| \le \omega(|u'(t)|)\,(|u'(t)| + h(t)) \quad \text{for a.e. } t \in [0,T] \text{ and for } |u'(t)| > 1. \tag{2.20}$$

Proof. Let $u \in \mathbb{AC}^1_{\mathrm{D}}[0,T]$ satisfy (2.20) and let $\|u\|_\infty \le r$. The Mean Value Theorem implies that there are $\xi_i \in (t_i, t_{i+1})$ such that

$$|u'(\xi_i)| \le \frac{2\,r}{\Delta} + 1, \quad i = 1,2,\dots,m, \tag{2.21}$$

where

$$\Delta = \min_{i=0,1,\dots,m} (t_{i+1} - t_i). \tag{2.22}$$

Put

$$c_0 = \frac{2\,r}{\Delta} + 1 \quad \text{and} \quad \rho = \|u'\|_\infty.$$

By replacing u by $-u$ if necessary, we may assume that $\rho > c_0$ and

$$\rho = \sup_{t \in (t_j, t_{j+1}]} u'(t) \quad \text{for some } j \in \{0,1,\dots,m\}.$$

Thus we have

$$\rho = u'(\alpha) \quad \text{for some } \alpha \in (t_j, t_{j+1}] \tag{2.23}$$

or

$$\rho = u'(\alpha+) \quad \text{with } \alpha = t_j. \tag{2.24}$$

By (2.21), there is $\beta \in (t_j, t_{j+1})$, $\beta \ne \alpha$, such that $u'(\beta) = c_0$ and $u'(t) \ge c_0$ for all t lying between α and β. Assume that (2.23) occurs. There are two cases to consider: $t_j < \beta < \alpha \le t_{j+1}$ or $t_j < \alpha < \beta < t_{j+1}$.

- Case 1. Let $t_j < \beta < \alpha \le t_{j+1}$. Since $u'(t) > 1$ on $[\beta,\alpha]$, (2.20) gives

$$u''(t) \le \omega(u'(t))\,(u'(t) + h(t)) \quad \text{for a.e. } t \in [\beta,\alpha]$$

and hence

$$\int_{c_0}^{\rho} \frac{\mathrm{d}s}{\omega(s)} = \int_{\beta}^{\alpha} \frac{u''(t)}{\omega(u'(t))}\,\mathrm{d}t \le \int_{\beta}^{\alpha} u'(t)\,\mathrm{d}t + \|h\|_1 \le 2\,r + \|h\|_1. \tag{2.25}$$

On the other hand, by (2.18), there is $r^* > c_0$ such that

$$\int_{c_0}^{r^*} \frac{\mathrm{d}s}{\omega(s)} > 2\,r + \|h\|_1, \tag{2.26}$$

which is possible only if $\rho < r^*$, i.e. if (2.19) holds.

- Case 2. Let $t_j < \alpha < \beta < t_{j+1}$. By (2.20), we get

$$-u''(t) \le \omega(u'(t))\,(u'(t) + h(t)) \quad \text{for a.e. } t \in [\alpha, \beta]$$

and

$$\int_{c_0}^{\rho} \frac{\mathrm{d}s}{\omega(s)} = -\int_{\alpha}^{\beta} \frac{u''(t)}{\omega(u'(t))}\,\mathrm{d}t \le 2\,r + \|h\|_1;$$

so the inequality (2.19) follows.

If (2.24) occurs, a similar argument to that in Case 2 applies and gives (2.19), as well. □

2.3 Remark. Notice, that the condition

$$\int_{1}^{\infty} \frac{\mathrm{d}s}{\omega(s)} = \infty$$

in (2.18) can be weakened. In particular, the estimate (2.19) holds whenever $r^* \in (0, \infty)$ is such that

$$\int_{c_0}^{r^*} \frac{\mathrm{d}s}{\omega(s)} > 2\,r + \|h\|_1.$$

3. Main results

The main existence result for problem (1.1) - (1.3) is provided by the following theorem.

3.1 Theorem. *Assume that* (1.10) – (1.13) *hold. Further, let*

$$|f(t,x,y)| \le \omega(|y|)\,(|y| + h(t)) \quad \text{for a.e. } t \in [0,T] \text{ and all } x \in [\sigma_1(t), \sigma_2(t)],\ |y| > 1, \tag{3.1}$$

where h *and* ω *fulfil* (2.17) *and* (2.18). *Then the problem* (1.1) – (1.3) *has a solution* u *satisfying* (2.8).

Before proving this theorem, we prove the next key proposition where we restrict ourselves to the case that f is bounded by a Lebesgue integrable function.

3.2 Proposition. *Assume that* (1.10) – (1.13) *hold. Further, let* $m \in \mathbb{L}[0,T]$ *be such that*

$$|f(t,x,y)| \le m(t) \quad \text{for a.e. } t \in [0,T] \text{ and all } (x,y) \in [\sigma_1(t), \sigma_2(t)] \times \mathbb{R}. \tag{3.2}$$

Then the problem (1.1) – (1.3) *has a solution* u *fulfilling* (2.8).

Proof.

- Step 1. *We construct a proper auxiliary problem.*

Let Δ be given by (2.22). Put

$$c = \|m\|_1 + \frac{\|\sigma_1\|_\infty + \|\sigma_2\|_\infty}{\Delta} + \|\sigma_1'\|_\infty + \|\sigma_2'\|_\infty \tag{3.3}$$

and for $t \in [0,T]$ and $(x,y) \in \mathbb{R}^2$ define

$$\alpha(t,x) = \begin{cases} \sigma_1(t) & \text{if } x < \sigma_1(t), \\ x & \text{if } \sigma_1(t) \le x \le \sigma_2(t), \\ \sigma_2(t) & \text{if } x > \sigma_2(t) \end{cases} \tag{3.4}$$

and

$$\beta(y) = \begin{cases} y & \text{if } |y| \le c, \\ c \operatorname{sgn} y & \text{if } |y| > c. \end{cases}$$

For a.e. $t \in [0,T]$ and all $(x,y) \in \mathbb{R}^2$, $\varepsilon \in [0,1]$ define functions

$$\omega_k(t,\varepsilon) = \sup_{y \in [\sigma_k'(t) - \varepsilon, \sigma_k'(t) + \varepsilon]} |f(t, \sigma_k(t), \sigma_k'(t)) - f(t, \sigma_k(t), y)|, \quad k = 1,2,$$

$$\text{(3.5)}\qquad \begin{cases} \widetilde{\mathrm{J}}_i(x) = x + \mathrm{J}_i(\alpha(t_i, x)) - \alpha(t_i, x), \\ \widetilde{\mathrm{M}}_i(y) = y + \mathrm{M}_i(\beta(y)) - \beta(y), \qquad\qquad i = 1, 2, \dots, m, \end{cases}$$

$$\text{(3.6)}\qquad \widetilde{f}(t,x,y) = \begin{cases} f(t,\sigma_1(t),y) - \omega_1(t, \dfrac{\sigma_1(t)-x}{\sigma_1(t)-x+1}) - \dfrac{\sigma_1(t)-x}{\sigma_1(t)-x+1} \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } \ x < \sigma_1(t), \\ f(t,x,y) \qquad\qquad\qquad\qquad\qquad \text{if } \ \sigma_1(t) \le x \le \sigma_2(t), \\ f(t,\sigma_2(t),y) + \omega_2(t, \dfrac{x-\sigma_2(t)}{x-\sigma_2(t)+1}) + \dfrac{x-\sigma_2(t)}{x-\sigma_2(t)+1} \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } \ x > \sigma_2(t). \end{cases}$$

We see that $\omega_k \in \mathrm{Car}([0,T] \times [0,1])$ are nonnegative and nondecreasing in the second variable and $\omega_k(0) = 0$ for $k = 1, 2$. Consequently, $\widetilde{f} \in \mathrm{Car}([0,T] \times \mathbb{R}^2)$. Furthermore, $\widetilde{\mathrm{J}}_i$, $\widetilde{\mathrm{M}}_i \in \mathbb{C}(\mathbb{R})$, $i = 1, 2, \dots, m$. The auxiliary problem is (2.5), (2.6), and

$$\text{(3.7)}\qquad u(0) = u(T) = \alpha(0, u(0) + u'(0) - u'(T)).$$

- Step 2. *We prove that problem* (2.5), (2.6), (3.7) *is solvable.*
  Let

$$G(t,s) = \begin{cases} \dfrac{t\,(s-T)}{T} & \text{if } \ 0 \le t \le s \le T, \\ \dfrac{s\,(t-T)}{T} & \text{if } \ 0 \le s < t \le T, \end{cases}$$

$$G_1(t,s) = \begin{cases} -\dfrac{t}{T} & \text{if } \ 0 \le t \le s \le T, \\ \dfrac{T-t}{T} & \text{if } \ 0 \le s < t \le T. \end{cases}$$

Define an operator $\widetilde{\mathrm{F}} : \mathbb{C}^1_{\mathrm{D}}[0,T] \mapsto \mathbb{C}^1_{\mathrm{D}}[0,T]$ by

$$\text{(3.8)}\qquad (\widetilde{\mathrm{F}}u)(t) = \alpha(0, u(0) + u'(0) - u'(T)) + \int_0^T G(t,s)\, \widetilde{f}(s, u(s), u'(s))\, \mathrm{d}s$$
$$+ \sum_{i=1}^m G_1(t,t_i)\, (\widetilde{\mathrm{J}}_i(u(t_i)) - u(t_i)) + \sum_{i=1}^m G(t,t_i)\, (\widetilde{\mathrm{M}}_i(u'(t_i)) - u'(t_i)).$$

As in [12, Lemma 3.1] we get that $\widetilde{\mathrm{F}}$ is completely continuous and u is a solution of (2.5), (2.6), (3.7) if and only if u is a fixed point of $\widetilde{\mathrm{F}}$.

Denote by I the identity operator on $\mathbb{C}^1_{\mathrm{D}}[0,T]$ and consider the parameter system of operator equations

$$(3.9) \qquad (\mathrm{I} - \lambda\,\widetilde{\mathrm{F}})u = 0, \quad \lambda \in [0,1].$$

For $R \in (0,\infty)$, define $\mathrm{B}(R) = \{u \in \mathbb{C}^1_{\mathrm{D}}[0,T] : \|u\|_{\mathrm{D}} < R\}$. By (3.2), (3.4) - (3.6) and (3.8), we can find $R_0 \in (0,\infty)$ such that $u \in \mathrm{B}(R_0)$ for each $\lambda \in [0,1]$ and each solution u of (3.9). So, for each $R \geq R_0$ the operator $\mathrm{I} - \lambda\,\widetilde{\mathrm{F}}$ is a homotopy on $\mathrm{cl}(\mathrm{B}(R)) \times [0,1]$ and its Leray-Schauder degree $\deg(\mathrm{I} - \lambda\,\widetilde{\mathrm{F}}, \mathrm{B}(R))$ has the same value for each $\lambda \in [0,1]$. Since $\deg(\mathrm{I}, \mathrm{B}(R)) = 1$, we conclude that

$$(3.10) \qquad \deg(\mathrm{I} - \widetilde{\mathrm{F}}, \mathrm{B}(R)) = 1 \ \text{ for } \ R \in [R_0, \infty).$$

By (3.10), there is at least one fixed point of $\widetilde{\mathrm{F}}$ in $\mathrm{B}(R)$. Hence there exists a solution of the auxiliary problem (2.5), (2.6), (3.7).

- Step 3. *We find estimates for solutions of the auxiliary problem.*

Let u be a solution of (2.5), (2.6), (3.7). We derive an estimate for $\|u\|_\infty$. By (3.5), (3.6) and (1.13), we obtain that $\widetilde{f}$, $\widetilde{\mathrm{J}}_i$, $\widetilde{\mathrm{M}}_i$, $i = 1,2,\ldots,m$, satisfy (2.1) - (2.3). Moreover, in view of (3.4) we have

$$\sigma_1(0) \leq \alpha(0, u(0) + u'(0) - u'(T)) \leq \sigma_2(0).$$

Thus u satisfies (2.8) by Lemma 2.1.

We find an estimate for $\|u'\|_\infty$. By the Mean Value Theorem and (2.8), there are $\xi_i \in (t_i, t_{i+1})$ such that

$$(3.11) \qquad |u'(\xi_i)| \leq \frac{\|\sigma_1\|_\infty + \|\sigma_2\|_\infty}{\Delta}, \quad i = 1,2,\ldots,m.$$

Moreover, by (2.8) and (3.6), u satisfies (1.1) for a.e. $t \in [0,T]$. Therefore, integrating (1.1) and using (3.2), (3.3) and (3.11), we obtain

$$(3.12) \qquad \|u'\|_\infty \leq |u'(\xi_i)| + \|m\|_1 < c.$$

Hence, by (3.5) and (3.7), we see that u fulfils (1.2) and $u(0) = u(T)$ (i.e. the first condition from (1.3) is satisfied).

- Step 4. *We verify that u fulfils the second condition in* (1.3).

We must prove that $u'(0) = u'(T)$. By (3.7), this is equivalent to

$$(3.13) \qquad \sigma_1(0) \leq u(0) + u'(0) - u'(T) \leq \sigma_2(0).$$

Suppose, on the contrary, that (3.13) is not satisfied. Let, for example,

$$u(0) + u'(0) - u'(T) > \sigma_2(0). \tag{3.14}$$

Then, by (3.4), we have $\alpha(0, u(0) + u'(0) - u'(T)) = \sigma_2(0)$. Together with (1.9) and (3.7), this yields

$$u(0) = u(T) = \sigma_2(0) = \sigma_2(T). \tag{3.15}$$

Inserting (3.15) into (3.14) we get

$$u'(0) > u'(T). \tag{3.16}$$

On the other hand, (3.15) together with (2.8) and (3.16) implies that

$$\sigma_2'(0) \ge u'(0) > u'(T) \ge \sigma_2'(T),$$

a contradiction to (1.9).

If we assume that $u(0) + u'(0) - u'(T) < \sigma_1(0)$, we can argue similarly and again derive a contradiction to (1.9).

So, we have proved that (3.13) is valid which means that $u'(0) = u'(T)$. Consequently, u is a solution of (1.1) - (1.3) satisfying (2.8). □

Proof of Theorem 3.1. Put

$$c = r^* + \|\sigma_1'\|_\infty + \|\sigma_2'\|_\infty,$$

where $r^* \in (0, \infty)$ is given by Lemma 2.2 for $r = \|\sigma_1\|_\infty + \|\sigma_2\|_\infty$. For a.e. $t \in [0, T]$ and all $(x, y) \in \mathbb{R}^2$ define a function

$$g(t,x,y) = \begin{cases} f(t,x,y) & \text{if } |y| \le c, \\ (2 - \dfrac{|y|}{c})\, f(t,x,y) & \text{if } c < |y| < 2\,c, \\ 0 & \text{if } |y| \ge 2\,c. \end{cases} \tag{3.17}$$

Then σ_1 and σ_2 are respectively lower and upper functions of the auxiliary problem (1.2), (1.3), and

$$u'' = g(t, u, u'). \tag{3.18}$$

There exists a function $m^* \in \mathbb{L}[0, T]$ such that

$$|f(t,x,y)| \le m^*(t)$$

for a.e. $t \in [0,T]$ and all $(x,y) \in [\sigma_1(t), \sigma_2(t)] \times [-2\,c, 2\,c]$. Hence

$$|g(t,x,y)| \le m^*(t) \ \text{ for a.e. } t \in [0,T] \ \text{ and all } (x,y) \in [\sigma_1(t), \sigma_2(t)] \times \mathbb{R}.$$

Since $g \in \mathrm{Car}([0,T] \times \mathbb{R}^2)$, we can apply Proposition 3.2 on problem (3.18), (1.2), (1.3) and get that this problem has a solution u fulfilling (2.8). Hence $\|u\|_\infty \le r$. Moreover, by (3.1), u satisfies (2.20). Therefore, by Lemma 2.2, $\|u'\|_\infty \le r^* \le c$. This implies that u is a solution of (1.1) - (1.3).

The next simple existence criterion, which follows from Theorem 3.1 and Remark 1.3, extends both [5, Theorem 4] and [12, Corollary 3.4].

3.3 Corollary. *Let* (1.10) *hold. Furthermore, assume that:*

(i) $\mathrm{M}_i(0) = 0$ *and* $y\,\mathrm{M}_i(y) \ge 0$ *for* $y \in \mathbb{R}$ *and* $i = 1,2,\dots,m$;

(ii) *there are* $r_1, r_2 \in \mathbb{R}$ *such that* $r_1 < r_2$, $f(t,r_1,0) \le 0 \le f(t,r_2,0)$ *for a.e.* $t \in [0,T]$, $\mathrm{J}_i(r_1) = r_1$, $\mathrm{J}_i(x) \in (r_1,r_2)$ *if* $x \in (r_1,r_2)$, $\mathrm{J}_i(r_2) = r_2$, $i = 1,2,\dots,m$.

(iii) *there are* h *and* ω *satisfying* (2.17) *and* (2.18) *with* $\sigma_1(t) \equiv r_1$ *and* $\sigma_2(t) \equiv r_2$ *and such that* (3.1) *holds.*

Then the problem (1.1) – (1.3) *has a solution* u *fulfilling* $r_1 \le u \le r_2$ *on* $[0,T]$.

3.4 Remark. Let $\sigma_1 < \sigma_2$ on $[0,T]$ and $\sigma_1(t_i+) < \sigma_2(t_i+)$ for $i = 1,2,\dots,m$. Having G and G_1 from the proof of Proposition 3.2, we define an operator $\mathrm{F} : \mathbb{C}^1_{\mathrm{D}}[0,T] \mapsto \mathbb{C}^1_{\mathrm{D}}[0,T]$ by

$$(3.19) \quad (\mathrm{F}\,u)(t) = u(0) + u'(0) - u'(T) + \int_0^T G(t,s)\, f(s,u(s),u'(s))\,\mathrm{d}s$$
$$+ \sum_{i=1}^m G_1(t,t_i)\,(\mathrm{J}_i(u(t_i)) - u(t_i)) + \sum_{i=1}^m G(t,t_i)\,(\mathrm{M}_i(u'(t_i)) - u'(t_i)).$$

Let r^* be given by Lemma 2.2 for $r = \|\sigma_1\|_\infty + \|\sigma_2\|_\infty$. Define a set

$$(3.20) \quad \Omega = \{u \in \mathbb{C}^1_{\mathrm{D}}[0,T] : \|u'\|_\infty < r^*,\ \sigma_1(t) < u(t) < \sigma_2(t) \ \text{ for } t \in [0,T],$$
$$\sigma_1(t_i+) < u(t_i+) < \sigma_2(t_i+) \ \text{ for } i = 1,2,\dots,m\}.$$

As in [12, Lemma 3.1] we get that F is completely continuous and u is a solution of (1.1)–(1.3) if and only if u is a fixed point of F . The proofs of Theorem 3.1 and of Proposition 3.2 yield the following result about the Leray-Schauder degree of the operator $\mathrm{I} - \mathrm{F}$ with respect to Ω.

3.5 Corollary. *Let $\sigma_1 < \sigma_2$ on $[0,T]$ and $\sigma_1(t_i+) < \sigma_2(t_i+)$ for $i = 1,2,\ldots,m$, and let the assumptions of Theorem* 3.1 *be satisfied. Further assume that* F *and* Ω *are respectively defined by* (3.18) *and* (3.19). *If* $\mathrm{F}\,u \neq u$ *for each* $u \in \partial\Omega$, *then*

$$\deg(\mathrm{I} - \mathrm{F}, \Omega) = 1.$$

Proof. Consider c and g from the proof of Theorem 3.1 and define $\widetilde{J}_i$, $\widetilde{M}_i$, $i = 1,2,\ldots,m$, and $\widetilde{f}$ by (3.5) and (3.6), where we insert g instead of f. Suppose that $\mathrm{F}\,u \neq u$ for each $u \in \partial\Omega$, define $\widetilde{\mathrm{F}}$ by (3.8) and put $\Omega_1 = \{u \in \Omega : \sigma_1(0) < u(0) + u'(0) - u'(T) < \sigma_2(0)\}$. We have

$$\mathrm{F} = \widetilde{\mathrm{F}} \quad \text{on} \quad \mathrm{cl}(\Omega_1) \tag{3.21}$$

and

$$(\mathrm{F}\,u = u \quad \text{and} \quad u \in \Omega) \implies u \in \Omega_1. \tag{3.22}$$

By the proof of Proposition 3.2, each fixed point u of $\widetilde{\mathrm{F}}$ satisfies (1.3), (2.8) and, consequently, $\|u\|_\infty \leq r$. Hence, in view of (3.1), (3.6) and (**??**), we have

$$|u''(t)| = |g(t, u(t), u'(t))| \leq \omega(|u'(t)|)\,(|u'(t)| + h(t))$$

for a.e. $t \in [0,T]$ and for $|u'(t)| > 1$. Therefore Lemma 2.2 implies that $\|u'\|_\infty \leq r^*$. So, $u \in \mathrm{cl}(\Omega)$ and, due to (1.3), $u \in \Omega_1$. Now, choose R in (3.10) so that $\mathrm{B}(R) \supset \Omega$. Then, by (3.21), (3.22) and by the excision property of the degree, we get

$$\deg(\mathrm{I} - \mathrm{F}, \Omega) = \deg(\mathrm{I} - \widetilde{\mathrm{F}}, \Omega_1) = \deg(\mathrm{I} - \widetilde{\mathrm{F}}, \Omega_1)) = \deg(\mathrm{I} - \widetilde{\mathrm{F}}, \mathrm{B}(R)) = 1. \quad \square$$

3.6 Remark. Following the ideas from [11] and [12], the evalution of $\deg(\mathrm{I} - \mathrm{F}, \Omega)$ enables us to prove the existence of solutions to the problem (1.1) - (1.3) also for nonordered lower/upper functions. This will be included in our next paper [13].

References

[1] D. Bainov and P. Simeonov. *Impulsive differential equations: periodic solutions and applications.* Pitman Monographs and Surveys in Pure and Applied Mathematics 66, Longman Scientific and Technical, Essex, England, 1993.

[2] A. Cabada and E. Liz. Boundary value problems for higher order ordinary differential equations with impulses. *Nonlinear Analysis, T.M.A.* **32** (1998), 775-786.

[3] A. Cabada, J. J. Nieto, D. Franco and S. I. Trofimchuk. A generalization of the monotone method for second order periodic boundary value problem with impulses at fixed points. *Dyn. Contin. Discrete Impulsive Syst.* **7**, (2000), 145-158.

[4] J. Cronin. *Fixed Points and Topological Degree in Nonlinear Analysis*, AMS, 1964.

[5] Yujun Dong. Periodic solutions for second order impulsive differential systems. *Nonlinear Analysis, T.M.A.* **27** (1996), 811-820.

[6] L. H. Erbe and Xinzhi Liu. Existence results for boundary value problems of second order impulsive differential equations. *J. Math. Anal. Appl.* **149** (1990), 56-69.

[7] Shouchuan Hu and V. Laksmikantham. Periodic boundary value problems for second order impulsive differential systems. *Nonlinear Analysis, T.M.A.* **13** (1989), 75-85.

[8] E. Liz and J. J. Nieto. Periodic solutions of discontinuous impulsive differential systems. *J. Math. Anal. Appl.* **161** (1991), 388-394.

[9] E. Liz and J. J. Nieto. The monotone iterative technique for periodic boundary value problems of second order impulsive differential equations. *Comment. Math. Univ. Carolinae* **34** (1993), 405-411.

[10] J. Mawhin. Topological degree and boundary value problems for nonlinear differential equations. M. Furi (ed.) et al., *Topological methods for ordinary differential equations.* Lectures given at the 1st session of the Centro Internazionale Matematico Estivo (C.I.M.E.) held in Montecatini Terme, Italy, June 24 - July 2, 1991. Berlin: Springer-Verlag, Lect. Notes Math. 1537, 74-142 (1993).

[11] I. Rachůnková and M. Tvrdý. Nonlinear systems of differential inequalities and solvability of certain nonlinear second order boundary value problems. *J. Inequal. Appl.* **6** (2001), 199-226.

[12] I. Rachůnková and M. Tvrdý. Impulsive Periodic Boundary Value Problem and Topological Degree. *Faculty of Science, Palacký Univ. Olomouc, Preprint* **9/2002** (2002), to appear in *Funct. Differ. Equ.* (*Israel Sem.*).

[13] I. Rachůnková and M. Tvrdý. Periodic Boundary Value Problem for Nonlinear Second Order Differential Equations with Impulses - Part II, in preparation.

[14] Zhitao Zhang. Existence of solutions for second order impulsive differential equations. *Appl. Math., Ser.B* (engl. Ed.) **12**, (1997), 307-320.

Irena Rachůnková,
Department of Mathematics, Palacký University,
779 00 OLOMOUC, Tomkova 40, Czech Republic
(e-mail: rachunko@risc.upol.cz)

Milan Tvrdý,
Mathematical Institute, Academy of Sciences of the Czech Republic,
115 67 PRAHA 1, Žitná 25, Czech Republic
(e-mail: tvrdy@math.cas.cz)