Generalized Differential Equations in the Space of Regulated Functions (Boundary Value Problems and Controllability)

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Abstract. The boundary value problem for generalized linear differential equation and the corresponding controllability problems are dealt with. The adjoint problems are given in such a way that the usual duality theorems are valid. As a special case the interface boundary value problems are included. In contrast to the earlier papers by the present author the right-hand side of the generalized differential equation as well as the solutions of this equation can be in general regulated functions (not necessarily of bounded variation). Similar problems in the space of regulated functions were treated e.g. by Ch. S. Hönig, L. Fichmann and L. Barbanti, who made use of the interior (Dushnik) integral. In this paper the integral is the Perron-Stieltjes (Kurzweil) integral.
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0 . Introduction

The paper is devoted to linear boundary value problems for generalized linear differential equations

$$x(t) - x(0) - \int_0^t \mathrm{d}[A(s)]\, x(s) = f(t) - f(0), \quad t \in [0,1], \tag{0.1}$$

$$Mx(0) + \int_0^1 K(s)\, \mathrm{d}[x(s)] = r \quad (\in \mathbb{R}^m) \tag{0.2}$$

and the corresponding controllability problems. In particular, we obtain the adjoints to these problems in such a way that the usual duality theory can be extended to them. In contrast to the papers [Tv73], [Tv74], [SchTv] (cf. also [SchTvVe]) the right-hand side of the equation (0.1) can be in general a regulated function (not necessarily of bounded variation). Similar problems in the space of regulated functions were treated e.g. by Ch. S. Hönig [Hö75], [Hö82], [Hö80], L. Fichmann [Fi] and L. Barbanti [Ba], where the interior (Dushnik) integral was used. In this paper the integral is the Perron-Stieljes integral and, in particular, we work with the equivalent Kurzweil definition (cf. e.g. [Ku57], [Ku80], [Sch85] and [SchTvVe]). Let us notice that by [Ka] and by the relationship between the interior and the Perron-Stieljes integrals (cf. [Hö80] and [Sch73]), the left-hand side of the additional condition (0.2) represents a general form of a linear bounded mapping of the space of functions regulated on the closed interval $[0,1]$ and left-continuous on its interior $(0,1)$, equipped with the supremal norm, into $\mathbb{R}^n$. For the direct proof of this assertion by means of the Kurzweil integral, see Theorem 3.8 in [Tv89].

1 . Preliminaries

1.1. Notation. Throughout the paper $\mathbb{R}^n$ denotes the space of real column n-vectors, $\mathbb{R}^1 = \mathbb{R}$. Given a $k \times n$-matrix M, its elements are denoted by $m_{i,j}$, i.e.

$$M = (m_{i,j})_{\substack{i=1,\dots,k\\ j=1,\dots,n}},$$

M^* stands for its transposition ($M^* = (m_{j,i})_{\substack{j=1,\dots,n\\ i=1,\dots,k}}$),

$$|M| = \max_{i=1,\dots,k} \sum_{j=1}^{n} |m_{i,j}|$$

is its norm, det M is its determinant and rank(M) denotes its rank. (In particular, $y^* = (y_1, y_2 \dots, y_n)$ for $y \in \mathbb{R}^n$.) The symbols I and 0 stand respectively for the identity and the zero matrix of the proper type.

If $-\infty < a < b < \infty$, then $[a,b]$ and (a,b) denote the corresponding closed and open intervals, respectively. Furthermore, $[a,b)$ and $(a,b]$ are the corresponding half-open intervals. Any function $f : [a,b] \mapsto \mathbb{R}^n$ which possesses finite limits

$$f(t+) = \lim_{\tau \to t+} f(\tau) \quad \text{and} \quad f(s-) = \lim_{\tau \to s-} f(\tau)$$

for all $t \in [a,b)$ and $s \in (a,b]$ is said to be regulated on $[a,b]$. An n-vector valued function $x : [a,b] \mapsto \mathbb{R}^n$ is said to be regulated on $[a,b]$ if all its components

$x_j (j = 1,2,\ldots,n)$ are regulated on $[a,b]$. The linear space of n-vector valued functions regulated on $[a,b]$ is denoted by $\mathbb{G}^n(a,b)$. $\mathbb{G}^n_L(a,b)$ stands for the space of all functions from $\mathbb{G}^n(a,b)$ which are left-continuous on (a,b). For $x \in \mathbb{G}^n(a,b)$ we put $\|x\| = sup_{t\in[0,1]}|x(t)|$. It is well known that both $\mathbb{G}^n(a,b)$ and $\mathbb{G}^n_L(a,b)$ are Banach spaces with respect to this norm (cf. [Hö75, Theorem I.3.6]). Given $f \in \mathbb{G}^n(a,b)$, $t \in [a,b)$ and $s \in (a,b]$, we put

$$\Delta^+ f(t) = f(t+) - f(t) \quad \text{and} \quad \Delta^- f(s) = f(s) - f(s-).$$

Moreover, we define

$$f(b+) = f(b), \quad f(a-) = f(a) \quad \text{and} \quad \Delta^- f(a) = \Delta^- f(b) = 0.$$

Given $M \subset \mathbb{R}$, χ_M denotes its characteristic function. $\mathbb{BV}^n(a,b)$ denotes the Banach space of column n-vector valued functions of bounded variation on $[a,b]$ (equipped with the norm

$$f \in \mathbb{BV}^n(a,b) \mapsto \|f\|_{\mathbb{BV}} = |f(a)| + \operatorname{var}_a^b f,$$

where $\operatorname{var}_a^b f$ stands for the variation of f on $[a,b]$). If, moreover, $-\infty < c < d < \infty$ and a matrix valued function U is defined on $[a,b] \times [c,d]$, then $\mathrm{v}_{[a,b]\times[c,d]}(U)$ denotes its two-dimensional Vitali variation on $[a,b] \times [c,d]$. (For the definition and basic properties, see [Hi, Section III.4].) Furthermore, for given $t \in [a,b]$ and $s \in [c,d]$, the symbols $U(t,.)$ and $U(.,s)$ denote the functions

$$U(t,.) : \tau \in [c,d] \mapsto U(t,\tau) \quad \text{and} \quad U(.,s) : \tau \in [a,b] \mapsto U(\tau,s),$$

respectively. In the case $[a,b] = [0,1]$ we write simply $\mathbb{G}^n$, $\mathbb{G}^n_L$ and $\mathbb{BV}^n$ instead of $\mathbb{G}^n(0,1)$, $\mathbb{G}^n_L(0,1)$ and $\mathbb{BV}^n(0,1)$, respectively. Analogously, we write $\mathrm{v}(U)$ instead of $\mathrm{v}_{[0,1]\times[0,1]}(U)$. (More details concerning regulated functions or functions of bounded variation may be found e.g. in [Au], [Hö75], [Fra] or [Hi], respectively.)

For given linear spaces $\mathbb{X}$ and $\mathbb{Y}$, the symbol $\mathscr{L}(\mathbb{X},\mathbb{Y})$ denotes the linear space of all linear mappings of $\mathbb{X}$ into $\mathbb{Y}$. If $\mathscr{A} \in \mathscr{L}(\mathbb{X},\mathbb{Y})$, then $\mathscr{R}(\mathscr{A})$, $\mathscr{N}(\mathscr{A})$ and $\mathscr{A}^*$ denote its range, null space and adjoint operator, respectively. For $P \subset \mathbb{Y}$ and $\mathscr{A} \in \mathscr{L}(\mathbb{X},\mathbb{Y})$, the symbol $\mathscr{A}_{-1}(P)$ denotes the set of all $x \in \mathbb{X}$ for which $\mathscr{A}x \in P$. If $\mathbb{X}$ is a Banach space and $M \subset \mathbb{X}$, then $\mathrm{cl}(M)$ stands for the closure of M in $\mathbb{X}$.

1.2. Integrals. The integrals which occur in this paper are the Perron-Stieltjes ones. For the original definition, see [Wa] or [Sa]. We use the equivalent summation definition due to J. Kurzweil (cf. [Ku57], [Ku80], [SchTvVe]). The basic properties of the Perron-Stieltjes integral with respect to scalar regulated functions were described in [Tv89]. Given a $p \times q$-matrix valued function F and a $q \times n$-matrix valued function G defined on $[a,b]$ and such that all integrals

$$\int_a^b f_{i,k}(t)\,\mathrm{d}[g_{k,j}(t)] \quad (i = 1,2,\dots,p;\ k = 1,2,\dots,q;\ j = 1,2,\dots,n)$$

exist (i.e. they have finite values), the symbol

$$\int_a^b F(t)\,\mathrm{d}[G(t)] \quad \text{(or more simply} \quad \int_a^b F\mathrm{d}G)$$

stands for the $p \times n$-matrix M with the entries

$$m_{i,j} = \sum_{k=1}^{q} \int_a^b f_{i,k}\mathrm{d}[g_{k,j}] \quad (i = 1,2,\dots,p; j = 1,2,\dots,n).$$

The integrals

$$\int_a^b \mathrm{d}[F]G \quad \text{and} \quad \int_a^b F\mathrm{d}[G]H$$

for matrix valued functions F, G, and H of proper types are defined analogously. The extension of the results obtained in [Tv89] for scalar functions to vector valued or matrix valued functions is obvious and hence for the basic facts concerning integrals with respect to regulated functions we shall refer to the corresponding assertions from [Tv89].

1.3. Lemma. *Let $W(t,s)$ be an $n \times n$-matrix valued function defined on $[0,1] \times [0,1]$ and such that*

$$v(W) + \mathrm{var}_0^1 W(0,.) < \infty. \tag{1.1}$$

Then for any $g \in \mathbb{G}^n$, the function

$$w(t) = \int_0^1 \mathrm{d}_s[W(t,s)]g(s) \tag{1.2}$$

is defined and has bounded variation on $[0,1]$,

$$w(t+) = \int_0^1 \mathrm{d}_s[W(t+,s)]g(s) \ \textit{if} \ \ t \in [0,1), \tag{1.3}$$

$$w(t-) = \int_0^1 \mathrm{d}_s[W(t-,s)]g(s) \ \textit{if} \ \ t \in (0,1]. \tag{1.4}$$

Proof. Let $g \in \mathbb{G}^n$ be given. Since (1.1) implies that $\mathrm{var}_0^1 W(t,.) < \infty$ for any $t \in [0,1]$ (cf. e.g. Lemma I.6.6 in [SchTvVe]), the function (1.2) is defined for any $t \in [0,1]$. Furthermore, let an arbitrary subdivision $\{0 = t_0 < t_1 < ... < t_k = 1\}$ of $[0,1]$ be given. Then by Lemmas I.4.16 and I.6.13 of [SchTvVe] we have

$$\sum_{j=1}^{k} |w(t_j) - w(t_{j-1})| \leq \sum_{j=1}^{k} \mathrm{var}_0^1 (W(t_j,.) - W(t_{j-1},.))\|g\| \leq v(W)\|g\|,$$

and consequently

$$\mathrm{var}_0^1 w \leq v(W)\|g\| < \infty.$$

In particular, $w \in \mathbb{G}^n$. Moreover, by [SchTvVe, Lemma I.6.14]) all the functions

$$W(t+,.) \quad \text{and} \quad W(s-,0), t \in [0,1), \ s \in (0,1]$$

are of bounded variation on $[0,1]$. Thus the integrals on the right-hand sides of (1.3) are well defined. As g is on $[0,1]$ a uniform limit of a sequence of finite step functions and any finite step function on $[0,1]$ is a linear combination of simple jump functions on $[0,1]$

$$\chi_{[0,\sigma]}, \chi_{[\sigma,1]}, \quad \sigma \in [0,1], \tag{1.5}$$

it is sufficient to verify the relations (1.3) for the case that g is a simple jump function of the type (1.5). Let $g = \chi_{[0,\sigma]}$, where $\sigma \in [0,1]$. Then for any $t \in [0,1]$ we have

$$w(t) = \int_0^1 \mathrm{d}_s W(t,s) + (W(t,\sigma+) - W(t,\sigma)) = W(t,\sigma+) - W(t,\sigma).$$

Consequently,

$$w(t+) = W(t+,\sigma+) - W(t+,0) \text{ if } \ t \in [0,1)$$

and

$$w(t-) = W(t-,\sigma+) - W(t-,0) \text{ if } \ t \in (0,1].$$

On the other hand, we have

$$\int_0^1 \mathrm{d}_s[W(t+,s)]g(s) = W(t+,\sigma+) - W(t+,0) \text{ if } \ t \in [0,1)$$

and

$$\int_0^1 \mathrm{d}_s[W(t-,s)]g(s) = W(t-,\sigma+) - W(t-,0) \text{ if } \ t \in (0,1]$$

This means that the function $g = \chi_{[0,\sigma]}$ satisfies the relations (1.2) for any $\sigma \in [0,1)$. Similarly we could verify that the function $g = \chi_{[\sigma,1]}$ satisfies (1.2) for any $\sigma \in [0,1]$, and this completes the proof. $\square$

2 . Boundary value problem

We will consider the boundary value problem of determining a function $x : [0,1] \mapsto \mathbb{R}^n$ fulfilling the generalized differential equation (0.1) and the additional condition (0.2).

Throughout the paper we assume

2.1. Assumptions. $A(t)$ is an $n \times n$-matrix valued function of bounded variation on $[0,1]$, left-continuous on $(0,1]$, right-continuous at 0, and such that

$$\det \left[I + \Delta^+ A(t)\right] \neq 0 \ \text{ on } \ [0,1]$$

(where $A(1+) = A(1)$, cf. Notation 1.1; $f : [0,1] \mapsto \mathbb{R}^n$ is regulated on $[0,1]$ and left-continuous on $[0,1)$; M is a constant $m \times n$-matrix; $K(t)$ is an $m \times n$-matrix valued function of bounded variation on $[0,1]$ and $r \in \mathbb{R}^m$.

2.2. Remark. Assumptions 2.1 ensure that

$$\mathscr{L} : x \in \mathbb{G}_L^n \mapsto x(t) - x(0) - \int_0^t \mathrm{d}[A(s)]\, x(s) \tag{2.1}$$

defines a linear bounded operator on $\mathbb{G}_L^n$ and

$$\mathscr{K} : x \in \mathbb{G}_L^n \mapsto Mx(0) + \int_0^1 K(s)\, \mathrm{d}[x(s)] \tag{2.2}$$

defines a linear bounded mapping of $\mathbb{G}_L^n$ into $\mathbb{R}^m$ (cf. [Tv89, Proposition 2.16 and Theorem 2.8]). Hence, by

$$\mathscr{A} : x \in \mathbb{G}_L^n \mapsto \begin{pmatrix} \mathscr{L}\, x \\ \mathscr{K}\, x \end{pmatrix} \in \mathbb{G}_L^n \times \mathbb{R}^m \tag{2.3}$$

we define a linear bounded mapping of $\mathbb{G}_L^n$ into $\mathbb{G}_L^n \times \mathbb{R}^m$.

2.3. Remark. It is well-known (cf. [SchTvVe, Theorem III.2.10]) that under our assumptions there exists a unique $n \times n$-matrix valued function $U(t,s)$ such that

$$U(t,s) = \mathrm{I} + \int_0^t \mathrm{d}[A(\tau)]U(\tau,s) \quad \text{for} \quad t,s \in [0,1]. \tag{2.4}$$

It is called the *fundamental matrix solution* of the homogeneous equation

$$x(t) - x(0) - \int_0^t \mathrm{d}[A(s)]\,x(s) = 0 \ \text{ on } \ [0,1] \tag{2.5}$$

and possesses the following properties

$$|U(t,s)| + \mathrm{var}_0^1 U(t,.) + \mathrm{var}_0^1 U(.,s) + \mathrm{v}(U) \le M < \infty \qquad \text{for } t,s \in [0,1], \tag{2.6}$$

$$U(t,\tau)U(\tau,s) = U(t,s) \qquad \text{for } t,s,\tau \in [0,1], \tag{2.7}$$

$$\det U(t,s) \ne 0 \qquad \text{for } t,s \in [0,1], \tag{2.8}$$

$$\begin{aligned} U(t+,s) &= [I + \Delta^+ A(t)]U(t,s) && \text{for } t \in [0,1), s \in [0,1], \\ U(t-,s) &= U(t,s) && \text{for } t \in (0,1], s \in [0,1], \\ U(t,s+) &= U(t,s)[I + \Delta^+ A(t)]^{-1} && \text{for } t \in [0,1], s \in [0,1), \\ U(t,s-) &= U(t,s) && \text{for } t \in [0,1], s \in (0,1]. \end{aligned} \tag{2.9}$$

For a given $c \in \mathbb{R}^n$, the equation (2.5) possesses a unique solution $x : [0,1] \mapsto \mathbb{R}^n$ on $[0,1]$ such that $x(0) = c$ and this solution is given by (cf. [SchTvVe, Theorem III.2.4])

$$x(t) = U(t,0)c, \quad t \in [0,1]. \tag{2.10}$$

It is well-known (cf. [SchTvVe, Theorem III.2.13]) that for any $f : [0,1] \mapsto \mathbb{R}^n$ of bounded variation on $[0,1]$ ($f \in \mathbb{BV}^n$) and any $c \in \mathbb{R}^n$ there exists a unique solution x of (0.1) on $[0,1]$ such that $x(0) = c$. This solution has a bounded variation on $[0,1]$ and is given on $[0,1]$ by

$$x(t) = U(t,0)c + f(t) - f(0) + \int_0^t \mathrm{d}_s[U(t,s)](f(s) - f(0)). \tag{2.11}$$

To extend this assertion also to equations (0.1) with right-hand sides $f \in \mathbb{G}_L^n$, the following lemma will be helpful.

2.4. Lemma. *For a given $f \in \mathbb{G}_L^n$, the function*

$$\psi(t) = f(t) - f(0) - \int_0^t \mathrm{d}_s[U(t,s)](f(s) - f(0)) \tag{2.12}$$

is defined and regulated on $[0,1]$ and left-continuous on $(0,1)$. The operator

$$\Psi : f \in \mathbb{G}_L^n \mapsto \psi \in \mathbb{G}_L^n \tag{2.13}$$

is linear and bounded.

Proof. The function ψ is obviously defined on $[0,1]$. Let us put

$$W(t,s) = U(t,s) \text{ if } t \geq s \quad \text{and} \quad W(t,s) = U(t,t) \text{ if } t < s. \tag{2.14}$$

Then

$$\int_0^t \mathrm{d}_s[U(t,s)](f(s) - f(0)) = \int_0^\tau \mathrm{d}_s[W(t,s)](f(s) - f(0)) \tag{2.15}$$

holds for all $t, \tau \in [0,1]$ such that $\tau \geq t$ and all $f \in \mathbb{G}_L^n$. Since obviously (2.4) implies that

$$\mathrm{v}(W) + \mathrm{var}_0^1 W(0,.) < \infty, \tag{2.16}$$

we may use Lemma 1.3 to show that $\psi \in \mathbb{G}_L^n$ for any $f \in \mathbb{G}_L^n$. The boundedness of the operator Ψ follows from the inequality

$$|\psi(t)| \leq 2(\mathrm{var}_0^1 W(t,.))\|f\| \leq 2(\mathrm{v}(W) + \mathrm{var}_0^1 W(0,.))\|f\|$$

(cf. [SchTvVe, Lemma I.6.6]). □

2.5. Proposition. *For any $f \in \mathbb{G}_L^n$ and any $c \in \mathbb{R}^n$ the equation (0.1) possesses on $[0,1]$ a unique solution $x \in \mathbb{G}_L^n$ such that $x(0) = c$. This solution belongs to $\mathbb{G}_L^n$ and is given by*

$$x = \Phi c + \Psi f, \tag{2.17}$$

where Ψ is the linear bounded operator on $\mathbb{G}_L^n$ given by (2.12) and (2.13) and Φ is the linear bounded mapping of $\mathbb{R}^n$ into $\mathbb{G}_L^n$ given by

$$\Phi :\ c \in \mathbb{R}^n \mapsto U(t,0)\, c. \tag{2.18}$$

Proof. Let $f \in \mathbb{G}_L^n$ and $c \in \mathbb{R}^n$ be given. Then by Lemma 2.4 the function x given by (2.17) is defined on $[0,1]$ and belongs to $\mathbb{G}_L^n$. Hence the integral

$$\int_0^t \mathrm{d}[A(s)]\, x(s)$$

is defined for any $t \in [0,1]$. Inserting (2.12) into this integral and taking into account (2.1) and (2.14)-(2.16) we obtain by Theorems 2.19 (*substitution*) and 2.20 (*change of the integration order*) of [Tv89]

$$\begin{aligned}
\int_0^t \mathrm{d}[A(s)]\, x(s) &= \int_0^t \mathrm{d}[A(s)]\, U(s,0)\, c + \int_0^t \mathrm{d}[A(s)]\, (f(s) - f(0)) \\
&\quad - \int_0^t \mathrm{d}[A(s)]\, \Big(\int_0^t \mathrm{d}_s[W(\tau,s)]\, (f(s) - f(0)) \Big) \\
&= [U(t,0) - I]\, c + \int_0^t \mathrm{d}[A(s)]\, (f(s) - f(0)) \\
&\quad - \int_0^t \mathrm{d} \left[\int_0^t \mathrm{d}[A(\tau)]\, W(\tau,s) \right] (f(s) - f(0)) \\
&= [U(t,0) - I]\, c - \int_0^t \mathrm{d}_s[U(t,s)]\, (f(s) - f(0)) \\
&= x(t) - x(0) - f(t) + f(0)
\end{aligned}$$

for any $t \in [0,1]$. Hence x is a solution of (0.1) on $[0,1]$. Obviously, $x(0) = c$. The uniqueness of this solution follows from the uniqueness of the zero solution to the equation

$$u(t) = \int_0^t \mathrm{d}[A(s)] u(s)$$

on $[0,1]$ (cf. [SchTvVe, Theorem III.1.4]). The boundedness of the operator Φ is evident and the boundedness of Ψ has been proved in Lemma 2.4. $\square$

Now, by a standard technique due to D.Wexler (cf. [We]) we may prove the normal solvability of the operator $\mathscr{A}$ given by (2.3).

2.6. Proposition. *The operator $\mathscr{A}$ has a closed range in $\mathbb{G}_L^n \times \mathbb{R}^m$.*

Proof. By (2.17) a couple $(f,r) \in \mathbb{G}_L^n \times \mathbb{R}^m$ belongs to the range $\mathscr{R}(\mathscr{A})$ of the operator $\mathscr{A}$ if and only if there exists a $c \in \mathbb{R}^n$ such that

$$(\mathscr{K}\Phi)c = r - (\mathscr{K}\Psi)f,$$

i.e. $\mathscr{R}(\mathscr{A}) = \Theta_{-1}(\mathscr{R}(\mathscr{K}\Phi))$, where

$$\Theta : (f, r) \in \mathbb{G}_L^n \times \mathbb{R}^m \mapsto r - (\mathscr{K}\Psi) f \in \mathbb{R}^m$$

is obviously a continuous operator. $\mathscr{R}(\mathscr{K}\Phi)$ being finite dimensional, it is closed and consequently $\mathscr{R}(\mathscr{A})$ is closed as well. □

2.7. The adjoint operator to $\mathscr{A}$. By [Tv89, Theorem 3.8] the dual space to $\mathbb{G}^n$ may be represented by the space $\mathbb{BV}^n \times \mathbb{R}^n$, while for $(y, \delta) \in \mathbb{BV}^n \times \mathbb{R}^n$ the corresponding linear bounded functional on $\mathbb{G}_L^n$ is given by

$$x \in \mathbb{G}_L^n \mapsto \langle x, (y, \delta)\rangle := \delta^* x(0) + \int_0^1 y^*(s)\,\mathrm{d}[x(s)] \in \mathbb{R}. \tag{2.19}$$

The adjoint operator $\mathscr{A}^*$ to $\mathscr{A}$ may be thus represented by the operator

$$\mathscr{A}^* : \mathbb{BV}^n \times \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{BV}^n \times \mathbb{R}^m$$

defined by the relation

$$\langle \mathscr{A}\, x, (y, \gamma, \delta)\rangle : = \langle \mathscr{L}\, x, (y, \gamma)\rangle + \delta^*(\mathscr{K}\, x) = \langle x, \mathscr{A}^*(y, \gamma, \delta)\rangle \quad \text{for any } \ x \in \mathbb{G}_L^n, y \in \mathbb{BV}^n, \gamma \in \mathbb{R}^n \text{ and } \ \delta \in \mathbb{R}^m. \tag{2.20}$$

The operator $\mathscr{A}^* : \mathbb{BV}^n \times \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{BV}^n \times \mathbb{R}^m$ fulfilling (2.20) will be called *the adjoint operator to* $\mathscr{A}$.

Let $x \in \mathbb{G}_L^n$, $y \in \mathbb{BV}^n$, $\gamma \in \mathbb{R}^n$ and $\delta \in \mathbb{R}^m$ be given. Inserting (2.1) and (2.2) into (2.20) and making use of the Substitution Theorem (cf. [Tv89, Theorem 2.19]) we obtain

$$\begin{aligned} &\langle \mathscr{A} x,(y, \gamma, \delta)\rangle \\ &\quad = \int_0^1 y^*(s)\mathrm{d}\left[x(t) - \int_0^t \mathrm{d}[A(s)]\, x(s)\right] \\ &\qquad + \delta^*\big(M\, x(0) + \int_0^1 K(t)\,\mathrm{d}[x(t)]\big) \\ &\quad = \int_0^1 \big(y^*(t) + \delta^* K(t)\big)\,\mathrm{d}[x(t)] \\ &\qquad + \delta^* M x(0) + \int_0^1 \mathrm{d}\left[\int_t^1 y^*(s)\,\mathrm{d}[A(s)]\right] x(t). \end{aligned} \tag{2.21}$$

Furthermore, integrating by parts (cf. [Tv89, Theorem 2.15]) we get

$$
\begin{aligned}
(2.22)\qquad & \int_0^1 \mathrm{d}\left[\int_t^1 y^*(s)\,\mathrm{d}[A(s)]\right] x(t) \\
&\quad = -\Big(\int_0^1 y^*(s)\,\mathrm{d}[A(s)]\Big)x(0) - \int_0^1 \Big(\int_t^1 y^*(s)\,\mathrm{d}[A(s)]\Big)\,\mathrm{d}[x(s)] \\
&\qquad + \sum_{0<t\le 1} \Delta^- w^*(t)\Delta^- x(t) - \sum_{0\le t<1} \Delta^+ w^*(t)\Delta^+ x(t),
\end{aligned}
$$

where

$$
w^*(t) = \int_t^1 y^*(s)\,\mathrm{d}[A(s)] \quad \text{for} \quad t \in [0,1].
$$

As

$$
\begin{aligned}
&\Delta^+ w^*(0) = -y^*(0)\Delta^+ A(0) = 0,\\
&\Delta^+ w^*(t) = -y^*(t)\Delta^+ A(t) \quad \text{for} \quad t \in (0,1)
\end{aligned}
$$

and

$$
\Delta^- w^*(t) = -y^*(t)\Delta^- A(t) = 0 \quad \text{for} \quad t \in (0,1],
$$

the relation (2.22) reduces to

$$
\begin{aligned}
&\int_0^1 \mathrm{d}\left[\int_t^1 y^*(s)\,\mathrm{d}[A(s)]\right] x(t) \\
&\quad = -\Big(\int_0^1 y^*(s)\,\mathrm{d}[A(s)]\Big)x(0) - \int_0^1 \Big(\int_t^1 y^*(s)\,\mathrm{d}[A(s)]\Big)\,\mathrm{d}[x(t)] \\
&\qquad + \sum_{0<t<1} y^*(t)\Delta^+ A(t)\Delta^+ x(t).
\end{aligned}
$$

Let us put $B(t) = \Delta^+ A(t)$ for $t \in [0,1)$ and $B(1) = 0$. Then $B(t+) = B(t-) = 0$ for $t \in (0,1)$, $B(0) = B(0+) = B(1-) = B(1) = 1$ and $B(t) \neq 0$ if and only if $\Delta^+ A(t) \neq 0$. Hence by [SchTvVe, Lemma I.4.23] or [Tv89, Corollary 2.14] we have

$$
\int_t^1 y^*(s)\,\mathrm{d}[B(s)] = \begin{cases} -y^*(t)\Delta^+ A(t) & \text{if } t < 1,\\ 0 & \text{if } t = 1 \end{cases}
$$

wherefrom by [Tv89, Proposition 2.12] the relation

$$
\int_0^1 \Big(\int_t^1 y^*(s)\,\mathrm{d}[B(s)]\Big)\,\mathrm{d}[x(t)] = -\sum_{0<t<1} y^*(t)\Delta^+ A(t)\Delta^+ x(t)
$$

follows. Consequently,

$$\int_0^1 \mathrm{d}\left[\int_t^1 y^*(s)\,\mathrm{d}[A(s)]\right] x(t) \tag{2.23}$$
$$= -\Big(\int_0^1 y^*(s)\,\mathrm{d}[A(s)]\Big)x(0) - \int_0^1 \Big(\int_t^1 y^*(s)\,\mathrm{d}[A(s+)]\Big)\,\mathrm{d}[x(t)]$$

where the convention $A(1+) = A(1)$ is used. Finally, inserting (2.23) into (2.21) we obtain

$$\langle \mathscr{A}x, (y,\gamma,\delta)\rangle = \int_0^1 \big(y^*(t) + \delta^*\,K(t) - \int_t^1 y^*(s)\,\mathrm{d}[A(s+)]\big)\,\mathrm{d}[x(t)]$$
$$+ \big(\delta^*\,M - \int_0^1 y^*(s)\,\mathrm{d}[A(s)]\big)x(0).$$

This proves the following theorem.

2.8. Theorem. *The operator*

$$\mathscr{A}^* : (y^*,\gamma^*,\delta^*) \in \mathbb{BV}^n \times \mathbb{R}^n \times \mathbb{R}^m \mapsto \tag{2.24}$$
$$\Big(y^*(t) + \delta^*\,K(t) - \int_t^1 y^*(t)\,\mathrm{d}[A(s+)],$$
$$\delta^* M - \int_0^1 y^*(s)\,\mathrm{d}[A(s)]\Big) \qquad \in \mathbb{BV}^n \times \mathbb{R}^n$$

(where $A(1+) = A(1)$*) is adjoint to* $\mathscr{A}$.

2.9. Corollary. *Let* $y \in \mathbb{BV}^n$, $\gamma \in \mathbb{R}^n$ *and* $\delta \in \mathbb{R}^m$. *Then* $(y,\gamma,\delta) \in \mathscr{N}(\mathscr{A}^*)$ *if and only if*

$$y^*(t) = y^*(1) + \int_t^1 y^*(s)\,\mathrm{d}[A(s+)] - \delta^*(K(t) - K(1)) \quad \textit{for} \quad t \in [0,1], \tag{2.25}$$

$$y^*(0) + \delta^*(K(0) - M) = 0, \quad y^*(1) + \delta^*\,K(1) = 0. \tag{2.26}$$

Proof. By (2.24) (y,γ,δ) belongs to $\mathscr{N}(\mathscr{A})$ if and only if

$$y^*(t) = \int_t^1 y^*(s)\,\mathrm{d}[A(s+)] - \delta^*\,K(t) \quad \text{on} \quad [0,1] \tag{2.27}$$

$$\delta^*\,M = \int_0^1 y^*(s)\mathrm{d}[A(s+)]. \tag{2.28}$$

For $t = 1$ the relation (2.27) yields $y^*(1) - \delta^* K(1) = 0$. Thus, (2.27) may be rewritten as (2.25). Furthermore, for $t = 0$ we get from (2.27)

$$y^*(0) = \int_0^1 y^*(s)\,\mathrm{d}[A(s+)] - \delta^* K(0). \tag{2.29}$$

Since

$$\int_0^1 y^*(s)\,\mathrm{d}[A(s+) - A(s)] = 0 \quad \text{for all} \quad y \in \mathbb{BV}^n,$$

the relation (2.29) reduces by (2.28) to

$$y^*(0) = \delta^*(M - K(0)).$$

This completes the proof. $\square$

2.10. Definition. The problem of determining a function $y : [0,1] \mapsto \mathbb{R}^n$ of bounded variation on $[0,1]$ and $\delta \in \mathbb{R}^m$ such that (2.25) and (2.26) hold is called the *adjoint problem* to the problem (0.1), (0.2).

By (2.19), Proposition 2.6 and Theorem 2.8 the linear operator equation

$$\mathscr{A}\, x = \begin{pmatrix} h \\ r \end{pmatrix},$$

where $h \in \mathbb{G}_L^n$ is given by $h(t) = f(t) - f(0)$ on $[0,1]$, fulfils the assumptions of the fundamental theorem on the *Fredholm alternative* for linear operator equations (cf. e.g. [Rud, Theorem 4.12]). Hence we have

2.11. Corollary. *The problem* (0.1), (0.2) *possesses a solution if and only if*

$$\int_0^1 y^*(t)\,\mathrm{d}[f(t)] + \delta^*\, r = 0 \tag{2.30}$$

holds for any solution (y, δ) *of the adjoint problem* (2.25), (2.26).

2.12. The adjoint problem. For any $\delta \in \mathbb{R}^m$ fixed, the equation (2.25) is a generalized linear differential equation which was treated in detail in Section III.4 in [SchTvVe]. Let us recall here some basic facts. For given $\delta \in \mathbb{R}^m$ and $\eta \in \mathbb{R}^n$, the equation (2.25) possesses a unique solution y on $[0,1]$ such that $y(1) = \eta$. This solution is given on $[0,1]$ by

(2.31)

$$y^*(t) = \eta^* V(1,t) - \delta^*(K(t) - K(1)) - \delta^* \int_t^1 (K(s) - K(1))\,\mathrm{d}_s[V(s,t)],$$

where V is an $n\times n$-matrix valued function uniquely determined on $[0,1]\times[0,1]$ by the relation

$$V(t,s) = \mathrm{I} + \int_s^t V(t,\tau)\,\mathrm{d}[A(\tau+)], \quad t,s\in[0,1].$$

The relationship of the matrix valued functions U and V is given by Theorem III.4.1 of [SchTvVe]. Under our Assumptions 2.1 we have according to this theorem

$$U(t,s) = V(t,s) + V(t,s)\Delta^+ A(s) + \Delta^+ A(t)U(t,s) \quad \text{for} \quad t,s\in[0,1]. \tag{2.32}$$

It is easy to verify that a couple $(y,\delta)\in \mathbb{BV}^n\times\mathbb{R}^m$ is a solution to the adjoint problem (2.25), (2.26) if and only if y is given by (2.31), where $\eta^* = -\delta^* K(1)$ and δ satisfies the algebraic equation

$$\delta^*\big(M + \int_0^1 K(s)\mathrm{d}_s[V(s,0)]\big) = 0. \tag{2.33}$$

Let us put $W(t) = V(t,0) - U(t,0)$. Then by (2.32) $W(t) = \Delta^+ A(t)U(t,0)$ and consequently

$$W(0) = W(0+) = W(t+) = W(t-) = W(1-) = W(1) = 0$$

holds for any $t\in(0,1)$. This implies that

$$\int_0^1 K(s)\mathrm{d}_s[V(s,0)] = \int_0^1 K(s)\mathrm{d}_s[U(s,0)]$$

holds, i.e. the equation (2.33) may be rewritten as

$$\delta^*\big(M + \int_0^1 K(s)\mathrm{d}_s[U(s,0)]\big) = 0. \tag{2.34}$$

Inserting $\eta^* = -\delta^* K(1)$ and

$$\int_t^1 K(1)\mathrm{d}_s[V(s,t)] = K(1)\big(V(1,t) - I\big)$$

into (2.31) we may now easily complete the proof of the following characterization of the adjoint problem to (0.1),(0.2).

2.13. Proposition. *A couple* $(y,\delta)\in\mathbb{BV}^n\times\mathbb{R}^m$ *is a solution to the problem* (2.25), (2.26) *(i.e.* $(y,\delta)\in\mathscr{N}(\mathscr{A}^*)$*) if and only if*

$$y^*(t) = -\delta^*\big(K(t) + \int_t^1 K(s)\mathrm{d}_s[V(s,t)]\big) \quad \textit{for} \quad t\in[0,1] \tag{2.35}$$

and δ verifies the equation (2.34). *Moreover, for the dimension* $\dim \mathscr{N}(\mathscr{A}^*)$ *of the null space* $\mathscr{N}(\mathscr{A}^*)$ *of the operator* $\mathscr{A}$ *the relation*

$$\dim \mathscr{N}(\mathscr{A}^*) = m - \operatorname{rank}\big(M + \int_0^1 K(s)\,\mathrm{d}_s[U(s,0)]\big) \tag{2.36}$$

is true.

Since, on the other hand, $x \in \mathbb{G}_L^n$ is a solution of the homogeneous boundary value problem (2.5),

$$Mx(0) + \int_0^1 K(s)\,\mathrm{d}[x(s)] = 0$$

(i.e. $x \in \mathscr{N}(\mathscr{A})$) if and only if $x(t) = U(t,0)c$ and

$$\big(\int_0^1 K(s)\,\mathrm{d}_s[U(s,0)]\big)c = 0,$$

the following assertion follows immediately from (2.36).

2.14. Proposition. $\dim \mathscr{N}(\mathscr{A}) - \dim \mathscr{N}(\mathscr{A}^*) = n - m.$

3 . Controllability type problem

In addition to Assumptions 2.1 let us assume

3.1. Assumptions. $\mathbb{U}$ is a linear space and $\mathscr{B} \in \mathscr{L}(\mathbb{U}, \mathbb{G}_L^n)$.

We will consider the problem of determining $x \in \mathbb{G}_L^n$ and $u \in \mathbb{U}$ such that

$$x(t) - x(0) - \int_0^t \mathrm{d}[A(s)]\,x(s) + (\mathscr{B}\,u)(t) - (\mathscr{B}\,u)(0) = f(t) - f(0) \tag{3.1}$$

and

$$M\,x(0) + \int_0^1 K(s)\,\mathrm{d}[x(s)] = r \tag{3.2}$$

hold.

3.2. Remark. If $m = n$,

$$M = \begin{pmatrix} \mathrm{I} \\ \mathrm{I} \end{pmatrix}, K(t) = \begin{pmatrix} 0 \\ \mathrm{I} \end{pmatrix} \quad \text{and} \quad r = \begin{pmatrix} x^0 \\ x^1 \end{pmatrix},$$

then the condition (3.2) reduces to the couple of conditions

$$x(0) = x^0, \quad x(1) = x^1. \tag{3.3}$$

Furthermore, if $\mathbb{U} = \mathbb{L}_2^n$ (the space of n-vector valued functions square integrable on $[0,1]$), P and q are Lebesgue integrable on $[0,1]$, Q is square integrable on $[0,1]$,

$$A(t) = \int_0^t P(s)\,\mathrm{d}s, \quad f(t) = \int_0^t q(s)\,\mathrm{d}s \quad \text{on} \quad [0,1]$$

and

$$\mathscr{B} : u \in \mathbb{L}_2^n \mapsto \int_0^t Q(s)u(s)\,\mathrm{d}s,$$

then the equation (3.1 reduces to the ordinary differential equation

$$x' = P(t)x + Q(t)u + q(t) \tag{3.4}$$

on $[0,1]$. Thus, the given problem (3.1), (3.2) is a generalization of the controllability problem for linear ordinary differential equations. The problem (3.1), (3.2) could be also considered as a (possibly infinite dimensional) perturbation of the boundary value problem (0.1), (0.2).

To obtain necessary and sufficient conditions for the solvability of the problem (3.1), (3.2) in the form of the Fredholm alternative the following abstract scheme will be applied.

3.3. Abstract controllability type problem. Let $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Y}^+$ and $\mathbb{U}$ be linear spaces and let

$$h \in \mathbb{Y}, y \in \mathbb{Y}^+ \mapsto \langle h, y\rangle_{\mathbb{Y}} \in \mathbb{R}$$

be a bilinear form on $\mathbb{Y} \times \mathbb{Y}^+$. For $M \subset \mathbb{Y}$ and $N \subset \mathbb{Y}^+$, let us denote

$$^{\perp}M = \{y \in \mathbb{Y}^+ : \langle m, y\rangle_{\mathbb{Y}} = 0 \quad \text{for all} \quad m \in M\}$$

and

$$N^{\perp} = \{h \in \mathbb{Y} : \langle h, y\rangle_{\mathbb{Y}} = 0 \quad \text{for all} \quad y \in N\}.$$

Let $\mathscr{A} \in \mathscr{L}(\mathbb{X},\mathbb{Y})$, $\mathscr{Q} \in \mathscr{L}(\mathbb{U},\mathbb{Y})$ and $h \in \mathbb{Y}$ be given and let us consider the operator equation for $(x,u) \in \mathbb{X} \times \mathbb{U}$

$$\mathscr{A}\,x + \mathscr{Q}\,u = h. \tag{3.5}$$

Let us denote

$$\mathscr{N}^+_{\mathscr{A}} = {}^{\perp}\mathscr{R}(\mathscr{A}) \quad \text{and} \quad \mathscr{N}^+_{\mathscr{Q}} = {}^{\perp}\mathscr{R}(\mathscr{Q}). \tag{3.6}$$

(obviously $\mathscr{N}^+_{\mathscr{A}}$ and $\mathscr{N}^+_{\mathcal{Q}}$ are linear subspaces of $\mathbb{Y}^+$.)

Let us assume that

$$({}^{\perp}\mathscr{R}(\mathscr{A}))^{\perp} = \mathscr{R}(\mathscr{A}) \quad \text{and} \quad \dim \mathscr{N}^+_{\mathscr{A}} < \infty. \tag{3.7}$$

In particular, we have (cf. (3.6)

$$\mathscr{R}(\mathscr{A}) = (\mathscr{N}^+_{\mathscr{A}})^{\perp}. \tag{3.8}$$

Furthermore, let $k = \dim\, \mathscr{N}^+_{\mathscr{A}}$ and let $\{y^1, y^2, \dots, y^k\}$ be a basis of $\mathscr{N}^+_{\mathscr{A}}$. In virtue of (3.8), the equation (3.5) possesses a solution in $\mathbb{X} \times \mathbb{U}$ if and only if there exists a solution $u \in \mathbb{U}$ to the equation

$$\mathscr{C} u = b, \tag{3.9}$$

where $\mathscr{C} \in \mathscr{L}(\mathbb{U}, \mathbb{R}^k)$ and $b \in \mathbb{R}^k$ are given by

$$\mathscr{C} : u \in \mathbb{U} \mapsto (\langle \mathscr{Q} u, y^j \rangle_{\mathbb{Y}})_{j=1,2,\dots,k} \in \mathbb{R}^k$$

and

$$b = (\langle h, y \rangle_Y)_{j=1,2,\dots,k} \in \mathbb{R}^k.$$

Since $\dim \mathscr{R}(\mathscr{C}) \le k < \infty$, it follows that $({}^{\perp}\mathscr{R}(\mathscr{C}))^{\perp} = \mathscr{R}(\mathscr{C})$ (cf. [Ru]) or, in other words, the equation (3.9) possesses a solution in $\mathbb{U}$ if and only if

$$v^* b = 0 \text{ for all } \ v \in \mathbb{R}^k \text{ such that } \ v^*(Cu) = 0 \text{ for all } \ u \in \mathbb{U}. \tag{3.10}$$

It is easy to verify that the condition (3.10) is equivalent to the condition

$$\langle h, y \rangle_{\mathbb{Y}} = 0 \quad \text{for all } \ y \in \mathscr{N}^+_{\mathscr{A}} \cap \mathscr{N}^+_{\mathscr{Q}}. \tag{3.11}$$

This completes the proof of the following proposition.

3.4. Proposition. *Under the assumption* (3.7), *the equation* (3.5) *possesses a solution in* $\mathbb{X} \times \mathbb{U}$ *if and only if* (3.11) *holds.*

Let us notice that up to now no assumptions on topologies in $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Y}^+$ and $\mathbb{U}$ and on the boundedness of the operators $\mathscr{A}$, $\mathscr{B}$ have been needed. Of course, the assumptions of the above proposition are fulfilled if $\mathbb{X}$ and $\mathbb{Y}$ are Banach spaces, $\mathbb{Y}^+$ is the dual space of $\mathbb{Y}$, $(\langle ., y \rangle_{\mathbb{Y}}$ for $y \in \mathbb{Y}^+$ are linear bounded functionals on $\mathbb{Y}$), $\mathscr{R}(\mathscr{A})$ is closed in $\mathbb{Y}$ and the null space $\mathscr{N}(\mathscr{A}^*)$ of the adjoint operator $\mathscr{A}^*$ to $\mathscr{A}$ has a finite dimension. (In this case $\mathscr{N}^+_{\mathscr{A}} = \mathscr{N}(\mathscr{A}^*)$.)

The given problem (3.1), (3.2) reduces to the operator equation (3.5) if we put $\mathbb{X} = \mathbb{G}^n_L$, $\mathbb{Y} = \mathbb{G}^n_L \times \mathbb{R}^n$, $\mathbb{Y}^+ = \mathbb{BV}^n \times \mathbb{R}^n \times \mathbb{R}^m$,

$$\langle (f,r),(y,\gamma,\delta)\rangle_{\mathbb{Y}} = \delta^*\, r + \gamma^* f(0) + \int_0^1 y^*(s)\, \mathrm{d}[f(s)]$$

for $f \in \mathbb{G}^n_L$, $r \in \mathbb{R}^m$, $y \in \mathbb{BV}^n$, $\gamma \in \mathbb{R}^n$ and $\delta \in \mathbb{R}^m$,

$$\mathscr{Q} : u \in \mathbb{U} \mapsto \begin{pmatrix} (\mathscr{B}u)(t) - (\mathscr{B}u)(0) \\ 0 \end{pmatrix} \in \mathbb{G}^n_L \times \mathbb{R}^m,$$

$$h(t) = \begin{pmatrix} f(t) - f(0) \\ 0 \end{pmatrix} \in \mathbb{G}^n_L \times \mathbb{R}^m,$$

and if we make use of (2.3) again. By Propositions 2.6 and 2.13 the assumptions of the above proposition are fulfilled and hence the following assertion is true (cf. Corollary 2.9).

3.5. Theorem. *The problem* (3.1), (3.2) *possesses a solution in* $\mathbb{G}^n_L \times \mathbb{U}$ *if and only if*

$$\int_0^1 y^*(t)\, \mathrm{d}[f(t)] + \delta^*\, r = 0 \tag{3.12}$$

holds for any solution (y,δ) *of the system* (2.25), (2.26) *such that*

$$\int_0^1 y^*(t)\, \mathrm{d}[(\mathscr{B}u)(t)] = 0 \quad \textit{for all} \quad u \in \mathbb{U}. \tag{3.13}$$

3.6. Corollary. *The problem* (3.1), (3.2) *possesses a solution in* $\mathbb{G}^n_L \times \mathbb{U}$ *for any* $f \in \mathbb{G}^n_L$ *and any* $r \in \mathbb{R}^m$ *if and only if the only solution* (y,δ) *of* (2.25), (2.26) *which fulfils* (3.13) *is the zero solution (i.e.* $y(t) \equiv 0$ *on* $[0,1]$, $\delta = 0$*).*

3.7. Remark. The system (3.1), (3.2) is sometimes (cf.[Hala], [Ma], [La]) called *completely* $(\mathscr{B}, M, K)$*-controllable*) if it possesses a solution in $\mathbb{G}^n_L \times \mathbb{U}$ for any $f \in \mathbb{G}^n_L$ and any $r \in \mathbb{R}^m$. The problem (2.25), (2.26), (3.13) adjoint to (3.1), (3.2) in the sense of Theorem 3.5 is a generalization of classical observability problems for linear ordinary differential equations and Corollary 3.6 is a generalization of the well known theorem (cf. e.g. [Russ], [Ro]) on the duality between controllability and observability problems for linear ordinary differential equations. Often, controllability is considered for homogeneous differential equations. In an analogous situation for the given problem (3.1), (3.2) (i.e. $f(t) \equiv f(0)$ on $[0,1]$) we obtain that the system

$$(3.14) \qquad x(t) - x(0) - \int_0^t \mathrm{d}[A(s)]\, x(s) + (\mathscr{B}u)(t) - (\mathscr{B}u)(0) = 0 \quad \text{on} \quad [0,1],$$

(3.2) possesses a solution in $\mathbb{G}_L^n \times \mathbb{U}$ for any $r \in \mathbb{R}^m$ if and only if the only couple $(y,\delta) \in \mathbb{BV}^n \times \mathbb{R}^m$ fulfilling (2.25), (2.26) and (3.13) is the zero one. In fact, it follows immediately from (3.12) that (3.14), (3.2) has a solution in $\mathbb{G}_L^n \times \mathbb{U}$ for any $r \in \mathbb{R}^m$ if and only if $\delta = 0$ holds for any couple $(y,\delta) \in \mathbb{BV}^n \times \mathbb{R}^m$ fulfilling (2.25), (2.26) and (3.13). By 2.12 this implies that $y(t) \equiv 0$ on $[0,1]$ for any such couple, of course.

3.8. Corollary. *If* $\mathbb{U} = \mathbb{G}^h{}_L$ *and*

$$\mathscr{B} : u \in \mathbb{G}^h{}_L \mapsto \int_0^t \mathrm{d}[B(s)]\, u(s), \quad t \in [0,1],$$

where $B(s)$ *is an* $n \times h$*-matrix valued function of bounded variation on* $[0,1]$, *right-continuous at* 0 *and left-continuous on* $(0,1]$, *then the problem* (3.1), (3.2) *has a solution if and only if* (3.12) *holds for any solution* (y,δ) *of the system* (2.25), (2.26) *such that*

$$\int_t^1 y^*(s)\, \mathrm{d}[B(s+)] = 0 \quad \textit{for all} \quad t \in [0,1].$$

Proof. follows from Theorem 3.5 and from the relation

$$\int_0^1 y^*(t)\,\mathrm{d}\left[\int_0^t \mathrm{d}[B(s)]\, u(s)\right] =$$
$$= \Big(\int_0^1 y^*(t)\, \mathrm{d}[B(t)]\Big) u(0) + \int_0^1 \Big(\int_t^1 y^*(s)\, \mathrm{d}[B(s+)]\Big)\, \mathrm{d}[u(t)]$$
$$\text{for all} \quad u \in \mathbb{G}^h{}_L \quad \text{and} \quad y \in \mathbb{BV}^n,$$

which can be verified analogously as the corresponding relation for the $n \times n$-matrix valued function $A(t)$ in the proof of Theorem 2.8. □

3.9. Corollary. *If* $\mathbb{U} = \mathbb{G}^h{}_L$ *and*

$$\mathscr{B} : u \in \mathbb{G}^h{}_L \mapsto \int_0^t B(s)\, \mathrm{d}[u(s)],$$

where $B(s)$ *is an* $n \times h$*-matrix valued function of bounded variation on* $[0,1]$, *then the problem* (3.1), (3.2) *has a solution if and only if* (3.12) *holds for any couple* $(y,\delta) \in \mathbb{BV}^n \times \mathbb{R}^m$ *fulfilling* (2.25), (2.26) *and* $y^*(t)B(t) = 0$ *on* $[0,1]$.

Proof. Since by the Substitution Theorem (cf. [Tv89, Theorem 2.19]) the relation

$$\int_0^1 y^*(t)\,\mathrm{d}[(Bu)(t)] = \int_0^1 y^*(t)\,\mathrm{d}\left[\int_0^t B(s)\,\mathrm{d}[u(s)]\right] = \int_0^1 y^*(t)B(t)\,\mathrm{d}[u(t)]$$

holds for all $y \in \mathbb{BV}^n$ and $u \in G_L^h$, the proof follows immediately from Theorem 3.5. $\square$

3.10. Definition. Let $T = \{t_1, t_2, \dots, t_\nu\}$ be such that

$$1 > t_1 > t_2 > \dots > t_\nu > 0. \tag{3.15}$$

Then by $\mathbb{U}_T$ we denote the subset of $\mathbb{G}_L^n$ consisting of all functions $u \in \mathbb{G}_L^n$ which are constant on each of the intervals

$$[0, t_\nu],\ (t_1, 1],\ (t_{k+1}, t_k], \quad k = 1, 2, \dots, \nu - 1.$$

3.11. Proposition. *Let $T = \{t_1, t_2, \dots, t_\nu\}$ fulfil* (3.15) *and let $\mathbb{U}_T$ be defined by Definition 3.10. Then $\mathbb{U}_T$ is a linear space. Furthermore, if $y \in \mathbb{BV}^n$, then the assertion*

$$\int_0^1 y^*(t)\,\mathrm{d}[u(t)] = 0 \quad \text{for all} \quad u \in \mathbb{U}_T \tag{3.16}$$

is true if and only if

$$y^*(\tau) = 0 \quad \text{for any} \quad \tau \in \mathbb{U}_T. \tag{3.17}$$

Proof. The first part of the proposition is evident. Let us suppose that (3.16) holds. Then for a given $\tau \in T$, the function $\chi_{(\tau,1]}$ belongs to $\mathbb{U}_T$ and (cf. e.g. [Tv89, Proposition 2.3])

$$\int_0^1 y^*(t)\,\mathrm{d}[\chi_{(\tau,1]}(t)] = y^*(\tau) = 0.$$

Analogously, $\chi_{[1]} \in \mathbb{U}_T$, while

$$\int_0^1 y^*(t)\,\mathrm{d}[\chi_{[1]}(t)] = y^*(1) = 0,$$

i.e. (3.17) is true.

On the other hand, since obviously $\mathbb{U}_T \subset \mathbb{BV}^n$, it follows from [SchTvVe, Lemma I.4.23] that (3.16) holds for any $y \in \mathbb{BV}^n$ satisfying (3.17) and any $u \in \mathbb{U}_T$. $\square$

3.12. Corollary. *Let* $T = \{t_k\}_{k=1}^{\nu}$ *be the set of points in* $(0,1)$ *such that* (3.15) *holds, and let* $\mathbb{U}_T$ *be defined by Definition* 3.10. *Let us put*

$$\mathscr{B} : u \in \mathbb{U}_T \mapsto u \in \mathbb{G}_L^n.$$

Then the problem (3.1), (3.2) *has a solution if and only if* (3.12) *holds for any couple* $(y, \delta) \in \mathbb{BV}^n \times \mathbb{R}^m$ *fulfilling* (2.25), (2.26) *and such that* $y(\tau) = 0$ *for any* $\tau \in T$.

Proof. follows immediately from Theorem 3.5 and Proposition 3.11. □

3.13. Remark. The case considered in Corollary 3.12 is a generalization of interface boundary value problems for ordinary differential equations which are usually defined (see e.g. [Br], [Co], [Sch80] or [Ze]) as follows:

Let $1 > t_1 > t_2 > \cdots > t_\nu > t_0 > 0$ and let $T = \{t_k\}_{k=1}^{\nu}$. Let $P(t)$ and an n-vector valued function $q(t)$ be Lebesgue integrable on $[0,1]$. Let an $m \times n$-matrix valued function $K(t)$ have bounded variation on $[0,1]$, let M_i, N_i $(i = 0, 1, \dots, \nu)$ be $m \times n$-matrices and let $r \in \mathbb{R}^m$. Then an n-vector valued function $x(t)$ is called a solution to the *interface* boundary value problem (3.18), (3.19) if it is regulated on $[0,1]$, left-continuous on $(0,1]$ (i.e. $x \in \mathbb{G}_L^n$) and absolutely continuous on every interval $(t_{k+1}, t_k]$,

$$x'(t) - P(t)x(t) = q(t) \quad \text{a.e. on } [0,1] \tag{3.18}$$

and

$$\mathscr{K}\, x := M_0 x(0) + N_0 x(1) + \sum_{i=1}^{\nu} [M_i x(t_i+) + N_i x(t_i-)] + \int_0^1 K_0(s)\, \mathrm{d}[x(s)] = r. \tag{3.19}$$

Indeed, let us put $\mathbb{U} = \mathbb{U}_T$, where $\mathbb{U}_T$ is defined by Definition 3.10 . Furthermore, let us put

$$M = \sum_{i=0}^{\nu} [M_i + N_i] \tag{3.20}$$

and

$$K(s) = K_0(s) + \sum_{i=1}^{\nu} [M_i\, \chi_{[0,t_i]}(s) + N_i\, \chi_{[0,t_i)}(s)] + N_0 \quad \text{for } s \in [0,1]. \tag{3.21}$$

Then

$$\mathscr{K}\, x = M\, x(0) + \int_0^1 K(s)\, \mathrm{d}[x(s)]$$

holds for any $x \in \mathbb{G}_L^n$ and $x \in \mathbb{G}_L^n$ is a solution of the interface boundary value problem (3.18), (3.19) if and only if there exists $u \in \mathbb{U}_T$ such that

(3.22)
$$x(t) - x(0) - \int_0^t \mathrm{d}[A(s)]\, x(s) - (u(t) - u(0)) = f(t) - f(0) \quad \text{on } [0,1]$$

and

$$M\, x(0) + \int_0^1 K(s)\, \mathrm{d}[x(s)] = r, \tag{3.23}$$

where

$$A(s) = \int_0^t P(\tau)\, \mathrm{d}\tau \quad \text{and} \quad f(s) = \int_0^t q(\tau)\, \mathrm{d}\tau \quad \text{for } s \in [0,1].$$

The problem (3.22), (3.23) obviously verifies the assumptions of this section. Since by (3.20) and (3.21)

$$K(0) = K_0(0) + M - M_0 \quad \text{and} \quad K(1) = K_0(1) + N_0,$$

the adjoint problem to (3.22), (3.23) is given by the system (2.25),

$$y^*(0) + \delta^*\big(K_0(0) - M_0\big) = 0, \quad y^*(1) + \delta^*\big(K_0(1) + N_0\big) \tag{3.24}$$

and

$$y^*(t_i) = 0 \quad \text{for } i = 1, 2, \ldots, \nu.$$

Furthermore, for any $t \in [0,1]$ we have

$$K(t) - K(1) = K_0(t) - K_0(1) + \sum_{i=1}^{\nu} [M_i\, \chi_{[0,t_i]}(t) + N_i\, \chi_{[0,t_i)}(t)]$$

Thus, it is easy to see that a couple $(y, \delta) \in \mathbb{BV}^n \times \mathbb{R}^m$ is a solution to the system adjoint to (3.22), (3.23) if and only if $y^* + \delta^* K_0$ is absolutely continuous on every interval $[\alpha, \beta]$ such that $[\alpha, \beta] \subset [0,1] \setminus T$,

$$\begin{aligned}
-(y^* + \delta^* K_0)'(t) + y^* P(t) &= 0 && \text{a.e. on } [0,1],\\
(y^* + \delta^* K_0)(0) &= \delta^* M_0,\\
(y^* + \delta^* K_0)(1) &= -\delta^* N_0,\\
\Delta^+(y^* + \delta^* K_0)(t_i) &= \delta^* M_i, && i = 1, 2, \ldots, \nu,\\
\Delta^-(y^* + \delta^* K_0)(t_i) &= \delta^* N_i, && i = 1, 2, \ldots, \nu,\\
y^*(t_i) &= 0, && i = 1, 2, \ldots, \nu.
\end{aligned}$$

References

[Au] Aumann G., *Reelle Funktionen*, (Springer-Verlag, Berlin, 1969).

[Ba] Barbanti L., Linear Volterra-Stieltjes integral equations and control, in *Equadiff 82. Proceedings, Würzburg 1982* (Lecture Notes in Mathematics 1017, Springer-Verlag, Berlin, 1983), pp. 67–72.

[Br] Bryan R. N., A nonhomogeneous linear differential system with interface conditions and their adjoints, *Proceedings A. M. S.*, **22** (1969), 270–276.

[Co] Conti R., On ordinary differential equations with interface conditions, *Journ. Diff. Eq.* **4** (1968), 4–11.

[Fi] Fichmann L., Volterra-Stieltjes integral equations and equations of the neutral type (in Portuguese), *Thesis, University of Sao Paulo*, 1984.

[Fra] Fraňková D., Regulated functions, *Math. Bohem.* **116** (1991), 20–59.

[Hala] Halanay A., Optimal control of periodic solutions, *Rev. Roum. Math. Pures et Appl.* **14** (1974), 3–16.

[Hi] Hildebrandt T. H., *Introduction to the Theory of Integration*, (Academic Press, New York-London, 1963).

[Hö75] Hönig Ch.S., *Volterra Stieltjes-Integral Equations*, (North Holland and American Elsevier, Mathematics Studies 16, Amsterdam and New York, 1975).

[Hö80] Hönig Ch.S., Volterra-Stieltjes integral equations, in *Functional Differential Equations and Bifurcation, Proceedings of the Sao Carlos Conference 1979* (Lecture Notes in Mathematics 799, Springer-Verlag, Berlin, 1980), pp. 173–216.

[Hö82] Hönig Ch.S., The adjoint equation of a linear Volterra-Stieltjes integral equation with a linear constraint, in *Differential Equations, Proceedings, 1981* (Lecture Notes in Mathematics 957, Springer-Verlag, Berlin, 1982), pp. 118–125.

[Ka] Kaltenborn H. S., Linear functional operations on functions having discontinuities of the first kind, *Bulletin A. M. S.* **40** (1934), 702–708.

[Ku57] Kurzweil J., Generalized ordinary differential equations and continuous dependence on a parameter, *Czechoslovak Math. J.* **7 (82)** (1957), 418–449.

[Ku80] Kurzweil J., *Nichtabsolute konvergente Integrale*, (BSB B. G. Teubner Verlagsgesselschaft, Leipzig, 1980).

[La] Lando Yu. K., Controllable integro-differential operators (in Russian), *Diff. Uravn.* **9** (1973), 2227–2230.

[Ma] Marchiò C., (M,N,F)-controllabilità completa, *Questioni di controllabilità, Istituto U. Dini, Firenze* **1973/2**, 14–26.

[Ro] Rolewicz S., *Functional Analysis and Control Analysis (Linear Systems)*, (Polish Scientific Publishers (PWN) and D.Reidel, Warszawa and Dordrecht, 1987).

[Ru] Rudin W., *Functional Analysis*, (McGraw-Hill, New York, 1973).

[Russ] Russell D. L., *Mathematics of Finite-Dimensional Control Systems (Theory and Design)* (LNPAM 43)(M. Dekker, New York and Basel, 1979).

[Sa] Saks S., *Theory of the Integral*, (Monografie Matematyczne, Warszawa, 1937),

[Sch73] Schwabik Š., On the relation between Young's and Kurzweil's concept of Stieltjes integral, *Časopis Pěst. Mat.* **98** (1973) 237–251.

[Sch80] Schwabik Š., Differential equations with interface conditions, *Časopis Pěst. Mat.* **105** (1980) 391–408.

[Sch85] Schwabik Š., *Generalized Differential Equations (Fundamental Results)*, (Rozpravy ČSAV, Řada MPV, 95 (6)) (Academia, Praha, 1985).

[SchTv] Schwabik Š., Tvrdý M., Boundary value problems for generalized linear differential equations, *Czechoslovak Math. J.* **29 (104)** (1974) 451–477.

[SchTvVe] Schwabik Š., Tvrdý M., Vejvoda O., *Differential and Integral Equations: Boundary Value Problems and Adjoints*, (Academia and D. Reidel, Praha and Dordrecht, 1979).

[Tv73] Tvrdý M., Boundary value problems for linear generalized differential equations and their adjoints, *Czechoslovak Math. J.* **23 (98)** (1973) 183–217.

[Tv74] Tvrdý M., Boundary value problems for generalized linear integrodifferential equations with left-continuous solutions, *Časopis Pěst. Mat.* **99** (1974) 147–157.

[Tv89] Tvrdý M., Regulated functions and the Perron-Stieltjes integral, *Časopis Pěst. Mat.* **114** (1989) 187–209.

[Wa] Ward A. J., The Perron-Stieltjes integral, *Math. Z.* **41** (1936), 578–604.

[We] Wexler D., On boundary value problems for an ordinary linear differential system, *Ann. Mat. pura ed appl.* **80** (1968) 123–134.

[Ze] Zettl A., Adjoint and self-adjoint boundary value problems with interface conditions, *Journ. Appl. Math.*, **16** (1968), 851–859.