Academy of Sciences of the Czech Republic
Mathematical Institute

Preprint Series
No.153/2003

Periodic Boundary Value Problems for Nonlinear Second Order Differential Equations with Impulses - Part III

Irena Rachůnková* and Milan Tvrdý†

May 7, 2003

Summary. This paper provides existence results for the nonlinear impulsive periodic boundary value problem

$$u'' = f(t, u, u'), \tag{1.1}$$

$$u(t_i+) = \mathrm{J}_i(u(t_i)), \quad u'(t_i+) = \mathrm{M}_i(u'(t_i)), \quad i = 1, 2, \dots, m, \tag{1.2}$$

$$u(0) = u(T), \quad u'(0) = u'(T), \tag{1.3}$$

where $f \in \mathrm{Car}([0,T] \times \mathbb{R}^2)$ and J_i, $\mathrm{M}_i \in \mathbb{C}(\mathbb{R})$. The basic assumption is the existence of lower/upper functions σ_1/σ_2 associated with the problem. Here we generalize and extend the existence results of our previous papers.

Mathematics Subject Classification 2000. 34B37, 34B15, 34C25

Keywords. Second order nonlinear ordinary differential equation with impulses, periodic solutions, lower and upper functions, Leray-Schauder topological degree, a priori estimates.

0. Introduction

This paper deals with the solvability of the nonlinear impulsive boundary value problem (1.1)–(1.3). The investigation of this problem was initiated by Hu and Lakshmiknatham in [3]. For the further development, see e.g. [1], [2], [4], [9] and the papers cited therein. We have already studied this problem in [7] and

*Supported by the grant No. 201/01/1451 of the Grant Agency of the Czech Republic and by the Council of Czech Government J14/98:153100011

†Supported by the grant No. 201/01/1199 of the Grant Agency of the Czech Republic

[8] under the assumption that there are lower/upper functions σ_1/σ_2 associated with the problem. In [7] we improved the already known results for the case that σ_1, σ_2 are well-ordered, i.e. $\sigma_1 \le \sigma_2$ on $[0,T]$. On the other hand, in [8] we have delivered the first existence results valid if σ_1, σ_2 are not well-ordered, i.e.

$$\sigma_1(\tau) > \sigma_2(\tau) \quad \text{for some} \quad \tau \in [0,T]. \tag{0.1}$$

The goal of this paper is to generalize the main existence results of [8], where we restricted our attention to impulsive functions M_i, $i = 1,2,\dots,m$, fulfilling the conditions

$$y\,\mathrm{M}_i(y) \ge 0 \text{ for } y \in \mathbb{R}, \quad i = 1,2,\dots,m. \tag{0.2}$$

Here we prove existence criteria without restriction (0.2).

Throughout the paper we keep the following notation and conventions: For a real valued function u defined a.e. on $[0,T]$, we put

$$\|u\|_\infty = \sup_{t\in[0,T]} |u(t)| \quad \text{and} \quad \|u\|_1 = \int_0^T |u(s)|\,\mathrm{d}s.$$

For a given interval $J \subset \mathbb{R}$, by $\mathbb{C}(J)$ we denote the set of real valued functions which are continuous on J. Furthermore, $\mathbb{C}^1(J)$ is the set of functions having continuous first derivatives on J and $\mathbb{L}(J)$ is the set of functions which are Lebesgue integrable on J.

Let $m \in \mathbb{N}$ and let $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = T$ be a division of the interval $[0,T]$. We denote $\mathrm{D} = \{t_1, t_2, \dots, t_m\}$ and define $\mathbb{C}^1_{\mathrm{D}}[0,T]$ as the set of functions $u : [0,T] \mapsto \mathbb{R}$ of the form

$$u(t) = \begin{cases} u_{[0]}(t) & \text{if } t \in [0,t_1], \\ u_{[1]}(t) & \text{if } t \in (t_1,t_2], \\ \dots & \quad \dots \\ u_{[m]}(t) & \text{if } t \in (t_m,T], \end{cases}$$

where $u_{[i]} \in \mathbb{C}^1[t_i,t_{i+1}]$ for $i = 0,1,\dots,m$. Moreover, $\mathbb{AC}^1_{\mathrm{D}}[0,T]$ stands for the set of functions $u \in \mathbb{C}^1_{\mathrm{D}}[0,T]$ having first derivatives absolutely continuous on each subinterval (t_i,t_{i+1}), $i = 0,1,\dots,m$. For $u \in \mathbb{C}^1_{\mathrm{D}}[0,T]$ and $i = 1,2,\dots,m+1$ we define

$$u'(t_i) = u'(t_i-) = \lim_{t\to t_i-} u'(t), \quad u'(0) = u'(0+) = \lim_{t\to 0+} u'(t) \tag{0.3}$$

and $\|u\|_{\mathrm{D}} = \|u\|_\infty + \|u'\|_\infty$. Note that the set $\mathbb{C}^1_{\mathrm{D}}[0,T]$ becomes a Banach space when equipped with the norm $\|.\|_{\mathrm{D}}$ and with the usual algebraic operations.

We say that $f : [0,T] \times \mathbb{R}^2 \mapsto \mathbb{R}$ satisfies the *Carathéodory conditions* on $[0,T] \times \mathbb{R}^2$ if (i) for each $x \in \mathbb{R}$ and $y \in \mathbb{R}$ the function $f(.,x,y)$ is measurable on $[0,T]$; (ii) for almost every $t \in [0,T]$ the function $f(t,.,.)$ is continuous on $\mathbb{R}^2$; (iii) for each compact set $K \subset \mathbb{R}^2$ there is a function $m_K(t) \in \mathbb{L}[0,T]$ such that $|f(t,x,y)| \le m_K(t)$ holds for a.e. $t \in [0,T]$ and all $(x,y) \in K$. The set of functions satisfying the Carathéodory conditions on $[0,T] \times \mathbb{R}^2$ will be denoted by $\mathrm{Car}([0,T] \times \mathbb{R}^2)$.

Given a Banach space $\mathbb{X}$ and its subset M, let $\mathrm{cl}(M)$ and ∂M denote the closure and the boundary of M, respectively.

Let Ω be an open bounded subset of $\mathbb{X}$. Assume that the operator $\mathrm{F} : \mathrm{cl}(\Omega) \mapsto \mathbb{X}$ is completely continuous and $\mathrm{F}\,u \neq u$ for all $u \in \partial\,\Omega$. Then $\deg(\mathrm{I}-\mathrm{F},\Omega)$ denotes the *Leray-Schauder topological degree* of $\mathrm{I}-\mathrm{F}$ with respect to Ω, where I is the identity operator on $\mathbb{X}$. For the definition and properties of the degree see e.g. [5].

1. Formulation of the problem and main assumptions

Here we study the existence of solutions to the problem

$$u'' = f(t,u,u'), \tag{1.1}$$

$$u(t_i+) = \mathrm{J}_i(u(t_i)), \quad u'(t_i+) = \mathrm{M}_i(u'(t_i)), \quad i = 1,2,\dots,m, \tag{1.2}$$

$$u(0) = u(T), \quad u'(0) = u'(T), \tag{1.3}$$

where $u'(t_i)$ are understood in the sense of (0.3), $f \in \mathrm{Car}([0,T] \times \mathbb{R}^2)$, $\mathrm{J}_i \in \mathbb{C}(\mathbb{R})$ and $\mathrm{M}_i \in \mathbb{C}(\mathbb{R})$.

1.1. Definition. By a *solution of the problem* (1.1)–(1.3) we understand a function $u \in \mathbb{AC}^1_{\mathrm{D}}[0,T]$ which satisfies the impulsive conditions (1.2), the periodic conditions (1.3) and for a.e. $t \in [0,T]$ fulfils the equation (1.1).

1.2. Definition. A function $\sigma_1 \in \mathbb{AC}^1_{\mathrm{D}}[0,T]$ is called a *lower function of the problem* (1.1)–(1.3) if

$$\sigma_1''(t) \ge f(t,\sigma_1(t),\sigma_1'(t)) \quad \text{for a.e.} \quad t \in [0,T], \tag{1.4}$$

$$\sigma_1(t_i+) = \mathrm{J}_i(\sigma_1(t_i)), \quad \sigma_1'(t_i+) \ge \mathrm{M}_i(\sigma_1'(t_i)), \quad i = 1,2,\dots,m, \tag{1.5}$$

$$\sigma_1(0) = \sigma_1(T), \quad \sigma_1'(0) \ge \sigma_1'(T). \tag{1.6}$$

Similarly, a function $\sigma_2 \in \mathbb{AC}^1_{\mathrm{D}}[0,T]$ is an *upper function of the problem* (1.1)–(1.3) if

$$\sigma_2''(t) \le f(t,\sigma_2(t),\sigma_2'(t)) \quad \text{for a.e.} \quad t \in [0,T], \tag{1.7}$$

$$\sigma_2(t_i+) = \mathrm{J}_i(\sigma_2(t_i)), \quad \sigma_2'(t_i+) \le \mathrm{M}_i(\sigma_2'(t_i)), \quad i = 1,2,\dots,m, \tag{1.8}$$

$$\sigma_2(0) = \sigma_2(T), \quad \sigma_2'(0) \le \sigma_2'(T). \tag{1.9}$$

1.3. Assumptions. In the paper we work with the following assumptions:

$$\begin{cases} 0 = t_0 < t_1 < \dots < t_m < t_{m+1} = T < \infty, \ \mathrm{D} = \{t_1, t_2, \dots, t_m\}, \\ f \in \mathrm{Car}([0,T] \times \mathbb{R}^2), \ \mathrm{J}_i \in \mathbb{C}(\mathbb{R}), \ \mathrm{M}_i \in \mathbb{C}(\mathbb{R}), \ i = 1,2,\dots,m; \end{cases} \tag{1.10}$$

(1.11) σ_1 and σ_2 are respectively lower and upper functions of (1.1)–(1.3);

$$\begin{cases} x > \sigma_1(t_i) \implies \mathrm{J}_i(x) > \mathrm{J}_i(\sigma_1(t_i)), \\ x < \sigma_2(t_i) \implies \mathrm{J}_i(x) < \mathrm{J}_i(\sigma_2(t_i)), \quad i = 1,2,\dots,m; \end{cases} \tag{1.12}$$

$$\begin{cases} y \le \sigma_1'(t_i) \implies \mathrm{M}_i(y) \le \mathrm{M}_i(\sigma_1'(t_i)), \\ y \ge \sigma_2'(t_i) \implies \mathrm{M}_i(y) \ge \mathrm{M}_i(\sigma_2'(t_i)), \quad i = 1,2,\dots,m. \end{cases} \tag{1.13}$$

1.4. Operator reformulation of (1.1)–(1.3).
Let $G(t,s)$ be the Green function for $u'' = 0$, $u(0) = u(T) = 0$ i.e.

$$G(t,s) = \begin{cases} \dfrac{t\,(s-T)}{T} & \text{if } \ 0 \le t \le s \le T, \\[2mm] \dfrac{s\,(t-T)}{T} & \text{if } \ 0 \le s < t \le T. \end{cases} \tag{1.14}$$

Furthermore, we define the operator $\mathrm{F} : \mathbb{C}^1_{\mathrm{D}}[0,T] \mapsto \mathbb{C}^1_{\mathrm{D}}[0,T]$ by

$$\begin{aligned} (\mathrm{F}\,x)(t) = x(0) + x'(0) - x'(T) + \int_0^T G(t,s)\, f(s,x(s),x'(s))\,\mathrm{d}s \\ - \sum_{i=1}^m \frac{\partial G}{\partial s}(t,t_i)\,(\mathrm{J}_i(x(t_i)) - x(t_i)) + \sum_{i=1}^m G(t,t_i)\,(\mathrm{M}_i(x'(t_i)) - x'(t_i)). \end{aligned} \tag{1.15}$$

As in [6, Lemma 3.1], where $m = 1$, we can prove (see Proposition 1.6 below) that F is completely continuous and that a function u is a solution of (1.1)–(1.3) if and only if u is a fixed point of F. To this aim we need the following lemma which extends Lemma 2.1 from [6].

1.5. Lemma. *For each* $h \in \mathbb{L}[0,T]$, $c, d_i, e_i \in \mathbb{R}$, $i = 1,2,\dots,m$, *there is a unique function* $x \in \mathbb{AC}^1_{\mathrm{D}}[0,T]$ *fulfilling*

$$\begin{cases} x''(t) = h(t) \ \text{a.e. on} \ [0,T], \\ x(t_i+) - x(t_i) = d_i, \ x'(t_i+) - x'(t_i) = e_i, \ i = 1,2,\dots,m, \end{cases} \tag{1.16}$$

$$x(0) = x(T) = c. \tag{1.17}$$

This function is given by

(1.18)

$$x(t) = c + \int_0^T G(t,s)\,h(s)\,ds - \sum_{i=1}^m \frac{\partial G}{\partial s}(t,t_i)\,d_i + \sum_{i=1}^m G(t,t_i)\,e_i \ \text{for} \ t \in [0,T],$$

where $G(t,s)$ *is defined by* (1.14).

Proof. It is easy to check that $x \in \mathbb{AC}^1_{\mathrm{D}}[0,T]$ fulfils (1.16) together with $x(0) = c$ if and only if there is $\widetilde{c} \in \mathbb{R}$ such that

$$\begin{aligned} x(t) = c + t\,\widetilde{c} + \sum_{i=1}^m \chi_{(t_i,T]}(t)\,d_i + \sum_{i=1}^m \chi_{(t_i,T]}(t)\,(t-t_i)\,e_i \\ + \int_0^t (t-s)\,h(s)\,\mathrm{d}s \ \text{for} \ t \in [0,T], \end{aligned} \tag{1.19}$$

where $\chi_{(t_i,T]}(t) = 1$ if $t \in (t_i,T]$ and $\chi_{(t_i,T]}(t) = 0$ if $t \in \mathbb{R}\setminus(t_i,T]$. Furthermore, $x(T) = c$ if and only if

$$\widetilde{c} = -\sum_{i=1}^m \frac{d_i}{T} - \sum_{i=1}^m \frac{T-t_i}{T}\,e_i - \int_0^T \frac{T-s}{T}\,h(s)\,\mathrm{d}s. \tag{1.20}$$

Inserting (1.20) into (1.19), we get

$$\begin{aligned} x(t) = \sum_{t_i<t} \frac{t_i\,(t-T)}{T}\,e_i + \sum_{t_i\ge t} \frac{t\,(t_i-T)}{T}\,e_i - \sum_{t_i<t} \frac{(t-T)}{T}\,d_i - \sum_{t_i\ge t} \frac{t}{T}\,d_i \\ + \int_0^t \frac{s\,(t-T)}{T}\,h(s)\,\mathrm{d}s + \int_t^T \frac{t\,(s-T)}{T}\,h(s)\,\mathrm{d}s, \quad t \in [0,T]. \end{aligned}$$

Hence, taking into account (1.14), we conclude that the function x given by (1.18) is the unique solution of (1.16), (1.17) in $\mathbb{AC}^1_{\mathrm{D}}[0,T]$. $\square$

1.6. Proposition. *Assume that* (1.10) *holds. Let the operator* $\mathrm{F} : \mathbb{C}^1_{\mathrm{D}}[0,T] \mapsto \mathbb{C}^1_{\mathrm{D}}[0,T]$ *be defined by* (1.14) *and* (1.15). *Then* F *is completely continuous and a function* u *is a solution of* (1.1)–(1.3) *if and only if* $u = \mathrm{F}\,u$.

Proof. Choose an arbitrary $y \in \mathbb{C}^1_{\mathrm{D}}[0,T]$ and put

$$(1.21)\quad \begin{cases} h(t) = f(t, y(t), y'(t)) \text{ for a.e. } t \in [0,T], \\ d_i = \mathrm{J}_i(y(t_i)) - y(t_i),\ e_i = \mathrm{M}_i(y'(t_i)) - y'(t_i), \quad i = 1, 2, = \dots, m, \\ c = y(0) + y'(0) - y'(T). \end{cases}$$

Then $h \in \mathbb{L}[0,T]$, $c, d_i, e_i \in \mathbb{R}$, $i = 1, 2, \dots, m$. By Lemma 1.5, there is a unique $x \in \mathbb{AC}^1_{\mathrm{D}}[0,T]$ fulfilling (1.16), (1.17) and it is given by (1.18). Due to (1.21), we have

$$x(t) = (\mathrm{F}\, y)(t) \text{ for } t \in [0,T].$$

Therefore, $u \in \mathbb{C}^1_{\mathrm{D}}[0,T]$ is a solution to (1.1)–(1.3) if and only if $u = \mathrm{F}\ u$. Define an operator $\mathrm{F}_1 : \mathbb{C}^1_{\mathrm{D}}[0,T] \mapsto \mathbb{C}^1_{\mathrm{D}}[0,T]$ by

$$(\mathrm{F}_1 y)(t) = \int_0^T G(t,s)\, f(s, y(s), y'(s))\, \mathrm{d}s, \quad t \in [0,T].$$

As F_1 is a composition of the Green type operator for the Dirichlet problem $u'' = 0$, $u(0) = u(T) = 0$, and of the superposition operator generated by $f \in \mathrm{Car}([0,T] \times \mathbb{R}^2)$, making use of the Lebesgue Dominated Convergence Theorem and the Arzelà-Ascoli Theorem, we get in a standard way that F_1 is completely continuous. Since J_i, M_i, $i = 1, 2, \dots, m$, are continuous, the operator $\mathrm{F}_2 = \mathrm{F} - \mathrm{F}_1$ is continuous, as well. Having in mind that F_2 maps bounded sets onto bounded sets and its values are contained in a $(2m+1)$-dimensional subspace of $\mathbb{C}^1_{\mathrm{D}}[0,T]$, we conclude that the operators F_2 and $\mathrm{F} = \mathrm{F}_1 + \mathrm{F}_2$ are completely continuous. $\square$

In the proof of our main result we will need the next proposition which concerns the case of well-ordered lower/upper functions and which follows from [7, Corollary 3.5].

1.7. Proposition. *Assume that* (1.10) *holds and let* α *and* β *be respectively lower and upper functions of* (1.1)–(1.3) *such that*

$$(1.22)\quad \alpha(t) < \beta(t) \textit{ for } t \in [0,T] \quad \textit{and} \quad \alpha(\tau+) < \beta(\tau+) \textit{ for } \tau \in \mathrm{D},$$

$$(1.23)\quad \alpha(t_i) < x < \beta(t_i) \implies \mathrm{J}_i(\alpha(t_i)) < \mathrm{J}_i(x) < \mathrm{J}_i(\beta(t_i)), \quad i = 1, 2, \dots, m$$

and

$$(1.24)\quad \begin{cases} y \le \alpha'(t_i) \implies \mathrm{M}_i(y) \le \mathrm{M}_i(\alpha'(t_i)), \\ y \ge \beta'(t_i) \implies \mathrm{M}_i(y) \ge \mathrm{M}_i(\beta'(t_i)), \quad i = 1, 2, \dots, m. \end{cases}$$

Further, let $h \in \mathbb{L}[0,T]$ *be such that*

$$(1.25)\quad |f(t,x,y)| \le h(t) \quad \textit{for a.e. } t \in [0,T] \textit{ and all } (x,y) \in [\alpha(t),\beta(t)] \times \mathbb{R}$$

and let the operator F *be defined by* (1.15). *Finally, for* $\gamma \in (0,\infty)$ *denote*

$$(1.26)\quad \Omega(\alpha,\beta,\gamma) = \{u \in \mathbb{C}^1_{\mathrm{D}}[0,T] : \alpha(t) < u(t) < \beta(t) \textit{ for } t \in [0,T],$$
$$\alpha(\tau+) < u(\tau+) < \beta(\tau+) \textit{ for } \tau \in \mathrm{D},\ \|u'\|_\infty < \gamma\}.$$

Then $\deg(\mathrm{I}-\mathrm{F},\Omega(\alpha,\beta,\gamma)) = 1$ *whenever* $\mathrm{F}\,u \ne u$ *on* $\partial\Omega(\alpha,\beta,\gamma)$ *and*

$$(1.27)\qquad \gamma > \|h\|_1 + \frac{\|\alpha\|_\infty + \|\beta\|_\infty}{\Delta}, \quad \textit{where } \Delta = \min_{i=1,2,\dots,m+1}(t_i - t_{i-1}).$$

Proof. Using the Mean Value Theorem, we can show that

$$(1.28)\qquad \|u'\|_\infty \le \|h\|_1 + \frac{\|\alpha\|_\infty + \|\beta\|_\infty}{\Delta}$$

holds for each $u \in \mathbb{C}^1_{\mathrm{D}}[0,T]$ fulfilling $\alpha(t) < u(t) < \beta(t)$ for $t \in [0,T]$ and $\alpha(\tau+) < u(\tau+) < \beta(\tau+)$ for $\tau \in \mathrm{D}$. Thus, if we denote by c the right-hand side of (1.28), we can follow the proof of [7, Corollary 3.5]. □

2. A priori estimates

In Section 3 we will need a priori estimates which are contained in Lemmas 2.1–2.3.

2.1. Lemma. *Let* $\rho_1 \in (0,\infty)$, $\widetilde{h} \in \mathbb{L}[0,T]$, $\mathrm{M}_i \in \mathbb{C}(\mathbb{R})$, $i = 1,2,\dots,m$. *Then there exists* $d \in (\rho_1,\infty)$ *such that the estimate*

$$(2.1)\qquad \|u'\|_\infty < d$$

is valid for each $u \in \mathbb{AC}^1_{\mathrm{D}}[0,T]$ *and each* $\widetilde{\mathrm{M}}_i \in \mathbb{C}(\mathbb{R})$, $i = 1,2,\dots,m$, *satisfying* (1.3),

$$(2.2)\qquad |u'(\xi_u)| < \rho_1 \quad \textit{for some } \xi_u \in [0,T],$$

$$(2.3)\qquad u'(t_i+) = \widetilde{\mathrm{M}}_i(u'(t_i)), \quad i = 1,2,\dots,m,$$

$$(2.4)\qquad |u''(t)| < \widetilde{h}(t) \textit{ for a.e. } t \in [0,T]$$

and

$$(2.5)\qquad \sup\{|\mathrm{M}_i(y)| : |y| < a\} < b \implies \sup\{|\widetilde{\mathrm{M}}_i(y)| : |y| < a\} < b$$
$$\textit{for } i = 1,2,\dots,m,\ a \in (0,\infty),\ b \in (a,\infty).$$

Proof. Suppose that $u \in \mathbb{AC}^1_{\mathrm{D}}[0,T]$ and $\widetilde{\mathrm{M}}_i \in \mathbb{C}(\mathbb{R})$, $i = 1,2,\dots,m$, satisfy (1.3) and (2.2)–(2.5). Due to (1.3), we can assume that $\xi_u \in (0,T]$, i.e. there is $j \in \{1,2,\dots,m+1\}$ such that $\xi_u \in (t_{j-1},t_j]$. We will distinguish 3 cases: either $j=1$ or $j=m+1$ or $1<j<m+1$.

Let $j=1$. Then, using (2.2) and (2.4), we obtain

$$|u'(t)| < a_1 \text{ on } [0,t_1], \tag{2.6}$$

where $a_1 = \rho_1 + \|\widetilde{h}\|_1$. Since $\mathrm{M}_1 \in \mathbb{C}(\mathbb{R})$, we can find $b_1(a_1) \in (a_1,\infty)$ such that $|\,\mathrm{M}_1(y)| < b_1(a_1)$ for all $y \in (-a_1,a_1)$. Hence, in view of (2.3) and (2.5), we have $|u'(t_1+)| < b_1(a_1)$, wherefrom, using (2.4), we deduce that $|u'(t)| < b_1(a_1)+\|\widetilde{h}\|_1$ for $t \in (t_1,t_2]$. Continuing by induction, we get $b_i(a_i) \in (a_i,\infty)$ such that $|u'(t)| < a_{i+1} = b_i(a_i) + \|\widetilde{h}\|_1$ on $(t_i,t_{i+1}]$ for $i = 2,\dots,m$, i.e.

$$\|u'\|_\infty < d := \max\{a_i : i = 1,2,\dots,m+1\}. \tag{2.7}$$

Assume that $j=m+1$. Then, using (2.2) and (2.4), we obtain

$$|u'(t)| < a_{m+1} \text{ on } (t_m,T], \tag{2.8}$$

where $a_{m+1} = \rho_1 + \|\widetilde{h}\|_1$. Furthermore, due to (1.3), we have $|u'(0)| < a_{m+1}$ which together with (2.4) yields that (2.6) is true with $a_1 = a_{m+1} + \|\widetilde{h}\|_1$. Now, proceeding as in the case $j=1$, we show that (2.7) is true also in the case $j=m+1$.

Assume that $1<j<m+1$. Then (2.2) and (2.4) yield $|u'(t)| < a_{j+1} = \rho_1 + \|\widetilde{h}\|_1$ on $(t_j,t_{j+1}]$. If $j<m$, then $|u'(t)| < a_{j+2} = b_{j+1}(a_{j+1}) + \|\widetilde{h}\|_1$ on $(t_{j+1},t_{j+2}]$, where $b_{j+1}(a_{j+1}) > a_{j+1}$. Proceeding by induction we get (2.8) with $a_{m+1} = b_m(a_m) + \|\widetilde{h}\|_1$ and $b_m(a_m) > a_m$, wherefrom (2.7) again follows as in the previous case. □

2.2. Lemma. *Let $\rho_0, d, q \in (0,\infty)$ and $\mathrm{J}_i \in \mathbb{C}(\mathbb{R})$, $i = 1,2,\dots,m$. Then there exists $c \in (\rho_0,\infty)$ such that the estimate*

$$\|u\|_\infty < c \tag{2.9}$$

is valid for each $u \in \mathbb{C}^1_{\mathrm{D}}[0,T]$ and each $\widetilde{\mathrm{J}}_i \in \mathbb{C}(\mathbb{R})$, $i = 1,2,\dots,m$, satisfying (1.3), (2.1),

$$u(t_i+) = \widetilde{\mathrm{J}}_i(u(t_i)), \quad i = 1,2,\dots,m, \tag{2.10}$$

$$|u(\tau_u)| < \rho_0 \quad \textit{for some} \quad \tau_u \in [0,T] \tag{2.11}$$

and

$$\sup\{|\,\mathrm{J}_i(x)| : |x| < a\} < b \implies \sup\{|\widetilde{\mathrm{J}}_i(x)| : |x| < a\} < b \tag{2.12}$$
$$\textit{for } \ i = 1,2,\dots,m, \ a \in (0,\infty), \ b \in (a+q,\infty).$$

Proof. We will argue similarly as in the proof of Lemma 2.1. Suppose that $u \in \mathbb{C}^1_{\mathrm{D}}[0,T]$ satisfies (1.3), (2.1), (2.10), (2.11) and that $\widetilde{\mathrm{J}}_i \in \mathbb{C}(\mathbb{R})$, $i = 1,2,\dots,m$, satisfy (2.12). Due to (1.3) we can assume that $\tau_u \in (0,T]$, i.e. there is $j \in \{1,2,\dots,m+1\}$ such that $\tau_u \in (t_{j-1},t_j]$. We will consider three cases: $j=1$, $j=m+1$, $1<j<m+1$. If $j=1$, then (2.1) and (2.11) yield $|u(t)| < a_1 = \rho_0 + d\,T$ on $[0,t_1]$. In particular, $|u(t_1)| < a_1$. Since $\mathrm{J}_1 \in \mathbb{C}(\mathbb{R})$, we can find $b_1(a_1) \in (a_1+q,\infty)$ such that $|\,\mathrm{J}_1(x)| < b_1(a_1)$ for all $x \in (-a_1,a_1)$ and consequently, by (2.12), also $|\widetilde{\mathrm{J}}_1(x)| < b_1(a_1)$ for all $x \in (-a_1,a_1)$. Therefore, by (2.1), $|u(t)| < |u(t_1+)| + d\,T = |\widetilde{\mathrm{J}}_1(u(t_1))| + d\,T < a_2 = b_1(a_1) + d\,T$ on $(t_1,t_2]$. Proceeding by induction we get $b_i(a_i) \in (a_i+q,\infty)$ such that $|u(t)| < a_{i+1} = b_i(a_i) + d\,T$ for $t \in (t_i,t_{i+1}]$ and $i=2,\dots,m$. As a result, (2.9) is true with $c = \max\{a_i : i=1,2,\dots,m+1\}$. Analogously we would proceed in the remaining cases $j=m+1$ or $1<j<m+1$. $\square$

Finally, we will need two estimates for functions u satisfying one of the following conditions:

$$u(s_u) < \sigma_1(s_u) \quad \text{and} \quad u(t_u) > \sigma_2(t_u) \quad \text{for some} \quad s_u, t_u \in [0,T], \tag{2.13}$$

$$u \ge \sigma_1 \text{ on } [0,T] \quad \text{and} \quad \inf_{t\in[0,T]} |u(t)-\sigma_1(t)| = 0, \tag{2.14}$$

$$u \le \sigma_2 \text{ on } [0,T] \quad \text{and} \quad \inf_{t\in[0,T]} |u(t)-\sigma_2(t)| = 0. \tag{2.15}$$

2.3. Lemma. *Assume that* σ_1, $\sigma_2 \in \mathbb{AC}^1_{\mathrm{D}}[0,T]$, J_i, M_i, $\widetilde{\mathrm{J}}_i$, $\widetilde{\mathrm{M}}_i \in \mathbb{C}(\mathbb{R})$, $i = 1,2,\dots,m$, *satisfy* (1.12), (1.13) *and*

$$\begin{cases} x > \sigma_1(t_i) \implies \widetilde{\mathrm{J}}_i(x) > \widetilde{\mathrm{J}}_i(\sigma_1(t_i)) = \mathrm{J}_i(\sigma_1(t_i)), \\ x < \sigma_2(t_i) \implies \widetilde{\mathrm{J}}_i(x) < \widetilde{\mathrm{J}}_i(\sigma_2(t_i)) = \mathrm{J}_i(\sigma_2(t_i)), \quad i=1,2,\dots,m \end{cases} \tag{2.16}$$

and

$$\begin{cases} y \le \sigma_1'(t_i) \implies \widetilde{\mathrm{M}}_i(y) \le \mathrm{M}_i(\sigma_1'(t_i)), \\ y \ge \sigma_2'(t_i) \implies \widetilde{\mathrm{M}}_i(y) \ge \mathrm{M}_i(\sigma_2'(t_i)), \quad i=1,2,\dots,m. \end{cases} \tag{2.17}$$

Define

$$B = \{u \in \mathbb{C}^1_{\mathrm{D}}[0,T] : u \textit{ satisfies } (1.3),(2.10),(2.3) \textit{ and one of the conditions } (2.13),(2.14),(2.15)\}. \tag{2.18}$$

Then each function $u \in B$ satisfies

$$\text{(2.19)} \qquad \begin{cases} |u'(\xi_u)| < \rho_1 \quad \textit{for some} \;\; \xi_u \in [0,T], \;\; \textit{where} \\[1ex] \rho_1 = \dfrac{2}{t_1}\,(\|\sigma_1\|_\infty + \|\sigma_2\|_\infty) + \|\sigma_1'\|_\infty + \|\sigma_2'\|_\infty + 1. \end{cases}$$

Proof. • PART 1. Assume that $u \in B$ satisfies (2.13). There are 3 cases to consider:

CASE A. If $\min\{\sigma_1(t), \sigma_2(t)\} \le u(t) \le \max\{\sigma_1(t), \sigma_2(t)\}$ for $t \in [0,T]$, then, by the Mean Value Theorem, there is $\xi_u \in (0, t_1)$ such that

$$\text{(2.20)} \qquad |u'(\xi_u)| \le \frac{2}{t_1}\,(\|\sigma_1\|_\infty + \|\sigma_2\|_\infty).$$

CASE B. Assume that $u(s) > \sigma_1(s)$ for some $s \in [0,T]$ and denote $v = u - \sigma_1$. Due to (2.13) we have

$$\text{(2.21)} \qquad v_* = \inf_{t\in[0,T]} v(t) < 0 \quad \text{and} \quad v^* = \sup_{t\in[0,T]} v(t) > 0.$$

We are going to prove that

$$\text{(2.22)} \qquad v'(\alpha) = 0 \text{ for some } \; \alpha \in [0,T] \text{ or } \; v'(\tau+) = 0 \text{ for some } \; \tau \in \mathrm{D}\,.$$

Suppose, on the contrary, that (2.22) does not hold.

Let $v'(0) > 0$. Then, according to (1.3) and (1.6), $v'(T) > 0$, as well. Due to the assumption that (2.22) does not hold, this together with (1.5) yields that

$$0 < v'(t_m+) = u'(t_m+) - \sigma_1'(t_m+) \le \widetilde{\mathrm{M}}_m(u'(t_m)) - \mathrm{M}_m(\sigma_1'(t_m)),$$

which is by (2.17) possible only if $u'(t_m) > \sigma_1'(t_m)$, i.e. $v'(t_m) > 0$. Continuing in this way on each $(t_i, t_{i+1}]$, $i = 0, 1, \ldots, m-1$, we get

$$\text{(2.23)} \qquad v'(t) > 0 \text{ for } t \in [0,T] \quad \text{and} \quad v'(\tau+) > 0 \text{ for } \tau \in \mathrm{D}\,.$$

If $v(0) \ge 0$, then $v(t) > 0$ on $(0, t_1]$ due to (2.23). Further, it follows by (1.5), (2.10) and (2.16) that $u(t_1+) > \sigma_1(t_1+)$, i.e. $v(t_1+) > 0$. Continuing by induction we deduce that $v \ge 0$ on $[0,T]$, contrary to (2.21).

If $v(0) < 0$, then by (1.3) and (1.6) we have $v(T) < 0$. Further, by virtue of (2.23) we obtain $v < 0$ on $(t_m, T]$ and, in particular, $v(t_m+) < 0$. So, $\widetilde{\mathrm{J}}_m(u(t_m)) < \mathrm{J}_m(\sigma_1(t_m))$ wherefrom $u(t_m) \le \sigma_1(t_m)$ follows, due to (2.16). Thus, we have $v < 0$ on (t_{m-1}, t_m). Continuing by induction we get $v \le 0$ on $[0,T]$, contrary to (2.21).

Now, assume that $v'(0) < 0$. Then $v'(t_1) < 0$, i.e. $u'(t_1) < \sigma_1'(t_1)$ wherefrom, by (1.5), (1.13) and the assumption that (2.22) does not hold, the inequality $v'(t_1+) = u'(t_1+) - \sigma_1'(t_1+) < 0$ follows. Similarly as in the proof of (2.23) we show that

$$v'(t) < 0 \text{ for } t \in [0,T] \quad \text{and} \quad v'(\tau+) < 0 \text{ for } \tau \in \mathrm{D}\,. \tag{2.24}$$

Now, having (2.24), we consider as above two cases: $v(0) \ge 0$ and $v(0) < 0$, and construct a contradiction by means of analogous arguments.

So we have proved that (2.22) is true, which yields the existence of $\xi_u \in [0,T]$ having the property

$$|u'(\xi_u)| < \|\sigma_1'\|_\infty + 1. \tag{2.25}$$

Case C. If $u(s) < \sigma_2(s)$ for some $s \in [0,T]$, we put $v = u - \sigma_2$ and, using the properties of σ_2 instead of σ_1, we can argue as in Case B and show that there exists $\xi_u \in [0,T]$ such that

$$|u'(\xi_u)| < \|\sigma_2'\|_\infty + 1. \tag{2.26}$$

Taking into account (2.20), (2.25) and (2.26) we conclude that (2.19) is valid for any $u \in B$ fulfilling (2.13).

• Part 2. Let $u \in B$ satisfy (2.14). Then $u \ge \sigma_1$ on $[0,T]$ and either there is $\alpha_u \in [0,T]$ such that $u(\alpha_u) = \sigma_1(\alpha_u)$ or there is $t_j \in \mathrm{D}$ such that $u(t_j+) = \sigma_1(t_j+)$.

Case A. Let the first possibility occur. If $\alpha_u \in (0,T) \setminus \mathrm{D}$, then necessarily $u'(\alpha_u) = \sigma_1'(\alpha_u)$. Consequently, the estimate (2.25) is valid. If $\alpha_u = 0$, then $\inf\{u(t) - \sigma_1(t) : t \in [0,T]\} = u(0) - \sigma_1(0) = u(T) - \sigma_1(T) = 0$, which, by virtue of (1.3) and (1.6), implies $0 \le u'(0) - \sigma_1'(0) \le u'(T) - \sigma_1'(T) \le 0$, i.e. $u'(0) = \sigma_1'(0)$ and the estimate (2.25) is valid with $\xi_u = 0$. If $\alpha_u = t_j$ for some $t_j \in \mathrm{D}$, then $0 = u(t_j) - \sigma_1(t_j) = u(t_j+) - \sigma_1(t_j+)$. Having in mind that $u \ge \sigma_1$ on $[0,T]$, we get $u'(t_j+) \ge \sigma_1'(t_j+)$ and $u'(t_j) \le \sigma_1'(t_j)$. On the other hand, with respect to (2.17), the last inequality gives also $\widetilde{\mathrm{M}}_j(u'(t_j)) \le \mathrm{M}_j(\sigma_1'(t_j))$, which leads to $\sigma_1'(t_j+) = u'(t_j+)$. Thus, (2.25) is fulfilled for some $\xi_u \in (t_j, t_{j+1})$ which is sufficiently close to t_j.

Case B. Let the second possibility occur, i.e. $u(t_j+) = \sigma_1(t_j+)$ for some $t_j \in \mathrm{D}$. According to (1.5) and (2.10), we have $\widetilde{\mathrm{J}}_j(u(t_j)) = \mathrm{J}_j(\sigma_1(t_j))$. Taking into account (2.16), we see that this can occur only if $u(t_j) \le \sigma_1(t_j)$. On the other hand, by the assumption (2.14) we have $u \ge \sigma_1$ on $[0,T]$. Hence we conclude that $u(t_j) = \sigma_1(t_j)$ and so, arguing as before, we get (2.25) again.

To summarize: (2.19) holds for any $u \in B$ fulfilling (2.14).

• Part 3. Let $u \in B$ satisfy (2.15). Then using the properties of σ_2 instead of σ_1, we argue analogously to Part 2 and prove that (2.26) is valid for each $u \in B$ which satisfies (2.15). In particular, (2.19) holds for any $u \in B$ fulfilling (2.15). □

3. Main result

Our main result consists in a generalization of [8, Theorem 3.1]. Particularly, we remove the condition (0.2), which was assumed in [8], and prove the following theorem.

3.1. Theorem. *Assume that* (1.10)–(1.13) *and* (0.1) *hold and let* $h \in \mathbb{L}[0,T]$ *be such that*

$$|f(t,x,y)| \le h(t) \text{ for a.e. } t \in [0,T] \text{ and all } (x,y) \in \mathbb{R}^2. \tag{3.1}$$

Then the problem (1.1)–(1.3) *has a solution* u *satisfying one of the conditions* (2.13)–(2.15).

Proof. • Step 1. *We construct a proper auxiliary problem.*

Let σ_1 and σ_2 be respectively lower and upper functions of (1.1)–(1.3) and let ρ_1 be associated with them as in (2.19). Put

$$\widetilde{h}(t) = 2\,h(t) + 1 \text{ for a.e. } t \in [0,T],$$
$$\widetilde{\rho} = \rho_1 + \sum_{i=1}^{m} \big(|\,\mathrm{M}_i(\sigma_1'(t_i))| + |\,\mathrm{M}_i(\sigma_2'(t_i))|\big).$$

By Lemma 2.1, find $d \in (\widetilde{\rho}, \infty)$ satisfying (2.1). Furthermore, put $\rho_0 = \|\sigma_1\|_\infty + \|\sigma_2\|_\infty + 1$ and

$$q = \frac{T}{m} \sum_{i=1}^{m} \max\{\max_{|y| \le d+1} |\,\mathrm{M}_i(y)|, d+1\} \tag{3.2}$$

and, by Lemma 2.2, find $c \in (\rho_0 + q, \infty)$ fulfilling (2.9). In particular, we have

$$c > \|\sigma_1\|_\infty + \|\sigma_2\|_\infty + q + 1, \quad d > \|\sigma_1'\|_\infty + \|\sigma_2'\|_\infty + 1. \tag{3.3}$$

Finally, for a.e. $t \in [0,T]$ and all $x, y \in \mathbb{R}$ and $i = 1, 2, \dots, m$, define functions

$$\widetilde{f}(t,x,y) = \begin{cases} f(t,x,y) - h(t) - 1 & \text{if } x \le -c-1, \\ f(t,x,y) + (x+c)\,(h(t)+1) & \text{if } -c-1 < x < -c, \\ f(t,x,y) & \text{if } -c \le x \le c, \\ f(t,x,y) + (x-c)\,(h(t)+1) & \text{if } c < x < c+1, \\ f(t,x,y) + h(t) + 1 & \text{if } x \ge c+1, \end{cases} \tag{3.4}$$

$$\widetilde{\mathrm{J}}_i(x) = \begin{cases} x+q & \text{if } x \le -c-1, \\ \mathrm{J}_i(-c)\,(c+1+x) - (x+q)\,(x+c) & \text{if } -c-1 < x < -c, \\ \mathrm{J}_i(x) & \text{if } -c \le x \le c, \\ \mathrm{J}_i(c)\,(c+1-x) + (x-q)\,(x-c) & \text{if } c < x < c+1, \\ x-q & \text{if } x \ge c+1, \end{cases} \tag{3.5}$$

$$\widetilde{\mathrm{M}}_i(y) = \begin{cases} y & \text{if } y \le -d-1, \\ \mathrm{M}_i(-d)\,(d+1+y) - y\,(y+d) & \text{if } -d-1 < y < -d, \\ \mathrm{M}_i(y) & \text{if } -d \le y \le d, \\ \mathrm{M}_i(d)\,(d+1-y) + y\,(y-d) & \text{if } d < y < d+1, \\ y & \text{if } y \ge d+1 \end{cases} \tag{3.6}$$

and consider the auxiliary problem

$$u'' = \widetilde{f}(t,u,u'), \quad (2.10), \quad (2.3), \quad (1.3). \tag{3.7}$$

Due to (1.10), $\widetilde{f} \in \mathrm{Car}([0,T] \times \mathbb{R})$ and $\widetilde{\mathrm{J}}_i, \widetilde{\mathrm{M}}_i \in \mathbb{C}(\mathbb{R})$ for $i = 1, 2, \dots, m$. According to (3.3)–(3.6) the functions σ_1 and σ_2 are respectively lower and upper functions of (3.7). By (3.1) we have

$$|\widetilde{f}(t,x,y)| \le \widetilde{h}(t) \text{ for a.e. } t \in [0,T] \text{ and all } (x,y) \in \mathbb{R}^2 \tag{3.8}$$

and

$$\begin{cases} \widetilde{f}(t,x,y) < 0 & \text{for a.e. } t \in [0,T] \text{ and all } (x,y) \in (-\infty, -c-1] \times \mathbb{R}, \\ \widetilde{f}(t,x,y) > 0 & \text{for a.e. } t \in [0,T] \text{ and all } (x,y) \in [c+1, \infty) \times \mathbb{R}. \end{cases} \tag{3.9}$$

- Step 2. *We show that* $\widetilde{\mathrm{J}}_i$ *and* $\widetilde{\mathrm{M}}_i$ *satisfy the assumptions of Lemmas* 2.1$-$2.3.
  Choose an arbitrary $i \in \{1, 2, \dots, m\}$.

(i) *Condition* (2.5). Let $a \in (0,\infty)$, $b \in (a,\infty)$ and $M_i^* = \sup\{|\,\mathrm{M}_i(y)| : |y| < a\} < b$. Then, by (3.6), we have $\sup\{|\widetilde{\mathrm{M}}_i(y)| : |y| < a\} \le \max\{a, M_i^*\} < b$.

(ii) *Condition* (2.12). Let $a \in (0,\infty)$, $b \in (a+q,\infty)$ and $J_i^* = \sup\{|\,\mathrm{J}_i(x)| : |x| < a\} < b$. Then, by (3.5), we have $\sup\{|\widetilde{\mathrm{J}}_i(x)| : |x| < a\} \le \max\{a+q, J_i^*\} < b$.

(iii) *Condition* (2.16). Due to (1.12), (3.3) and (3.5), we see that (2.16) holds if $|x| \le c$. Assume that $x > c$. Then $x > \max\{\sigma_1(t_i), \sigma_2(t_i)\}$ which means that the second condtion in (2.16) need not be considered in this case. Since $|\sigma_1(t_i)| < c$, we have $\widetilde{\mathrm{J}}_i(\sigma_1(t_i)) = \mathrm{J}_i(\sigma_1(t_i))$. Furthermore, due to (3.3), $x - q > \|\sigma_1\|_\infty + \|\sigma_2\|_\infty + 1$. If $x \ge c+1$, then $\widetilde{\mathrm{J}}_i(x) = x - q > \sigma_1(t_i+) = \mathrm{J}_i(\sigma_1(t_i))$. Finally, if $x \in (c, c+1)$, then $\widetilde{\mathrm{J}}_i(x) = \mathrm{J}_i(c)\,(c+1-x) + (x-q)\,(x-c) > \mathrm{J}_i(\sigma_1(t_i))$ because $\mathrm{J}_i(c) > \mathrm{J}_i(\sigma_1(t_i))$ by (1.12). For $x < (\infty, -c)$ we can argue similarly.

(iv) *Condition* (2.17). Due to (1.13), (3.3) and (3.6), we see that (2.17) holds for $|y| < d$. Assume that $y > d$. Then $y > \max\{\sigma_1'(t_i), \sigma_2'(t_i)\}$ which means that the first condition in (2.17) need not be considered in this case. Since $d > \widetilde{\rho} > \mathrm{M}_i(\sigma_2'(t_i))$, we have $\widetilde{\mathrm{M}}_i(y) = y > \mathrm{M}_i(\sigma_2'(t_i))$ if $y > d+1$ and $\widetilde{\mathrm{M}}_i(y) = \mathrm{M}_i(d)\,(d+1-y) + y\,(y-d) > \mathrm{M}_i(\sigma_2'(t_i))$ if $y \in (d, d+1)$. Hence the second condition in (2.17) is satisfied for $y \in (d,\infty)$. Similarly we can verify the first condition in (2.17) for $y \in (-\infty, -d)$.

• Step 3. *We construct a well-ordered pair of lower/upper functions for* (3.7). Put

$$A^* = q + \sum_{i=1}^{m} \max_{|x| \le c+1} |\widetilde{\mathrm{J}}_i(x)| \tag{3.10}$$

and

$$\begin{cases} \sigma_4(0) = A^* + m\,q, \\ \sigma_4(t) = A^* + (m-i)\,q + \frac{m\,q}{T}\,t \text{ for } t \in (t_i, t_{i+1}],\ i = 0,1,\dots,m, \\ \sigma_3(t) = -\sigma_4(t) \text{ for } t \in [0,T]. \end{cases} \tag{3.11}$$

Then $\sigma_3, \sigma_4 \in \mathbb{AC}^1_{\mathrm{D}}[0,T]$ and, by (3.5) and (3.10),

$$\sigma_3(t) < -A^* < -c-1, \quad \sigma_4(t) > A^* > c+1 \text{ for } t \in [0,T]. \tag{3.12}$$

In view of (3.2),

$$\sigma_3'(t) = -\frac{m\,q}{T} \le -(d+1) \quad \text{and} \quad \sigma_4'(t) = \frac{m\,q}{T} \ge d+1 \text{ for } t \in [0,T]. \tag{3.13}$$

Now, we prove that σ_4 is an upper function of (3.7):
By (3.9) and (3.12), we have

$$0 = \sigma_4''(t) < \widetilde{f}(t, \sigma_4(t), \sigma_4'(t)) \text{ for a.e. } t \in [0,T].$$

Furthermore, by (3.5),

$$\sigma_4(t_i+) = A^* + (m-i)\,q + \frac{m\,q}{T}t_i = \sigma_4(t_i) - q = \widetilde{\mathrm{J}}_i(\sigma_4(t_i)).$$

By virtue of (3.2) and (3.6), we get

$$\sigma_4'(t_i+) = \frac{m\,q}{T} = \sigma_4'(t_i) = \widetilde{\mathrm{M}}_i(\sigma_4'(t_i)) \text{ for } i = 1, 2, \dots, m.$$

Finally, $\sigma_4(0) = A^* + m\,q = \sigma_4(T)$ and $\sigma_4'(0) = \frac{m\,q}{T} = \sigma_4'(T)$, i.e. σ_4 is an upper function of (3.7). Since $\sigma_3 = -\sigma_4$, we can see that σ_3 is a lower function of (3.7). Clearly,

$$\sigma_3 < \sigma_4 \text{ on } [0,T] \quad \text{and} \quad \sigma_3(\tau+) < \sigma_4(\tau+) \text{ for } \tau \in \mathrm{D}\,. \tag{3.14}$$

Having G from (1.15), define an operator $\widetilde{\mathrm{F}} : \mathbb{C}^1_{\mathrm{D}}[0,T] \mapsto \mathbb{C}^1_{\mathrm{D}}[0,T]$ by

$$\begin{aligned}(\widetilde{\mathrm{F}}u)(t) = u(0) + u'(0) - u'(T) + \int_0^T G(t,s)\,\widetilde{f}(s, u(s), u'(s))\,\mathrm{d}s \\ - \sum_{i=1}^{m} \frac{\partial G}{\partial s}(t, t_i)(\widetilde{\mathrm{J}}_i(u(t_i)) - u(t_i)) \\ + \sum_{i=1}^{m} G(t, t_i)(\widetilde{\mathrm{M}}_i(u'(t_i)) - u'(t_i)), \quad t \in [0,T].\end{aligned} \tag{3.15}$$

By Proposition 1.6, $\widetilde{\mathrm{F}}$ is completely continuous and u is a solution of (3.7) whenever $\widetilde{\mathrm{F}}u = u$.

- STEP 4. *We prove the first a priori estimate for solutions of* (3.7).
  Define

$$\begin{aligned}\Omega_0 = \{u \in \mathbb{C}^1_{\mathrm{D}}[0,T] :\ \|u'\|_\infty < C^*,\ \sigma_3 < u < \sigma_4 \text{ on } [0,T], \\ \sigma_3(\tau+) < u(\tau+) < \sigma_4(\tau+) \text{ for } \tau \in \mathrm{D}\},\end{aligned} \tag{3.16}$$

where

$$C^* = 1 + \|\widetilde{h}\|_1 + \frac{\|\sigma_3\|_\infty + \|\sigma_4\|_\infty}{\Delta} \tag{3.17}$$

and Δ is defined in (1.27). We are going to prove that for each solution u of (3.7) the estimate

$$u \in \mathrm{cl}(\Omega_0) \implies u \in \Omega_0 \tag{3.18}$$

is true. To this aim, suppose that u is a solution of (3.7) and $u \in \mathrm{cl}(\Omega_0)$, i.e. $\|u'\|_\infty \le C^*$ and

$$\sigma_3 \le u \le \sigma_4 \text{ on } [0, T]. \tag{3.19}$$

By the Mean Value Theorem, there are $\xi_i \in (t_i, t_{i+1})$, $i = 1, 2, \dots, m$, such that

$$|u'(\xi_i)| \le \frac{\|\sigma_3\|_\infty + \|\sigma_4\|_\infty}{\Delta}.$$

Hence, by (3.8), we get

$$\|u'\|_\infty < C^*, \tag{3.20}$$

where C^* is defined in (3.17). It remains to show that $\sigma_3 < u < \sigma_4$ on $[0, T]$ and $\sigma_3(\tau+) < u(\tau+) < \sigma_4(\tau+)$ for $\tau \in \mathrm{D}$. Assume the contrary. Then there exists $k \in \{3, 4\}$ such that

$$u(\xi) = \sigma_k(\xi) \qquad \text{for some } \ \xi \in [0, T] \tag{3.21}$$

or

$$u(t_i+) = \sigma_k(t_i+) \quad \text{for some } \ t_i \in \mathrm{D}\,. \tag{3.22}$$

Case A. Let (3.21) hold for $k = 4$.

(i) If $\xi = 0$, then $u(0) = \sigma_4(0) = \sigma_4(T) = u(T) = A^* + q\,m$ which gives, in view of (1.3), (3.13) and (3.19),

$$u'(0) = u'(T) = \frac{m\,q}{T} = \sigma_4'(t) \text{ for } t \in [0, T].$$

Further, due to (3.9) and (3.12), we can find $\delta > 0$ such that $u > c+1$ on $[0, \delta]$ and

$$u'(t) - u'(0) = \int_0^t \widetilde{f}(s, u(s), u'(s))\,\mathrm{d}s > 0 \text{ for } t \in [0, \delta].$$

Hence $u'(t) > u'(0) = \sigma_4'(t)$ on $(0, \delta]$ which implies that $u > \sigma_4$ on $(0, \delta]$, contrary to (3.19).

(ii) If $\xi \in (t_i, t_{i+1})$ for some $t_i \in \mathrm{D}$, then $u'(\xi) = \sigma_4'(\xi) = \frac{m\,q}{T} = \sigma_4'(t)$ for $t \in [0,T]$ and we reach a contradiction as above.

(iii) If $\xi = t_i \in \mathrm{D}$, then $u(t_i) = \sigma_4(t_i)$ and, by (3.5) and (3.12),

$$u(t_i+) = \sigma_4(t_i+) = \sigma_4(t_i) - q > c + 1 - q > \|\sigma_1\|_\infty + \|\sigma_2\|_\infty.$$

By virtue of (3.19) we have $u'(t_i+) \le \sigma_4'(t_i+)$ and $u'(t_i) \ge \sigma_4'(t_i)$. Now, since the last inequality together with (3.6) and (3.13) yield $u'(t_i+) \ge \sigma_4'(t_i+)$, we get $u'(t_i+) = \sigma_4'(t_i+) = \frac{m\,q}{T} = \sigma_4'(t)$ for $t \in [0,T]$. Similarly as above, this leads again to a contradiction.

Case B. Let (3.22) hold for $k = 4$, i.e. $u(t_i+) = \sigma_4(t_i+)$. By (3.5) and (3.12), $\widetilde{\mathrm{J}}_i(u(t_i)) = \sigma_4(t_i+) = \sigma_4(t_i) - q > A^* - q$, wherefrom, with respect to (3.10), we get $u(t_i) > c+1$ and hence $\widetilde{\mathrm{J}}_i(u(t_i)) = u(t_i) - q$. Therefore $u(t_i) = \sigma_4(t_i)$ and we can continue as in Case A (iii).

If (3.21) or (3.22) hold for $k = 3$, then we use analogical arguments as in Case A or Case B.

- Step 5. *We prove the second a priori estimate for solutions of* (3.7).

Define sets

$$\Omega_1 = \{u \in \Omega_0 :\, u(t) > \sigma_1(t) \text{ for } t \in [0,T],\ u(\tau+) > \sigma_1(\tau+) \text{ for } \tau \in \mathrm{D}\},$$

$$\Omega_2 = \{u \in \Omega_0 :\, u(t) < \sigma_2(t) \text{ for } t \in [0,T],\ u(\tau+) < \sigma_2(\tau+) \text{ for } \tau \in \mathrm{D}\}$$

and $\widetilde{\Omega} = \Omega_0 \setminus \mathrm{cl}(\Omega_1 \cup \Omega_2)$. Then, by (0.1), $\Omega_1 \cap \Omega_2 = \emptyset$ and

$$\widetilde{\Omega} = \{u \in \Omega_0 :\, u \text{ satisfies (2.13)}\}. \tag{3.23}$$

Furthermore, with respect to (1.26), (3.16) and (3.11) we have

$$\Omega_0 = \Omega(\sigma_3, \sigma_4, C^*),\ \Omega_1 = \Omega(\sigma_1, \sigma_4, C^*) \quad \text{and} \quad \Omega_2 = \Omega(\sigma_3, \sigma_2, C^*).$$

Consider c from Step 1. We are going to prove that the estimates

$$u \in \mathrm{cl}(\widetilde{\Omega}) \implies \|u\|_\infty < c, \quad \|u'\|_\infty < d \tag{3.24}$$

are valid for each solution u of (3.7). So, assume that u is a solution of (3.7) and $u \in \mathrm{cl}(\widetilde{\Omega})$. Then, due to (3.18), u fulfils one of the conditions (2.13), (2.14), (2.15) and so, by (2.18), $u \in B$. Since we have already proved that (2.16) and (2.17) hold, we can use Lemma 2.3 and get $\xi_u \in [0,T]$ such that (2.19) is true. Further, since $\widetilde{\mathrm{M}}_i$, $i = 1, 2, \ldots, m$, fulfil (2.5) and since (1.3), (2.3) and (3.8) are valid, we can apply Lemma 2.1 to show that u satisfies the estimate (2.1). Finally, by [8, Lemma 2.4], u satisfies (2.11) with ρ_0 defined in Step 1. Moreover, let us recall that $\widetilde{\mathrm{J}}_i$, $i = 1, 2, \ldots, m$, verify the condition (2.12). Hence, by Lemma 2.2, we have (2.9), i.e. each solution u of (3.7) satisfies (3.24).

- Step 6. *We prove the existence of a solution to the problem* (1.1)–(1.3).

Consider the operator $\widetilde{\mathrm{F}}$ defined by (3.15). We distinguish two cases: either $\widetilde{\mathrm{F}}$ has a fixed point in $\partial\widetilde{\Omega}$ or it has no fixed point in $\partial\widetilde{\Omega}$.

Assume that $\widetilde{\mathrm{F}}\,u = u$ for some $u \in \partial\widetilde{\Omega}$. Then u is a solution of (3.7) and, with respect to (3.24), we have $\|u\|_\infty < c$, $\|u'\|_\infty < d$, which means, by (3.4)–(3.6), that u is a solution of (1.1)–(1.3). Furthermore, due to (3.18), u satisfies (2.14) or (2.15).

Now, assume that $\widetilde{\mathrm{F}}\,u \neq u$ for all $u \in \partial\widetilde{\Omega}$. Then $\widetilde{\mathrm{F}}u \neq u$ for all $u \in \partial\Omega_0 \cup \partial\Omega_1 \cup \partial\Omega_2$. If we replace f, h, J_i, M_i, $i = 1, 2, \ldots, m$, α, β and γ respectively by $\widetilde{f}$, $\widetilde{h}$, $\widetilde{\mathrm{J}}_i$, $\widetilde{\mathrm{M}}_i$, $i = 1, 2, \ldots, m$, σ_3, σ_4 and C^* in Proposition 1.7, we see that the assumptions (1.22)–(1.25) and (1.27) are satisfied. Thus, by Proposition 1.7, we obtain that

$$\deg(\mathrm{I} - \widetilde{\mathrm{F}}, \Omega(\sigma_3, \sigma_4, C^*)) = \deg(\mathrm{I} - \widetilde{\mathrm{F}}, \Omega_0) = 1. \tag{3.25}$$

Similarly, we can apply Proposition 1.7 to show that

$$\deg(\mathrm{I} - \widetilde{\mathrm{F}}, \Omega(\sigma_1, \sigma_4, C^*)) = \deg(\mathrm{I} - \widetilde{\mathrm{F}}, \Omega_1) = 1 \tag{3.26}$$

and

$$\deg(\mathrm{I} - \widetilde{\mathrm{F}}, \Omega(\sigma_3, \sigma_2, C^*)) = \deg(\mathrm{I} - \widetilde{\mathrm{F}}, \Omega_2) = 1. \tag{3.27}$$

Using the additivity property of the Leray-Schauder topological degree we derive from (3.25)–(3.27) that

$$\deg(\mathrm{I} - \widetilde{\mathrm{F}}, \widetilde{\Omega}) = \deg(\mathrm{I} - \widetilde{\mathrm{F}}, \Omega_0) - \deg(\mathrm{I} - \widetilde{\mathrm{F}}, \Omega_1) - \deg(\mathrm{I} - \widetilde{\mathrm{F}}, \Omega_2) = -1.$$

Therefore, $\widetilde{\mathrm{F}}$ has a fixed point $u \in \widetilde{\Omega}$. By (3.24) we have $\|u\|_\infty < c$ and $\|u'\|_\infty < d$. This together with (3.4)–(3.6) and (3.23) yields that u is a solution to (1.1)–(1.3) fulfilling (2.13). □

References

[1] A. Cabada, J. J. Nieto, D. Franco and S. I. Trofimchuk. A generalization of the monotone method for second order periodic boundary value problem with impulses at fixed points. *Dynam. Contin. Discrete Impuls. Systems* **7** (2000), 145-158.

[2] Dong Yujun. Periodic solutions for second order impulsive differential systems. *Nonlinear Anal.* **27** (1996), 811-820.

[3] Hu Shouchuan and V. Laksmikantham. Periodic boundary value problems for second order impulsive differential systems. *Nonlinear Anal.* **13** (1989), 75-85.

[4] E. Liz and J. J. Nieto. Periodic solutions of discontinuous impulsive differential systems. *J. Math. Anal. Appl.* **161** (1991), 388-394.

[5] J. Mawhin. Topological Degree and Boundary Value Problems for Nonlinear Differential Equations. in: *Topological methods for ordinary differential equations.* (M. Furi and P. Zecca, eds.) Lect. Notes Math. 1537, Springer, Berlin, 1993, pp. 73-142.

[6] I. Rachůnková and M. Tvrdý. Impulsive periodic boundary value problem and topological degree. *Funct. Differ. Equ.* **9** (2002), no. 3-4, 471-498.

[7] I. Rachůnková and M. Tvrdý. Periodic boundary value problems for nonlinear second order differential equations with impulses - Part I. *Mathematical Institute of the Academy of Sciences of the Czech Republic*, Preprint **148/2002**.

[8] I. Rachůnková and M. Tvrdý. Periodic boundary value problems for nonlinear second order differential equations with impulses - Part II. *Mathematical Institute of the Academy of Sciences of the Czech Republic*, Preprint **151/2002**.

[9] Zhang Zhitao. Existence of solutions for second order impulsive differential equations. *Appl. Math., Ser. B (Engl. Ed.)* **12**, (1997), 307-320.

Irena Rachůnková, Department of Mathematics, Palacký University, 779 00 OLOMOUC, Tomkova 40, Czech Republic (`e-mail: rachunko@risc.upol.cz`)

Milan Tvrdý, Mathematical Institute, Academy of Sciences of the Czech Republic, 115 67 PRAHA 1, Žitná 25, Czech Republic (`e-mail: tvrdy@math.cas.cz`)