ON SYSTEMS OF LINEAR ALGEBRAIC EQUATIONS IN THE COLOMBEAU ALGEBRA

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Abstract. From the fact that the unique solution of a homogeneous linear algebraic system is the trivial one we can obtain the existence of a solution of the nonhomogeneous system. Coefficients of the systems considered are elements of the Colombeau algebra $\overline{\mathbb{R}}$ of generalized real numbers. It is worth mentioning that the algebra $\overline{\mathbb{R}}$ is not a field.

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1. Introduction

We shall consider the systems of linear equations

$$
\begin{array}{ccccc}
a_{11}x_1+ & a_{12}x_2+ & \dots & + & a_{1m}x_m = & b_1, \\
a_{21}x_1+ & a_{22}x_2+ & \dots & + & a_{2m}x_m = & b_2, \\
\vdots & \vdots & \ddots & & \vdots & \vdots \\
a_{n1}x_1+ & a_{n2}x_2+ & \dots & + & a_{nm}x_m = & b_n,
\end{array}
\tag{1.1}
$$

where a_{ij} $(i = 1, 2, \dots, n,\ j = 1, 2, \dots, m)$, b_i $(i = 1, 2, \dots, n)$ and x_j $(j = 1, 2, \dots, m)$ are elements of the Colombeau algebra $\overline{\mathbb{R}}$ of generalized real numbers. The coefficients a_{ij}, $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$, and b_i, $i = 1, 2, \dots, n$, are given, while $x_1, x_2, \dots, x_m$ are to be found. The multiplication, the summation and the equality of two elements from $\overline{\mathbb{R}}$ are meant in the Colombeau algebra sense. After extending these operations in a natural way to matrices and vectors with entries from $\overline{\mathbb{R}}$ we can rewrite the system (1.1) in the equivalent matrix form

$$
Ax = b. \tag{1.2}
$$

It is well-known that $\overline{\mathbb{R}}$ is a commutative algebra with the unit element and it is also well-known (cf. [4, pp. 6–7] or [3, Section 37]) that most of the theory known for determinants of matrices of real or complex numbers are applicable to determinants with elements in commutative rings with the unit element. In particular, if $\mathbf{X}$ is a commutative ring with the unit element, $\mathbf{X}^n$ is the space of column n-vectors with entries from $\mathbf{X}$, A is an $n \times m$-matrix whose columns are elements of $\mathbf{X}^n$ and $b \in \mathbf{X}^n$, then the determinant $\det(A)$ of A is defined in such a way that the following assertions are true:

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1.1. Proposition. *If* $\det(A)$ *possesses an inverse element* $(\det(A))^{-1}$ *in* $\mathbf{X}$, *then the given nonhomogeneous system* (1.1) *has a unique solution* x *for any right-hand side and this solution is given by*

$$x_i = \det(A_i)(\det(A))^{-1}, \quad i = 1, 2, \ldots, m,$$

where A_i *stands for the matrix obtained from* A *by replacing the* i*-th column by the column* b.

(For the *proof* see [4, p.6].) □

1.2. Proposition. *If the homogeneous system*

$$Ax = 0 \tag{1.3}$$

possesses a nontrivial solution, then $\det(A)$ *is not invertible in* $\mathbf{X}$.

(For the *proof* see [4, Proposition 1.1.2].) □

1.3. Proposition. *The system* (1.3) *possesses a nonzero solution if and only if there is a nonzero element* λ *of* $\mathbf{X}$ *such that* $\lambda \det(A) = 0$ *(i.e.* $\det(A)$ *is a divisor of the zero element* 0 *in* $\mathbf{X}$*).*

(For the *proof* see [3, Corollary of Theorem 51].) □

The aim of this paper is to prove some additional theorems on existence and uniqueness of solutions of the system (1.2). In particular, from the fact that the unique solution of the system (1.3) is the trivial one we obtain the existence and uniqueness of solutions of the system (1.2). The results of this paper will be applied in the investigation of boundary value problems for generalized differential equations in the Colombeau algebra (see [2]).

2. Algebra of generalized numbers

Let us recall here some basic facts concerning the Colombeau algebra of generalized numbers which are needed later on. For more details see e.g. [1].

As usual, we denote the space of real numbers by $\mathbb{R}$, while $\mathbb{N}$ stands for the set of natural numbers ($\mathbb{N} = \{1, 2, \ldots\}$).

Let $\mathcal{D}(\mathbb{R})$ be the set of all C^∞ functions $\mathbb{R} \mapsto \mathbb{R}$ with a compact support. For a given $q \in \mathbb{N}$ we denote by $\mathscr{A}_q$ the set of all functions $\varphi \in \mathcal{D}(\mathbb{R})$ such that the relations

$$\int_{-\infty}^{\infty} \varphi(t)\,\mathrm{d}t = 1, \quad \text{and} \quad \int_{-\infty}^{\infty} t^k \varphi(t)\,\mathrm{d}t = 0 \quad \text{for any} \quad 1 \le k \le q$$

hold. We have

$$\mathscr{A}_q \supsetneqq \mathscr{A}_{q+1} \quad \text{for any} \quad q \in \mathbb{N} \quad \text{and} \quad \bigcap_{q=1}^{\infty} \mathscr{A}_q = \emptyset. \tag{1.4}$$

For given $\varphi \in \mathcal{D}(\mathbb{R})$ and $\varepsilon > 0$, φ_ε is defined by

$$\varphi_\varepsilon(t) = \tfrac{1}{\varepsilon}\varphi\big(\tfrac{t}{\varepsilon}\big).$$

Now, we denote by $\mathscr{E}_0$ the set of all mappings from $\mathscr{A}_1$ into $\mathbb{R}$. Obviously, when equipped with naturally defined operations, $\mathscr{E}_0$ is a commutative algebra over the field $\mathbb{R}$ of real numbers and the mapping $\varphi \in \mathscr{A}_1 \mapsto 1 \in \mathbb{R}$ is its unit element. In particular, the product $R_1 \cdot R_2$ of the elements R_1 and R_2 of $\mathscr{E}_0$ is given by

$$R_1 \cdot R_2 : \ \varphi \in \mathscr{A}_1 \mapsto R_1(\varphi) R_2(\varphi) \in \mathbb{R}.$$

Furthermore, we denote by $\mathscr{E}_M$ the set of all moderate elements of $\mathscr{E}_0$ defined by

$$\begin{aligned}\mathscr{E}_M = \big\{R \in \mathscr{E}_0 : \quad & \exists\,(N \in \mathbb{N})\ \forall\,(\varphi \in \mathscr{A}_N)\ \exists\,(c > 0, \mu_0 > 0)\\ & \forall\,(\varepsilon \in (0, \mu_0)) \quad \big|R(\varphi_\varepsilon)\big| \le c\,\varepsilon^{-N}\big\}.\end{aligned} \tag{1.5}$$

Clearly $\mathscr{E}_M$ is a linear subspace and a subalgebra of $\mathscr{E}_0$.

By Γ we denote the set of all increasing mappings $\alpha : \mathbb{N} \mapsto \mathbb{R}^+$ such that

$$\lim_{q\to\infty} \alpha(q) = \infty$$

and we define an ideal $\mathscr{T}$ of $\mathscr{E}_M$ by

$$\begin{aligned}\mathscr{T} = \big\{R \in \mathscr{E}_0 : \quad & \exists\,(N \in \mathbb{N}, \alpha \in \Gamma)\ \forall\,(q \ge N, \varphi \in \mathscr{A}_q)\\ & \exists\,(c > 0, \mu_0 > 0)\ \forall\,(\varepsilon \in (0, \mu_0))\\ & \big|R(\varphi_\varepsilon)\big| \le c\,\varepsilon^{\alpha(q)-N}\big\}.\end{aligned} \tag{1.6}$$

The factor algebra

$$\overline{\mathbb{R}} = \frac{\mathscr{E}_M}{\mathscr{T}}$$

is called the *algebra of generalized numbers* (cf. [1, Sec.2.1]). For a given $x \in \overline{\mathbb{R}}$ we denote by R_x its representative ($R_x \in \mathscr{E}_M$) and write usually $x = [R_x]$ ($x = R_x + \mathscr{T}$). Obviously, $\overline{\mathbb{R}}$ is a commutative algebra with the unit element $1 = [R_1]$, where $R_1(\varphi) \equiv 1$ for any $\varphi \in \mathscr{A}_1$, and the zero element $0 = [R_0]$, where $R_0(\varphi) \equiv 0$ for any $\varphi \in \mathscr{A}_1$. Let us recall that for given $x, y \in \overline{\mathbb{R}}$ we have

$$xy = [R_x \cdot R_y] = R_x \cdot R_y + \mathscr{T}.$$

Furthermore, it is worth mentioning that $\overline{\mathbb{R}}$ possesses nonzero divisors of the zero element. In fact, let $a = [R_a] \in \overline{\mathbb{R}}$ and $a^* = [R_{a^*}] \in \overline{\mathbb{R}}$ be given by

$$R_a(\varphi) = \begin{cases} 1 & \text{if } \ \varphi \in \mathscr{A}_{2k-1} \setminus \mathscr{A}_{2k} \quad \text{for some} \quad k \in \mathbb{N},\\ 0, & \text{otherwise}\end{cases} \tag{2.1}$$

and

$$R_{a^*}(\varphi) = \begin{cases} 0 & \text{if } \ \varphi \in \mathscr{A}_{2k-1} \setminus \mathscr{A}_{2k} \quad \text{for some} \quad k \in \mathbb{N},\\ 1 & \text{otherwise.}\end{cases} \tag{2.2}$$

Obviously $R_a \cdot R_{a^*} \in \mathscr{T}$ and $R_{a^*} \cdot R_a \in \mathscr{T}$, i.e. $aa^* = a^*a = 0$, while both a and a^* are nonzero. It follows immediately that $\overline{\mathbb{R}}$ is not a field. In fact, let a and $a^* \in \overline{\mathbb{R}}$ be given respectively by (2.1) and (2.2) and let $x \in \overline{\mathbb{R}}$ be such that $ax = 1$. Then $0 = (a^*a)x = a^*(ax) = a^*$ would hold, while $a^* \ne 0$ according to the definition (2.2).

On the other hand, the algebra $\overline{\mathbb{R}}$ possesses the following helpful property.

2.1. Proposition. *If $a \in \overline{\mathbb{R}}$ is not invertible, then a is a divisor of the zero element 0 of $\overline{\mathbb{R}}$.*

For the proof of Proposition 2.1 the following lemma is helpful

2.2. Lemma. *Let us assume that*

$$\begin{aligned}& \exists\,(q^* \in \mathbb{N})\ \forall\,(\varphi \in \mathcal{A}_{q^*})\ \exists\,(d_{q^*,\varphi} > 0)\ \exists\,(\eta_{q^*,\varphi} > 0)\\ & \qquad \forall\,(\varepsilon \in (0, \eta_{q^*,\varphi})) \quad \big|R_a(\varphi_\varepsilon)\big| \ge d_{q^*,\varphi}\,\varepsilon^{q^*}.\end{aligned} \tag{2.3}$$

Then the element $a = [R_a] \in \overline{\mathbb{R}}$ is invertible in $\overline{\mathbb{R}}$.

Proof. Let the assumptions of the lemma be satisfied. Let us put

$$R_{a^*}(\varphi) = \begin{cases} \frac{1}{R_a(\varphi)} & \text{if } \varphi = \psi_\varepsilon \quad \text{for some } \psi \in \mathscr{A}_{q^*} \text{ and } \varepsilon \in (0, \eta_{q^*,\varphi}), \\ 1 & \text{otherwise.} \end{cases}$$

We shall show that then

$$R_a \cdot R_{a^*} - R_1 \in \mathscr{T},$$

i.e.

$$\begin{aligned} &\exists\, (N \in \mathbb{N}, \alpha \in \Gamma)\ \forall\, (q \geq N, \psi \in \mathscr{A}_q)\ \exists\, (c > 0, \eta > 0) \\ &\qquad \forall\, (\varepsilon \in (0, \eta)) \quad \big|R_a(\psi_\varepsilon) R_{a^*}(\psi_\varepsilon) - 1\big| \leq c\, \varepsilon^{\alpha(q) - N}. \end{aligned}$$

Indeed, let us put $N = q^*$ and let α be an arbitrary element of Γ. Then for any $q \geq N$ and any $\psi \in \mathscr{A}_q$ we have $\psi \in \mathscr{A}_{q^*}$. By the assumptions of the lemma there is an $\eta_{q^*,\varphi} > 0$ such that

$$R_a(\psi_\varepsilon) R_{a^*}(\psi_\varepsilon) = 1 \quad \text{for any } \varepsilon \in (0, \eta_{q^*,\psi}).$$

Thus, if we put

$$c = \frac{1}{d_{q^*,\psi}} \quad \text{and} \quad \eta = \eta_{q^*,\psi},$$

we complete the proof of the lemma. □

Proof of Proposition 2.1. Let us assume that (2.3) does not hold, i.e.

$$\begin{aligned} &\forall\, (m \in \mathbb{N})\ \exists\, (\varphi^{[m]} \in \mathcal{A}_m)\ \forall\, (c > 0, \eta > 0) \\ &\qquad \exists\, (\varepsilon \in (0, \eta)) \quad \big|R_a\big(\varphi_\varepsilon^{[m]}\big)\big| < \tfrac{1}{c}\, \varepsilon^m. \end{aligned} \tag{2.4}$$

As $\bigcap_{q=1}^\infty \mathscr{A}_q$ is empty (cf. (1.4)), for any $m \in \mathbb{N}$ there exists $r_m \in \mathbb{N} \cup \{0\}$ such that

$$\varphi^{[m]} \in \mathscr{A}_{m+r_m} \setminus \mathscr{A}_{m+r_m+1}.$$

Obviously, for any $m \in \mathbb{N}$ there is an $\overline{r_m} \in \mathbb{N} \cup \{0\}$ such that $0 \leq \overline{r_m} \leq r_m$ and

$$\varphi^{[m]}, \varphi^{[m+1]}, \ldots, \varphi^{[m+\overline{r_m}]} \in \mathscr{A}_{m+r_m} \setminus \mathscr{A}_{m+r_m+1},$$

while

$$\varphi^{[m+\overline{r_m}+1]} \notin \mathscr{A}_{m+r_m} \setminus \mathscr{A}_{m+r_m+1}.$$

Let us put

$$\psi^{[m]} = \varphi^{[m+\overline{r_m}]} \quad \text{for } m \in \mathbb{N}. \tag{2.5}$$

Clearly, since according to our definition

$$\varphi^{[m+\overline{r_m}+1]} \neq \psi^{[m]} \quad \text{for any } m \in N,$$

we have

$$\psi^{[m+1]} \neq \psi^{[m]} \quad \text{for any } m \in \mathbb{N}.$$

Furthermore, (2.4) implies that

$$\begin{aligned}&\forall\,(m\in\mathbb{N}, c>0, \eta\in(0,1))\ \exists\,(\beta_m\in(0,\eta))\\ &\big|R_a\big(\psi^{[m]}_{\beta_m}\big)\big| < \tfrac{1}{c}\,(\beta_m)^{m+\overline{r_m}}.\end{aligned} \tag{2.6}$$

(Let us notice that without any loss of generality we can assume that the relations

$$\varphi^{[m]}=\varphi^{[m+1]}=\dots=\varphi^{[m+\overline{r_m}]}=\psi^{[m]}$$

hold as well.)

Now, let us define a sequence $\{m_\ell\}_{\ell=1}^\infty$ by

$$m_\ell=\begin{cases}1 & \text{if } \ \ell=1,\\ m_{\ell-1}+\overline{r_{m_{\ell-1}}}+1 & \text{if } \ \ell\in\mathbb{N} \text{ and } \ \ell\ge 2.\end{cases}$$

Clearly, $m_{\ell+1}>m_\ell$ holds for any $\ell\in\mathbb{N}$ and

$$\lim_{\ell\to\infty} m_\ell=\infty.$$

Furthermore, for any $\ell\in\mathbb{N}$ we have

$$\begin{aligned}&\varphi^{[m_\ell]}=\varphi^{[m_\ell+1]}=\dots=\varphi^{[m_\ell+\overline{r_{m_\ell}}]}=\psi^{[m_\ell]},\\ &\psi^{[m_\ell]}\in\mathscr{A}_{m_\ell+\overline{r_{m_\ell}}}\setminus\mathscr{A}_{m_\ell+r_{m_\ell}+1}\subset\mathscr{A}_{m_\ell+r_{m_\ell}}\setminus\mathscr{A}_{m_\ell+r_{m_\ell}+1},\\ &\varphi^{[m_\ell+\overline{r_{m_\ell}}+1]}\notin\mathscr{A}_{m_\ell+r_{m_\ell}}\setminus\mathscr{A}_{m_\ell+r_{m_\ell}+1}\end{aligned}$$

and in virtue of (2.6) the sequence $\{\psi^{[m_\ell]}\}_{\ell=1}^\infty$ possesses the following property:

$$\begin{aligned}&\forall\,(\ell\in\mathbb{N}, c>0, \eta\in(0,1))\ \exists\,(\beta_{m_\ell}\in(0,\eta))\\ &\big|R_a\big(\psi^{[m_\ell]}_{\beta_{m_\ell}}\big)\big| < \tfrac{1}{c}\,(\beta_{m_\ell})^{m_\ell+\overline{r_{m_\ell}}}.\end{aligned} \tag{2.7}$$

Let us put $c=2$ and $\eta=\frac{1}{2}$ and let $\{\beta_{m_\ell}\}_{\ell=1}^\infty$ be the corresponding sequence from (2.7). Let us define a mapping $R_\lambda:\mathscr{A}_1\mapsto\mathbb{R}$ as follows:

$$R_\lambda:\ \varphi\in\mathscr{A}_1, \varepsilon>0\mapsto R_\lambda(\varphi_\varepsilon)=\begin{cases}1 & \text{if } \ \varphi=\psi^{[m_\ell]}_{\beta_{m_\ell}} \text{ and}\\ & \big|R_a\big(\psi^{[m_\ell]}_{\varepsilon\beta_{m_\ell}}\big)\big|<\tfrac{1}{2}\,(\varepsilon\beta_{m_\ell})^{m_\ell+\overline{r_{m_\ell}}}\\ & \text{for some } \ \ell\in\mathbb{N},\\ 0 & \text{otherwise.}\end{cases} \tag{2.8}$$

We claim that

$$R_\lambda\notin\mathscr{T}. \tag{2.9}$$

Indeed, if $R_\lambda\in\mathscr{T}$ then

$$\begin{aligned}&\exists\,(N\in\mathbb{N},\alpha\in\Gamma)\ \forall\,(q\ge N,\varphi\in\mathscr{A}_q)\ \exists\,(\overline{c},\overline{\eta}>0)\\ &\forall\,(\varepsilon\in(0,\overline{\eta}))\quad \big|R_\lambda\big(\varphi_\varepsilon\big)\big|<\overline{c}\,\varepsilon^{\alpha(q)-N}.\end{aligned} \tag{2.10}$$

Let arbitrary fixed $N \in \mathbb{N}$ and $\alpha \in \Gamma$ be given such that (2.10) holds. Without any loss of generality we may assume that

$$\alpha(N) > N \tag{2.11}$$

is true as well. Let $\ell_0 \in \mathbb{N}$ be such that

$$m_\ell + \overline{r_{m_\ell}} \geq N \quad \text{for any} \quad \ell \in \mathbb{N},\ \ell \geq \ell_0. \tag{2.12}$$

Then for any $\ell \in \mathbb{N}$ such that $\ell \geq \ell_0$ we have

$$\begin{aligned} &\exists\, (\overline{c_\ell} \geq 2, \overline{\eta}_\ell \in (0, \tfrac{1}{2}))\ \forall\, (\varepsilon \in (0, \overline{\eta}_\ell)) \\ &\qquad \big|R_\lambda\big((\psi_\varepsilon^{[m_\ell]})_{\beta_{m_\ell}}\big)\big| < \overline{c_\ell}\, (\varepsilon\beta_{m_\ell})^{\alpha(m_\ell + \overline{r_{m_\ell}}) - N}. \end{aligned} \tag{2.13}$$

Now, let $\{\eta_k\}_{k=1}^\infty$ be an arbitrary decreasing sequence in $(0,1)$ such that

$$\lim_{k\to\infty} \eta_k = 0. \tag{2.14}$$

According to (2.7) we have

$$\begin{aligned} &\forall\, (\ell \in \mathbb{N}, k \in \mathbb{N})\ \exists\, (\beta_{m_\ell}^{[k]} \in (0, \eta_k)) \\ &\qquad \big|R_a(\psi^{[m_\ell]}_{\beta_{m_\ell}^{[k]}})\big| < \tfrac{1}{\overline{c_\ell}}\, (\beta_{m_\ell}^{[k]})^{m_\ell + \overline{r_{m_\ell}}}. \end{aligned} \tag{2.15}$$

In particular, the relations (2.14) and (2.15) imply that

$$\lim_{k\to\infty} \beta_{m_\ell}^{[k]} = 0 \quad \text{for any} \quad \ell \in \mathbb{N},\ \ell \geq \ell_0, \quad \text{fixed}. \tag{2.16}$$

Thus, if we put

$$\varepsilon_{k,\ell} = \frac{\beta_{m_\ell}^{[k]}}{\beta_{m_\ell}} \qquad \text{for} \quad k, \ell \in \mathbb{N}$$

we obtain

$$\begin{aligned} \big|R_\lambda\big((\psi^{[m_\ell]}_{\beta_{m_\ell}})_{\varepsilon_{k,\ell}}\big)\big| &= \big|R_\lambda(\psi^{[m_\ell]}_{\beta^{[k]}_{m_\ell}})\big| \\ &< \tfrac{1}{\overline{c_\ell}}\, (\beta_{m_\ell}^{[k]})^{m_\ell + \overline{r_{m_\ell}}} \leq \tfrac{1}{2}\, (\beta_{m_\ell}^{[k]})^{m_\ell + \overline{r_{m_\ell}}}. \end{aligned}$$

According to the definition (2.8) this means that for all $\varphi = \psi^{[m_\ell]}_{\beta_{m_\ell}}$, $k \in \mathbb{N}$ and $\ell \in \mathbb{N}$ such that $\ell \geq \ell_0$ we have

$$R_\lambda(\varphi_{\varepsilon_{k,\ell}}) = R_\lambda\big((\psi^{[m_\ell]}_{\beta_{m_\ell}})_{\varepsilon_{k,\ell}}\big) = R_\lambda(\psi^{[m_\ell]}_{\beta^{[k]}_{m_\ell}}) = 1. \tag{2.17}$$

On the other hand, (2.13) yields

$$\big|R_\lambda(\varphi_{\varepsilon_{k,\ell}})\big| < \overline{c_\ell}\, (\beta_{m_\ell}^{[k]})^{\alpha(m_\ell + \overline{r_{m_\ell}}) - N} \quad \text{for all} \quad k, \ell \in \mathbb{N} \text{ such that } \ell \geq \ell_0.$$

Consequently, as by (2.11) and (2.12) we have $\alpha(m_\ell + \overline{r_{m_\ell}}) > N$ and thus by (2.16)

$$\lim_{k\to\infty} \overline{c_\ell}(\beta_{m_\ell}^{[k]})^{\alpha(m_\ell + \overline{r_{m_\ell}}) - N} = 0,$$

we obtain that for any $\ell \geq \ell_0$ there is a k_0 such that

$$\left|R_\lambda(\varphi_{\varepsilon_{k,\ell}})\right| < 1 \quad \text{for any} \quad k \geq k_0,$$

which contradicts (2.17). This proves the relation (2.9).

Now, we will prove that the relation

$$R_\lambda \cdot R_a \in \mathscr{T} \tag{2.18}$$

is true as well. To this purpose let us define a mapping $\alpha^* : \mathbb{N} \mapsto \mathbb{R}^+ = (0, \infty)$ as follows:

$$\alpha^*(q) = \begin{cases} \dfrac{1}{2}\left(1 + \dfrac{q-1}{m_2}\right) & \text{if } \ 1 \leq q \leq m_2, \\ m_{\ell-1} + \dfrac{(q-m_\ell)(m_{\ell+1}-m_\ell)}{m_{\ell+1}-m_\ell} & \text{if } \ m_\ell < q \leq m_{\ell+1} \\ & \qquad \text{and } \ \ell \geq 2. \end{cases}$$

Since obviously

$$\alpha^*(m_1) = \alpha^*(1) = \tfrac{1}{2} \quad \text{and} \quad \alpha^*(m_\ell) = m_{\ell-1} \quad \text{for} \quad \ell = 2, 3, \dots,$$

it is easy to verify that $\alpha^* \in \Gamma$. Furthermore, according to the definition (2.8) we have for any $\varphi \in \mathscr{A}_1$

$$R_\lambda(\varphi_\varepsilon) R_a(\varphi_\varepsilon) = \begin{cases} R_a(\varphi_\varepsilon) & \text{if } \ \varphi = \psi^{[m_\ell]}_{\beta_{m_\ell}} \text{ and} \\ & \left|R_a\big(\psi^{[m_\ell]}_{\varepsilon\beta_{m_\ell}}\big)\right| < \tfrac{1}{2}\left(\varepsilon\beta_{m_\ell}\right)^{m_\ell + \overline{r_{m_\ell}}} \\ & \text{for some } \ \ell \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Let arbitrary $N, q \in \mathbb{N}$ be given such that $q \geq N$, then there is a unique $\ell \in \mathbb{N}$ such that $q \in \mathbb{N} \cap (m_\ell, m_{\ell+1}]$. Since $\beta_{m_\ell} < 1$, it follows that for all $\varphi \in \mathscr{A}_q$ and $\varepsilon \in (0,1)$ we have

$$\left|R_\lambda(\varphi_\varepsilon) R_a(\varphi_\varepsilon)\right| \leq \varepsilon^{m_\ell + \overline{r_{m_\ell}}} \leq \varepsilon^{m_{\ell-1}} \leq \varepsilon^{\alpha^*(q) - N}.$$

(Let us recall that $\mathscr{A}_q \subset \mathscr{A}_{m_{\ell-1}}$ in such a case.) Consequently, if we choose $N \in \mathbb{N}$ arbitrarily (e.g. $N = 1$) then for all $q \in \mathbb{N}$ such that $q \geq N$, any $\varphi \in \mathscr{A}_q$ and any $\varepsilon \in (0,1)$ we get

$$\left|R_\lambda(\varphi_\varepsilon) R_a(\varphi_\varepsilon)\right| \leq \varepsilon^{\alpha^*(q) - N},$$

i.e. (2.18) is true. $\square$

2.3. Vectors and matrices of generalized numbers. Let us put

$$\overline{\mathbb{R}}^n = \underbrace{\overline{\mathbb{R}} \times \cdots \times \overline{\mathbb{R}}}_{n \text{ times}}.$$

The elements of $\overline{\mathbb{R}}^n$ will be considered as column n-vectors, i.e. $1 \times n$-matrices of generalized numbers. For a given $n \times m$-matrix A of generalized numbers, its entries will be denoted by a_{ij} ($A = (a_{ij}) = (a_{ij})_{\substack{i=1,\dots,n \\ j=1,\dots,m}}$.) Given an $n \times m$-matrix A of generalized numbers and an $m \times k$-matrix

B of generalized numbers, their product AB is the $n \times k$-matrix of generalized numbers defined in the natural way and the transpose of A is denoted as usual by A^T.

Obviously, if $A = (a_{ij})_{\substack{i=1,\dots,n\\ j=1,\dots,m}}$ is a given matrix of generalized numbers, then

$$x = (x_1, x_2, \dots, x_m)^T \in \overline{\mathbb{R}}^n$$

is a solution of the system (1.2) if and only if it satisfies the system of relations

$$R_{a_{i1}} \cdot R_{x_1} + R_{a_{i2}} \cdot R_{x_2} + \dots + R_{a_{im}} \cdot R_{x_m} - R_{b_i} \in \mathscr{T}, \quad i = 1, 2, \dots, n.$$

For a given $q \in \mathbb{N}$, the symbol $\mathbb{N}_q$ denotes the subset $\{1, 2, \dots, q\}$ of $\mathbb{N}$. For a given subset M of $\mathbb{N}$, we will denote by $\nu(\mathrm{M})$ the number of its elements. Let $A = (a_{ij})$ be an $n \times m$-matrix $(n, m > 1)$ of generalized numbers and let $\mathrm{U} \subset \mathbb{N}_n$ and $\mathrm{V} \subset \mathbb{N}_m$ be given such that $\nu(\mathrm{U}) \le n - 1$ and $\nu(\mathrm{V}) \le m - 1$. Then the symbol $A_{\mathrm{U},\mathrm{V}}$ stands for the matrix obtained from the matrix A by deleting the rows with the indices $i \in \mathfrak{U}$ and the columns with the indices $j \in \mathrm{V}$. If $\mathrm{U} = \{i\}$ and $\mathrm{V} = \{j\}$, then we write

$$A_{\mathrm{U},\mathrm{V}} = A_{i,j}.$$

We say that the minor $\det(A_{\mathrm{U},\mathrm{V}})$ of the matrix A is of the k-th order if $k > 0$ and $n - \nu(\mathrm{U}) = m - \nu(\mathrm{V}) = k$. For a given $r \in \mathbb{N}$ such that $1 \le r \le \min(n, m)$, the symbol $A^{(r)}$ stands for the submatrix $(a_{ij})_{\substack{i=1,\dots,r\\ j=1,\dots,r}}$ of the matrix $A = (a_{ij})_{\substack{i=1,\dots,n\\ j=1,\dots,m}}$. Let an $n \times n$-matrix A and a couple $i, j \in \mathbb{N}_n$ of indices be given. Then we define the cofactor $\mathcal{D}_{i,j}$ of a_{ij} in A by

$$\mathcal{D}_{i,j} = (-1)^{i+j} \det(A_{i,j}).$$

The $n \times (m+1)$-matrix obtained when we attach a column $b \in \overline{\mathbb{R}}^m$ to the columns of a given $n \times m$-matrix A of generalized numbers will be denoted by (A, b).

If A has not only zero elements, then the highest order r of nonzero minors is called the rank of A and will be denoted by rank (A). If A is the zero matrix, we put rank $(A) = 0$.

3. Main results

Before formulating the main results of the paper let us give several simple examples indicating that under our assumptions the situation is even in the case $m = n = 1$ more complicated than in the classical case.

Let $a \in \overline{\mathbb{R}}$ and $b \in \overline{\mathbb{R}}$ be given and let us consider the equations

$$ax = b \tag{3.1}$$

and

$$ax = 0. \tag{3.2}$$

a) If a is given by (2.1), then $a \neq 0$ and as mentioned above there exists a nonzero generalized number $a^* \in \overline{\mathbb{R}}$ (cf. (2.2)) such that $aa^* = a^*a = 0$. This shows that the homogeneous equation (3.2) with $a \neq 0$ may in general possess nonzero solutions.

b) Furthermore, it was also mentioned above that if a is given by (2.1), then a is noninvertible, i.e. the equation (3.2) possesses for $b = 1$ no solutions, though a is nonzero. Let us notice that in this case we have
$$\text{rank } (A) = \text{rank } (A, b) = 1.$$

c) Let a be given by (2.1) and let $b = a$. Then $(A, b) = (a, a)$, rank $(A) = rank(A, b) = 1$ and $x = 1$ is evidently a solution to the equation (3.1) (i.e. $ax = a$).

Our main results are the following theorems.

Theorem 3.1. *Let $m = n$ and let the zero vector be the unique solution of the system* (1.3) *in* $\overline{\mathbb{R}}^n$. *Then the system* (1.2) *has exactly one solution x in $\overline{\mathbb{R}}^n$ for any $b \in \overline{\mathbb{R}}^n$.*

Theorem 3.2. *Let us assume that* $\operatorname{rank}(A) = \operatorname{rank}(A, b) = r \geq 1$ *and that there are subsets* U *and* V *of the set* $\mathbb{N}_{\min(n,m)}$ *such that* $\nu(\mathrm{U}) = \nu(\mathrm{V}) = r$ *and* $\det(A_{\mathrm{U},\mathrm{V}})$ *is invertible in* $\overline{\mathbb{R}}$. *Then the system* (1.2) *has at least one solution* $x \in \overline{\mathbb{R}}^m$.

Theorem 3.3. *Let us assume that the system* (1.2) *has a solution* $x \in \overline{\mathbb{R}}^m$. *Then*

$$\operatorname{rank}(A) = \operatorname{rank}(A, b). \tag{3.3}$$

4. Proofs

Proof of Theorem 3.1. Let $m = n$ and let $x = 0 \in \overline{\mathbb{R}}^n$ be the only solution of the homogeneous system (1.3).

Let us assume that $\det(A)$ is not invertible in $\overline{\mathbb{R}}$. Then by Proposition 2.1, $\det(A)$ is a divisor of the zero element in $\overline{\mathbb{R}}$ and hence by Proposition 1.3 the system (1.3) possesses a nonzero solution. This being contradictory to our assumptions, it follows immediately that under the assumptions of the theorem $\det(A)$ has to be invertible in $\overline{\mathbb{R}}$. The proof of Theorem 3.1 is now easily completed by making use of Proposition 1.1. □

4.2. Proof of Theorem 3.2. Without any loss of generality we may assume that $\det(A^{(r)}) \neq 0$ and $\det(A^{(r)})$ is invertible in $\overline{\mathbb{R}}$. Furthermore, let us assume that $r < m$. The modification of the proof in the case $r = m$ is obvious.

Let an arbitrary vector $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{m-r})^T \in \overline{\mathbb{R}}^{m-r}$ be given. Let us denote

$$\widetilde{b}_i = b_i - \sum_{k=r+1}^{m} a_{ik} \lambda_{k-r} \quad \text{for} \quad i = 1, 2, \ldots, n$$

and

$$\widetilde{b} = (\widetilde{b}_1, \widetilde{b}_2, \ldots, \widetilde{b}_r)^T.$$

By Proposition 1.1 there exists the unique solution $y = (y_1, y_2, \ldots, y_r)^T$ to the system

$$A^{(r)} y = \widetilde{b}$$

and this solution is given by

$$y_j = \det(A_j^{(r)})(\det(A^{(r)}))^{-1}, \quad j = 1, 2, \ldots, r,$$

where for a given $j = 1, 2, \ldots, r$, the symbol $A_j^{(r)}$ denotes the matrix obtained from the matrix $A^{(r)}$ by replacing the j-th column by the vector $\widetilde{b}$. If $r = n$, then $x = y$ is a solution of the given system (1.2) and the proof of the assertion of the theorem is obvious, of course. If $r < n$ then analogously to the classical case (when $a_{ij}, b_i \in \mathbb{R}$) for any $i = r+1, r+2, \ldots, n$ and any $\lambda \in \overline{\mathbb{R}}^{m-r}$ we obtain

$$\begin{aligned} &\Big(\sum_{j=1}^{r} a_{ij} y_j - \widetilde{b}_i\Big) \det(A^{(r)}) \\ &\quad = \sum_{j=1}^{r} a_{ij} \det(A_j^{(r)}) - \widetilde{b}_i \det(A^{(r)}) = -\det(A^{r,i}, \widetilde{b}^i), \end{aligned} \tag{4.1}$$

where the $(r+1)\times(r+1)$-matrix $(A^{r,i},\widetilde{b^i})$ is given by

$$(A^{r,i},\widetilde{b^i}) = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1r} & \widetilde{b_1} \\ a_{21} & a_{22} & \dots & a_{2r} & \widetilde{b_2} \\ \vdots & & \ddots & & \vdots \\ a_{r1} & a_{r2} & \dots & a_{rr} & \widetilde{b_r} \\ a_{i1} & a_{i2} & \dots & a_{ir} & \widetilde{b_i} \end{pmatrix}$$

It is easy to verify that if we denote by $(A^{r,i},\overline{b^i})$ the $(r+1)\times(r+1)$-matrix given by

$$(A^{r,i},\overline{b^i}) = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1r} & b_1 \\ a_{21} & a_{22} & \dots & a_{2r} & b_2 \\ \vdots & & \ddots & & \vdots \\ a_{r1} & a_{r2} & \dots & a_{rr} & b_r \\ a_{i1} & a_{i2} & \dots & a_{ir} & b_i \end{pmatrix},$$

then the following relation is true:

$$\det(A^{r,i},\widetilde{b^i}) = \det(A^{r,i},\overline{b^i}).$$

By the assumption of the theorem we have

$$\det(A^{r,i},\overline{b^i}) = 0,$$

of course. Consequently, since $\det(A^{(r)})$ is assumed to be invertible, it follows easily from the relation (4.1) that the relations

$$a_{i1}y_1 + a_{i2}y_2 + \cdots + a_{ir}y_r = \widetilde{b}_i, \quad i = 1,2,\dots,n$$

are true. Thus, if we set

$$x_i = y_i \quad \text{for} \quad i = 1,2,\dots,r \quad \text{and} \quad x_i = \lambda_{i-r} \quad \text{for} \quad i = r+1, r+2,\dots,n,$$

then the vector $x = (x_1,x_2,\dots,x_n)^T$ is the desired solution to the given system (1.2). □

Proof of Theorem 3.3. Let us assume that the system (1.2) possesses a solution $x = (x_1,x_2,\dots,x_m)^T \in \overline{\mathbb{R}}^m$. Let us put again $r = \operatorname{rank}(A)$. If $r = m$ or $r = 0$, then the proof of the theorem is obvious. Let us assume $0 < r < m$. Furthermore, without any loss of generality we can assume that

$$\det(A^{(r)}) \neq 0$$

holds.

Obviously we have

$$\operatorname{rank}(A,b) \geq r. \tag{4.2}$$

Let us denote $y = (x_1,x_2\dots,x_r)^T$ and $\overline{b} = (b_1,b_2,\dots,b_r)^T$. Then the relation

$$A^{(r)}y = \widetilde{b} = \overline{b} - (\sum_{j=r+1}^{m} a_{ij}x_j)_{i=1,\dots,r}$$

is true. Analogously to the proof of Theorem 3.2 we could show that for any $i = r+1, r+2,\dots,m$ the determinant of the matrix $(A^{r,i},\overline{b^i})$ vanishes. Consequently, we have $\operatorname{rank}(A,b) \leq r$ wherefrom with respect to (4.2) our assertion immediately follows. □

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