Periodic problem with quasilinear differential operator and weak singularity

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Abstract. We study the singular periodic boundary value problem of the form

$$\left(|u'|^{p-2}\,u'\right)' = f(t,u), \quad u(0) = u(T), \quad u'(0) = u'(T),$$

where $p \in (1,\infty)$ and $f \in Car([0,T] \times (0,\infty)$ can have a repulsive space singularity at $x = 0$. Contrary to previous results by Mawhin and Jebelean, Liu Bing and Rachůnková and Tvrdý, we need not assume any strong force conditions. Our main existence results rely on a new antimaximum principle for periodic quasilinear periodic problem, which has an independent meaning.

Keywords. Singular problem, periodic problem, Dirichlet problem, p-Laplacian, repulsive singularity, weak singularity, lower and upper functions, antimaximum principle, quasilinear equation

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1. Introduction

This paper deals with singular periodic problems of the form

$$(\phi_p(u'))' = f(t,u), \tag{1.1}$$

$$u(0) = u(T), \quad u'(0) = u'(T), \tag{1.2}$$

where

$$0 < T < \infty, \quad p \in (1,\infty), \quad \phi_p(y) = |y|^{p-2}\,y \quad \text{for} \quad y \in \mathbb{R} \tag{1.3}$$

and f satisfies the Carathéodory conditions on $[0,T]\times(0,\infty)$, i.e. f has the following properties: (i) for each $x \in (0,\infty)$ the function $f(.,x)$ is measurable on $[0,T]$; (ii) for

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almost every $t \in [0,T]$ the function $f(t,.)$ is continuous on $(0,\infty)$; (iii) for each compact set $K \subset (0,\infty)$ the function $m_{\mathrm{K}}(t) = \sup_{x\in K} |f(t,x)|$ is Lebesgue integrable on $[0,T]$.

Second order nonlinear differential equations or systems with singularities appear naturally in the description of particles submitted to Newtonian type forces or to forces caused by compressed gases, see e.g. [12], [15] or [16]. The mathematical interest in periodic singular problems increased when the paper [22] by Lazer and Solimini appeared in 1987. Motivated by the model equation $u'' = a\,u^{-\alpha} + e(t)$ with $\alpha > 0, a \neq 0$ and e integrable on $[0,T]$, they investigated the existence of positive solutions to the Duffing equation $u'' = g(u)+e(t)$ using topological arguments and the lower and upper functions method. The restoring force g was allowed to have an attractive singularity or a strong repulsive singularity at origin. The results by Lazer and Solimini have been generalized or extended e.g. by Habets and Sanchez [18], Mawhin [27], del Pino, Manásevich and Montero [10], Omari and Ye [29], Zhang [42] and [44], Ge and Mawhin [17], Rachůnková and Tvrdý [32] or Rachůnková, Tvrdý and Vrkoč [37]. All of these papers, when dealing with the repulsive singularity, supposed that the strong force condition is satisfied. For the case of the weak singularity, first results were delivered by Rachůnková, Tvrdý and Vrkoč in [36]. Further results were delivered later also by Bonheure and De Coster [2] and Torres [39].

Regular periodic problems with $\phi-$ or p-Laplacian on the left hand side were considered by several authors, see e.g. del Pino, Manásevich and Murúa [11] or Yan [41]. General existence principles for the regular vector problem, based on the homotopy to the averaged nonlinearity, were presented by Manásevich and Mawhin in [25] (see also Mawhin [28]).

In the well-ordered case, the lower/upper functions method was extended to periodic problems with a ϕ-Laplacian operator on the left hand side by Cabada and Pouso in [5], Jiang and Wang in [21] and Staněk in [38]. The general existence principle valid also when lower/upper functions are non-ordered was given by Rachůnková and Tvrdý in [34] and, for the case when impulses are admitted, also in [33].

The singular periodic problem for the Liénard type equation

$$\big(|u'|^{p-2}\,u'\big)' + h(u)\,u' = g(u) + e(t) \tag{1.4}$$

with g having either an attractive singularity or a strong repulsive singularity at $x = 0$ was treated by Liu [24], Jebelean and Mawhin [19] and [20] and Rachůnková and Tvrdý [35].

Let us recall that a function g is said to have an *attractive* singularity at $x = 0$ if

$$\liminf_{x\to 0+} g(x) = -\infty.$$

Alternatively, we say that g has a *repulsive* singularity at the origin if

$$\limsup_{x\to 0+} g(x) = +\infty \tag{1.5}$$

and g has a *strong repulsive* singularity at the origin if

$$\lim_{x\to 0+} \int_x^1 g(s)\,\mathrm{d}s = +\infty. \tag{1.6}$$

For a more detailed survey of the recent development we refer to [31, Section 5].

The main goal of this paper is a new existence result, Theorem 4.4, for problem (1.1), (1.2). As in [36, Theorem 2.5] (see also [31, Theorem 5.26]), where the classical case $p = 2$ was treated, we need not to assume that f satisfies any strong force condition. Our main tools are the lower and upper function method and a generalization of a classical antimaximum principle to the quasilinear periodic problem

$$(\phi_p(u'))' + \lambda\,\phi_p(u) = e(t), \quad u(0) = u(T), \quad u'(0) = u'(T) \tag{1.7}$$

established below in Theorem 3.2.

Our main result applies, in particular, to the Duffing type model problem

$$(\phi_p(u'))' + \mu_p\, u^{p-1} = a\, u^{-\alpha} + e(t), \quad u(0) = u(T), \quad u'(0) = u'(T), \tag{1.8}$$

where $a > 0,\ \alpha > 0,\ 1 < p \le 2$ and $\mu_p = (\pi_p/T)^p$ is the least eigenvalue of a homogeneous Dirichlet problem related to (1.7) and $e \in L_1[0,T]$. In particular, we get that problem (1.8) has a positive solution if $\inf \operatorname{ess}_{t\in[0,T]} e(t) > 0$. It is worth mentioning that for $\alpha \in (0,1)$ the function $g(x) = a\,x^{-\alpha}$ does not satisfy the strong force condition (1.6).

2. Preliminaries

As usual, for an arbitrary subinterval I of $\mathbb{R}$ we denote by $C(I)$ the set of functions $x : I \to \mathbb{R}$ which are continuous on I, $C^1[0,T]$ stands for the set of functions $x \in C[0,T]$ with the first derivative continuous on $[0,T]$. Further, $L_1[0,T]$ is the set of functions $x : [0,T] \to \mathbb{R}$ which are measurable and Lebesgue integrable on $[0,T]$. $AC[0,T]$ is the set of functions absolutely continuous on $[0,T]$. For $x \in L_1[0,T]$ we put

$$\|x\|_\infty = \sup_{t\in[0,T]} \operatorname{ess} |x(t)| \quad \text{and} \quad \overline{x} = \frac{1}{T}\int_0^T x(s)\,\mathrm{d}s.$$

If $f : [0,T] \times (0,\infty) \to \mathbb{R}$ satisfies the Carathéodory conditions on $[0,T]\times(0,\infty)$ we write

$$f \in Car([0,T]\times(0,\infty)). \tag{2.1}$$

2.1 . Definition. A function $u : [0,T] \to \mathbb{R}$ is a *solution* to problem (1.1,) (1.2) if $\phi_p(u') \in AC[0,T]$, $u > 0$ on $[0,T]$, $(\phi_p(u'(t)))' = f(t,u(t))$ for a.e. $t \in [0,T]$, $u(0) = u(T)$ and $u'(0) = u'(T)$.

Notice that the requirement $\phi(u') \in AC[0,T]$ implies that $u \in C^1[0,T]$.

The singular problem (1.1), (1.2) will be also investigated through regular auxiliary problems of the form

$$(\phi_p(u'))' = \widetilde{f}(t,u), \quad u(0) = u(T), \quad u'(0) = u'(T), \tag{2.2}$$

or

$$(\phi_p(u'))' = \widetilde{f}(t,u), \quad u(a) = u(b) = 0, \tag{2.3}$$

where $\widetilde{f} \in Car([0,T] \times \mathbb{R})$ and $a, b \in \mathbb{R}$, $a < b$. As usual, by a *solution of problem* (2.2) we understand a function u such that $\phi_p(u') \in AC[0,T]$, (1.2) is true and $(\phi_p(u'(t)))' = \widetilde{f}(t,u(t))$ for a.e. $t \in [0,T]$. Analogously, u is a solution to (2.3) if $\phi_p(u') \in AC[a,b]$, $u(a) = u(b) = 0$ and $(\phi_p(u'(t)))' = \widetilde{f}(t,u(t))$ for a.e. $t \in [a,b]$.

The lower and upper functions method combined with the topological degree argument is an important tool for proofs of solvability of boundary value problems. For our purposes the following definitions of lower and upper functions associated with problems (2.2) or (2.3) are suitable.

2.2 . Definition. Let $\widetilde{f} \in Car([0,T] \times \mathbb{R})$. We say that a function $\sigma \in C[0,T]$ is a *lower function* of problem (2.2) if $\phi_p(\sigma') \in AC([0,T])$ and

$$\begin{cases} (\phi_p(\sigma'(t)))' \geq \widetilde{f}(t,\sigma(t)) & \text{for a.e. } t \in [0,T], \\ \sigma(0) = \sigma(T), \quad \sigma'(0) \geq \sigma'(T). \end{cases} \tag{2.4}$$

Analogously, $\sigma \in C[0,T]$ is a *lower function* of (2.3) if $\phi_p(\sigma') \in AC([a,b])$ and

$$\begin{cases} (\phi_p(\sigma'(t)))' \geq \widetilde{f}(t,\sigma(t)) & \text{for a.e. } t \in [a,b], \\ \sigma(a) \leq 0, \quad \sigma(b) \leq 0. \end{cases} \tag{2.5}$$

If the inequalities in (2.4) or (2.5) are reversed, then σ is called an *upper function* of (2.2) or of (2.3), respectively.

The next two assertions based on the lower and upper functions method will be useful for our purposes.

2.3 . Proposition. ([34, Theorem 3.2] or [31, Lemma 5.9]) *Assume* (1.3) *and* $\widetilde{f} \in Car([0,T] \times \mathbb{R})$. *Furthermore, let* σ_1 *and* σ_2 *be a lower and an upper function of* (2.2) *and let there be* $m \in L_1[0,T]$ *such that*

$$\widetilde{f}(t,x,y) > m(t) \ (\textit{or } \widetilde{f}(t,x) < m(t)) \ \textit{ for a.e. } t \in [0,T] \textit{ and all } x \in \mathbb{R}.$$

Then problem (2.2) *has a solution* u *such that*

$$\min\{\sigma_1(\tau_u), \sigma_2(\tau_u)\} \leq u(\tau_u) \leq \max\{\sigma_1(\tau_u), \sigma_2(\tau_u)\} \quad \textit{for some } \tau_u \in [0,T].$$

2.4. Proposition. ([6, Theorem 2.1] or [40] or [30, Lemma 3.2] or [7, Theorem 3.5]) *Assume* (1.3), $\widetilde{f} \in Car([0,T] \times \mathbb{R})$ *and let* $a, b \in \mathbb{R}$, $a < b$ *be given. Furthermore, let* σ_1 *and* σ_2 *be a lower and an upper function of* (2.3) *such that* $\sigma_1 \le \sigma_2$ *on* $[a,b]$.

Then problem (2.3) *has a solution* u *such that* $\sigma_1 \le u \le \sigma_2$ *on* $[a,b]$.

3. Sign properties of quasilinear periodic problems

First, let us recall some basic known facts concerning initial value problems of the form

$$(\phi_p(u'))' + \lambda\, \phi_p(u) = 0, \tag{3.1}$$

$$u(t_0) = 0, \quad u'(t_0) = d, \tag{3.2}$$

where $p \in (1, \infty)$, $t_0 \in \mathbb{R}$, $\lambda \in \mathbb{R}$ and $d \in \mathbb{R}$. As in [8] (see also e.g. [1], [9], [13], [14], [43], [45], [26]), let us put

$$\pi_p = 2\,(p-1)^{1/p} \int_0^1 (1 - s^p)^{-1/p}\, \mathrm{d}s.$$

Clearly, $\pi_2 = \pi$. Furthermore, it is known that

$$\pi_p = 2\,(p-1)^{\frac{1}{p}}\, \frac{(\pi/p)}{\sin(\pi/p)}\,.$$

(See [14, Sec. 1.1.2], but take into account that our definition differs slightly from that used in [14], where $\pi_p = 2 \int_0^1 (1 - s^p)^{-1/p}\, \mathrm{d}s$.) It is known (see [14, Theorem 1.1.1]) that for each $t_0 \in \mathbb{R}$, $\lambda \in \mathbb{R}$ and $d \in \mathbb{R}$ problem (3.1), (3.2) has a unique solution u on $\mathbb{R}$ which can be, by [8, sec. 3]), expressed as

$$u(t) = d\,\lambda^{-1/p} \sin_p(\lambda^{1/p}\,(t - t_0)) \quad \text{for } t \in \mathbb{R},$$

where the function $\sin_p : \mathbb{R} \to [-(p-1)^{1/p}, (p-1)^{1/p}]$ is defined as follows.

Let $w :\, [0, \pi_p/2] \to [0, (p-1)^{1/p}]$ be the inverse function to

$$x \in [0, \pi_p/2] \mapsto \int_0^x \frac{\mathrm{d}s}{(1 - \frac{s^p}{p-1})^{1/p}} \in [0, (p-1)^{1/p}].$$

Further, put $\widetilde{w}(t) = w(\pi_p - t)$ for $t \in [\pi_p/2, \pi_p]$ and then $\widetilde{w}(t) = -\widetilde{w}(-t)$ for $t \in [-\pi_p, 0]$. Finally, we define $\sin_p : \mathbb{R} \to \mathbb{R}$ as the $2\,\pi_p-$ periodic extension of $\widetilde{w}$ to the whole $\mathbb{R}$. In particular, if $d = 0$, then $u \equiv 0$ on $\mathbb{R}$. Obviously, we have

$$\begin{aligned} &\sin_p(t) = 0 && \text{if and only if } t = n\,\pi_p,\ n \in \mathbb{N} \cup \{0\}, \\ &\sin_p(t) = (p-1)^{1/p} && \text{if and only if } t = (2\,n+1)\,\frac{\pi_p}{2},\ n \in \mathbb{N} \cup \{0\}, \end{aligned}$$

and

$$\sin_p(t) > 0 \qquad\qquad \text{for } \ t \in (2\,n\,\pi_p, (2\,n+1)\,\pi_p), \ \ n \in \mathbb{N} \cup \{0\}.$$

As a corollary, we immediately obtain that for given $a, b \in \mathbb{R}$, $a < b$, the corresponding Dirichlet problem

$$(\phi_p(u'))' + \lambda\,\phi_p(u) = 0, \quad u(a) = u(b) = 0 \tag{3.3}$$

possesses a nontrivial solution, i.e. λ is an eigenvalue for (3.3), if and only if

$$\lambda \in \left\{ \left(\frac{n\,\pi_p}{b-a} \right)^p : \ n \in \mathbb{N} \cup \{0\} \right\}. \tag{3.4}$$

In particular,

$$\mu_p = \left(\frac{\pi_p}{T}\right)^p. \tag{3.5}$$

is the least eigenvalue for (3.3) with $b - a = T$, wherefrom the following assertion follows.

3.1. Proposition. *Let* $p \in (1,\infty)$, $a, b \in \mathbb{R}$, $a < b$, *and let* $\lambda = \mu_p$, *where* μ_p *is given by* (3.5). *Then problem* (3.3) *has a nontrivial solution if and only if* $b - a \geq T$.

It is easy to check that the function

$$G(t,s) = \frac{T}{2\,\pi}\,\sin\left(\frac{\pi}{T}\,|t-s|\right), \quad t, s \in [0,T],$$

is the Green function for $v'' + \mu_2\,v = 0$, $v(0) = v(T)$, $v'(0) = v'(T)$ and $G(t,s)$ is nonnegative on $[0,T] \times [0,T]$. Hence, for classical linear second order periodic problems, the following *antimaximum principle* is true:

for each $h \in L_1[0,T]$ *such that* $h \geq 0$ *a.e. on* $[0,T]$, *all solutions* v *of the problem*

$$v'' + \mu_2\,v = h(t), \quad v(0) = v(T), \quad v'(0) = v'(T)$$

are nonnegative on $[0,T]$.

3.2. Theorem. *Let* $p \in (1,\infty)$ *and let* $\mu \in L_1[0,T]$ *be such that*

$$0 \leq \mu \leq \mu_p \quad \text{a.e. on} \quad [0,T] \quad \text{and} \quad \overline{\mu} > 0, \tag{3.6}$$

and let $v \in C^1[0,T]$ *be such that* $\phi_p(v') \in AC[0,T]$,

$$(\phi_p(v'(t)))' + \mu(t)\,\phi_p(v(t)) \geq 0 \ \text{for a.e.}\ t \in [0,T] \tag{3.7}$$

and

$$v(0) = v(T), \quad v'(0) = u'(T). \tag{3.8}$$

Then $v \geq 0$ *on* $[0,T]$.

Proof. Let $v \in C^1[0,T]$ be such that $\phi_p(v') \in AC[0,T]$ and (3.6) and (3.7) hold. Without any loss of generality we may assume that v is does not vanish on $[0,T]$.

Step 1. First, we show that

$$\max\{v(t) : t \in [0,T]\} > 0. \tag{3.9}$$

Assuming, on the contrary, that $v \le 0$ on $[0,T]$, we get by (3.7)

$$(\phi_p(v'(t)))' \ge -\mu(t)\,\phi_p(v(t)) \ge 0 \text{ for a.e. } t \in [0,T].$$

Therefore, v' is nondecreasing on $[0,T]$ and, taking into account (3.8), we deduce that $v' = 0$ on $[0,T]$. Consequently, $v(t) \equiv v(0) \le 0$ on $[0,T]$. Hence, (3.7) reduces to

$$-\mu(t)\,(-v(0))^{p-1} \ge 0 \text{ for a.e. } t \in [0,T].$$

However, as $\mu \ge 0$ a.e. on $[0,T]$ and $\overline{\mu} > 0$, this is possible if and only if $v(0) = 0$, i.e. $v \equiv 0$ on $[0,T]$, which contradicts our assumption that v does not vanish identically on $[0,T]$.

Step 2. Assume that $\min\{v(t) : t \in [0,T]\} < 0$. Let us extend v and μ to T-periodic functions on $\mathbb{R}$. In view of *Step* 1, there are a, $b \in \mathbb{R}$ such that $v \ge 0$ on (a,b), $v(a) = v(b) = 0$ and

$$0 < b - a < T. \tag{3.10}$$

Moreover, $v > 0$ on (a,b). Indeed, if $v(\tau) = 0$ for some $\tau \in (a,b)$, then necessarily also $v'(\tau) = 0$, i.e. $v \equiv 0$ on $\mathbb{R}$. In virtue of (3.6) and (3.7), we have

$$(\phi_p(v'(t)))' + \mu_p\,\phi_p(v(t)) \ge (\phi_p(v'(t)))' + \mu(t)\,\phi_p(v(t)) \ge 0 \text{ for a.e. } t \in [a,b]. \tag{3.11}$$

Furthermore, put

$$a_0 = a - \frac{T-b+a}{2}, \quad b_0 = a_0 + T$$

and

$$\sigma_2(t) = d\,\mu_p^{-1/p}\,\sin_p\big(\mu_p^{1/p}\,(t-a_0)\big) \quad \text{for } t \in \mathbb{R}$$

with $d > 0$ large enough, i.e. such that $\sigma_2(t) > v(t) \ge 0$ on $[a,b]$. We have

$$(\phi_p(\sigma_2'(t)))' + \mu_p\,\phi_p(\sigma_2(t)) = 0 \text{ for a.e. } t \in [a,b]. \tag{3.12}$$

Thus, σ_2 is an upper function for (3.3). Moreover, in view of (3.11), $\sigma_1 = v$ is a lower function for (3.3). Hence, by Proposition 2.4, where we put $\widetilde{f}(t,x) = -\mu_p\,\phi_p(x)$ for t, $x \in \mathbb{R}$, there exists a nontrivial solution u to (3.3). This, due to (3.10), contradicts Proposition 3.1. $\square$

4. Main results

First, let us recall the following a priori estimate (see [34, Lemma 2.4] or [31, Lemma 5.8]).

4.1. Lemma. *Let* $p \in (1,\infty)$ *and let* $\psi \in L_1[0,T]$. *Then*

$$\|v'\|_\infty \le \phi_p^{-1}(\|\psi\|_1) \tag{4.1}$$

holds for each $v \in C^1[0,T]$ *fulfilling* $\phi_p(v') \in AC[0,T]$, $v(0)=v(T)$, $v'(0)=v'(T)$ *and* $(\phi_p(v'(t)))' \ge \psi(t)$ *(or* $(\phi_p(v'(t)))' \le \psi(t)$*) for a.e.* $t \in [0,T]$.

Next, we prove an existence principle which relies on the comparison of the given problem (1.1), (1.2) with the related quasilinear problem fulfilling the antimaximum principle.

4.2. Theorem. *Assume* (1.3), (2.1) *and* $p \in [2,\infty)$. *Furthermore, let* $r > 0$, $A \ge r$, $\mu, \beta \in L_1[0,T]$ *be such that* $\mu(t) \ge 0$ *a.e. on* $[0,T]$, $\overline{\mu} > 0$,

$$\overline{\beta} \le 0 \quad \text{and} \quad f(t,x) \le \beta(t) \quad \text{for a.e. } t \in [0,T] \text{ and all } x \in [A,B] \tag{4.2}$$

and

$$f(t,x) + \mu(t)\,\phi_p(x-r) \ge 0 \quad \text{for a.e. } t \in [0,T] \text{ and all } x \in [r,B], \tag{4.3}$$

where

$$B - A \ge \frac{T}{2}\,\phi_p^{-1}(\|m\|_1),$$

$$m(t) = \max\big\{\sup\{f(t,x) : x \in [r,A]\}, \beta(t), 0\big\} \quad \text{for a.e. } t \in [0,T]$$

and

$$\left\{\begin{array}{l} v \ge 0 \text{ on } [0,T] \text{ holds for each } v \in C^1[0,T] \text{ such that} \\ \quad \phi_p(v') \in AC[0,T], \\ \quad (\phi_p(v'(t)))' + \mu(t)\,\phi_p(v(t)) \ge 0 \text{ for a.e. } t \in [0,T], \\ \quad v(0) = v(T), \quad v'(0) = v'(T). \end{array}\right. \tag{4.4}$$

Then problem (1.1), (1.2) *has a solution* u *such that*

$$r \le u \le B \quad \text{on } [0,T] \quad \text{and} \quad \|u'\|_\infty < \phi_p^{-1}(\|m\|_1). \tag{4.5}$$

Proof. Part I. First, assume that $\overline{\beta} < 0$.
Step 1. Put

$$\widetilde{f}(t,x) = \left\{\begin{array}{ll} f(t,r) - \mu(t)\,\phi_p(x-r) & \text{if } x \le r, \\ f(t,x) & \text{if } x \in [r,B], \\ f(t,B) & \text{if } x \ge B \end{array}\right. \tag{4.6}$$

and consider problem (2.2). We have $\widetilde{f} \in Car([0,T] \times \mathbb{R})$. Furthermore, by (4.2)–(4.6), the inequalities

$$\widetilde{f}(t,x) \le \beta(t) \quad \text{if} \quad x \ge A \tag{4.7}$$

and

$$\widetilde{f}(t,x) + \mu(t)\,\phi_p(x-r) \ge 0 \quad \text{for all} \quad x \in \mathbb{R} \tag{4.8}$$

are valid for a.e. $t \in [0,T]$. In particular, in view of (4.6), we have

$$\widetilde{f}(t,x) \ge h(t) := -\mu(t)\,\phi_p(B-r) \quad \text{for a.e.} \quad t \in [0,T] \text{ and all } \; x \in \mathbb{R}, \tag{4.9}$$

with $h \in L_1[0,T]$.

By (4.8), $\sigma_2 \equiv r$ is an upper function of (2.2). Further, if $b = \beta - \overline{\beta}$, then $b \in L_1[0,T]$ and $\overline{b} = 0$ and it is easy to see that there is a uniquely defined $\sigma_0 \in C^1[0,T]$ such that $\phi_p(\sigma_0') \in AC[0,T]$,

$$(\phi_p(\sigma_0(t)'))' = b(t) \quad \text{for a.e.} \quad t \in [0,T] \quad \text{and} \quad \sigma_0(0) = \sigma_0(T) = 0.$$

Now, let us choose $c^* > 0$ such that $c^* + \sigma_0 \ge A$ on $[0,T]$ and define $\sigma_1 = c^* + \sigma_0$. By (4.7) we have

$$\sigma_1(0) = \sigma_1(T) = c^*,$$
$$(\phi_p(\sigma_1'(t)))' = \beta(t) - \overline{\beta} > \beta(t) \ge \widetilde{f}(t, \sigma_1(t)) \text{ for a.e. } t \in [0,T],$$

and

$$\phi_p(\sigma_0'(T)) - \phi_p(\sigma_0'(0)) = T\,\overline{b} = 0.$$

Consequently, σ_1 is a lower function of (2.2). Therefore, by (4.9) and by Proposition 2.3, the regular problem (2.2) has a solution u such that $u(t_u) \ge r$ for some $t_u \in [0,T]$.

Step 2. We show that

$$u \ge r \text{ on } [0,T]. \tag{4.10}$$

To this aim, set $v = u - r$. By virtue of (4.8), we have

$$\big(\phi_p'(v'(t))\big)' + \mu(t)\,\phi_p(v(t)) = \widetilde{f}(t,u(t)) + \mu(t)\,\phi_p(u(t)-r) \ge 0$$

for a.e. $t \in [0,T]$. By (4.4) it follows that $v(t) \ge 0$ on $[0,T]$, i.e. (4.10) is true.

Step 3. We show that

$$u \le B \text{ on } [0,T]. \tag{4.11}$$

Indeed, by the definition of m and by (4.6) and (4.7) we have

$$\widetilde{f}(t,x) \le m(t) \quad \text{for a.e. } t \in [0,T] \text{ and all } x \ge r.$$

Hence, we can use Lemma 4.1 to get the estimate

$$\|u'\|_\infty \le \phi_p^{-1}(\|m\|_1). \tag{4.12}$$

If $u \ge A$ were valid on $[0,T]$, then taking into account the periodicity of u' and (4.7) we would get

$$0 = \int_0^T \widetilde{f}(t,u(t))\,\mathrm{d}t \le \int_0^T \beta(t)\,\mathrm{d}t = T\,\overline{\beta} < 0,$$

a contradiction. Hence,

$$\min\{u(s) :\ s \in [0,T]\} < A.$$

Now, assume that

$$u^* := \max\{u(s) :\ s \in [0,T]\} > A$$

and extend u to be T-periodic on $\mathbb{R}$. There are s_1, s_2 and $s^* \in \mathbb{R}$ such that

$$s_1 < s^* < s_2, \quad s_2 - s_1 < T, \quad u(s_1) = u(s_2) = A \quad \text{and} \quad u(s^*) = u^* > A.$$

In particular, due to (4.12),

$$2\,(u(s^*) - A) = \int_{s_1}^{s^*} u'(s)\,\mathrm{d}s + \int_{s_2}^{s^*} u'(s)\,\mathrm{d}s \le T\,\phi_p^{-1}(\|m\|_1),$$

wherefrom the estimate

$$u(t) - A \le \frac{T}{2}\,\phi_p^{-1}(\|m\|_1) \le B - A \text{ on } [0,T]$$

follows. Thus, (4.11) is true.

Step 4. The estimates (4.10) and (4.11) mean that $r \le u \le B$ holds on $[0,T]$. In view of (4.6), we conclude that u is a solution to (1.1), (1.2).

Part II. Now, let $\overline{\beta} = 0$. Put $n_0 = \max\{\frac{1}{r}, \frac{1}{B-A}, 3\}$. For an arbitrary $n \in \mathbb{N}$, define

$$\widetilde{f}_n(t,x) = \begin{cases} f(t,r) & \text{if } x \le r,\\[1ex] f(t,x) & \text{if } x \in [r,A],\\[1ex] f(t,x) - \mu(t)\,\phi_p\left(\frac{1}{n}\,\frac{x-A}{x-A+1}\right) & \text{if } x \in (A,B],\\[1ex] f(t,B) - \mu(t)\,\phi_p\left(\frac{1}{n}\,\frac{B-A}{B-A+1}\right) & \text{if } x \ge B. \end{cases} \tag{4.13}$$

Taking into account (4.2), we get

$$\begin{aligned} \widetilde{f}_n(t,x) &= f(t,x) - \mu(t)\,\phi_p\left(\frac{1}{n}\,\frac{x-A}{x-A+1}\right) \le \beta(t) - \mu(t)\,\phi_p\left(\frac{1}{n}\,\frac{x-A}{x-A+1}\right)\\ &\le \beta(t) - \mu(t)\,\phi_p\Big(\frac{1}{2n^2}\Big) \qquad \text{if } \ x \in [A + \frac{1}{n}, B] \end{aligned}$$

and

$$\widetilde{f}_n(t,x) = f(t,B) - \mu(t)\,\phi_p\left(\frac{1}{n}\,\frac{B-A}{B-A+1}\right) \le \beta(t) - \mu(t)\,\phi_p(\frac{1}{2n^2}) \quad \text{if} \;\; x \ge B$$

for a.e. $t \in [0,T]$ and all $n \in \mathbb{N}$ such that $n \ge n_0$. Thus,

$$\left\{ \begin{array}{l} \widetilde{f}_n(t,x) \le \beta_n(t) := \beta(t) - \mu(t)\,\phi_p(\frac{1}{2n^2}) \\ \qquad \text{for} \;\; x \ge A + \frac{1}{n}\,, \;\; \text{for a.e.} \;\; t \in [0,T] \quad \text{and all} \;\; n \ge n_0. \end{array} \right. \tag{4.14}$$

Clearly,

$$\overline{\beta_n} < 0 \quad \text{and} \quad \beta_n(t) \le \beta(t) \text{ for a.e. } t \in [0,T]. \tag{4.15}$$

Furthermore, by (4.3) and (4.13), we have

$$\widetilde{f}_n(t,x) + \mu(t)\,\phi_p\left(x - (r - \frac{1}{n})\right) \ge f(t,r) \ge 0 \quad \text{if} \;\; x \in [r - \frac{1}{n}, r],$$

$$\widetilde{f}_n(t,x) + \mu(t)\,\phi_p\left(x - (r - \frac{1}{n})\right) = f(t,x) + \mu(t)\,\phi_p\,(x-r) \ge 0 \quad \text{if} \;\; x \in [r,A],$$

and, taking into account that

$$\xi^\alpha + \eta^\alpha \le (\xi+\eta)^\alpha \quad \text{holds for all} \;\; \xi,\eta \ge 0 \quad \text{and each} \;\; \alpha \ge 1,$$

$$\begin{aligned} &\widetilde{f}_n(t,x) + \mu(t)\,\phi_p\left(x - (r - \frac{1}{n})\right) \\ &\qquad = f(t,x) - \mu(t)\,\phi_p\left(\frac{1}{n}\,\frac{x-A}{x-A+1}\right) + \mu(t)\,\phi_p\left(x - r + \frac{1}{n}\right) \\ &\qquad \ge f(t,x) + \mu(t)\,\phi_p\,(x-r) \ge 0 \quad \text{if} \;\; x \in [A,B], \end{aligned}$$

and

$$\begin{aligned} &\widetilde{f}_n(t,x) + \mu(t)\,\phi_p\left(x - (r - \frac{1}{n})\right) \\ &\qquad = f(t,B) - \mu(t)\,\phi_p\left(\frac{1}{n}\,\frac{B-A}{B-A+1}\right) + \mu(t)\,\phi_p\left(x - (r - \frac{1}{n})\right) \\ &\qquad \ge f(t,B) + \mu(t)\,\phi_p\,(B-r) \ge 0 \quad \text{if} \;\; x \ge B. \end{aligned}$$

To summarize,

$$\widetilde{f}_n(t,x) + \mu(t)\,\phi_p\left(x - (r - \frac{1}{n})\right) \ge 0 \quad \text{for all} \;\; x \ge r - \frac{1}{n}. \tag{4.16}$$

For a.e. $t \in [0,T]$ and all $n \in \mathbb{N}$, put

$$\widetilde{m}_n(t) := \max\big\{\sup\{\widetilde{f}_n(t,x) : x \in [r - \frac{1}{n}, A + \frac{1}{n}]\}, b_n(t)\big\}.$$

In view of (4.13) and (4.15), we have

$$\widetilde{m}_n(t) \le m(t) \quad \text{for a.e. } t \in [0,T] \quad \text{and} \quad n > \frac{1}{B-A}\,.$$

This together with (4.14)-(4.16) means that, for each $n \in \mathbb{N}$ large enough, *Part* I of this proof ensures the existence of a solution u_n to the auxiliary problem

$$(\phi_p(u_n'))' = \widetilde{f}_n(t,u_n), \quad u_n(0) = u_n(T), \quad u_n'(0) = u_n'(T)$$

which satisfies the estimates

$$r - \frac{1}{n} \le u_n(t) \le B + \frac{1}{n} \quad \text{on} \quad [0,T] \quad \text{and} \quad \|u_n'\|_\infty \le \phi_p^{-1}(\|m\|_1).$$

Now, notice that

$$|\widetilde{f}_n(t,x) - \widetilde{f}(t,x)| \le \mu(t)\,\phi_p\left(\frac{1}{n}\right) \quad \text{for a.e. } t \in [0,T], \text{ all } x \in \mathbb{R} \text{ and all } n \in \mathbb{N},$$

where

$$\widetilde{f}(t,x) = \begin{cases} f(t,r) & \text{if } x \le r, \\ f(t,x) & \text{if } x \in [r,B], \\ f(t,B) & \text{if } x \ge B. \end{cases}$$

Thus, in a standard way (using the Arzelá-Ascoli and the Lebesgue Dominated Convergence Theorem) we can show that the sequence $\{u_n\}_{n=1}^{\infty}$ contains a subsequence which converges in $C^1[0,T]$ to a solution u of the problem

$$(\phi_p(u'))' = \widetilde{f}(t,u), \quad u(0) = u(T), \quad u'(0) = u'(T)$$

which satisfies necessarily the estimate (4.5), i.e. solves also (1.1), (1.2). $\square$

The next supplementary assertion concerning the case $p \in (1,2)$ follows immediately from *Part* I of the previous proof.

4.3. Theorem. *Let all assumptions of Theorem* 4.2 *be satisfied, with the exceptions that* $p \in (1,2)$ *is allowed and* $\overline{\beta} < 0$ *is required in* (4.2).

Then problem (1.1), (1.2) *has a solution* u *such that* (4.5) *is true.*

Theorems 3.2, 4.2 and 4.3 yield the following new existence criterion.

4.4. Theorem. *Assume* (1.3) *and* (2.1). *Furthermore, let* $p \in (1,\infty)$, $\mu_p = (\pi_p/T)^p$ *and let* $r > 0$, $A \ge r$, $B > A$ *and* $\beta \in L_1[0,T]$ *be such that* (4.2), *with* $\overline{\beta} < 0$, *if* $p \in (1,2)$, *and* (4.3) *hold, where*

$$B - A \ge \frac{T}{2}\,\phi_p^{-1}(\|m\|_1)$$

and

$$m(t) = \max\big\{\sup\{f(t,x) : x \in [r,A]\}, \beta(t), 0\big\} \quad \textit{for a.e. } t \in [0,T].$$

Then problem (1.1), (1.2) *has a solution* u *such that* (4.5) *is true.*

In particular, for the Duffing type equation $(\phi_p(u'))' = g(u) + e(t)$ we have

4.5. Corollary. *Let* $p \in (1,\infty)$. *Suppose that* $f(t,x) = g(x) + e(t)$ *for* $x \in (0,\infty)$ *and a.e.* $t \in [0,T]$, *where* $g \in C(0,\infty)$, $e \in L_1[0,T]$, *and*

$$\overline{e} + \limsup_{x\to\infty} g(x) < 0 \tag{4.17}$$

and there is $r > 0$ *such that*

$$e(t) + g(x) + \mu_p\,(x^{p-1} - r^{p-1}) \ge 0 \quad \textit{for a.e. } t \in [0,T] \textit{ and all } x \ge r. \tag{4.18}$$

Then problem (1.1), (1.2) *has a solution* u *such that* $u(t) \ge r$ *on* $[0,T]$.

Proof. Due to (4.17), we can find $A \ge r$ such that

$$g(x) + \overline{e} < \frac{1}{2}\left(\overline{e} + \limsup_{x\to\infty} g(x)\right) < 0 \quad \text{for } x \in [A,\infty).$$

Consequently,

$$f(t,x) = g(x) + e(t) = (g(x) + \overline{e}) + (e(t) - \overline{e}) < \frac{1}{2}\left(\overline{e} + \limsup_{x\to\infty} g(x)\right) + e(t) - \overline{e}$$

for a.e. $t \in [0,T]$ and all $x \in [A,\infty)$. Therefore, (4.2) holds with

$$\beta(t) := e(t) + \tfrac{1}{2}\left(\limsup_{x\to\infty} g(x) - \overline{e}\right),$$

$\overline{\beta} < 0$ and $B > A$ arbitrarily large. Finally, by virtue of (4.18), f satisfies (4.3) with $B > r$ arbitrarily large. Now, the assertion follows by Theorem 4.4. □

For the case $p \ge 2$ we have an existence result under weaker assumptions.

4.6. Corollary. *Let* $p \ge 2$. *Assume the hypothesis of the previous result, replacing condition (4.17) by the following one:*

There is $B > 0$ *such that* $\overline{e} + g(x) \le 0$ *for all* $x \ge B$.

Then problem (1.1), (1.2) *has a solution* u *such that* $u(t) \ge r$ *on* $[0,T]$.

4.7. Example. Consider the problem

$$(\phi_p(u'))' = g(u) + e(t), \quad u(0) = u(T), \quad u'(0) = u'(T), \tag{4.19}$$

with $1 < p < \infty$; $g(x) := -k\,x^{p-1} + \frac{a}{x^\alpha}$ for $x > 0$; $a > 0$, $\alpha > 0$, $k \ge 0$; and $e \in L_1[0,T]$ essentially bounded below, i.e.

$$e_* := \inf \operatorname{ess}\{e(t) : t \in [0,T]\} > -\infty.$$

We need to find conditions under which assumptions (4.17) and (4.18) are satisfied.

It is easy to see that if $k > 0$, then the assumption (4.17) of Corollary 4.5 is satisfied for all $e \in L_1[0,T]$, while in the case $k = 0$ this condition holds whenever $\overline{e} < 0$.

Denote $\mu = \left(\frac{\pi_p}{T}\right)^p$,

$$h_r(x) := \frac{a}{x^\alpha} + \mu\,(x-r)^{p-1} - k\,x^{p-1} \quad \text{for } r > 0 \text{ and } x \ge r \quad \text{or } r = 0 \text{ and } x > r.$$

We can see that to verify condition (4.18) it suffices to show that

$$\exists\, r > 0 \quad \text{such that} \quad e_* + h_r(x) \ge 0 \quad \text{for all} \quad x \ge r. \tag{4.20}$$

Since $\lim_{x\to\infty} h_r(x) = -\infty$ if $k > \mu$, $p > 1$ and $r \ge 0$ and also if $k = \mu$, $p > 2$ and $r > 0$, condition (4.20) can not be satisfied in these cases. It remains to consider the following two possibilities:

(i) $1 < p \le 2$, $0 \le k \le \mu$,

(ii) $2 < p < \infty$, $0 \le k < \mu$.

Case (i). If $1 < p \le 2$, then the inequality $(x-r)^{p-1} \ge x^{p-1} - r^{p-1}$ holds for all $r \ge 0$ and $x \ge r$. Therefore, $h_r(x) \ge h_0(x) - \mu\,r^{p-1}$ for all $x \ge r$, i.e.

$$\varkappa(r) \ge \varkappa(0) - \mu\,r^{p-1} \quad \text{for all} \quad k \in [0,\mu),$$

where $\varkappa(r)$ stands for

$$\varkappa(r) := \inf\{h_r(x) : x \in (r,\infty)\} \quad \text{for} \quad r \ge 0.$$

It follows that condition (4.20) is satisfied provided $e_* + \varkappa(0) > 0$ and $r = \left(\frac{e_* + \varkappa(0)}{\mu}\right)^{\frac{1}{p-1}}$. Notice that

$$\begin{cases} \varkappa(0) = a\,\left(\dfrac{\alpha+p-1}{p-1}\right)\left(\dfrac{(p-1)\;(\mu-k)}{\alpha\,a}\right)^{\frac{\alpha}{\alpha+p-1}} & \text{if} \quad k \in [0,\mu), \\ \varkappa(0) = 0 & \text{if} \quad k = \mu. \end{cases} \tag{4.21}$$

In particular, if $k=\mu$, problem (4.19) possesses a positive solution whenever $e_*>0$.

Case (ii). Let $p>2$. First, assume that $k=0$. Then, for each $r\ge 0$ there is exactly one $\widetilde{x}_r\in(0,\infty)$ such that $\varkappa(r)=h_r(\widetilde{x}_r)$. We can check that

$$\lim_{r\to 0+}\widetilde{x}_r=\widetilde{x}_0=\left(\frac{\alpha\, a}{(p-1)\,\mu}\right)^{\frac{1}{\alpha+p-1}}\quad\text{and}\quad\lim_{r\to 0+}\varkappa(r)=\lim_{r\to 0+}h_r(\widetilde{x}_r)=h_0(\widetilde{x}_0)=\varkappa(0),$$

where $\varkappa(0)$ is given by (4.21). In particular, if $\varkappa(0)+e_*>0$, then there is $r>0$ such that $\varkappa(r)+e_*>0$, which means that (4.20) and hence also (4.18) are satisfied.

Now, assume that $0<k<\mu$ and let $e_*\ge 0$. Denote

$$\widetilde{g}_r(x)=\mu\,(x-r)^{p-1}-k\,x^{p-1}\quad\text{for } r>0 \quad\text{and}\quad x\ge r.$$

We have $\widetilde{g}\,'_r(x)=0$ if and only if $x=\widetilde{x}_r$, where $\widetilde{x}_r:=r\Big(1-\Big(\dfrac{k}{\mu}\Big)^{\frac{1}{p-2}}\Big)^{-1}$ and

$$\widetilde{g}_r(\widetilde{x}_r)=\mu\, r^{p-1}\left(1-\Big(\frac{k}{\mu}\Big)^{\frac{1}{p-2}}\right)^{1-p}\left(\Big(\frac{k}{\mu}\Big)^{\frac{p-1}{p-2}}-k\right)=\inf\left\{\widetilde{g}_r(x)\colon x\in(0,\infty)\right\}.$$

Furthermore, $\widetilde{g}_r$ is strictly increasing on $[\widetilde{x}_r,\infty)$ and $\widetilde{g}_r(x)\ge 0$ for all $x\ge\xi_r$, where $\xi_r:=r\Big(1-\Big(\dfrac{k}{\mu}\Big)^{\frac{1}{p-1}}\Big)^{-1}\in(\widetilde{x}_r,\infty)$. Thus, $e_*+h_r(x)=e_*+\frac{a}{x^\alpha}+\widetilde{g}_r(x)\ge 0$ for $x\ge\xi_r$. On the other hand, for $x\in[r,\xi_r]$ we have $e_*+h_r(x)\ge e_*+\frac{a}{\xi_r{}^\alpha}+\widetilde{g}_r(\widetilde{x}_r)$. Consequently, condition (4.20) is satisfied whenever

$$e_*+\frac{a}{\xi_r^{\,\alpha}}+\widetilde{g}_r(\widetilde{x}_r)\ge 0. \tag{4.22}$$

Now, since $\lim_{r\to 0+}\widetilde{x}_r=\lim_{r\to 0+}\xi_r=0$, we have

$$\lim_{r\to 0+}\widetilde{g}_r(\widetilde{x}_r)=0\quad\text{and}\quad\lim_{r\to 0+}\frac{a}{\xi_r{}^{\alpha}}=\infty.$$

Thus, we can see that (4.22) and hence also (4.20) hold.

Finally, let us consider the case that $p>2$, $0<k<\mu$ and $e_*=-\eta_*<0$. We want again to show that (4.20) is true. To this aim let us rewrite the inequality $e_*+h_r(x)\ge 0$ as follows

$$\widetilde{h}_r(x):=-\frac{\eta_*}{x^{p-1}}+\frac{a}{x^{\alpha+p-1}}+\mu\left(1-\frac{r}{x}\right)^{p-1}\ge k.$$

First, notice that $\lim_{x\to\infty}\widetilde{h}_r(x)=\mu>k$ and there is $\delta>0$ such that

$$\widetilde{h}_r(r)=\frac{a-\eta_*\,r^\alpha}{r^{\alpha+p-1}}\ge k\quad\text{for all } r\in(0,\delta).$$

Indeed, it suffices to choose $\delta>0$ in such a way that both inequalities

$$\delta^{\alpha+p-1}\le\frac{a}{2k}\qquad\text{and}\qquad \eta_*\,\delta^{\alpha}\le\frac{a}{2}$$

hold. Furthermore, $\widetilde{h}\,'_r(x)=0$ if and only if

$$\rho_r(x)=\sigma(x) \tag{4.23}$$

holds, where

$$\rho_r(x):=r\,\mu\,(p-1)\left(1-\frac{r}{x}\right)^{p-2}\qquad\text{and}\quad \sigma(x):=\frac{a\,(\alpha+p-1)}{x^{\alpha+p-2}}-\frac{\eta_*\,(p-1)}{x^{p-2}}.$$

We can see that, for each $r>0$, the function ρ_r is strictly increasing on $[r,\infty)$, while $\rho_r(r)=0$ and $\lim_{x\to\infty}\rho_r(x)=r\mu\,(p-1)>0$. On the other hand, $\sigma'(x)=0$ if and only if $x=\xi_*$, where

$$\xi_*=\left(\frac{a\,(\alpha+p-1)\,(\alpha+p-2)}{\eta_*\,(p-1)\,(p-2)}\right)^{\frac{1}{\alpha}}.$$

Furthermore, σ is strictly decreasing on $(0,\xi_*]$, strictly increasing on $[\xi_*,\infty)$, $\sigma(\xi_*)<0$, $\lim_{x\to\infty}\sigma(x)=0$ and $\sigma(x)=0$ if and only if $x=\xi$, where

$$\xi=\left(\frac{a\,(\alpha+p-1)}{\eta_*\,(p-1)}\right)^{\frac{1}{\alpha}}.$$

As a result, for each $r\in(0,\xi)$ there is exactly one point $\widetilde{x}_r\in(r,\xi)$ such that (4.23) holds for $x=\widetilde{x}_r$. Consequently, $\widetilde{h}_r(\widetilde{x}_r)=\inf\{\widetilde{h}_r(x):\,x\in[r,\infty)\}$. Now, we can show that

$$\lim_{r\to0+}\widetilde{h}_r(\widetilde{x}_r)=\widetilde{h}_0(\xi)=\mu-\left(\frac{\alpha\,a}{p-1}\right)\left(\frac{\eta_*\,(p-1)}{a\,(\alpha+p-1)}\right)^{\frac{\alpha+p-1}{\alpha}}.$$

Therefore, if

$$\mu-k>\left(\frac{\alpha\,a}{p-1}\right)\left(\frac{\eta_*\,(p-1)}{a\,(\alpha+p-1)}\right)^{\frac{\alpha+p-1}{\alpha}}, \tag{4.24}$$

then there exist $r\in(0,\min\{\delta,\xi\})$ such that $\widetilde{h}_r(x)\ge k$ holds for each $x\ge r$. In other words, provided (4.24) is true, condition (4.20) is satisfied. Now, it is a question of routine, to verify that condition (4.24) is equivalent to condition $\varkappa(0)+e_*>0$ with $\varkappa(0)$ given by (4.21).

To summarize, by Corollary 4.5 problem (4.19) has a positive solution if:

$$k=0,\ 1<p<\infty,\ \overline{e}<0\quad\text{and}\quad e_*>-a\left(\frac{\alpha+p-1}{p-1}\right)\left(\frac{(p-1)\,\mu}{\alpha\,a}\right)^{\frac{\alpha}{\alpha+p-1}}$$

or

$$0<k<\mu,\ 1<p\le 2 \qquad \text{and} \quad e_* > -a\left(\frac{\alpha+p-1}{p-1}\right)\left(\frac{(p-1)\,(\mu-k)}{\alpha\, a}\right)^{\frac{\alpha}{\alpha+p-1}}$$

or

$$0<k=\mu,\ 1<p\le 2 \qquad \text{and} \quad e_*\ge 0.$$

In particular, if $k=\mu$ and $1<p\le 2$, then problem (4.19) has a positive solution whenever $e_*>0$.

References

[1] P. Binding, P. Drábek and Y.X. Huang, On the Fredholm alternative for the p-Laplacian. *Proc. Amer. Math. Soc.* **125** (1997), 3555-3559.

[2] D. Bonheure, C. De Coster, Forced singular oscillators and the method of lower and upper functions. *Topol. Methods Nonlinear Anal.* **22** (2003), 297–317.

[3] A. Cabada, P. Habets and R. Pouso, Lower and upper solutions for the periodic problem associated with a ϕ-Laplacian equation. In: *EQUADIFF 1999 - International Conference on Differential Equations*, Vol. 1, 2 (Berlin, 1999), World Sci. Publishing, River Edge, NJ, 2000, pp. 491–493.

[4] A. Cabada, P. Habets and R. Pouso, Optimal existence conditions for ϕ-Laplacian equations with upper and lower solutions in the reversed order. *J. Differential Equations* **166** (2000), 385–401.

[5] A. Cabada and R. Pouso, Existence result for the problem $(\phi(u'))'=f(t,u,u')$ with periodic and Neumann boundary conditions. *Nonlinear Anal., Theory Methods Appl.* **30** (1997), 1733–1742.

[6] A. Cabada and R. Pouso, Existence result for the problem $(\phi(u'))'=f(t,u,u')$ with nonlinear boundary conditions. *Nonlinear Anal., Theory Methods Appl.* **35** (1999), 221–231.

[7] A. Cabada, D. O'Regan and R.. Pouso, Second Order Problems with Functional Conditions Including Sturm-Liouville and Multipoint Conditions. *Mathematische Nachrichten*, to appear.

[8] M. del Pino, M. Elgueta and R. Manásevich, A homotopic deformation along p of a Leray-Schauder degree result and existence for $|u'|^{p-2}u')'+f(t,u)=0,\quad u(0)=u(T)=0,\quad p>1$. *J. Differential Equations* **80** (1989), 1–13.

[9] M. del Pino, P. Drábek and R. Manásevich, The Fredholm alternative at the first eigenvalue for the one dimensional p-Laplacian. *J. Differential Equations* **151** (1999), 386–419.

[10] M. del Pino, R. Manásevich and A. Montero, *T-periodic solutions for some second order differential equations with singularities*, Proc. Roy. Soc. Edinburgh Sect. A **120** (1992), 231–243.

[11] M. del Pino, R. Manásevich and A. Murúa, Existence and multiplicity of solutions with prescribed period for a second order quasilinear O.D.E. *Nonlinear Anal., Theory Methods Appl.* **18** (1992), 79–92.

[12] L. Derwidué, Systèmes differérentiels non lineáires ayant des solutions périodiques. Acad. R. Belgique, Bull. Cl. Sci. (5) **49** (1963), 11–32; (5) **50** (1964), 928–942; (5) **50** (1964), 1130–1142.

[13] O. Došlý, *Half-Linear Differential Equations.* In: *Handbook of Differential Equations, Ordinary Differential Equations* Vol. 1, A. Cañada, P. Drábek, A. Fonda, eds., Elsevier, North Holland, Amsterdam (2004), Chapter 3, pp. 161–357.

[14] O. Došlý and Řehák, *Half-Linear Differential Equations.* North-Holland Mathematics Studies 202, Elsevier, Amsterdam (2005).

[15] R. Faure, Solutions périodiques d'equations différentielles et mèthode de Leray-Schauder (Cas des vibrations forceés). Ann. Inst. Fourier (Grenoble) **14** (1964), 195–204.

[16] N. Forbat and A. Huaux, Détermination approachée et stabilité locale de la solution périodique d'une equation différentielle non linéaire. Mém. Public. Soc. Sci. Arts Letters Hainaut **76** (1962), 3–13.

[17] W.G. Ge and J. Mawhin, Positive solutions to boundary value problems for second order ordinary differential equations with singular nonlinearities. *Result. Math.* **34** (1998), 108–119.

[18] P. Habets and L. Sanchez, Periodic solutions of some Liénard equations with singularities. *Proc. Amer. Math. Soc.* **109** (1990), 1035–1044.

[19] P. Jebelean and J. Mawhin, Periodic solutions of singular nonlinear perturbations of the ordinary p-Laplacian. *Adv. Nonlinear Stud.* **2** (2002), 299–312.

[20] P. Jebelean and J. Mawhin, Periodic solutions of forced dissipative $p-$ Liénard equations with singularities. *Vietnam J. Math.* **32** (2004), 97–103.

[21] D.Q. Jiang and J.Y. Wang, A generalized periodic boundary value problem for the one-dimensional p-Laplacian. *Ann. Polon. Math.* **65** (1997), 265–270.

[22] A.C. Lazer and S. Solimini, On periodic solutions of nonlinear differential equations with singularities. *Proc. Amer. Math. Soc.* **99** (1987), 109–114.

[23] I.T. Kiguradze, *Boundary value problems for systems of ordinary differential equations.* J. Sov. Math. **43** (1988), 2259–2339; translation from Itogi Nauki Tekh., Ser. Sovrem. Probl. Mat., Novejshie Dostizh. **30** (1987) 3–103.

[24] Liu Bing, Periodic solutions of dissipative dynamical systems with singular potential and p-Laplacian. *Ann. Polon. Math.* **79** (2) (2002), 109–120.

[25] R. Manásevich and J. Mawhin, Periodic solutions for nonlinear systems with p-Laplacian-like operators. *J. Differential Equations* **145** (1998), 367–393.

[26] R. Manásevich and J. Mawhin, The spectrum of p-Laplacian systems under Dirichlet, Neumann and periodic boundary conditions. In: *Morse theory, minimax theory and their applications to nonlinear differential equations*, Proceedings of the workshop, H. Brezis, S.J. Li, J.Q Liu, P.H Rabinowitz (eds.), Beijing, China, April 1-September 30, 1999. Somerville, International Press. New Stud. Adv. Math. 1 (2003) pp. 201–216.

[27] J. Mawhin, *Topological Degree and Boundary Value Problems for Nonlinear Differential Equations.* In: *Topological methods for ordinary differential equations*, M. Furi and P. Zecca, eds., Lecture Notes in Mathematics, Vol. 1537, Springer-Verlag, Berlin (1993), pp. 74–142.

[28] J. Mawhin, *Periodic Solutions of Systems with p-Laplacian-like Operators.* In: *Nonlinear Analysis and its Applications to Differential Equations*, M.R. Grossinho, M. Ramos, C. Rebelo and L. Sanchez, eds., Progress in Nonlinear Differerential Equations and their Applications, Vol. 43, Birkhäuser, Boston (2001), pp. 37–63.

[29] P. Omari and W.Y. Ye, Necessary and sufficient conditions for the existence of periodic solutions of second order ordinary differential equations with singular nonlinearities. *Differ. Integral Equ.* **8** (1995), 1843–1858.

[30] V. Polášek and I. Rachůnková, Singular Dirichlet problem for ordinary differential equations with ϕ-Laplacian. *Math. Bohem.* **130** (2005), 409–425.

[31] I. Rachůnková, S. Staněk and M. Tvrdý, *Singularities and Laplacians in Boundary Value Problems for Nonlinear Ordinary Differential Equations.* In: *Handbook of Differential Equations, Ordinary Differential Equations* Vol. 3 (Elsevier), A. Cañada, P. Drábek, A. Fonda, eds., Elsevier, North Holland, Amsterdam (2006), Chapter 6, pp. 605–721.

[32] I. Rachůnková and M. Tvrdý, Construction of lower and upper functions and their application to regular and singular boundary value problems. *Nonlinear Anal., Theory Methods Appl.* **47** (2001), 3937–3948.

[33] I. Rachůnková and M. Tvrdý, Second-order periodic problem with ϕ-Laplacian and impulses. *Nonlinear Anal., Theory Methods Appl.* **63** (2005), e257-e266.

[34] I. Rachůnková and M. Tvrdý, Periodic problems with ϕ - Laplacian involving non-ordered lower and upper functions. *Fixed Point Theory* **6** (2005), 99–112.

[35] I. Rachůnková and M. Tvrdý, Periodic singular problem with quasilinear differential operator. *Math. Bohem.* **131** (2006), 321–336.

[36] I. Rachůnková and M. Tvrdý and I. Vrkoč, Existence of nonnegative and nonpositive nolutions for second order periodic boundary value problems. *J. Differential Equations* **176** (2001), 445–469.

[37] I. Rachůnková and M. Tvrdý and I. Vrkoč, Resonance and multiplicity in periodic BVPs with singularity. *Math. Bohem.* **128** (2003), 45–70.

[38] S. Staněk, Periodic boundary value problem for second order functional differential equations. *Math. Notes (Miskolc)* **1** (2000), 63–81.

[39] P.J. Torres, Existence of one-signed periodic solutions of some second-order differential equations via a Krasnoselskii fixed point theorem. *J. Differential Equations* **190** (2003), 643–662.

[40] J. Wang, W. Gao and Z. Lin, Boundary value problems for general second order equations and similarity solutions to the Rayleigh problem. *Tôhoku Math. J.* **47** (1995), 327–344.

[41] P. Yan, Nonresonance for one-dimensional p-Laplacian with regular restoring. *J. Math. Anal. Appl.* **285** (2003), 141–154.

[42] M.R. Zhang, Periodic solutions of Liénard equations with singular forces of repulsive type. *J. Math. Anal. Appl.* **203** (1996), 254–269.

[43] M.R. Zhang, Nonuniform nonresonance at the first eigenvalue of the p-Laplacian. *Nonlinear Anal., Theory Methods Appl.* **29** (1997), 41–51.

[44] M.R. Zhang, A relationship between the periodic and the Dirichlet BVPs of singular differential equations. *Proc. Roy. Soc. Edinburgh Sect. A* **128A** (1998), 1099–1114.

[45] M.R. Zhang, Nonuniform nonresonance of semilinear differential equations. *J. Differential Equations* **166** (2000), 33–50.

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