Localization of Nonsmooth Lower and Upper Functions for Periodic Boundary Value Problems

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Summary. In this paper we present conditions ensuring the existence and localization of lower and upper functions of the periodic boundary value problem $u'' + k\,u = f(t,u), u(0) = u(2\,\pi), u'(0) = u'(2\pi)$, $k \in \mathbb{R}$, $k \neq 0$. These functions are constructed as solutions of some related generalized linear problems and can be nonsmooth in general.

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1 . Introduction

Theorems about the existence of solutions of boundary value problems for ordinary differential equations often suppose the existence of lower and upper functions to the problem studied. For such theorems concerning periodic boundary value problems we can refer e.g. to the papers [1]-[3], [5], [6], [8]-[14] and [15]. We can decide whether the problem has constant lower and upper functions (see e.g. [2], [5]) and to find them provided they exist. In general, however, it is easy neither to find nonconstant lower and upper functions nor to prove their existence which can make the application of such theorems difficult. One possibility how to get nonconstant and possibly nonsmooth lower and upper functions to the periodic boundary value problem

$$u'' + k\,u = f(t,u), \quad u(0) = u(2\pi),\ u'(0) = u'(2\pi), \quad k \in \mathbb{R},\ k \neq 0 \tag{1.1}$$

is shown here. (The case $k = 0$ is solved in [13] and [14].) We make use of fairly general definitions of these notions introduced in [12] and construct them as solutions of generalized periodic boundary value problems for linear differential equations in Sections 2 and 3. (Essentially they are solutions of linear generalized differential

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equations in the sense of J. Kurzweil, cf. e.g. [4], [17], [18] and [19].) Our next paper [16] will show new effective existence criteria for the problem (1.1). The proofs of them are based on the theorems about the existence and localization of lower and upper functions from this paper.

Throughout the paper we assume that $f : [0, 2\pi] \times \mathbb{R} \mapsto \mathbb{R}$ fulfils the *Carathéodory conditions* on $[0, 2\pi] \times \mathbb{R}$, i.e. f has the following properties: (i) for each $x \in \mathbb{R}$ the function $f(., x)$ is measurable on $[0, 2\pi]$; (ii) for almost every $t \in [0, 2\pi]$ the function $f(t, .)$ is continuous on $\mathbb{R}$; (iii) for each compact set $\mathrm{K} \subset \mathbb{R}$ the function $m_{\mathrm{K}}(t) = \sup_{x \in \mathrm{K}} |f(t, x)|$ is Lebesgue integrable on $[0, 2\pi]$.

For a given subinterval J of $\mathbb{R}$ (possibly unbounded), $\mathbb{C}(J)$ denotes the set of functions continuous on J, $\mathbb{L}[0, 2\pi]$ stands for the set of functions (Lebesgue) integrable on $[0, 2\pi]$, $\mathbb{L}_2[0, 2\pi]$ is the set of functions square integrable on $[0, 2\pi]$, $\mathbb{L}_\infty[0, 2\pi]$ is the set of functions essentially bounded on $[0, 2\pi]$, $\mathbb{AC}[0, 2\pi]$ denotes the set of functions absolutely continuous on $[0, 2\pi]$ and $\mathbb{BV}[0, 2\pi]$ is the set of functions of bounded variation on $[0, 2\pi]$. For $x \in \mathbb{L}_\infty[0, 2\pi]$, $y \in \mathbb{L}[0, 2\pi]$ and $z \in \mathbb{L}_2[0, 2\pi]$ we denote $\|x\|_\infty = \sup \operatorname{ess}_{t \in [0, 2\pi]} |x(t)|$,

$$\overline{y} = \frac{1}{2\pi} \int_0^{2\pi} y(s)\, \mathrm{d}s, \;\; \|y\|_1 = \int_0^{2\pi} |y(t)|\, \mathrm{d}t \text{ and } \|z\|_2 = \Big(\int_0^{2\pi} z^2(t)\, \mathrm{d}t \Big)^{\frac{1}{2}}.$$

If $x \in \mathbb{BV}[0, 2\pi]$, $s \in (0, 2\pi]$ and $t \in [0, 2\pi)$, then the symbols $x(s-)$, $x(t+)$ and $\Delta^+ x(t)$ are defined respectively by $x(s-) = \lim_{\tau \to s-} x(\tau)$, $x(t+) = \lim_{\tau \to t+} x(\tau)$ and $\Delta^+ x(t) = x(t+) - x(t)$. Furthermore, x^{ac} and x^{sing} stand for the absolutely continuous part of x and the singular part of x, respectively. We suppose $x^{\mathrm{sing}}(0) = 0$. $\mathbb{L}^n[0, 2\pi]$, $\mathbb{AC}^n[0, 2\pi]$ and $\mathbb{BV}^n$ are the sets of column n-vector valued functions with elements from $\mathbb{L}[0, 2\pi]$, $\mathbb{AC}[0, 2\pi]$ and $\mathbb{BV}[0, 2\pi]$. For a subset M of $\mathbb{R}$, χ_M denotes the characteristic function of M. For $x \in \mathbb{L}[0, 2\pi]$, x^+ and x^- stand respectively for its nonnegative and nonpositive parts.

By a *solution of* (1.1) we understand a function $u : [0, 2\pi] \mapsto \mathbb{R}$ such that $u' \in \mathbb{AC}[0, 2\pi]$, $u(0) = u(2\pi)$, $u'(0) = u'(2\pi)$ and

$$u''(t) + k\, u(t) = f(t, u(t)) \quad \text{for a.e.} \quad t \in [0, 2\pi].$$

1.1. Definition. Functions $(\sigma_1, \rho_1) \in \mathbb{AC}[0, 2\pi] \times \mathbb{BV}[0, 2\pi]$ are called *lower functions of the problem* (1.1), if the singular part ρ_1^{sing} of ρ_1 is nondecreasing on $[0, 2\pi]$, $\sigma_1'(t) = \rho_1(t)$ and $\rho_1'(t) + k\, \sigma_1(t) \geq f(t, \sigma_1(t))$ a.e. on $[0, 2\pi]$, $\sigma_1(0) = \sigma_1(2\pi)$ and $\rho_1(0+) \geq \rho_1(2\pi-)$.

Similarly, functions $(\sigma_2, \rho_2) \in \mathbb{AC}[0, 2\pi] \times \mathbb{BV}[0, 2\pi]$ are *upper functions of* (1.1), if the singular part ρ_2^{sing} of ρ_2 is nonincreasing on $[0, 2\pi]$, $\sigma_2'(t) = \rho_2(t)$ and $\rho_2'(t) + k\, \sigma_2(t) \leq f(t, \sigma_2(t))$ a.e. on $[0, 2\pi]$, $\sigma_2(0) = \sigma_2(2\pi)$ and $\rho_2(0+) \leq \rho_2(2\pi-)$.

2 . Periodic solutions of certain generalized linear differential problems

We want to show that if for a.e. $t \in [0, 2\pi]$ and all $x \in I_t$, where I_t is a subinterval of $\mathbb{R}$, the function f fulfils a condition of the form

$$f(t,x) \le \beta(t) \tag{2.1}$$

or

$$f(t,x) \ge \beta(t) \tag{2.2}$$

with $\beta \in \mathbb{L}[0, 2\pi]$, then it is possible to construct lower or upper functions for the problem (1.1), respectively.

It is known that if $k \neq n^2$ for all $n \in \mathbb{N} \cup \{0\}$, then the problem

$$\sigma' = \rho, \quad \rho' + k\,\sigma = \beta(t) \quad \text{a.e. on} \quad [0, 2\pi], \tag{2.3}$$

$$\sigma(0) = \sigma(2\pi), \quad \rho(0) = \rho(2\pi) \tag{2.4}$$

possesses a unique solution $(\sigma, \rho) \in \mathbb{AC}[0, 2\pi] \times \mathbb{AC}[0, 2\pi]$ for any $\beta \in \mathbb{L}[0, 2\pi]$. Consequently, if we have in addition

$$\sigma(t) \in I_t \quad \text{for all } t \in [0, 2\pi], \tag{2.5}$$

then the functions (σ, ρ) are lower or upper functions of (1.1) (according to whether (2.1) or (2.2) is satisfied). In general the relation (2.5) need not be true, of course. However, if we admit a more general notion of a solution to the linear problem (2.3), (2.4) and if the intervals I_t of validity of (2.1) or (2.2) are large enough, we can always use the problem (2.3), (2.4) for the construction of lower or upper functions for (1.1). To show this, let us consider a linear differential system on $[0, 2\pi]$

$$\xi' = P(t)\,\xi + q(t), \tag{2.6}$$

where P is an $n \times n$-matrix valued function whose entries are from $\mathbb{L}[0, 2\pi]$ and $q \in \mathbb{L}^n[0, 2\pi]$. By a solution of (2.6) on $[0, 2\pi]$ we mean a function $\xi \in \mathbb{AC}^n[0, 2\pi]$ satisfying (2.6) a.e. on $[0, 2\pi]$. The corresponding normalized fundamental matrix solution of the system $\xi' = P(t)\,\xi$ is denoted by X, i.e. $X \in \mathbb{AC}^{nn}[0, 2\pi]$ and

$$X(t) = \mathrm{I} + \int_0^t P(s)\,X(s)\,\mathrm{d}s \quad \text{on } [0, 2\pi],$$

where I stands, as usual, for the identity $n \times n$-matrix. The inverse matrix $X^{-1}(t)$ is defined for any $t \in [0, 2\pi]$, $X^{-1} \in \mathbb{AC}^{n \times n}[0, 2\pi]$ and if

$$\det\big(X^{-1}(2\pi) - \mathrm{I}\big) \neq 0 \tag{2.7}$$

holds, then for any $q \in \mathbb{L}^n[0, 2\,\pi]$ there is a unique solution $\xi \in \mathbb{AC}^n[0, 2\,\pi]$ of (2.6) on $[0, 2\,\pi]$ such that

$$\xi(0) = \xi(2\pi). \tag{2.8}$$

This solution can be written in the form

$$\xi(t) = \int_0^{2\pi} G(t,s)\,q(s)\,\mathrm{d}s \quad \text{on } [0, 2\,\pi],$$

where

$$G(t,s) = X(t) \left\{ \begin{array}{cl} \big(X^{-1}(2\pi) - \mathrm{I}\big)^{-1} & \text{for } \ t \le s \\ \mathrm{I} + \big(X^{-1}(2\pi) - \mathrm{I}\big)^{-1} & \text{for } \ s < t \end{array} \right\} X^{-1}(s) \tag{2.9}$$

is the Green function of the problem (2.6), (2.8).

2.1. Definition. Let $\tau \in [0, 2\pi)$ and $d \in \mathbb{R}^n$ be given. By a solution of the problem (2.6), (2.8),

$$\Delta^+\xi(\tau) = d \tag{2.10}$$

we mean a function $\xi \in \mathbb{BV}^n[0, 2\,\pi]$ such that the relations (2.8) and

$$\xi'(t) = P(t)\,\xi(t) + q(t) \quad \text{a.e. on } [0, 2\,\pi] \tag{2.11}$$

are satisfied and $\xi - d\,\chi_{(\tau, 2\pi]} \in \mathbb{AC}^n[0, 2\,\pi]$.

2.2. Proposition. *Assume* (2.7). *Then for any $\tau \in [0, 2\pi)$, any $d \in \mathbb{R}^n$ and any $q \in \mathbb{L}^n[0, 2\,\pi]$, the problem* (2.6), (2.8), (2.10) *possesses a unique solution ξ and this solution is given by*

$$\xi(t) = G(t,\tau)\,d + \int_0^{2\pi} G(t,s)\,q(s)\,\mathrm{d}s \quad \textit{on } [0, 2\,\pi], \tag{2.12}$$

where G is defined by (2.9).

Proof. For any $c \in \mathbb{R}^n$ the functions

$$x(t) = X(t)\,c + X(t) \int_0^t X^{-1}(s)\,q(s)\,\mathrm{d}s, \quad t \in [0, 2\,\pi],$$

and

$$y(t) = X(t)X^{-1}(2\pi)\,c - X(t) \int_t^{2\pi} X^{-1}(s)\,q(s)\,\mathrm{d}s \quad t \in [0, 2\,\pi],$$

are the unique solutions of (2.6) on $[0,2\pi]$ such that $x(0)=c$ and $y(2\pi)=c$, respectively. Define $\xi(t)=x(t)$ for $0\le t\le\tau$ and $\xi(t)=y(t)$ for $\tau<t\le 2\pi$. Then $\xi\in\mathbb{BV}^n[0,2\pi]$ fulfils (2.8), (2.11) and

$$\Delta^+\xi(\tau)=X(\tau)\,[X^{-1}(2\pi)-\mathrm{I}]\,c-X(\tau)\int_0^{2\pi}X^{-1}(s)\,q(s)\,\mathrm{d}s.$$

Consequently, if we put

$$c=M^{-1}\Big(X^{-1}(\tau)\,d+\int_0^{2\pi}X^{-1}(s)\,q(s)\,\mathrm{d}s\Big),$$

where $M=X^{-1}(2\pi)-\mathrm{I}$, then ξ verifies (2.10). Moreover, $\xi(t)-d\,\chi_{(\tau,2\pi]}(t)=x(t)$ holds on $[0,2\pi]$ and hence $\xi-d\,\chi_{(\tau,2\pi]}\in\mathbb{AC}^n[0,2\pi]$. Finally, using the relation $X^{-1}(2\pi)\,M^{-1}=\mathrm{I}+M^{-1}$, we get

$$\xi(t)=X(t)\Big(M^{-1}\big[X^{-1}(\tau)\,d+\int_0^{2\pi}X^{-1}(s)\,q(s)\,\mathrm{d}s\big]+\int_0^{t}X^{-1}(s)\,q(s)\,\mathrm{d}s\Big)$$

for $0\le t\le\tau$ and

$$\xi(t)=X(t)\,\Big([\mathrm{I}+M^{-1}]\,\big[X^{-1}(\tau)\,d+\int_0^{2\pi}X^{-1}(s)\,q(s)\,\mathrm{d}s\big]-\int_t^{2\pi}X^{-1}(s)\,q(s)\,\mathrm{d}s\Big)$$

for $\tau<t\le 2\pi$, wherefrom the representation (2.12) of ξ follows. $\square$

2.3. Remark. Clearly, for any solution ξ of (2.6), (2.8), (2.10) we have $\xi^{\mathrm{ac}}=\xi-d\,\chi_{(\tau,2\pi]}$, $\xi^{\mathrm{sing}}=d\,\chi_{(\tau,2\pi]}$ and ξ is left-continuous on $(0,2\pi]$.

2.4. Remark. The problem (2.6), (2.8), (2.10) can be rewritten as the integral equation

$$\xi(t)=\xi(0)+\int_0^t P(s)\,\xi(s)\,\mathrm{d}s+\int_0^t q(s)\,\mathrm{d}s+d\,(\chi_{(\tau,2\pi]}(t)-\chi_{(\tau,2\pi]}(0)),\quad t\in[0,2\pi],$$

which is a very special case of generalized differential equations introduced by J. Kurzweil in [4].

Now, we will apply Proposition 2.2 to the problem (2.3), (2.4) generalized in the sense of Definition 2.1. In the case $k=-\alpha^2$ we get the following result:

2.5. Corollary. *Let $k=-\alpha^2$, $\alpha\in(0,\infty)$, $\tau\in[0,2\pi)$, $\delta\in\mathbb{R}$ and $\beta\in\mathbb{L}[0,2\pi]$. Then the problem*

$$\begin{aligned}&\sigma'=\rho, && \rho'+k\,\sigma=\beta(t),\\ &\sigma(0)=\sigma(2\pi), && \rho(0)=\rho(2\pi), \qquad \Delta^+\sigma(\tau)=0,\quad \Delta^+\rho(\tau)=\delta\end{aligned}\tag{2.13}$$

possesses a unique solution (σ,ρ). *Moreover,* $\sigma\in\mathbb{AC}[0,2\pi]$, $\rho^{sing}=\delta\,\chi_{(\tau,2\pi]}$ *and*

$$\sigma(t)=g(|t-\tau|)\,\delta+\int_0^{2\pi} g(|t-s|)\,\beta(s)\,\mathrm{d}s \quad \textit{on } [0,2\pi], \tag{2.14}$$

where

$$g(t)=-\frac{\cosh(\alpha\,(\pi-t))}{2\,\alpha\,\sinh(\alpha\pi)} \quad \textit{on } [0,2\pi]. \tag{2.15}$$

Proof. The fundamental matrix solution X of the corresponding homogeneous system $\sigma'=\rho$, $\rho'=\alpha^2\,\sigma$ is given by

$$X(t)=\begin{pmatrix} \cosh(\alpha\,t) & , & \dfrac{\sinh(\alpha\,t)}{\alpha} \\ \alpha\,\sinh(\alpha t) & , & \cosh(\alpha\,t)\end{pmatrix} \quad \text{on } [0,2\pi]$$

and $\det\big(X^{-1}(2\pi)-\mathrm{I}\big)=-4\,\sinh(\alpha\pi)\neq 0$.

Thus, we can apply Proposition 2.2 to the problem (2.6), (2.8), (2.10) with

$$\xi(t)=\begin{pmatrix}\sigma\\ \rho\end{pmatrix},\quad q(t)=\begin{pmatrix}0\\ \beta\end{pmatrix}\quad\text{and}\quad d=\begin{pmatrix}0\\ \delta\end{pmatrix},$$

obtaining that the problem (2.13) possesses a unique solution (σ,ρ). Since, in particular, $\Delta^+\sigma(\tau)=0$ and $\Delta^+\rho(\tau)=\delta$, it follows from Definition 2.1 that $\sigma\in\mathbb{AC}[0,2\pi]$ and $\rho-\delta\,\chi_{(\tau,2\pi]}\in\mathbb{AC}[0,2\pi]$ (i.e. $\rho^{\mathrm{sing}}=\delta\,\chi_{(\tau,2\pi]}$). Furthermore, substituting for X into (2.9), we get

$$G(t,s)=\begin{cases}\begin{pmatrix} -\dfrac{\sinh(\alpha(\pi+t-s))}{2\,\sinh(\alpha\pi)} & , & -\dfrac{\cosh(\alpha(\pi+t-s))}{2\,\alpha\,\sinh(\alpha\pi)} \\ -\dfrac{\alpha\,\cosh(\alpha(\pi+t-s))}{2\,\sinh(\alpha\pi)} & , & -\dfrac{\sinh(\alpha(\pi+t-s))}{2\,\sinh(\alpha\pi)}\end{pmatrix} & \text{if } 0\le t\le s\le 2\pi, \\[2ex] \begin{pmatrix} \dfrac{\sinh(\alpha(\pi+s-t))}{2\,\sinh(\alpha\pi)} & , & -\dfrac{\cosh(\alpha(\pi+s-t))}{2\,\alpha\,\sinh(\alpha\pi)} \\ -\dfrac{\alpha\,\cosh(\alpha(\pi+s-t))}{2\,\sinh(\alpha\pi)} & , & \dfrac{\sinh(\alpha(\pi+s-t))}{2\,\sinh(\alpha\pi)}\end{pmatrix} & \text{if } 0\le s< t\le 2\pi,\end{cases}$$

which implies that σ has the form (2.14), where g is defined in (2.15). $\square$

2.6. Remark. We can verify that for any $\alpha \in (0,\infty)$ the Green function g from (2.15) satisfies the relations

$$-\frac{\coth(\alpha\pi)}{2\,\alpha} \le g(t) \le -\frac{1}{2\,\alpha\,\sinh(\alpha\pi)} < 0 \qquad \text{for } t \in [0, 2\,\pi] \tag{2.16}$$

and

$$\int_0^{2\pi} g(|t-s|)\,\mathrm{d}s = -\frac{1}{\alpha^2} \qquad \text{for } t \in [0, 2\,\pi]. \tag{2.17}$$

The next result concerns the case $k = \alpha^2$.

2.7. Corollary. *Let $k = \alpha^2$, $\alpha \in (0,\infty) \setminus \mathbb{N}$, $\tau \in [0, 2\pi)$, $\delta \in \mathbb{R}$ and $\beta \in \mathbb{L}[0, 2\,\pi]$. Then the problem* (2.13) *possesses a unique solution* (σ, ρ). *Moreover,* $\sigma \in \mathbb{AC}[0, 2\,\pi]$, $\rho^{sing} = \delta\,\chi_{(\tau,2\pi]}$ *and σ has the form* (2.14), *where*

$$g(t) = \frac{\cos(\alpha\,(\pi - t))}{2\,\alpha\,\sin(\alpha\pi)} \quad \textit{on } [0, 2\,\pi]. \tag{2.18}$$

Proof. Substituting X in (2.9) by

$$X(t) = \begin{pmatrix} \cos(\alpha\,t) & , & \dfrac{\sin(\alpha\,t)}{\alpha} \\ -\alpha\,\sin(\alpha\,t) & , & \cos(\alpha\,t) \end{pmatrix}, \quad t \in [0, 2\,\pi],$$

we get

$$G(t,s) = \begin{cases} \begin{pmatrix} -\dfrac{\sin(\alpha(\pi+t-s))}{2\,\sin(\alpha\pi)} & , & \dfrac{\cos(\alpha(\pi+t-s))}{2\,\alpha\,\sin(\alpha\pi)} \\ -\alpha\,\dfrac{\cos(\alpha(\pi+t-s))}{2\,\sin(\alpha\pi)} & , & -\dfrac{\sin(\alpha(\pi+t-s))}{2\,\sin(\alpha\pi)} \end{pmatrix} \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } 0 \le t \le s \le 2\pi, \\ \begin{pmatrix} \dfrac{\sin(\alpha(\pi+s-t))}{2\,\sin(\alpha\pi)} & , & \dfrac{\cos(\alpha(\pi+s-t))}{2\,\alpha\,\sin(\alpha\pi)} \\ -\dfrac{\alpha\,\cos(\alpha(\pi+s-t))}{2\,\sin(\alpha\pi)} & , & \dfrac{\sin(\alpha(\pi+s-t))}{2\,\sin(\alpha\pi)} \end{pmatrix} \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } 0 \le s < t \le 2\pi \end{cases}$$

and since under our assumptions we have $\det\big(X^{-1}(2\pi) - \mathrm{I}\big) = 4\,\sin^2(\alpha\pi) \ne 0$, the proof follows from Proposition 2.2 similarly as the proof of Corollary 2.5. $\square$

2.8. Remark. For the function g from (2.18) and any $\alpha \in (0,\infty)\setminus \mathbb{N}$ we have

$$\|g^-\|_\infty \le \|g^+\|_\infty = \|g\|_\infty = \frac{1}{2\,\alpha\,|\sin(\alpha\pi)|} \tag{2.19}$$

and

$$\int_0^{2\pi} g(|t-s|)\,\mathrm{d}s = \frac{1}{\alpha^2} \qquad \text{for } t \in [0,2\,\pi]. \tag{2.20}$$

Furthermore, if $\alpha \in (0,\frac{1}{2}]$, then

$$0 \le \frac{\operatorname{cotan}(\alpha\pi)}{2\,\alpha} \le g(t) \le \frac{1}{2\,\alpha\,\sin(\alpha\pi)} \quad \text{for } t \in [0,2\,\pi]. \tag{2.21}$$

In the rest of this section we will derive some estimates for solutions of the problem

$$\sigma'' + k\,\sigma = b(t), \quad \sigma(0) = \sigma(2\pi),\ \sigma'(0) = \sigma'(2\pi), \tag{2.22}$$

which will be useful for the proofs of Section 3.

2.9. Lemma. *Let $p \in \mathbb{L}_\infty[0,2\,\pi]$, $b \in \mathbb{L}[0,2\,\pi]$ and $\overline{b} = 0$. Then*

$$\Big|\int_0^{2\pi} p(s)\,b(s)\,\mathrm{d}s\Big| \le p^*\,\|b\|_1, \quad \textit{where} \quad p^* = \frac{\|p^+\|_\infty + \|p^-\|_\infty}{2}.$$

Proof. First, notice that $\overline{b^+} = \overline{b^-}$ and $\|b\|_1 = 4\pi\overline{b^+} = 4\,\pi\overline{b^-}$ holds whenever $b \in \mathbb{L}[0,2\,\pi]$ and $\overline{b} = 0$. Thus, if $p(s) \ge 0$ a.e. on $[0,2\,\pi]$, then

$$\Big|\int_0^{2\pi} p(s)\,b(s)\,\mathrm{d}s\Big| \le \|p\|_\infty\,2\pi\,\overline{b^+} = \|p^+\|_\infty \frac{\|b\|_1}{2}.$$

In the general case, we have

$$\begin{aligned}\Big|\int_0^{2\pi} p(s)\,b(s)\,\mathrm{d}s\Big| &= \Big|\int_0^{2\pi} p^+(s)\,b(s)\,\mathrm{d}s + \int_0^{2\pi} p^-(s)\,(-b(s))\,\mathrm{d}s\Big| \\ &\le (\|p^+\|_\infty + \|p^-\|_\infty)\,\frac{\|b\|_1}{2} = p^*\,\|b\|_1.\end{aligned}$$

□

2.10. Lemma. *Let $k \ne 0$, $b \in \mathbb{L}[0,2\,\pi]$ and $\overline{b} = 0$. Then $\overline{\sigma} = 0$ holds for any solution σ of* (2.22).

Proof. In virtue of the periodicity conditions we have $\overline{\sigma''} = 0$. Thus, integrating the differential equation from (2.22) over $[0,2\,\pi]$, we get $0 = \overline{\sigma''} = -k\,\overline{\sigma} + \overline{b}$, i.e. $\overline{\sigma} = 0$. □

2.11. Lemma. *Let $k = -\alpha^2$, $\alpha \in (0,\infty)$, $b \in \mathbb{L}[0,2\pi]$, $\overline{b} = 0$ and let σ be a solution of* (2.22). *Then $\|\sigma\|_\infty \le \Phi(\alpha)\,\|b\|_1$, where*

$$\Phi(\alpha) = \min\{\frac{\pi}{6}, \frac{\coth(\alpha\pi)}{4\,\alpha}\}. \tag{2.23}$$

If $b \in \mathbb{L}_2[0,2\pi]$, then σ moreover fulfils $\|\sigma\|_\infty \le \sqrt{\frac{\pi}{6}}\,\|b\|_2$.

Proof. Multiplying the relation

$$-\sigma''(t) = -\alpha^2\,\sigma(t) - b(t) \quad \text{for a.e. } t \in [0,2\pi] \tag{2.24}$$

by $\sigma(t)$, integrating it over $[0,2\pi]$ and using the Hölder inequality, we get

$$\|\sigma'\|_2^2 = -\alpha^2\,\|\sigma\|_2^2 + \int_0^{2\pi} b(t)\,\sigma(t)\,\mathrm{d}t \le \|b\|_1\,\|\sigma\|_\infty.$$

Furthermore, by Lemma 2.10 we have $\overline{\sigma} = 0$, and thus σ satisfies the Sobolev inequality (see [7, Proposition 1.3])

$$\|\sigma\|_\infty \le \sqrt{\frac{\pi}{6}}\,\|\sigma'\|_2. \tag{2.25}$$

Hence $\|\sigma'\|_2 \le \sqrt{\frac{\pi}{6}}\,\|b\|_1$ and, using the Sobolev inequality (2.25) once more, we obtain

$$\|\sigma\|_\infty \le \frac{\pi}{6}\,\|b\|_1. \tag{2.26}$$

On the other hand, by Corollary 2.5,

$$\sigma(t) = \int_0^{2\pi} g(|t-s|)\,b(s)\,\mathrm{d}s \quad \text{on } [0,2\pi] \tag{2.27}$$

with g defined by (2.15). Hence, according to Lemma 2.9 and (2.16) we also have

$$\|\sigma\|_\infty \le \frac{\coth(\alpha\pi)}{4\,\alpha}\,\|b\|_1.$$

This together with (2.26) completes the proof of the first assertion of the lemma.

Now, suppose $b \in \mathbb{L}_2[0,2\pi]$. Using the Schwarz inequality, we deduce from (2.24) that $\|\sigma'\|_2^2 \le \|b\|_2\,\|\sigma\|_2$. Since σ satisfies the Wirtinger inequality (see [7, Proposition 1.3])

$$\|\sigma\|_2 \le \|\sigma'\|_2, \tag{2.28}$$

we get $\|\sigma'\|_2 \le \|b\|_2$. This together with the Sobolev inequality (2.25) completes the proof of the second assertion of the lemma. $\square$

2.12. Remark. Since the function $\varphi(\alpha)=\frac{\coth(\alpha\pi)}{4\,\alpha}$ is decreasing on $(0,\infty)$, $\varphi(0+)=\infty$ and $\lim_{x\to\infty}\varphi(x)=0$, there is exactly one $\alpha^*\in(0,\infty)$ ($\alpha^*\approx 0.51624$) such that $\varphi(\alpha^*)=\frac{\pi}{6}$, while $\varphi(\alpha)>\frac{\pi}{6}$ for $\alpha\in(0,\alpha^*)$ and $\varphi(\alpha)<\frac{\pi}{6}$ for $\alpha\in(\alpha^*,\infty)$. This means that the function Φ from (2.23) can be described by $\Phi(\alpha)=\frac{\pi}{6}$ if $\alpha\in(0,\alpha^*]$ and $\Phi(\alpha)=\varphi(\alpha)$ if $\alpha\in(\alpha^*,\infty)$.

2.13. Lemma. *Let* $k=\alpha^2$, $\alpha\in(0,\infty)\setminus\mathbb{N}$, $b\in\mathbb{L}[0,2\,\pi]$, $\overline{b}=0$ *and let* σ *be a solution of* (2.22). *Then* $\|\sigma\|_\infty\le\Psi(\alpha)\,\|b\|_1$, *where*

$$\Psi(\alpha)=\begin{cases}\min\{\dfrac{\pi}{6(1-\alpha^2)},\dfrac{1}{4\,\alpha\,\sin(\alpha\pi)}\} & \textit{if}\ \ \alpha\in(0,\frac{1}{2}],\\ \min\{\dfrac{\pi}{6(1-\alpha^2)},\dfrac{1}{2\,\alpha\,\sin(\alpha\pi)}\} & \textit{if}\ \ \alpha\in(\frac{1}{2},1),\\ \dfrac{1}{2\,\alpha\,|\sin(\alpha\pi)|} & \textit{if}\ \ \alpha\in(1,\infty)\setminus\mathbb{N}.\end{cases}\tag{2.29}$$

If $b\in\mathbb{L}_2[0,2\,\pi]$ *and* $\alpha\in(0,1)$, *then* σ *moreover fulfils* $\|\sigma\|_\infty\le\sqrt{\frac{\pi}{6}}\,\dfrac{\|b\|_2}{1-\alpha^2}$.

Proof. We have $-\sigma''(t)=\alpha^2\,\sigma(t)-b(t)$ a.e. on $[0,2\,\pi]$. According to Lemma 2.10, σ satisfies both the Sobolev inequality (2.25) and the Wirtinger inequality (2.28). Thus, proceeding similarly as in the proof of Lemma 2.11, we get $\|\sigma'\|_2^2\le\alpha^2\,\|\sigma'\|_2^2+\sqrt{\dfrac{\pi}{6}}\,\|b\|_1\,\|\sigma'\|_2$, i.e. $\|\sigma'\|_2\,(1-\alpha^2)\le\sqrt{\frac{\pi}{6}}\,\|b\|_1$. If $\alpha\in(0,1)$, then using (2.25) once more, the relation

$$\|\sigma\|_\infty\le\frac{\pi}{6\,(1-\alpha^2)}\,\|b\|_1$$

follows.

Further, σ has the form (2.27), where g is given by (2.18) and satisfies (2.21) if $\alpha\in(0,\frac{1}{2}]$ and (2.19) for $\alpha\in(\frac{1}{2},\infty)\setminus\mathbb{N}$. Therefore Lemma 2.9 implies that

$$\|\sigma\|_\infty\le\frac{\|b\|_1}{4\,\alpha\,\sin(\alpha\pi)}\qquad\text{if }\alpha\in(0,\frac{1}{2}]$$

and

$$\|\sigma\|_\infty\le\frac{\|b\|_1}{2\,\alpha\,|\sin(\alpha\pi)|}\qquad\text{if }\alpha\in(\frac{1}{2},\infty)\setminus\mathbb{N}.$$

Finally, let $b\in\mathbb{L}_2[0,2\,\pi]$ and $\alpha\in(0,1)$. Then we can argue as in the proof of Lemma 2.11 and derive the inequalities $\|\sigma'\|_2^2\le\alpha^2\,\|\sigma'\|_2^2+\|b\|_2\|\sigma\|_2$ and $\|\sigma'\|_2\,(1-\alpha^2)\le\|b\|_2$, wherefrom the second assertion of the lemma follows. □

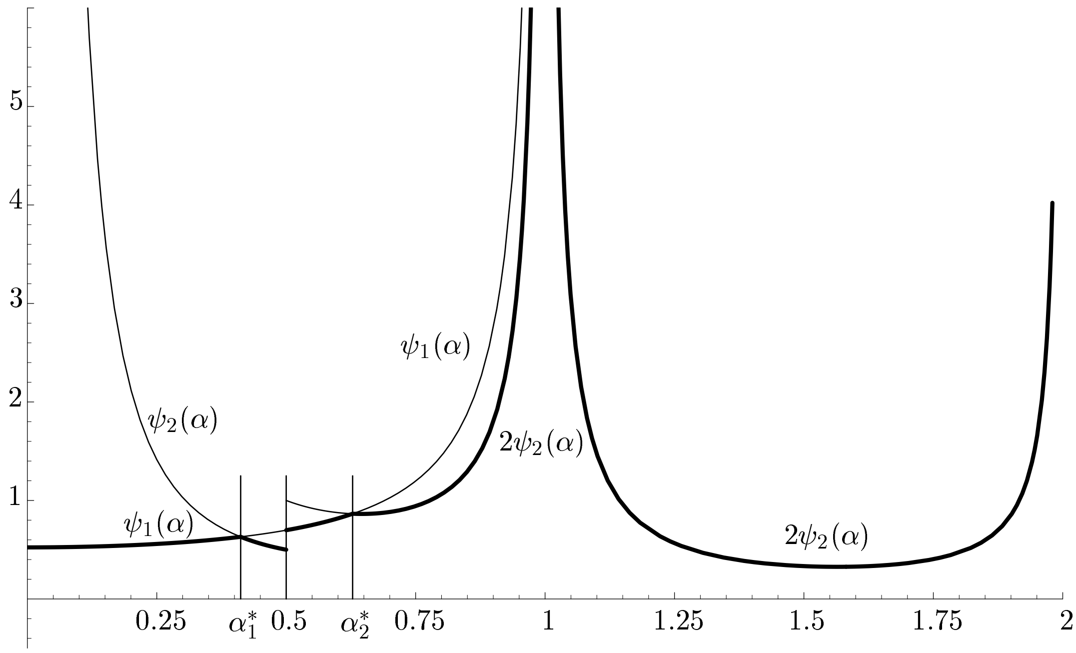

Figure 1:

2.14. Remark. For $\alpha \in (0,1)$ denote $\psi_1(\alpha) = \frac{\pi}{6(1-\alpha^2)}$ and $\psi_2(\alpha) = \frac{1}{4\,\alpha\,\sin(\alpha\pi)}$. It can be verified that there is exactly one $\alpha_1^* \in (0,1)$ ($\alpha_1^* \approx 0.412036$) such that $\psi_1(\alpha_1^*) = \psi(\alpha_1^*)$, $\psi_1(\alpha) > \psi_2(\alpha)$ for $\alpha \in (0,\alpha_1^*)$ and $\psi_1(\alpha) < \psi_2(\alpha)$ for $\alpha \in (\alpha_1^*, 1)$. Similarly, there is exactly one $\alpha_2^* \in (0,1)$ ($\alpha_2^* \approx 0.628308$) such that $\psi_1(\alpha_2^*) = 2\,\psi(\alpha_2^*)$, $\psi_1(\alpha) < 2\,\psi_2(\alpha)$ for $\alpha \in (\frac{1}{2}, \alpha_1^*)$ and $\psi_1(\alpha) > 2\,\psi_2(\alpha)$ for $\alpha \in (\alpha_2^*, 1)$. This means that for the function Ψ from (2.29) we have $\Psi(\alpha) = \psi_1(\alpha)$ if $\alpha \in (0,\alpha_1^*]$ or $\alpha \in (\frac{1}{2}, \alpha_2^*]$, $\Psi(\alpha) = \psi_2(\alpha)$ if $\alpha \in (\alpha_1^*, \frac{1}{2}]$ and $\Psi(\alpha) = 2\,\psi_2(\alpha)$ if $\alpha \in (\alpha_2^*, \infty) \setminus \mathbb{N}$. (Cf. Fig. 1).

3 . Lower and upper functions

Consider the problem (1.1) and suppose that $k > 0$ and that there exist $a \in \mathbb{R}$, $A_1 \in (0,\infty)$, $b \in \mathbb{L}_\infty[0, 2\,\pi]$ such that

$$f(t,x) \le a + b(t) \tag{3.1}$$

holds for a.e. $t \in [0, 2\,\pi]$ and all $x \in (A_1, \infty)$. Then we can find $r_1 \in (A_1, \infty)$ such that $k\,r_1 \ge a + \|b\|_\infty \ge f(t, r_1)$ for a.e. $t \in [0, 2\,\pi]$, which means that if we put $\sigma_1(t) = r_1$ and $\rho_1(t) = \sigma_1'(t) = 0$ on $[0, 2\,\pi]$, the functions (σ_1, ρ_1) are lower functions for (1.1). Similarly, if

$$f(t,x) \ge a + b(t) \tag{3.2}$$

holds for a.e. $t \in [0, 2\pi]$ and all $x \in (-\infty, -A_2)$, we can find $r_2 \in (-\infty, -A_2)$ such that $k\, r_2 \le a - \|b\|_\infty \le f(t, r_2)$ for a.e.$t \in [0, 2\pi]$, which means that the functions $(\sigma_2(t), \rho_2(t)) = (r_2, 0)$ on $[0, 2\pi]$ are upper functions of (1.1). We see that the constant function σ_1 has to be quite large and positive and the constant function σ_2 has to be sufficiently small negative. Similar observations can be done for $k < 0$.

In the case that b is not essentially bounded or if we need other localization of σ_1 or σ_2, this approach fails. However, we can construct and localize lower and upper functions by means of the results from Section 2.

If $k < 0$, we can write (1.1) in the form

$$u'' - \alpha^2 u = f(t, u), \quad u(0) = u(2\pi),\ u'(0) = u'(2\pi). \tag{3.3}$$

3.1. Theorem. *Let $\alpha \in (0, \infty)$ and let $a \in \mathbb{R}$, $\tau \in [0, 2\pi)$, $\delta \in [0, \infty)$, $b \in \mathbb{L}[0, 2\pi]$ be such that $\overline{b} = 0$ and (3.1) holds for a.e. $t \in [0, 2\pi]$ and all $x \in [A(t), B(t)]$, where*

$$A(t) = g(|t-\tau|)\delta - \frac{a}{\alpha^2} - \Phi(\alpha)\|b\|_1 \ \text{ and } \ B(t) = A(t) + 2\,\Phi(\alpha)\|b\|_1 \ \text{ on } [0, 2\pi] \tag{3.4}$$

and g and Φ are given by (2.15) and (2.23), respectively.

Then there exist lower functions (σ, ρ) of (3.3) such that

$$A(t) \le \sigma(t) \le B(t) \quad \text{on } [0, 2\pi]. \tag{3.5}$$

Proof. Let us put $\beta(t) = a + b(t)$ for a.e. $t \in [0, 2\pi]$. By Corollary 2.5 there is a unique solution (σ, ρ) of (2.13), where $k = -\alpha^2$, and, in view of (2.17), we have

$$\sigma(t) = g(|t-\tau|)\,\delta - \frac{a}{\alpha^2} + \int_0^{2\pi} g(|t-s|)\, b(s)\, \mathrm{d}s \quad \text{on } [0, 2\pi], \tag{3.6}$$

where g is given by (2.15). Denote

$$\sigma_0(t) = \int_0^{2\pi} g(|t-s|)\, b(s)\, \mathrm{d}s \quad \text{for } t \in [0, 2\pi]. \tag{3.7}$$

Then σ_0 is a solution to (2.22) and by Lemma 2.11 the estimate

$$\|\sigma_0\|_\infty \le \Phi(\alpha)\, \|b\|_1 \tag{3.8}$$

is true. Substituting (3.7) into (3.6) and using (3.4) and (3.8), we get (3.5). This together with (3.1) and (2.13) means that $\rho'(t) - \alpha^2\, \sigma(t) = \beta(t) = a + b(t) \ge f(t, \sigma(t))$ is true a.e. on $[0, 2\pi]$, i.e. (σ, ρ) are lower functions of (3.3). □

In the case $k > 0$ we will write the problem (1.1) in the form

$$u'' + \alpha^2 u = f(t, u), \quad u(0) = u(2\pi),\ u'(0) = u'(2\pi). \tag{3.9}$$

3.2. Theorem. *Let* $\alpha \in (0,\infty)\setminus\mathbb{N}$ *and let* $a \in \mathbb{R}$, $\tau \in [0,2\pi)$, $\delta \in [0,\infty)$, $b \in \mathbb{L}[0,2\pi]$ *be such that* $\overline{b} = 0$ *and* (3.1) *holds for a.e.* $t \in [0,2\pi]$ *and all* $x \in [A(t),B(t)]$, *where*

$$A(t) = g(|t-\tau|)\,\delta + \frac{a}{\alpha^2} - \Psi(\alpha)\,\|b\|_1 \ \text{ and } \ B(t) = A(t) + 2\Psi(\alpha)\,\|b\|_1 \tag{3.10}$$

and g *and* Ψ *are given by* (2.18) *and* (2.29), *respectively.*

Then there exist lower functions (σ,ρ) *of* (3.9) *fulfilling* (3.5).

Proof. By Corollary 2.7, the problem (2.13) with $k=\alpha^2$ and $\beta(t) = a + b(t)$ a.e. on $[0,2\pi]$ has a unique solution (σ,ρ) and, with respect to (2.14), (2.20),

$$\sigma(t) = g(|t-\tau|)\,\delta + \frac{a}{\alpha^2} + \sigma_0(t) \quad \text{on } [0,2\pi], \tag{3.11}$$

where σ_0 and g are defined by (3.7) and (2.18), respectively. Thus, σ_0 is a solution of (2.22). By Lemma 2.13 we have

$$\|\sigma_0\|_\infty \le \Psi(\alpha)\,\|b\|_1. \tag{3.12}$$

Furthermore, since (3.10)-(3.12) yield (3.5), according to (3.1) and (2.13) we have $\rho'(t) + \alpha^2\sigma(t) = a + b(t) \ge f(t,\sigma(t))$ a.e. on $[0,2\pi]$, i.e. (σ,ρ) are lower functions of (3.9). □

The next two assertions are dual respectively to Theorems 3.1 and 3.2 and their proofs can be omitted.

3.3. Theorem. *Let* $\alpha \in (0,\infty)$ *and let* $a \in \mathbb{R}$, $\tau \in [0,2\pi)$, $\delta \in (-\infty,0]$, $b \in \mathbb{L}[0,2\pi]$ *be such that* $\overline{b} = 0$ *and* (3.2) *holds for a.e.* $t \in [0,2\pi]$ *and all* $x \in [A(t),B(t)]$, *where* $A(t)$, $B(t)$ *are defined by* (3.4), (2.15) *and* (2.23).

Then there exist upper functions (σ,ρ) *of* (3.3) *fulfilling* (3.5).

3.4. Theorem. *Let* $\alpha \in (0,\infty)\setminus\mathbb{N}$ *and let* $a \in \mathbb{R}$, $\tau \in [0,2\pi)$, $\delta \in (-\infty,0]$, $b \in \mathbb{L}[0,2\pi]$ *be such that* $\overline{b} = 0$ *and* (3.2) *holds for a.e.* $t \in [0,2\pi]$ *and all* $x \in [A(t),B(t)]$, *where* $A(t)$, $B(t)$ *are defined by* (3.10), (2.18) *and* (2.29).

Then there exist upper functions (σ,ρ) *of* (3.9) *fulfilling* (3.5).

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