Satisfied with Physics: Glassy States in Satisfiability Problem

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Algorithm Complexity

I can't find an efficient algorithm, I guess I'm just too dumb.

Adapted from: Garey M. R., Johnson D. S., *Computers and Intractability: A Guide to the Theory of NP-Completeness* (Freeman, New York, 1979)

Algorithm Complexity

I can't find an efficient algorithm, because no such algorithm is possible!

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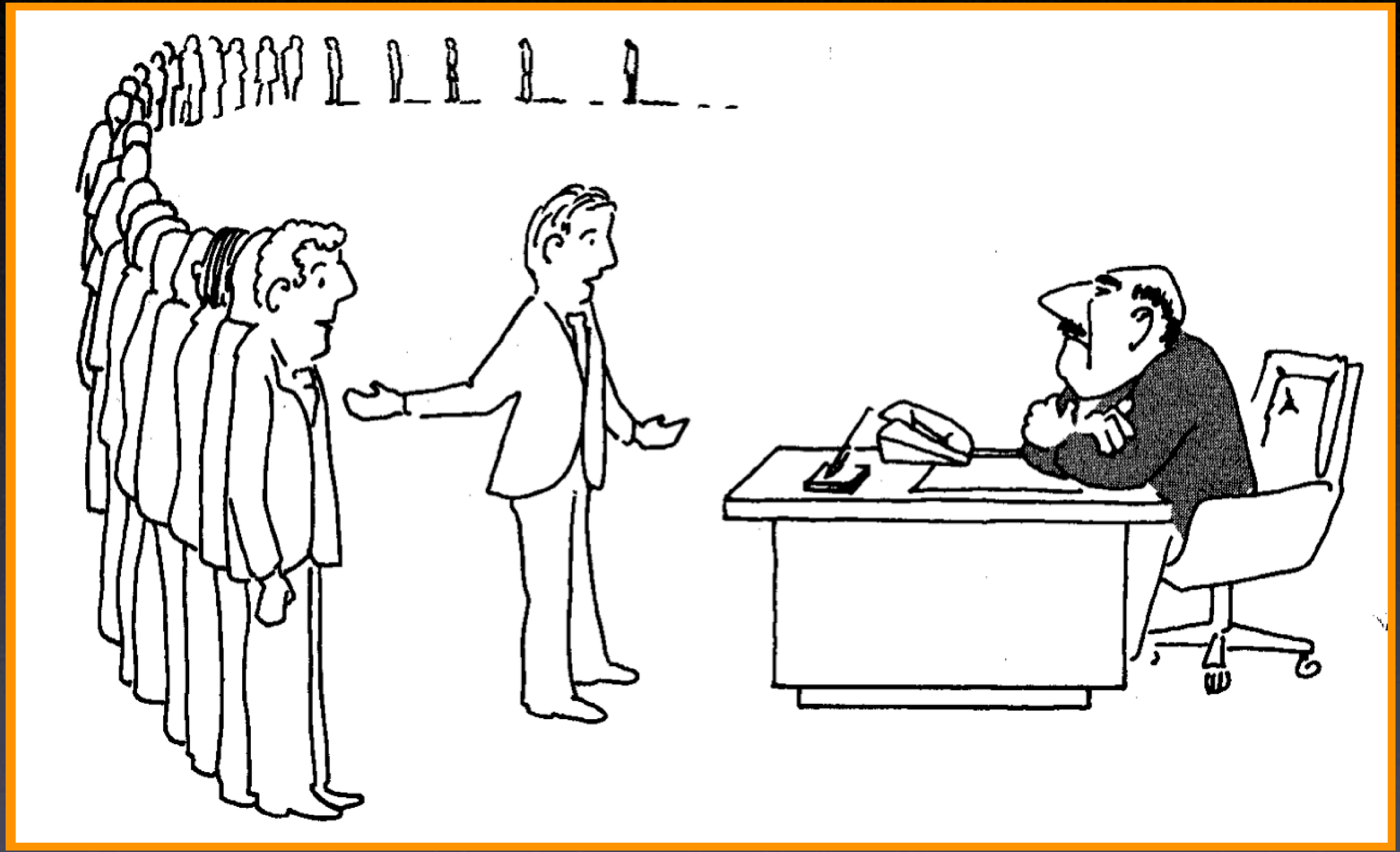
Algorithm Complexity



I can't find an efficient algorithm, but neither can all these famous people.

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Algorithm Complexity

NP - Completeness

- Travelling Salesman Problem
- Graph Colouring
- Planning and Scheduling
- Boolean Satisfiability Problem

A Satisfying Hamiltonian

- K-SAT: find a satisfying assignment for CNF

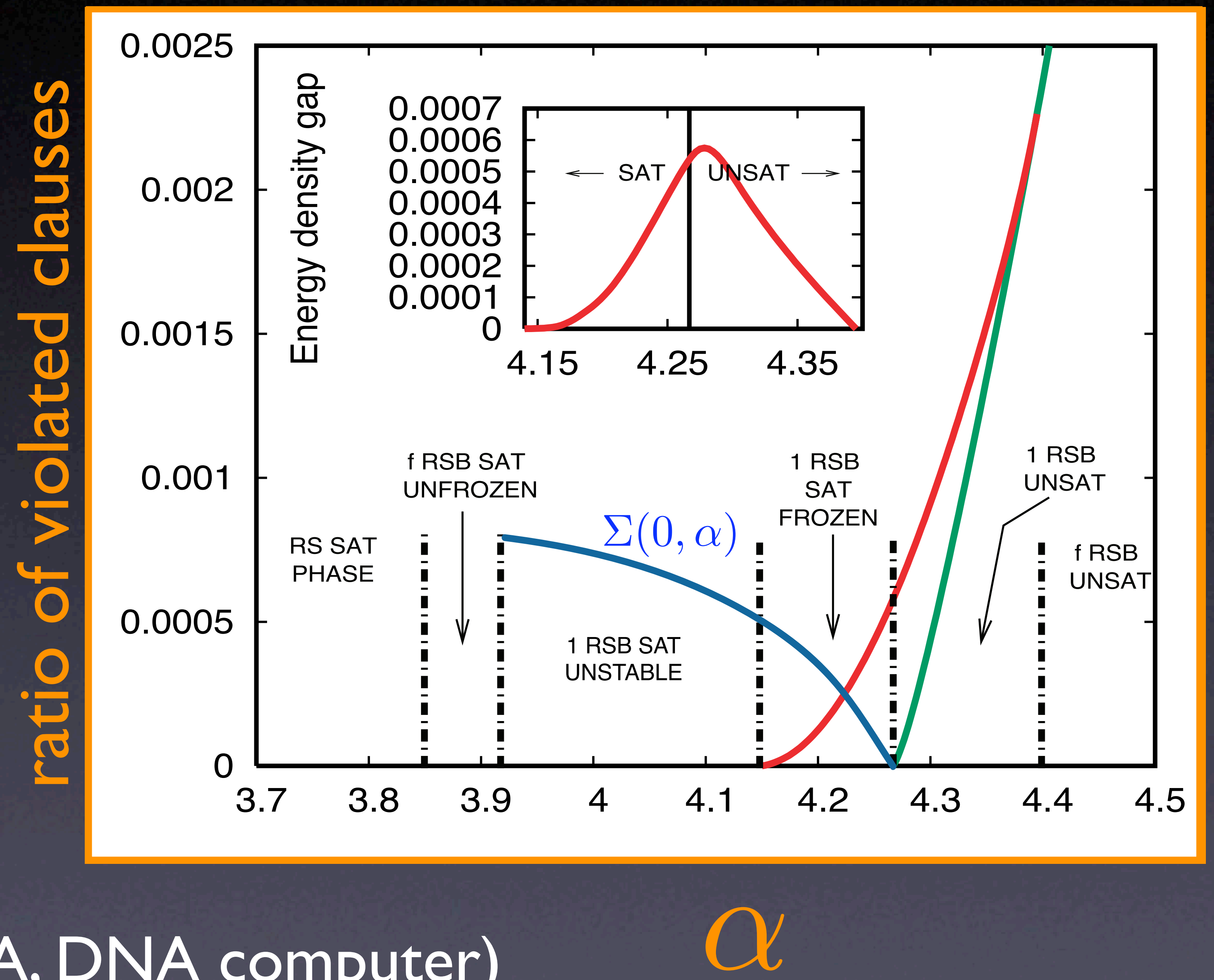
$$\ldots \wedge (A \vee B \vee C) \wedge \left(D \vee \bar{A} \vee E\right) \wedge \ldots$$

- Mapping to a spin model
  - ↑ ≡ TRUE, ↓ ≡ FALSE
  - Energy ≡ number of violated clauses

$$H = \sum_{c=1}^{M} \mathcal{E}_c = \sum_{c=1}^{M} \prod_{l=1}^{K} \frac{1 + J_{cl}\sigma_{(c,l)}}{2}$$

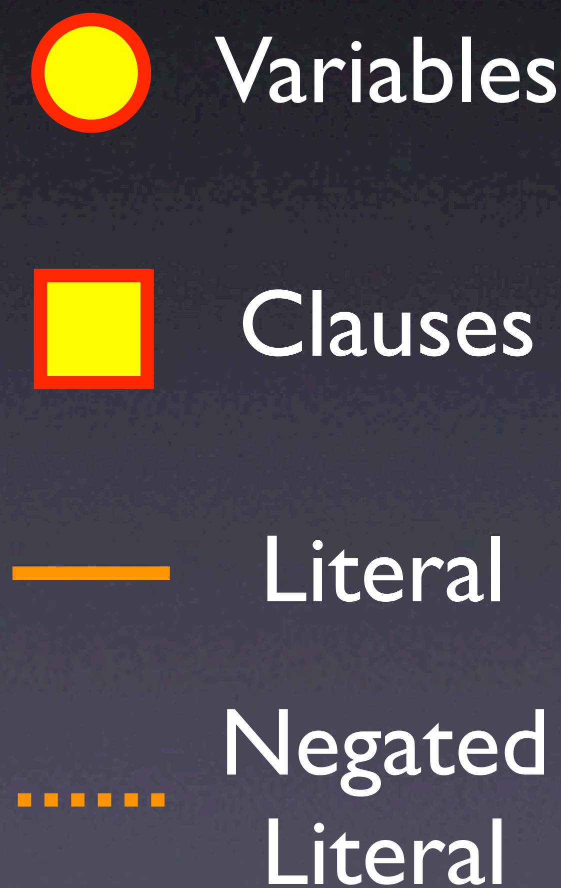
SATisfiable or not?

- Phase diagram
  - SAT / UNSAT
  - easy / hard
- Heuristics
  - Simulated Annealing WalkSAT
  - Exotic approaches (QA, DNA computer)
- Belief Propagation, Survey Propagation

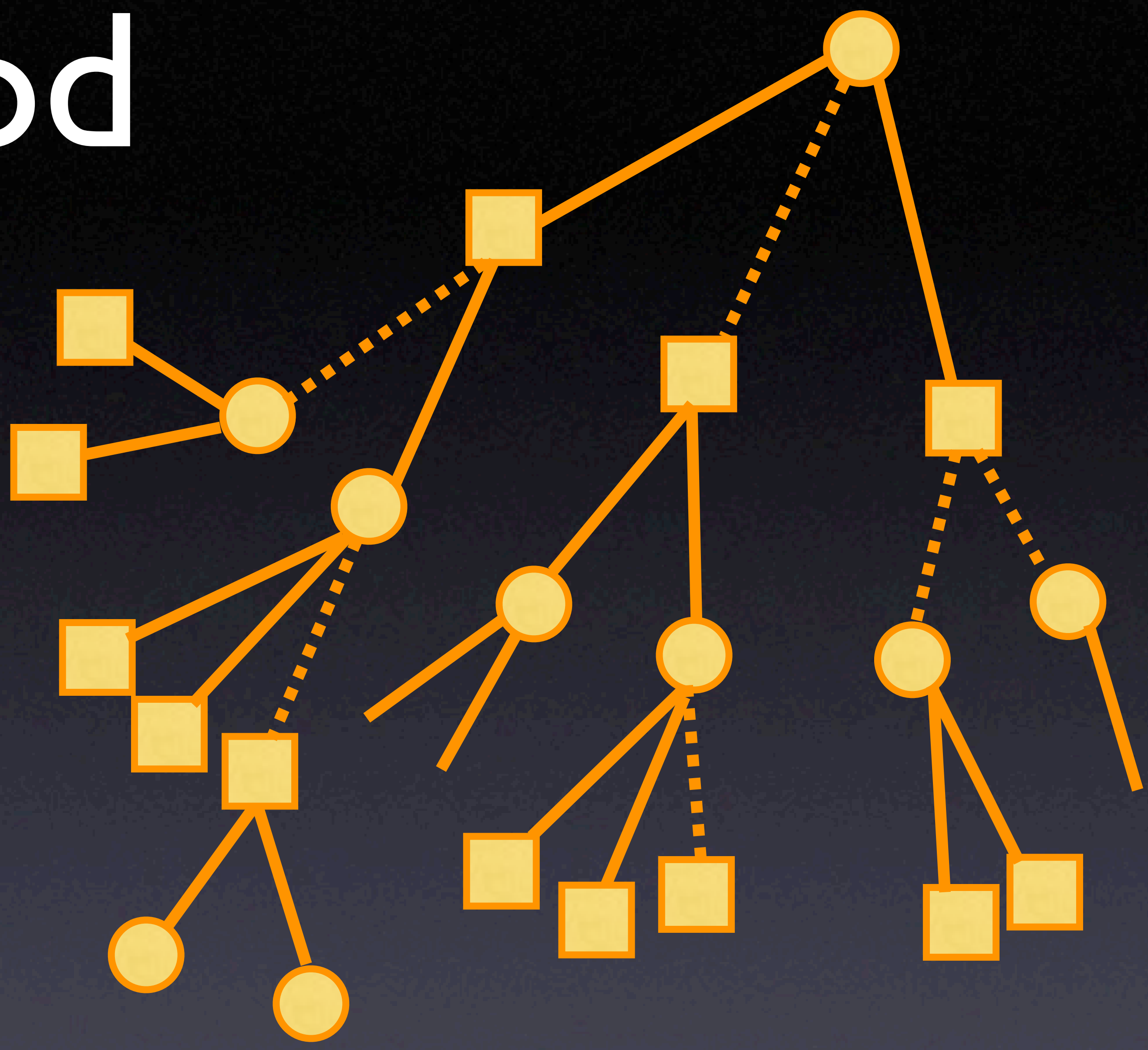


Adapted from Mézard M., Parisi G., Zecchina R., Science 2002

Factor Graph representation



Cavity method



- Remove a spin and relax the system
- Add again the spin and relax the system

$$E^{N+1} = E_0 + \sum_{c=1}^{\gamma} \min_{\{\sigma_c, \tau_c\}} \left[-\sigma_c g_c - \tau_c h_c + \mathcal{E}_c \left(\sigma_0, \sigma_c, \tau_c \right) \right]$$

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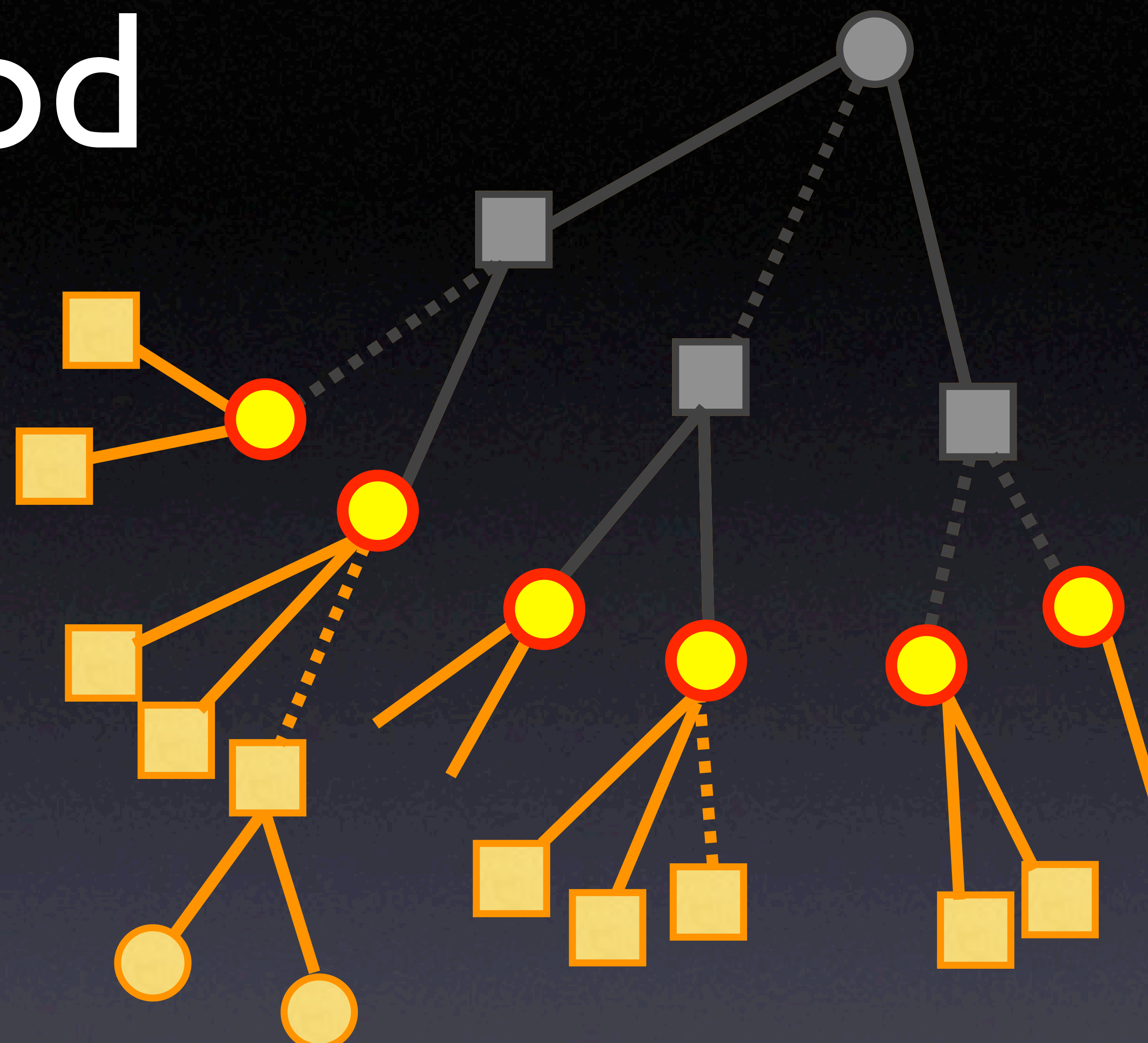

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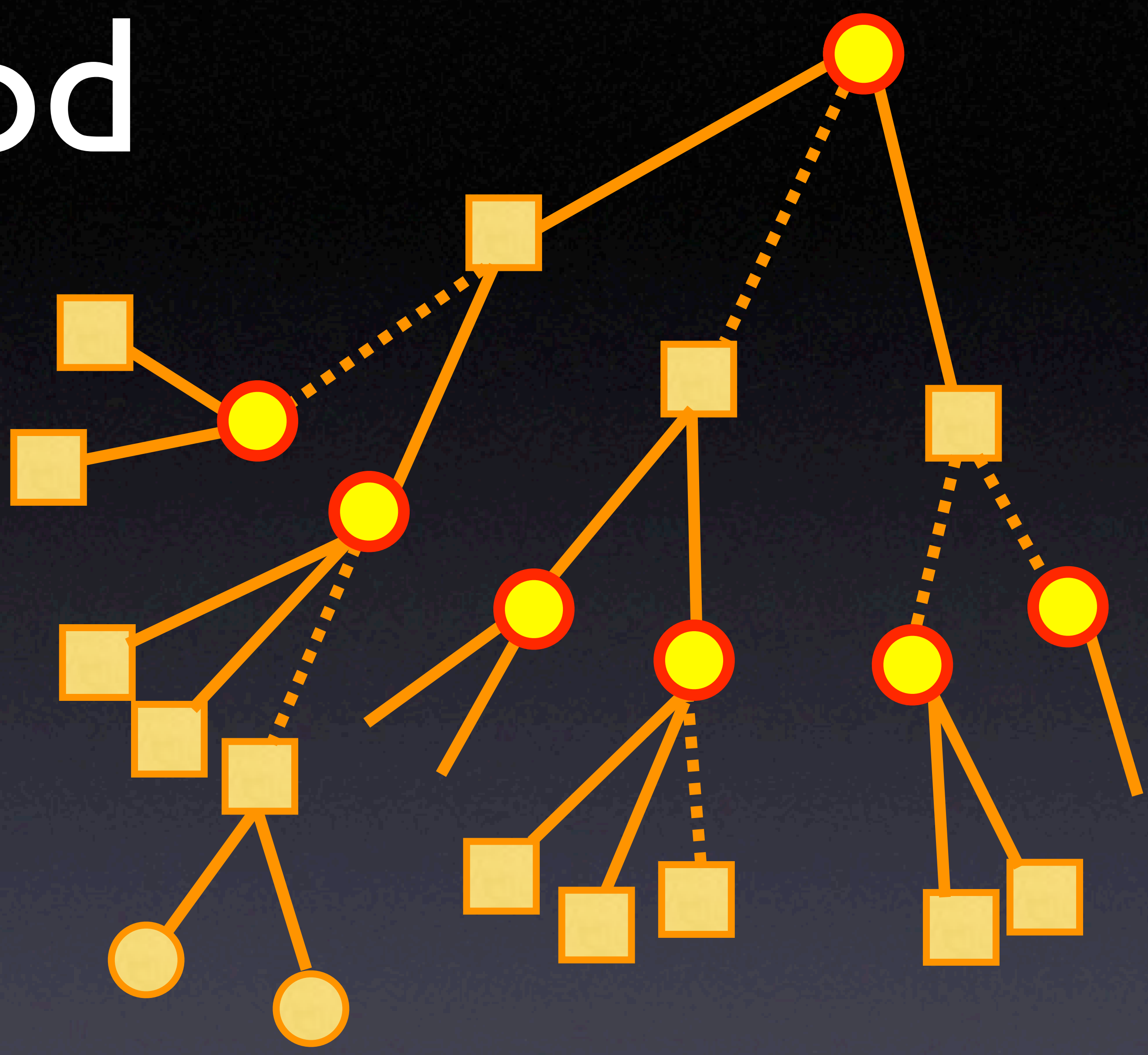

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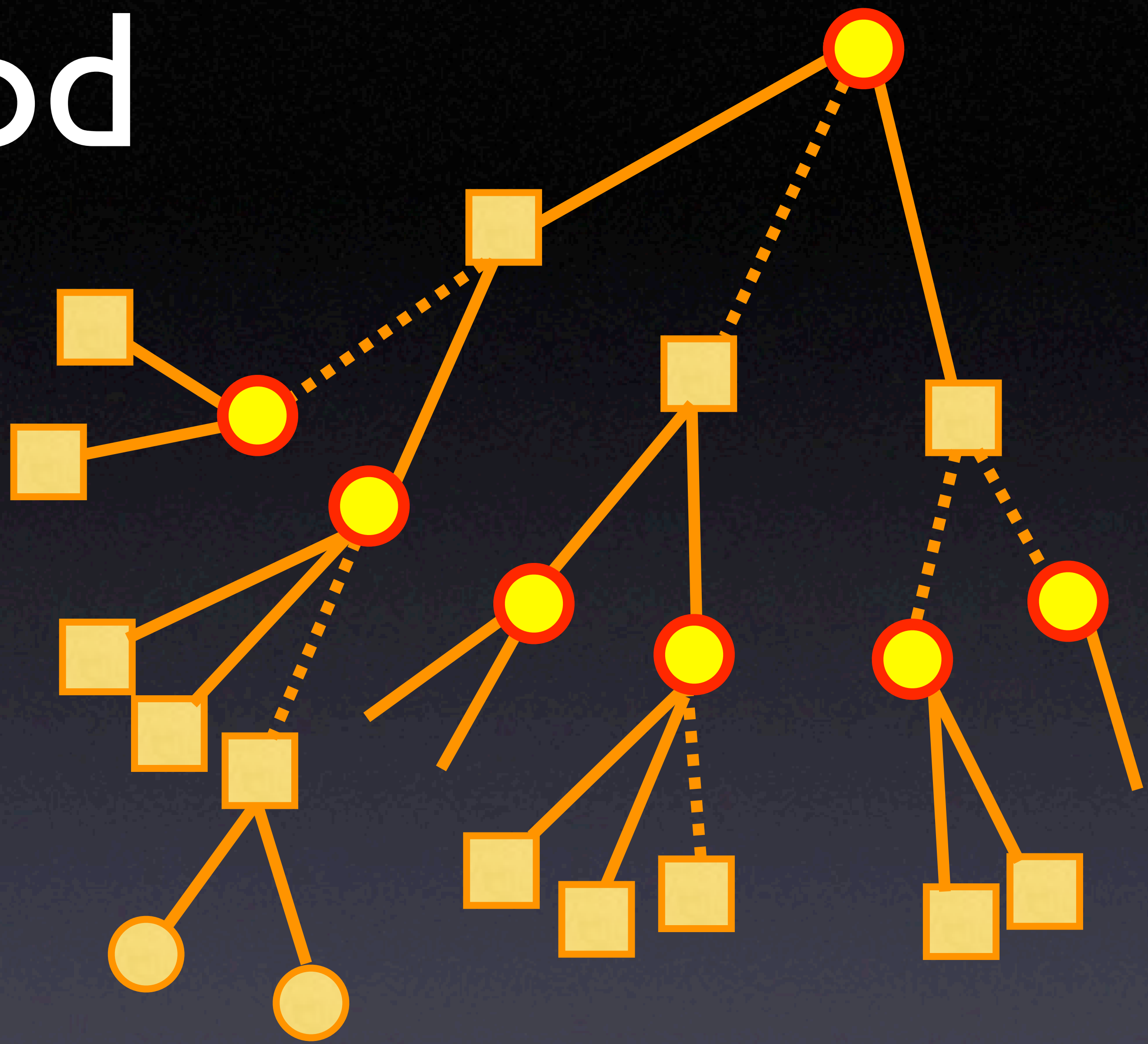


$$E^{N+1} = E_0 - \sum_{c=1}^{\gamma} \left[w_c\left(g_c, h_c\right) + \sigma_0 u_c\left(g_c, h_c\right) \right]$$

Cavity method

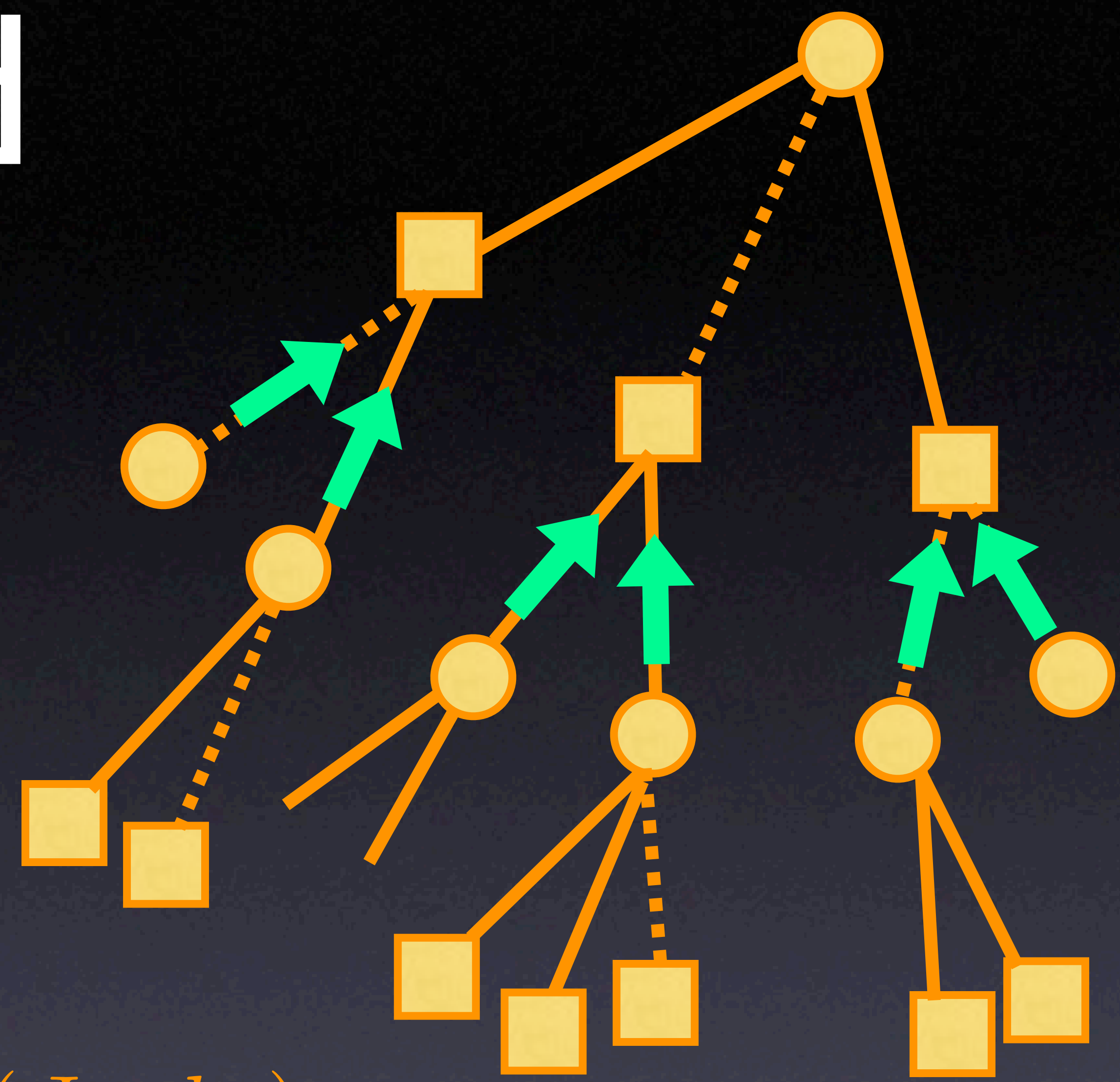
- Replica symmetric solution
- Interpretation in term of signals (Belief Propagation)

$$u_c\left(g_c, h_c\right) = -J_{c,0}\,\theta\left(J_{c,g} g_c\right)\theta\left(J_{c,h} h_c\right)$$

$$w_c\left(g_c, h_c\right) = \left|g_c\right| + \left|h_c\right| - \theta\left(J_{c,g} g_c\right)\theta\left(J_{c,h} h_c\right)$$

$$h_c = \sum_{c'=1}^{\gamma'-1} u_{c'}$$

Cavity method



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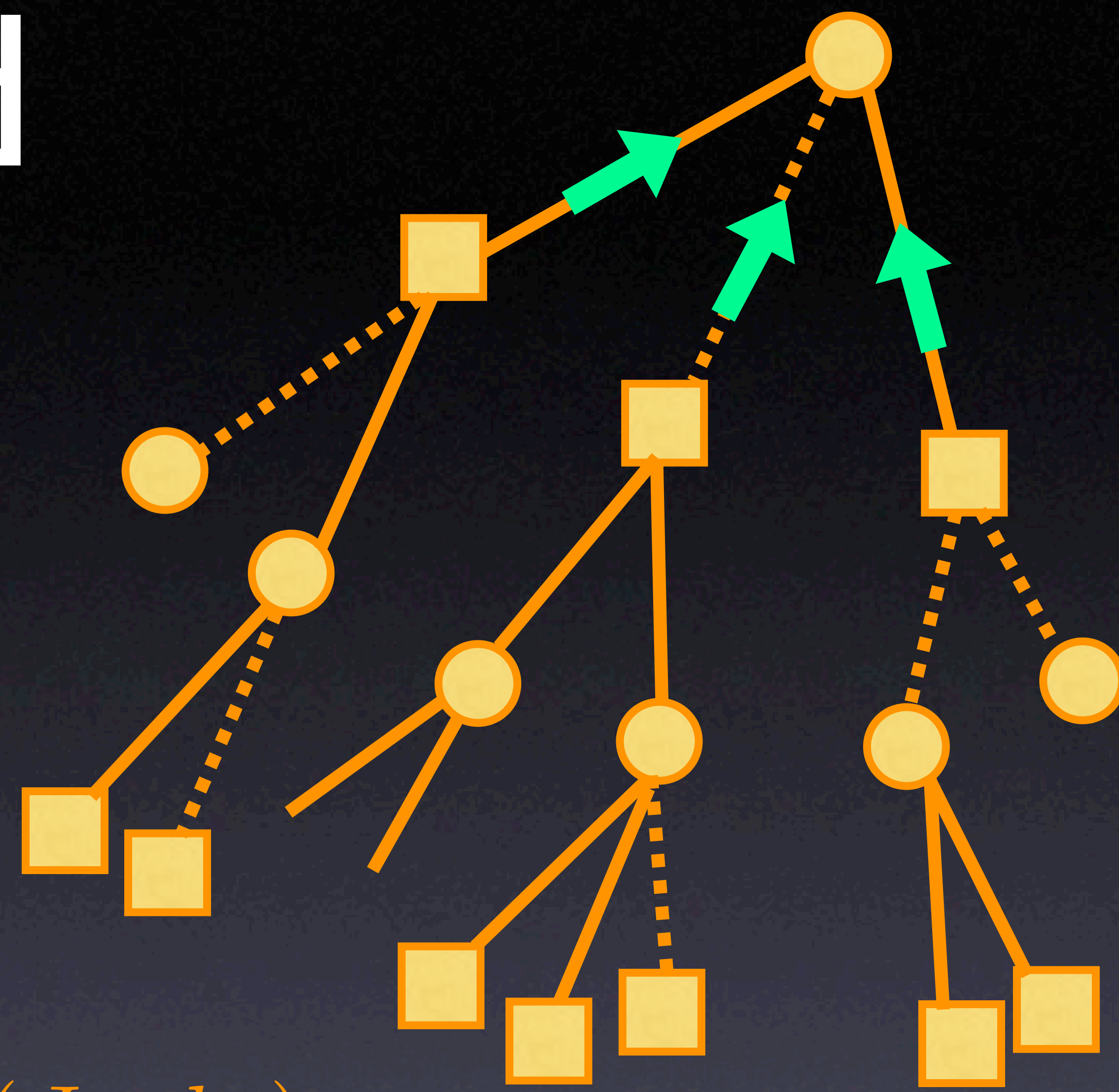
$$h_c = \sum_{c'=1}^{\gamma'-1} u_{c'}$$

With *h* variable says to clause:
"I can (or I can't) satisfy you!"

Cavity method



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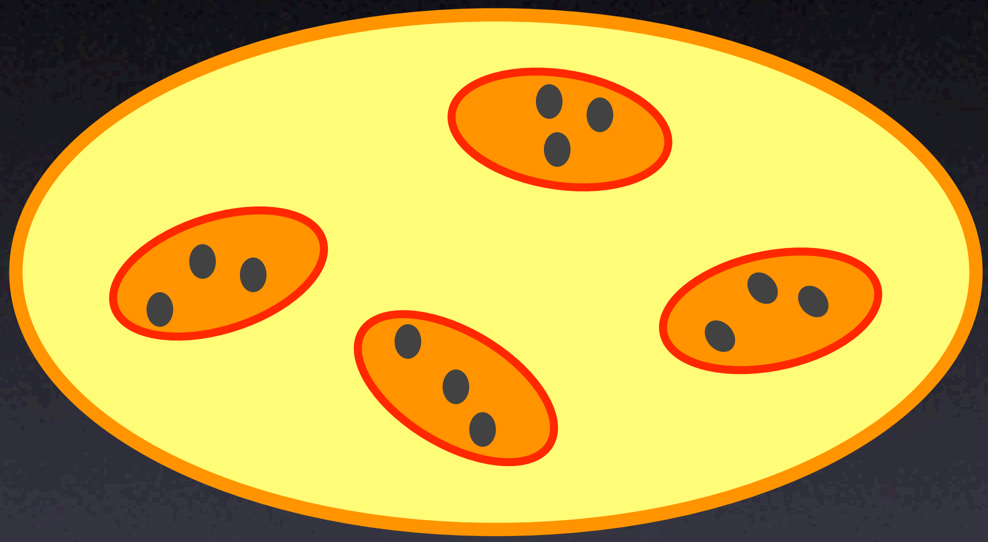
$$h_c = \sum_{c'=1}^{\gamma'-1} u_{c'}$$

With *u* clause says to variable:
"Please, make me true!"

Survey Propagation

- One step RSB
  - Clustering of states
  - Complexity
- From Belief Propagation to Survey Propagation



$$N_{\text{states}} = \exp\{N\Sigma\}$$

u-survey: $Q(u) = \eta_0\,\delta(u) + \eta_+\,\delta(u-1) + \eta_-\,\delta(u+1)$

Finite y SP Algorithm

- Pseudo-temperature:

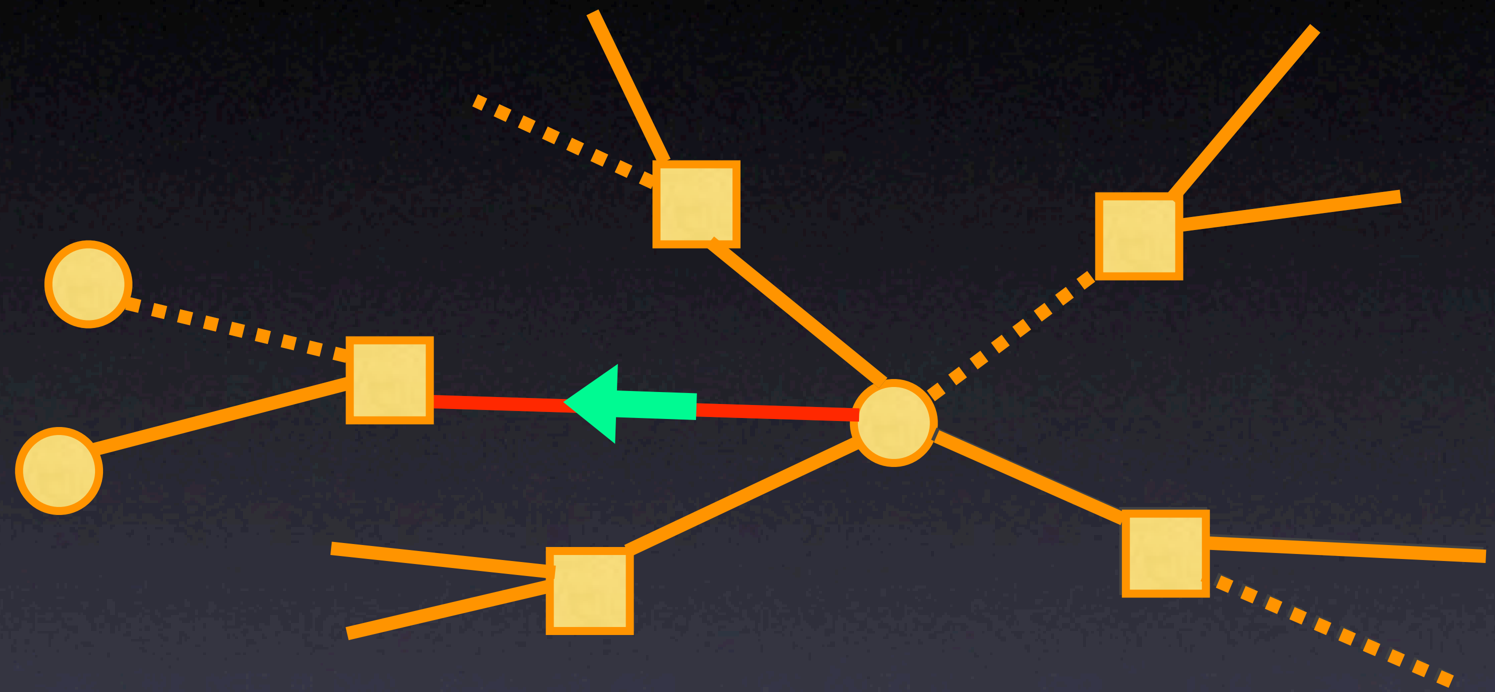
$$\frac{1}{y} = \left(\frac{\partial \Sigma}{\partial E}\right)$$

- Population dynamics equations:

$$P(h) \propto \int \mathcal{D}Q(u)\delta\left(h - \sum u\right)\exp\left[y\left|\sum u\right| - y\sum |u|\right]$$

$$Q(u) = \int dg\, dh\, P(g)P(h)\,\delta\left(u - u(g,h)\right)$$

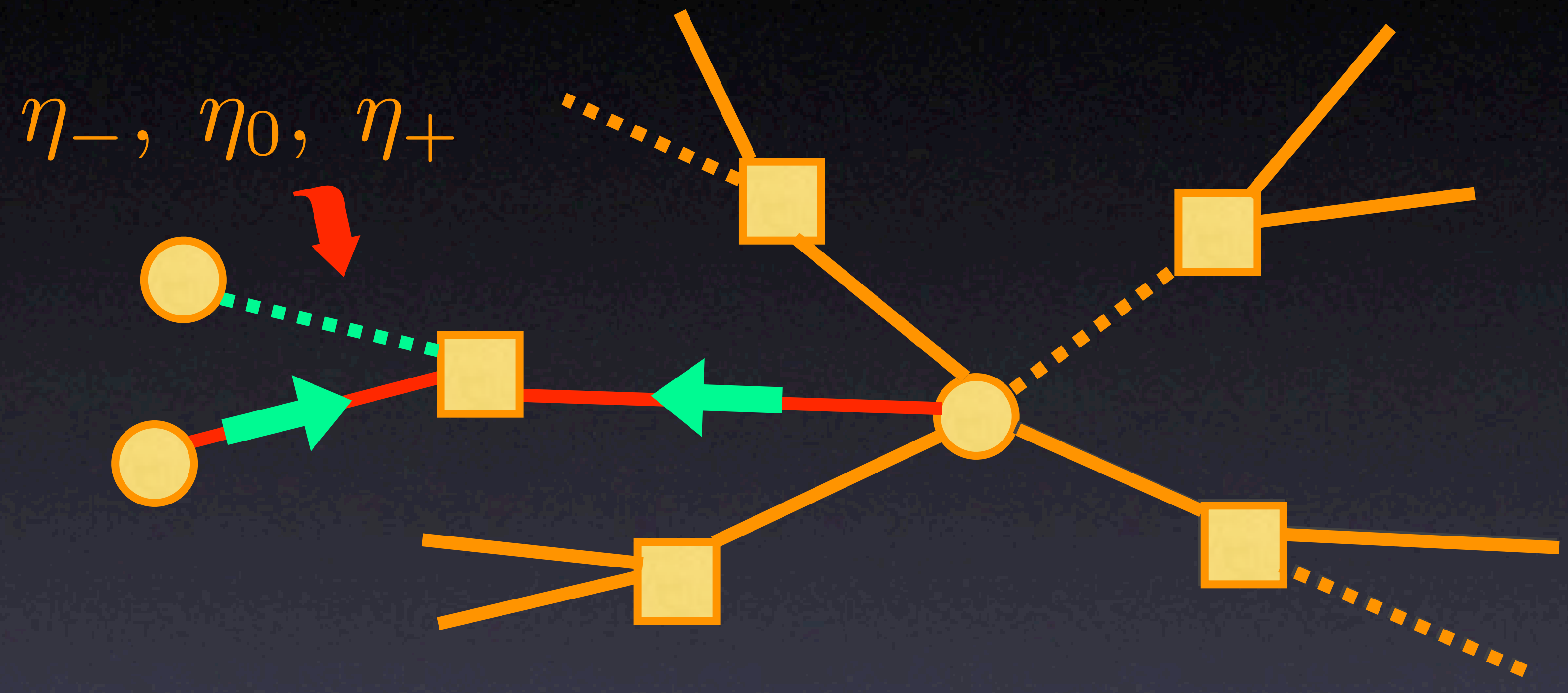
Finite y SP Algorithm



$$\frac{P^{(1)}}{C^{(1)}} = \eta_0\,\delta(h) + \eta_+\,\delta(h-1) + \eta_-\,\delta(h+1)$$

$$\frac{P^{(\gamma-1)}(h)}{C^{(\gamma-1)}} = \eta_0 \frac{P^{(\gamma-2)}(h)}{C^{(\gamma-2)}} + \eta_+ \mathrm{e}^{-2y\,\hat{\theta}(-h)} \frac{P^{(\gamma-2)}(h-1)}{C^{(\gamma-2)}} + \eta_- \mathrm{e}^{-2y\,\hat{\theta}(h)} \frac{P^{(\gamma-2)}(h+1)}{C^{(\gamma-2)}}$$

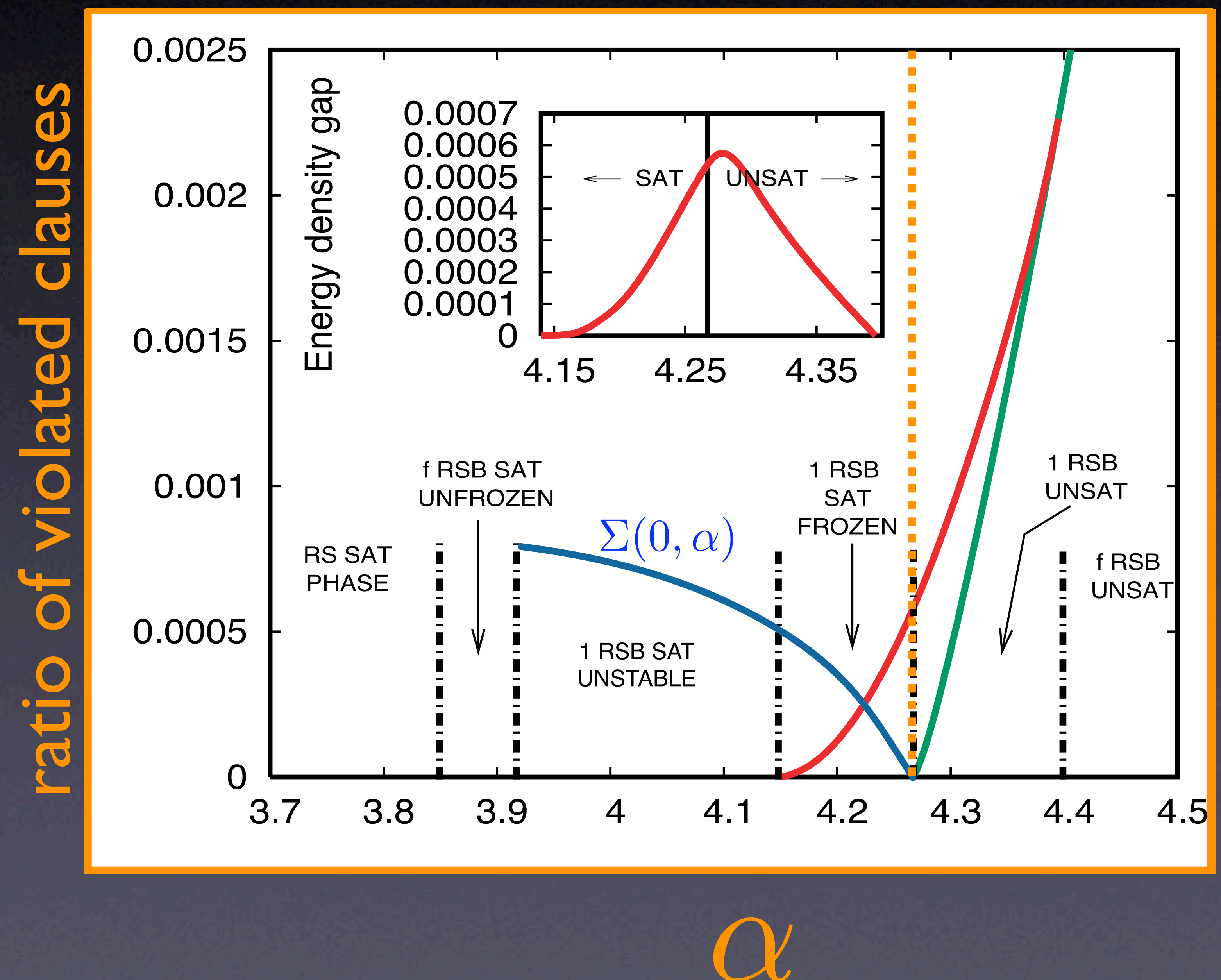
Finite y SP Algorithm



1. Compute two *h*-surveys ➠ Update one η
2. Update all η's until reaching convergence (*probability of probability distribution*)
3. Fix spins using local field information ➠ Decimate the formula

Exploring MAX-K-SAT problem

- $y = \infty$ filters out all contradictory assignments
- Finite y allows to study the UNSAT region
  - MAX-SAT problem
  - Improvements over heuristics

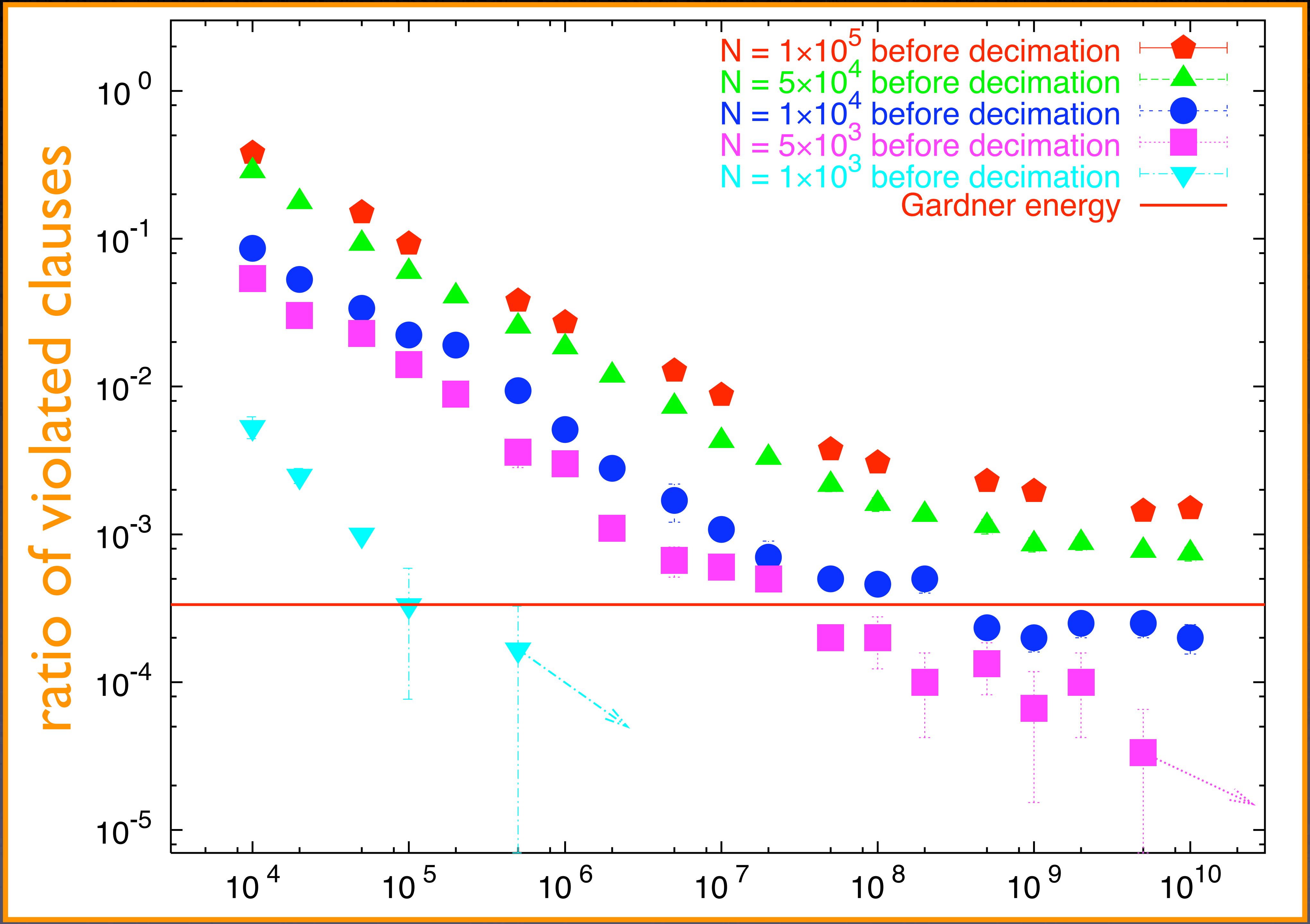


Adapted from Mézard M., Parisi G., Zecchina R., Science 2002

Threshold states

- $y = \infty$ filters out all contradictory assignments
- Finite y allows to study the UNSAT region
  - MAX-SAT problem
  - Improvements over heuristics

WalkSAT performance, $\alpha = 4.24$

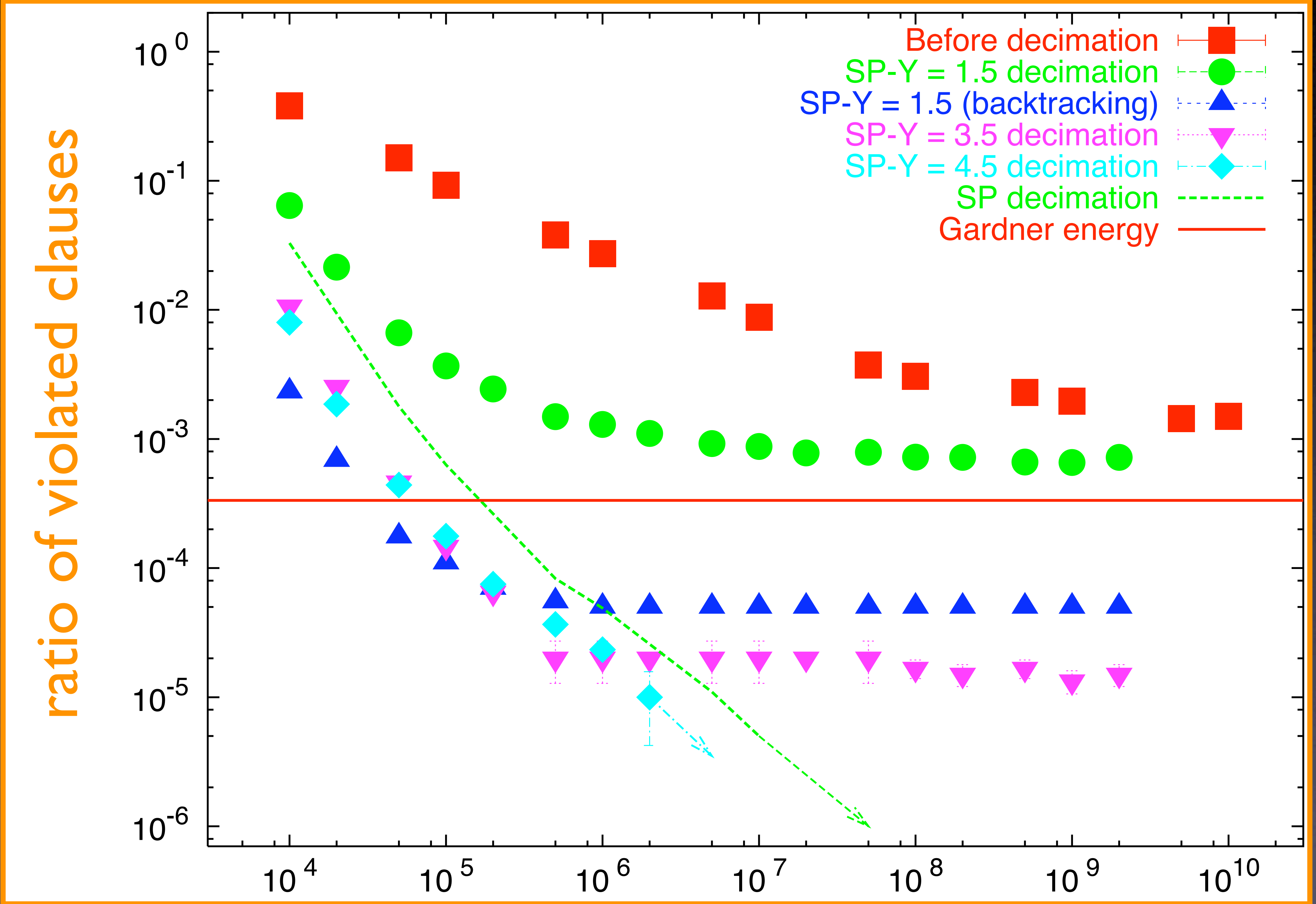


Battaglia D., Kolář M., Zecchina R., Phys.Rev.E 2004

Exploring SAT region

- $y = \infty$ filters out all contradictory assignments
- Finite y allows to study the UNSAT region
  - MAX-SAT problem
  - Improvements over heuristics

Survey Propagation performance, $\alpha = 4.24$



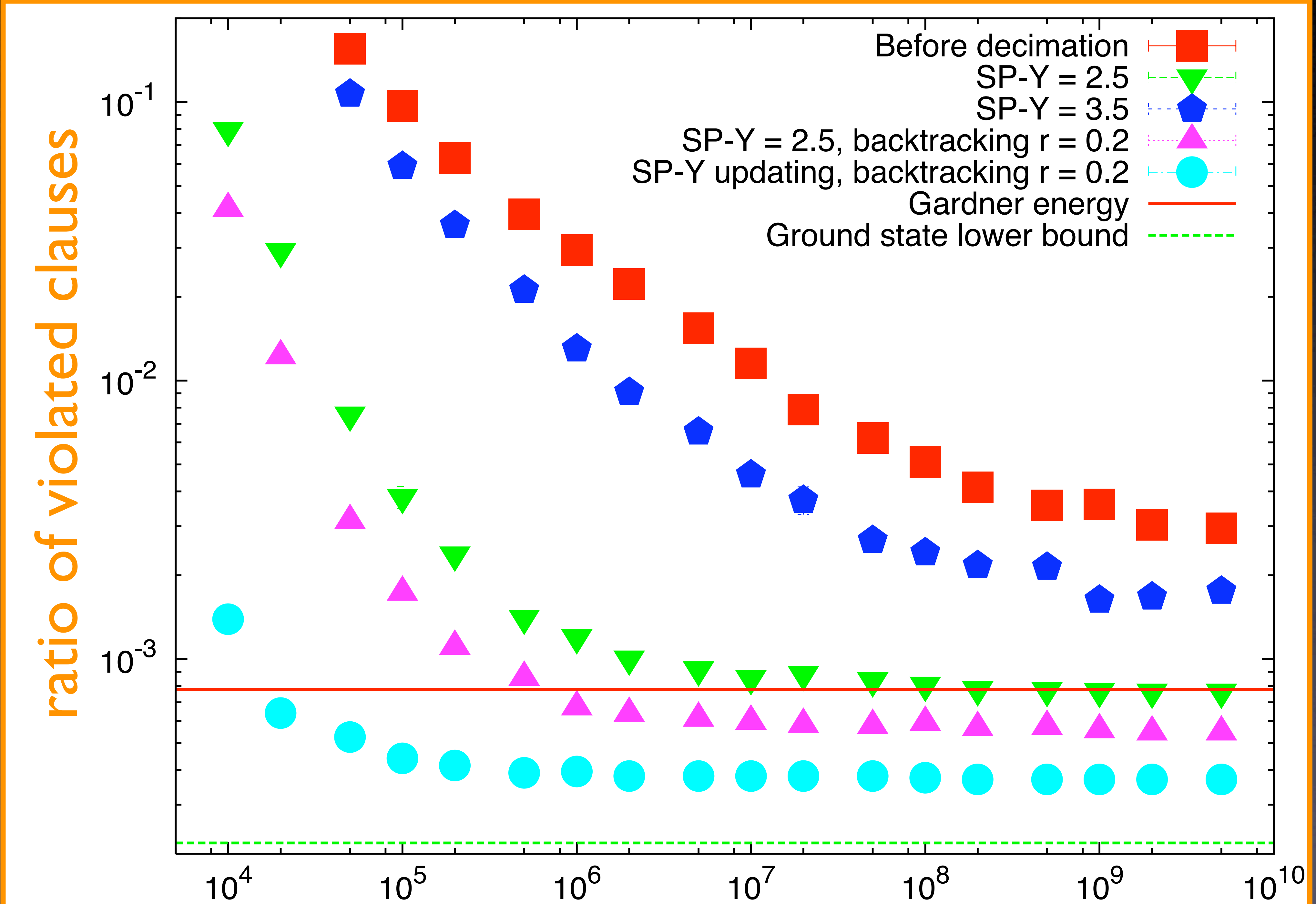
WalkSAT cutoff

Battaglia D., Kolář M., Zecchina R., Phys.Rev.E 2004

Exploring UNSAT region

- $y = \infty$ filters out all contradictory assignments
- Finite y allows to study the UNSAT region
  - MAX-SAT problem
  - Improvements over heuristics

Survey Propagation performance, $\alpha = 4.29$



WalkSAT cutoff

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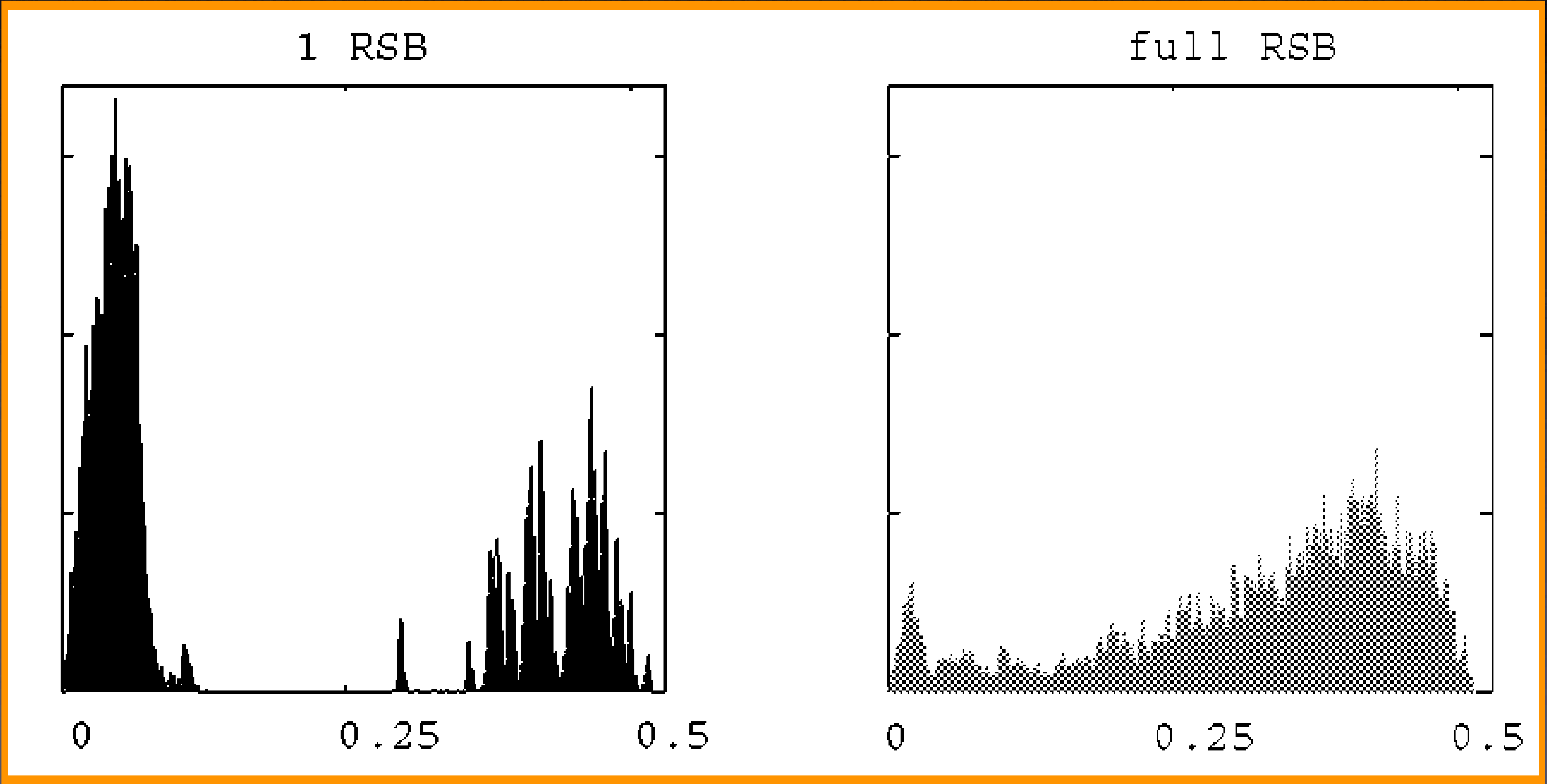
Solutions are Clustered

Hamming distances of solutions in 1RSB and full RSB phase



Battaglia D., Kolář M., Zecchina R., Int.J.Mod.Phys.B 2006

Conclusions

- New Survey Propagation Algorithm for MAX-3-SAT Problem
  - Threshold states observed and crossed
  - Heuristics Improvement: SP with finite y approaches closly the GS energy
- Clusterisation of solutions observed in SAT phase (1RSB, fullRSB)

Thank You for Your Attention :-)

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