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# Second Order Periodic Problem with $\phi$-Laplacian and Impulses - Part I

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**Abstract.** Existence principles for the BVP $(\phi(u'))' = f(t,u,u')$, $u(t_i+) = J_i(u(t_i))$, $u'(t_i+) = M_i(u'(t_i))$, $i = 1,2,\dots,m$, $u(0) = u(T)$, $u'(0) = u'(T)$ are presented. They are based on the method of lower/upper functions which are well-ordered.

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## 1. Formulation of the problem

Let $m \in \mathbb{N}$, $0 = t_0 < t_1 < \dots < t_m < t_{m+1} = T$ and $\mathrm{D} = \{t_1, t_2, \dots, t_m\}$. Define $\mathbb{C}_\mathrm{D}$ (or $\mathbb{C}^1_\mathrm{D}$) as the sets of functions $u : [0,T] \mapsto \mathbb{R}$,

$$u(t) = \begin{cases} u_{[0]}(t) & \text{if } t \in [0,t_1], \\ u_{[1]}(t) & \text{if } t \in (t_1,t_2], \\ \dots & \quad \dots \\ u_{[m]}(t) & \text{if } t \in (t_m,T], \end{cases}$$

where $u_{[i]}$ is continuous on $[t_i, t_{i+1}]$ (or continuously differentiable on $[t_i, t_{i+1}]$) for $i = 0,1,\dots,m$. We put $\|u\|_\mathrm{D} = \|u\|_\infty + \|u'\|_\infty$, where $\|u\|_\infty = \sup \mathrm{ess}_{t\in[0,T]} |u(t)|$. Then $\mathbb{C}_\mathrm{D}$ and $\mathbb{C}^1_\mathrm{D}$ respectively with the norms $\|.\|_\infty$ and $\|.\|_\mathrm{D}$ become Banach spaces. Further, $\mathbb{AC}_\mathrm{D}$ is the set of functions $u \in \mathbb{C}_\mathrm{D}$ which are absolutely continuous on each subinterval $(t_i, t_{i+1})$, $i = 0,1,\dots,m$. As usual, $\mathbb{L}_1$ denotes the Banach space of Lebesgue integrable functions on $[0,T]$ with the norm $\|f\|_1 = \int_0^T |f(t)|\,\mathrm{d}t$.

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We consider the problem

$$(\phi(u'(t)))' = f(t, u(t), u'(t)) \quad \text{a.e. on} \quad [0, T], \tag{1.1}$$

$$u(t_i+) = J_i(u(t_i)), \quad u'(t_i+) = M_i(u'(t_i)), \quad i = 1, 2, \ldots, m, \tag{1.2}$$

$$u(0) = u(T), \quad u'(0) = u'(T), \tag{1.3}$$

where $u'(t_i) = u'(t_i-) = \lim_{t \to t_i-} u'(t)$ for $i = 1, 2, \ldots, m+1$, $u'(0) = u'(0+) = \lim_{t \to 0+} u'(t)$, and $f$ is an $\mathbb{L}_1$-Carathéodory function on $[0, T] \times \mathbb{R}^2$ (i.e. for each $x \in \mathbb{R}$ and $y \in \mathbb{R}$ the function $f(., x, y)$ is measurable on $[0, T]$; for almost every $t \in [0, T]$ the function $f(t, ., .)$ is continuous on $\mathbb{R}^2$; for each compact set $K \subset \mathbb{R}^2$ there is a function $m_K(t) \in \mathbb{L}_1$ such that $|f(t, x, y)| \le m_K(t)$ holds for a.e. $t \in [0, T]$ and all $(x, y) \in K$.) Further we assume that functions $J_i$, $M_i$ are continuous on $\mathbb{R}$ and $\phi$ is an increasing homeomorphism such that $\phi(0) = 0$ and $\phi(\mathbb{R}) = \mathbb{R}$. A typical example of a proper function $\phi$ is the $p$-Laplacian $\phi_p(y) = |y|^{p-2}\, y$, where $p > 1$.

Clearly, if $J_i(x) = x$, $M_i(x) = x$ for all $x \in \mathbb{R}$, $i = 1, 2, \ldots, m$, we get the problem (1.1), (1.3) (a periodic problem without impulses).

**1.1. Definition.** A *solution* of the problem (1.1)–(1.3) is a function $u \in \mathbb{C}^1_{\mathrm{D}}$ such that $\phi(u') \in \mathbb{AC}_{\mathrm{D}}$ and (1.1)–(1.3) hold.

**1.2. Definition.** A function $\sigma_1 \in \mathbb{C}^1_{\mathrm{D}}$ is called a *lower function of the problem* (1.1)–(1.3) if $\phi(\sigma_1') \in \mathbb{AC}_{\mathrm{D}}$ and

$$(\phi(\sigma_1'(t)))' \ge f(t, \sigma_1(t), \sigma_1'(t)) \quad \text{for a.e.} \quad t \in [0, T], \tag{1.4}$$

$$\sigma_1(t_i+) = J_i(\sigma_1(t_i)), \quad \sigma_1'(t_i+) \ge M_i(\sigma_1'(t_i)), \quad i = 1, 2, \ldots, m, \tag{1.5}$$

$$\sigma_1(0) = \sigma_1(T), \quad \sigma_1'(0) \ge \sigma_1'(T). \tag{1.6}$$

Similarly, a function $\sigma_2 \in \mathbb{C}^1_{\mathrm{D}}$ is an *upper function of the problem* (1.1)–(1.3) if $\phi(\sigma_2') \in \mathbb{AC}_{\mathrm{D}}$ and

$$(\phi(\sigma_2'(t)))' \le f(t, \sigma_2(t), \sigma_2'(t)) \quad \text{for a.e.} \quad t \in [0, T], \tag{1.7}$$

$$\sigma_2(t_i+) = J_i(\sigma_2(t_i)), \quad \sigma_2'(t_i+) \le M_i(\sigma_2'(t_i)), \quad i = 1, 2, \ldots, m, \tag{1.8}$$

$$\sigma_2(0) = \sigma_2(T), \quad \sigma_2'(0) \le \sigma_2'(T). \tag{1.9}$$

**1.3. Remark.** If $M_i(0) = 0$ for $i = 1, 2, \ldots, m$ and $r_1 \in \mathbb{R}$ is such that $J_i(r_1) = r_1$ for $i = 1, 2, \ldots, m$ and

$$f(t, r_1, 0) \le 0 \quad \text{for a.e.} \quad t \in [0, T],$$

then $\sigma_1(t) \equiv r_1$ on $[0, T]$ is a lower function of the problem (1.1)–(1.3). Similarly, if $r_2 \in \mathbb{R}$ is such that $J_i(r_2) = r_2$ for all $i = 1, 2, \ldots, m$ and

$$f(t, r_2, 0) \ge 0 \quad \text{for a.e.} \quad t \in [0, T],$$

then $\sigma_2(t) \equiv r_2$ is an upper function of the problem (1.1)–(1.3).

The aim of this paper is to offer existence principles for problem (1.1)–(1.3) in terms of lower/upper functions. Hence our basic assumption is the existence of lower/upper functions. We will suppose that $\sigma_1/\sigma_2$ are well-ordered, i.e. that the condition

$$\begin{aligned} &\sigma_1 \text{ and } \sigma_2 \text{ are respectively lower and upper functions of (1.1)–(1.3)} \\ &\text{such that } \sigma_1 \le \sigma_2 \text{ on } [0,T] \end{aligned} \tag{1.10}$$

is true.

Note that problems with $\phi$-Laplacians and impulses have not been studied yet. As concerns problem (1.1), (1.3) (without impulses), there are various results about its solvability. For example the papers [3] and [20] present some results about the existence or multiplicity of periodic solutions of the equation

$$(\phi_p(u'))' = f(t,u)$$

under non resonance conditions imposed on $f$. The paper [10] presents general existence principles for the vector problem (1.1), (1.3). Using this the authors provide various existence theorems and illustrative examples. The vector case is also considered in [8], [11] and [13]. The existence of periodic solutions of the Liénard type equations with $p$-Laplacians has been proved in the resonance case under the Landesman-Lazer conditions in [4] and [5]. Multiplicity results of the Ambrosetti-Prodi type for this problem (with a real parameter) can be found in [7].

For the problem (1.1), (1.3), the lower/upper functions method with well-ordered $\sigma_1/\sigma_2$ has been justified by the papers [1] and [2] which study the problem (1.1), (1.3) under the Nagumo type two-sided growth conditions and in the paper [18] where the second order equation with a $\phi$-Laplacian is considered provided a functional right-hand side of this equation fulfils one-sided growth conditions of the Nagumo type. The significance of the lower/upper functions method is shown in the papers [6] and [19] where this method is used in the investigation of singular periodic problems with a $\phi$-Laplacian.

We will impose the following assumptions on the impulse functions $J_i$, $M_i$:

$$\sigma_1(t_i) \le x \le \sigma_2(t_i) \implies J_i(\sigma_1(t_i)) \le J_i(x) \le J_i(\sigma_2(t_i)), \quad i = 1,2,\dots,m; \tag{1.11}$$

$$\begin{cases} y \le \sigma_1'(t_i) \implies M_i(y) \le M_i(\sigma_1'(t_i)), \\ y \ge \sigma_2'(t_i) \implies M_i(y) \ge M_i(\sigma_2'(t_i)), \end{cases} \qquad i = 1,2,\dots,m. \tag{1.12}$$

## 2 . A priori estimates

Consider a class of auxiliary Dirichlet problems:

$$(\phi(u'))' = \widetilde{f}(t,u,u'), \tag{2.1}$$

$$u(t_i+) = \widetilde{J}_i(u(t_i)), \quad u'(t_i+) = \widetilde{M}_i(u'(t_i)), \quad i = 1,2,\dots,m, \tag{2.2}$$

$$u(0) = u(T) = d, \tag{2.3}$$

where $d \in \mathbb{R}$, $\widetilde{f}$ is an $\mathbb{L}_1$-Carathéodory function on $[0,T]\times\mathbb{R}^2$, $\widetilde{J}_i$, $\widetilde{M}_i$, $i = 1,2,\dots,m$, are continuous on $\mathbb{R}$ and such that

$$\begin{cases} \widetilde{f}(t,x,y) < \ f(t,\sigma_1(t),\sigma_1'(t)) \ \text{ for a.e. } \ t\in[0,T], \ \text{all} \ \ x\in(-\infty,\sigma_1(t)) \\ \qquad\qquad \text{and all } \ y\in\mathbb{R} \text{ such that } \ |y-\sigma_1'(t)| \le \dfrac{\sigma_1(t)-x}{\sigma_1(t)-x+1}, \\ \widetilde{f}(t,x,y) > \ f(t,\sigma_2(t),\sigma_2'(t)) \ \text{ for a.e. } \ t\in[0,T], \ \text{all} \ \ x\in(\sigma_2(t),\infty) \\ \qquad\qquad \text{and all } \ y\in\mathbb{R} \text{ such that } \ |y-\sigma_2'(t)| \le \dfrac{x-\sigma_2(t)}{x-\sigma_2(t)+1}, \end{cases} \tag{2.4}$$

$$\begin{cases} \widetilde{J}_i(x) < J_i(\sigma_1(t_i)) & \text{if } \ x < \sigma_1(t_i) \\ \widetilde{J}_i(x) = J_i(x) & \text{if } \ x \in [\sigma_1(t_i),\sigma_2(t_i)] \\ \widetilde{J}_i(x) > J_i(\sigma_2(t_i)) & \text{if } \ x > \sigma_2(t_i), \qquad i = 1,2,\dots,m, \end{cases} \tag{2.5}$$

$$\begin{cases} \widetilde{M}_i(y) \le M_i(\sigma_1'(t_i)) & \text{if } \ y \le \sigma_1'(t_i) \\ \widetilde{M}_i(y) \ge M_i(\sigma_2'(t_i)) & \text{if } \ y \ge \sigma_2'(t_i), \ \ i = 1,2,\dots,m, \end{cases} \tag{2.6}$$

and

$$\sigma_1(0) \le d \le \sigma_2(0). \tag{2.7}$$

Due to the assumption (1.10) and the properties of the lower and upper functions associated with the given problem (1.1)–(1.3), we can derive uniform estimates for the solutions of the class (2.1)–(2.3).

**2.1. Lemma.** *Let* (1.10), (1.11) *and* (2.4)–(2.7) *hold. Then every solution* $u$ *of* (2.1)–(2.3) *satisfies*

$$\sigma_1 \le u \le \sigma_2 \quad \text{on} \ \ [0,T]. \tag{2.8}$$

*Proof.* Let $u$ be a solution of (2.1)–(2.3). Put $v(t) = u(t) - \sigma_2(t)$ for $t \in [0,T]$. Then, by (2.7), we have

$$v(0) = v(T) \le 0. \tag{2.9}$$

So, it remains to prove that $v \le 0$ on $(0,T)$.

• PART (i). First, we show that $v$ does not have a positive local maximum at any point of $(0,T) \setminus \mathrm{D}$. Assume, on the contrary, that there is $\alpha \in (0,T) \setminus \mathrm{D}$ such that $v$ has a positive local maximum at $\alpha$; i.e.,

$$v(\alpha) > 0 \quad \text{and} \quad v'(\alpha) = 0. \tag{2.10}$$

This guarantees the existence of $\beta$ such that $[\alpha,\beta] \subset (0,T) \setminus \mathrm{D}$ and

$$v(t) > 0 \quad \text{and} \quad |v'(t)| < \frac{v(t)}{v(t)+1} < 1 \tag{2.11}$$

for $t \in [\alpha,\beta]$. Using (1.7), (2.4) and (2.11), we get

$$\begin{aligned}(\phi(u'(t)))' - (\phi(\sigma_2'(t)))' &= \widetilde{f}(t,u(t),u'(t)) - (\phi(\sigma_2'(t)))' \\ &> f(t,\sigma_2(t),\sigma_2'(t)) - (\phi(\sigma_2'(t)))' \ge 0\end{aligned}$$

for a.e. $t \in [\alpha,\beta]$. Hence, by (2.10),

$$0 < \int_\alpha^t (\phi(u'(s)))' - (\phi(\sigma_2'(s)))' \,\mathrm{d}s = \phi(u'(t)) - \phi(\sigma_2'(t))$$

for all $t \in (\alpha,\beta]$. Therefore $v'(t) = u'(t) - \sigma_2'(t) > 0$ for all $t \in (\alpha,\beta]$. This contradicts that $v$ has a local maximum at $\alpha$.

• PART (ii). Now, assume that there is $t_j \in \mathrm{D}$ such that

$$\max_{t \in (t_{j-1},t_j]} v(t) = v(t_j) > 0.$$

Then $v'(t_j) \ge 0$. By (2.5) and (2.6), we get

$$\widetilde{J}_j(u(t_j)) > J_j(\sigma_2(t_j)) \quad \text{and} \quad \widetilde{M}_j(u'(t_j)) \ge M_j(\sigma_2'(t_j));$$

by (2.2) and (1.8), the relations

$$v(t_j+) > 0 \quad \text{and} \quad v'(t_j+) \ge 0 \tag{2.12}$$

follow. If $v'(t_j+) > 0$, then there is $\beta \in (t_j, t_{j+1})$ such that

$$v'(t) > 0 \ \text{ on } \ (t_j,\beta]. \tag{2.13}$$

If $v'(t_j+) = 0$, then we can find $\beta$ such that $(t_j, \beta] \subset (0,T) \setminus \mathrm{D}$ and (2.11) is satisfied on $(t_j, \beta]$. Consequently, (2.13) is valid in this case, as well. In the both cases we have

$$v'(t) \ge 0 \quad \text{on} \quad (t_j, t_{j+1}), \tag{2.14}$$

because $v'$ cannot change its sign on $(t_j, t_{j+1})$, due to Part (i). Now, by (2.12)–(2.14) we get

$$\max_{t \in (t_j, t_{j+1}]} v(t) = v(t_{j+1}) > 0.$$

Continuing inductively we get $v(T) > 0$, contrary to (2.9).

• Part (iii). Finally, assume that

$$\sup_{t \in (t_j, t_{j+1}]} v(t) = v(t_j+) > 0 \tag{2.15}$$

for some $t_j \in \mathrm{D}$. In view of (2.5), this is possible only if

$$\widetilde{J}_j(u(t_j)) > J_j(\sigma_2(t_j)). \tag{2.16}$$

If $u(t_j) \in [\sigma_1(t_j), \sigma_2(t_j)]$, then by (2.5) and (1.11) we have

$$\widetilde{J}_j(u(t_j)) = J_j(u(t_j)) \le J_j(\sigma_2(t_j)),$$

contrary to (2.16). If $u(t_j) < \sigma_1(t_j)$, then by (2.5), (1.10) and (1.11) we get

$$\widetilde{J}_j(u(t_j)) < J_j(\sigma_1(t_j)) \le J_j(\sigma_2(t_j)),$$

which contradicts (2.16) again. Therefore $u(t_j) > \sigma_2(t_j)$, i.e. $v(t_j) > 0$. Further, (2.15) gives $v'(t_j+) \le 0$. If $v'(t_j+) = 0$, then, as in Part (ii), we get (2.13), which contradicts (2.15). Therefore $v'(t_j+) < 0$. This with (2.6) imply that $v'(t_j) < 0$. Thus, in view of Part (i), we deduce that $v' \le 0$ on $(t_{j-1}, t_j)$; i.e., $\sup_{t \in (t_{j-1}, t_j]} v(t) = v(t_{j-1}+) > 0$. Continuing inductively we get $v(0) > 0$, contradicting (2.9).

To summarize: we have proved that $v \le 0$ on $[0,T]$ which means that $u \le \sigma_2$ on $[0,T]$.

If we put $v = \sigma_1 - u$ on $[0,T]$ and use the properties of $\sigma_1$ instead of $\sigma_2$, we can prove $\sigma_1 \le u$ on $[0,T]$ by an analogous argument. □

A priori estimates for derivatives of solutions are provided by the next lemma. In its proof and in what follows, we will use the following notation:

$$\left\{ \begin{array}{l} \text{if } \ \psi \in \mathbb{C}(\mathbb{R}) \text{ is increasing on } \ \mathbb{R} \text{ and } \ x \in \mathbb{R}, \text{ then} \\ \qquad \{x\}_\psi = \max\{|\psi(-x)|, |\psi(x)|\}. \end{array} \right. \tag{2.17}$$

**2.2. Lemma.** *Assume that $r \in (0,\infty)$ and that*

(2.18) $$h \in \mathbb{L}_1 \quad \textit{is nonnegative a.e. on } [0,T],$$

(2.19) $$\omega \quad \textit{is continuous and positive on } [0,\infty) \textit{ and } \int_0^\infty \frac{ds}{\omega(s)} = \infty.$$

*Then there exists $r^* \in (1,\infty)$ such that the estimate*

(2.20) $$\|u'\|_\infty \le r^*$$

*holds for each function $u \in \mathbb{C}^1_{\mathrm{D}}$ satisfying $\phi(u') \in \mathbb{AC}_{\mathrm{D}}$, $\|u\|_\infty \le r$ and*

(2.21) $$\left\{ \begin{array}{l} |(\phi(u'(t)))'| \le \omega(|\phi(u'(t))|)\,(|u'(t)| + h(t)) \\ \qquad\qquad \textit{for a.e. } t \in [0,T] \textit{ and for } |u'(t)| \ge 1. \end{array} \right.$$

*Proof.* Let $u$ satisfy the assumptions of Lemma 2.2. The Mean Value Theorem implies that there are $\xi_i \in (t_i, t_{i+1})$ such that

$$|u'(\xi_i)| < \frac{2\,r}{\Delta} + 1 \ \text{ for } i = 0,1,\dots,m, \quad \text{where } \Delta = \min_{i=0,1,\dots,m} (t_{i+1} - t_i).$$

Put

$$c_0 = \left\{\frac{2\,r}{\Delta} + 1)\right\}_\phi, \quad \rho = \|\phi(u')\|_\infty$$

and assume that $\rho > c_0$ and

$$\rho = \sup_{t \in (t_j, t_{j+1}]} \phi(u'(t)) \quad \text{for some } j \in \{0,1,\dots,m\}.$$

We have either

(2.22) $$\rho = \phi(u'(\alpha)) \quad \text{for some } \alpha \in (t_j, t_{j+1}]$$

or

(2.23) $$\rho = \phi(u'(\alpha+)) \quad \text{with } \alpha = t_j.$$

In both cases, there is $\beta \in (t_j, t_{j+1})$, $\beta \ne \alpha$, such that $\phi(u'(\beta)) = c_0$ and $\phi(u'(t)) \ge c_0$ for all $t$ lying between $\alpha$ and $\beta$. Assume that (2.22) occurs. There are two possibilities: $t_j < \beta < \alpha \le t_{j+1}$ or $t_j < \alpha < \beta < t_{j+1}$. If $t_j < \beta < \alpha \le t_{j+1}$, then, since $u'(t) > 1$ on $[\beta, \alpha]$, (2.21) gives

$$(\phi(u'(t)))' \le \omega(\phi(u'(t)))\,(u'(t) + h(t)) \ \text{ for a.e. } t \in [\beta, \alpha].$$

Consequently,

$$\int_{c_0}^{\rho} \frac{\mathrm{d}s}{\omega(s)} = \int_{\beta}^{\alpha} \frac{(\phi(u'(t)))'}{\omega(\phi(u'(t)))}\,\mathrm{d}t \le \int_{\beta}^{\alpha} u'(t)\,\mathrm{d}t + \|h\|_1 \le 2\,r + \|h\|_1,$$

i.e.

$$\int_{c_0}^{\rho} \frac{\mathrm{d}s}{\omega(s)} \le 2\,r + \|h\|_1. \tag{2.24}$$

Similarly, if $t_j < \alpha < \beta < t_{j+1}$, then, using (2.21), we get

$$-(\phi(u'(t)))' \le \omega(\phi(u'(t)))\,(u'(t) + h(t)) \quad \text{for a.e. } t \in [\alpha, \beta],$$

wherefrom the inequality (2.24) again follows. On the other hand, by (2.19), there is $r_0 > c_0$ such that

$$\int_{c_0}^{r_0} \frac{\mathrm{d}s}{\omega(s)} > 2\,r + \|h\|_1, \tag{2.25}$$

which together with (2.24) may occur only if $\rho < r_0$. Therefore, (2.20) holds for $r^* = \phi^{-1}(r_0)$.

If (2.23) or $\rho = \sup_{t \in (t_j, t_{j+1}]}(-\phi(u'(t)))$ for some $j \in \{0, 1, \dots, m\}$ are true, then similar arguments apply and yield (2.20), as well. $\square$

**2.3. Remark.** Notice, that the condition

$$\int_0^{\infty} \frac{\mathrm{d}s}{\omega(s)} = \infty$$

in (2.19) can be weakened. In particular, the estimate (2.20) holds whenever $r^* \in (0, \infty)$ is such that

$$\int_{c_0}^{r^*} \frac{\mathrm{d}s}{\omega(s)} > 2\,r + \|h\|_1.$$

# 3. A fixed point operator

We will transform the problem (1.1)–(1.3) into a fixed point problem in $\mathbb{C}^1_{\mathrm{D}}$. As usual $\chi_M$ will denote the characteristic functions of the set $M \subset \mathbb{R}$. First, we will consider the following auxiliary Dirichlet problem

$$(\phi(u'(t)))' = h(t) \quad \text{a.e. on } [0, T], \tag{3.1}$$

$$u(t_i+) - u(t_i) = d_i, \quad \phi(u'(t_i+)) - \phi(u'(t_i)) = e_i, \quad i = 1, 2, \dots, m, \tag{3.2}$$

$$u(0) = u(T) = 0, \tag{3.3}$$

where $h \in \mathbb{L}_1$, $d_i$, $e_i \in \mathbb{R}$, $i = 1, \dots, m$.

**3.1. Lemma.** *A function $u \in \mathbb{C}^1_{\mathrm{D}}$ is a solution of* (3.1)–(3.3) *if and only if $u$ satisfies conditions*

$$\left\{ \begin{aligned} u(t) = {}& \textstyle\int_0^t \phi^{-1}\big(\phi(u'(0)) + H(s) + \sum_{j=1}^m e_j\,\chi_{(t_j,T]}(s)\big)\,ds \\ & + \textstyle\sum_{j=1}^m d_j\,\chi_{(t_j,T]}(t) \quad \textit{on} \;\; [0,T] \end{aligned} \right. \tag{3.4}$$

*and*

$$\sum_{j=1}^m d_j + \int_0^T \phi^{-1}\big(\phi(u'(0)) + H(s) + \sum_{j=1}^m e_j\chi_{(t_j,T]}(s)\big)\,ds = 0, \tag{3.5}$$

*where $H(s) = \int_0^s h(\tau)\,d\tau$.*

*Proof.* (i) Let $u$ be a solution (3.1)–(3.3). We will integrate (3.1) from 0 to $t$. In view of the second condition in (3.2) we obtain

$$\phi(u'(t)) = \phi(u'(0)) + H(t) + \sum_{j=1}^m e_j\chi_{(t_j,T]}(t) \text{ on } [0,T]. \tag{3.6}$$

Hence

$$u'(t) = \phi^{-1}\big(\phi(u'(0)) + H(t) + \sum_{j=1}^m e_j\chi_{(t_j,T]}(t)\big) \text{ on } [0,T]. \tag{3.7}$$

Integrating (3.7) and using the first condition in (3.2) we get (3.4). By (3.3) we see that for $t = T$ the equation (3.4) has the form (3.5).

(ii) Let $u \in \mathbb{C}^1_{\mathrm{D}}$ satisfy (3.4) and (3.5). Then, by (3.4),

$$u(t_i+) - u(t_i) = \sum_{j=1}^m d_j(\chi_{(t_j,T]}(t_i+) - \chi_{(t_j,T]})(t_i)) = d_i, \quad i = 1,\dots,m$$

and $u(0) = 0$. Moreover, according to (3.5), $u(T) = 0$. Further, (3.4) implies that (3.7) and consequently (3.6) hold. Therefore $\phi(u') \in \mathbb{AC}_{\mathrm{D}}$ and

$$\phi(u'(t_i+)) - \phi(u'(t_i)) = \sum_{j=1}^m e_j(\chi_{(t_j,T]}(t_i+) - \chi_{(t_j,T]})(t_i)) = e_i, \quad i = 1,\dots,m.$$

□

Now, we borrow some ideas from [10] to get the following two lemmas.

**3.2. Lemma.** *For each* $\ell \in \mathbb{C}_{\mathrm{D}}$ *and* $d \in \mathbb{R}$, *the function*

$$\Psi_{\ell,d} : \mathbb{R} \mapsto \mathbb{R}, \quad \Psi_{\ell,d}(a) = d + \int_0^T \phi^{-1}\big(a + \ell(t)\big)\, dt$$

*has exactly one zero point* $a(\ell, d)$ *in* $\mathbb{R}$.

*Proof.* Choose $\ell \in \mathbb{C}_{\mathrm{D}}$ and $d \in \mathbb{R}$. Since $\Psi_{\ell,d}$ is continuous, increasing on $\mathbb{R}$ and $\Psi_{\ell,d}(\mathbb{R}) = \mathbb{R}$, there is a unique real number $a(\ell, d)$ such that

$$\Psi_{\ell,d}(a(\ell,d)) = 0. \tag{3.8}$$

□

**3.3. Lemma.** *The mapping* $a : \mathbb{C}_{\mathrm{D}} \times \mathbb{R} \mapsto \mathbb{R}$ *defined by* (3.8) *is continuous and maps bounded sets into bounded sets.* [1]

*Proof.* (i) Assume that $\mathcal{A} \subset \mathbb{C}_{\mathrm{D}} \times \mathbb{R}$ and $\gamma \in (0, \infty)$ are such that $\|\ell\|_\infty + |d| \le \gamma$ for each $(\ell, d) \in \mathcal{A}$ and that there is a sequence $\{a(\ell_n, d_n)\}_{n=1}^\infty \subset a(\mathcal{A})$ such that $\lim_{n\to\infty} a(\ell_n, d_n) = \infty$ or $\lim_{n\to\infty} a(\ell_n, d_n) = -\infty$. Let the former possibility occur. Then, by (3.8), we have $0 = \lim_{n\to\infty} \Psi_{\ell_n,d_n}(a(\ell_n, d_n)) \ge \lim_{n\to\infty}(-\gamma + T\phi^{-1}\big(a(\ell_n, d_n) - \gamma)\big) = \infty$, a contradiction. The latter possibility can be argued similarly.
(ii) Let $\lim_{n\to\infty}(\ell_n, d_n) = (\ell_0, d_0)$ in $\mathbb{C}_{\mathrm{D}} \times \mathbb{R}$. By (i) the sequence $\{a(\ell_n, d_n)\}_{n=1}^\infty$ is bounded and hence we can choose a subsequence such that $\lim_{n\to\infty} a(\ell_{k_n}, d_{k_n}) = a_0 \in \mathbb{R}$. By (3.8), we get

$$0 = \Psi_{\ell_{k_n},d_{k_n}}(a(\ell_{k_n}, d_{k_n})) = d_{k_n} + \int_0^T \phi^{-1}\big(a(\ell_{k_n}, d_{k_n}) + \ell_{k_n}(t)\big)\, \mathrm{d}t,$$

which, for $n \to \infty$, yields

$$0 = d_0 + \int_0^T \phi^{-1}\big(a_0 + \ell_0(t)\big)\, \mathrm{d}t.$$

Thus, with respect to Lemma 3.2, we have $a_0 = a(\ell_0, d_0) = \lim_{n\to\infty} a(\ell_n, d_n)$. □

**3.4. Lemma.** *The operator* $\mathcal{N} : \mathbb{C}^1_{\mathrm{D}} \mapsto \mathbb{C}_{\mathrm{D}}$ *given by*

$$(\mathcal{N}(x))(t) = \int_0^t f(s, x(s), x'(s))\, ds + \sum_{i=1}^m \big[\phi\big(M_i(x'(t_i))\big) - \phi\big(x'(t_i)\big)\big]\, \chi_{(t_i,T]}(t) \tag{3.9}$$

*is absolutely continuous.*

---

[1] The norm of $(\ell, d) \in \mathbb{C}_{\mathrm{D}} \times \mathbb{R}$ is defined by $\|\ell\|_\infty + |d|$.

*Proof.* The continuity of $\mathcal{N}$ follows from the continuity of all the mappings involved in the right-hand side of (3.9). Furthermore, let $\mathcal{H} \subset \mathbb{C}^1_{\mathrm{D}}$ be bounded. We need to show that the closure $\mathrm{cl}(\mathcal{N}(\mathcal{H})]$ of $\mathcal{N}(\mathcal{H})$ in $\mathbb{C}_{\mathrm{D}}$ is compact. To this aim, let $\|x\|_{\mathrm{D}} \le \gamma < \infty$ for each $x \in \mathcal{H}$. Then there are $c \in (0,\infty)$ and $h \in \mathbb{L}_1$ such that

$$\sum_{i=1}^{m} \big[\phi\big(M_i(x'(t_i))\big) - \phi\big(x'(t_i)\big)\big] \le c \quad \text{and} \quad |f(t,x(t),x'(t))| \le h(t) \quad \text{a.e. on } [0,T]$$

for all $x \in \mathcal{H}$. Therefore

$$\|\mathcal{N}(x)\|_\infty \le \|h\|_1 + c \quad \text{for each} \;\; x \in \mathcal{H}. \tag{3.10}$$

Put $(\mathcal{N}_1(x))(t) = \int_0^t f(s,x(s),x'(s))\,\mathrm{d}s$. Then, for $t_1$, $t_2 \in [0,T]$, we have

$$|(\mathcal{N}_1(x))(t_2) - (\mathcal{N}_1(x))(t_1)| \le \Big|\int_{t_1}^{t_2} h(s)\,\mathrm{d}s\Big|,$$

wherefrom, by (3.10), we deduce that the functions in $\mathcal{N}_1(\mathcal{H})$ are uniformly bounded and equicontinuous on $[0,T]$. Hence, making use of the Arzelà-Ascoli Theorem in the space of functions continuous on $[0,T]$ with the norm $\|.\|_\infty$, we get that each sequence in $\mathcal{N}_1(\mathcal{H})$ contains a subsequence convergent with respect to the norm $\|.\|_\infty$. This shows that $\mathrm{cl}(\mathcal{N}_1(\mathcal{H}))$ is compact in $\mathbb{C}_{\mathrm{D}}$. We know that the operator $\mathcal{N}_2 = \mathcal{N} - \mathcal{N}_1$ is continuous. By (3.10), it maps bounded sets into bounded sets. Moreover, its values are contained in an $m$-dimensional subspace of $\mathbb{C}_{\mathrm{D}}$. Thus, $\mathrm{cl}(\mathcal{N}_2(\mathcal{H}))$ is compact in $\mathbb{C}_{\mathrm{D}}$. □

**3.5. Theorem.** *Let $a : \mathbb{C}_{\mathrm{D}} \times \mathbb{R} \mapsto \mathbb{R}$ and $\mathcal{N} : \mathbb{C}^1_{\mathrm{D}} \mapsto \mathbb{C}_{\mathrm{D}}$ be respectively defined by* (3.8) *and* (3.9). *Furthermore define $\mathcal{J} : \mathbb{C}^1_{\mathrm{D}} \mapsto \mathbb{C}^1_{\mathrm{D}}$ by*

$$(\mathcal{J}(x))(t) = \sum_{i=1}^{m} \big[J_i(x(t_i)) - x(t_i)\big]\, \chi_{(t_i,T]}(t) \tag{3.11}$$

*and*

$$\begin{aligned}(\mathcal{F}(x))(t) = \int_0^t \phi^{-1}\Big(a\big(\mathcal{N}(x),(\mathcal{J}(x))(T)\big) + (\mathcal{N}(x))(s)\Big)\, ds \\ + x(0) + x'(0) - x'(T) + (\mathcal{J}(x))(t).\end{aligned} \tag{3.12}$$

*Then $\mathcal{F} : \mathbb{C}^1_{\mathrm{D}} \mapsto \mathbb{C}^1_{\mathrm{D}}$ is an absolutely continuous operator. Moreover, $u$ is a solution of the problem* (1.1)–(1.3) *if and only if $\mathcal{F}(u) = u$.*

*Proof.* For $x \in \mathbb{C}^1_{\mathrm{D}}$ and $t \in [0,T]$, we have

$$(\mathcal{F}(x))'(t) = \phi^{-1}\big(a(\mathcal{N}(x),(\mathcal{J}(x))(T)) + (\mathcal{N}(x))(t)\big). \tag{3.13}$$

Since the mappings $a$, $\mathcal{N}$ and $\mathcal{J}$ included in (3.12) and (3.13) are continuous, it follows that $\mathcal{F}$ is continuous in $\mathbb{C}^1_{\mathrm{D}}$.

Choose an arbitrary bounded set $\mathcal{H} \subset \mathbb{C}^1_{\mathrm{D}}$. We will show that then the set $\mathrm{cl}(\mathcal{F}(\mathcal{H}))$ is compact in $\mathbb{C}^1_{\mathrm{D}}$. Let a sequence $\{v_n\} \subset \mathcal{F}(\mathcal{H})$ be given. It suffices to show that it contains a subsequence convergent in $\mathbb{C}^1_{\mathrm{D}}$. Let $\{x_n\} \subset \mathcal{H}$ be such that $v_n = \mathcal{F}(x_n)$ for $n \in \mathbb{N}$. By Lemma 3.4, there is a subsequence $\{x_{k_n}\}$ such that $\{\mathcal{N}(x_{k_n})\}$ is convergent in $\mathbb{C}_{\mathrm{D}}$. According to (3.10) and (3.11), there exists $\gamma \in (0,\infty)$ such that $\|\mathcal{N}(x)\|_\infty + |(\mathcal{J}(x))(T)| \le \gamma$ for all $x \in \mathcal{H}$. Hence, by Lemma 3.3, the sequence $\{a(\mathcal{N}(x_{k_n}), (\mathcal{J}(x_{k_n}))(T))\} \subset \mathbb{R}$ is bounded and we can choose a subsequence $\{x_{\ell_n}\} \subset \{x_{k_n}\}$ such that $\{a\big(\mathcal{N}(x_{\ell_n}), (\mathcal{J}(x_{\ell_n}))(T)\big) + \mathcal{N}(x_{\ell_n})\}$ is convergent in $\mathbb{C}_{\mathrm{D}}$. Consequently, $\{(\mathcal{F}(x_{\ell_n}))'\}$ and $\{\mathcal{F}(x_{\ell_n})\}$ are convergent in $\mathbb{C}_{\mathrm{D}}$, as well. So, we have proved the $\mathcal{F}$ is absolutely continuous in $\mathbb{C}^1_{\mathrm{D}}$.

To prove the last assertion of Theorem 3.5 we will write conditions (1.2),(1.3) in the equivalent form

$$\begin{aligned} u(t_i+) - u(t_i) &= J_i(u(t_i)) - u(t_i), \\ \phi(u'(t_i+)) - \phi(u'(t_i)) &= \phi(M_i(u'(t_i))) - \phi(u'(t_i)), \quad i = 1\dots,m, \\ u(0) = u(T) &= u(0) + u'(0) - u'(T). \end{aligned}$$

Denote $\int_0^s f(\tau,u(\tau),u'(\tau))\,\mathrm{d}\tau = F(s)$. Then, by Lemma 3.1, we get that u is a solution of (1.1)–(1.3) if and only if u satisfies

$$\begin{aligned} u(t) = u(0) &+ u'(0) - u'(T) + \sum_{j=1}^m (J_j(u(t_j)) - u(t_j))\chi_{(t_j,T]}(t) \\ &+ \int_0^t \phi^{-1}\big(\phi(u'(0)) + F(s) + \sum_{j=1}^m [\phi(M_j(u'(t_j))) - \phi(u'(t_j))]\,\chi_{(t_j,T]}(s)\big)\,\mathrm{d}s \end{aligned}$$

and

$$\begin{aligned} 0 = &\sum_{j=1}^m (J_j(u(t_j)) - u(t_j)) \\ &+ \int_0^T \phi^{-1}\Big(\phi(u'(0)) + F(s) + \sum_{j=1}^m [\phi(M_j(u'(t_j))) - \phi(u'(t_j))]\,\chi_{(t_j,T]}(s)\Big)\,\mathrm{d}s. \end{aligned}$$

These two conditions can be written by (3.9) and (3.11) in the form

$$u(t) = u(0) + u'(0) - u'(T) + (\mathcal{J}(u))(t) + \int_0^t \phi^{-1}\big(\phi(u'(0)) + (\mathcal{N}(u))(s)\big)\,\mathrm{d}s$$

and

$$0 = (\mathcal{J}(u))(T) + \int_0^T \phi^{-1}\big(\phi(u'(0)) + (\mathcal{N}(u))(s)\big)\,\mathrm{d}s.$$

By virtue of Lemma 3.2, the last equality yields that $\phi(u'(0)) = a(\mathcal{N}(u), (\mathcal{J}(u)(T)),$ which means that $u = \mathcal{F}(u)$. $\square$

# 4. Main results

The main existence result for problem (1.1)–(1.3) is provided by the following theorem. Its proof is based on the topological degree arguments. Let us recall that if $\Omega$ is an open bounded subset of a Banach space $\mathbb{X}$ and an operator $\mathcal{F} : cl(\Omega) \mapsto \mathbb{X}$ is completely continuous and $\mathcal{F}(u) \neq u$ for all $u \in \partial\,\Omega$, then we can define the *Leray-Schauder topological degree* $\deg(\mathrm{I} - \mathcal{F}, \Omega)$. Here I is the identity operator on $\mathbb{X}$ and $\mathrm{cl}(\Omega)$ and $\partial\Omega$ denote the closure and the boundary of $\Omega$, respectively. For a definition and properties of the degree see e.g. [9] or [12].

**4.1. Theorem.** *Assume that* (1.10), (1.11) *and* (1.12) *hold. Further, let*

$$(4.1)\qquad \begin{cases} |f(t,x,y)| \le \omega(|\phi(y)|)\,(|y| + h(t)) \\ \qquad\qquad \text{for a.e. } t \in [0,T] \ \text{ and all } x \in [\sigma_1(t), \sigma_2(t)],\ |y| \ge 1, \end{cases}$$

*where $h$ and $\omega$ fulfil* (2.18) *and* (2.19). *Then the problem* (1.1)–(1.3) *has a solution $u$ satisfying* (2.8).

Before proving this theorem, we prove the next key proposition where we restrict ourselves to the case that $f$ is bounded by a Lebesgue integrable function.

**4.2. Proposition.** *Assume that* (1.10), (1.11) *and* (1.12) *hold. Further, let $m \in \mathbb{L}_1$ be such that*

$$(4.2)\qquad |f(t,x,y)| \le m(t) \ \text{ for a.e. } t \in [0,T] \ \text{ and all } (x,y) \in [\sigma_1(t), \sigma_2(t)] \times \mathbb{R}.$$

*Then the problem* (1.1)–(1.3) *has a solution $u$ fulfilling* (2.8).

*Proof.*
•Step 1. *We construct a proper auxiliary problem.* Put $r = \|\sigma_1\|_\infty + \|\sigma_2\|_\infty$ and

$$\Delta = \min\{(t_{i+1} - t_i) : i = 0, 1, \ldots, m\}, \quad c_0 = \Big\{\frac{2r}{\Delta} + 1\Big\}_\phi, \quad c_1 = \big\{c_0 + \|m\|_1\big\}_{\phi^{-1}},$$

where we make use of the notation introduced in (2.17). Further, for $t \in [0,T]$ and $(x,y) \in \mathbb{R}^2$, define

$$(4.3)\qquad \alpha(t,x) = \begin{cases} \sigma_1(t) & \text{if } x < \sigma_1(t), \\ x & \text{if } \sigma_1(t) \le x \le \sigma_2(t), \\ \sigma_2(t) & \text{if } x > \sigma_2(t) \end{cases}$$

and

$$\beta(y) = \begin{cases} y & \text{if } |y| \le c, \\ c\, sgn\ y & \text{if } |y| > c, \end{cases} \tag{4.4}$$

where

$$c = c_1 + \|\sigma_1'\|_\infty + \|\sigma_2'\|_\infty, \tag{4.5}$$

Finally, for a.e. $t \in [0,T]$ and all $(x,y) \in \mathbb{R}^2$, $\varepsilon \in [0,1]$, put

$$\omega_k(t,\varepsilon) = \sup_{y \in [\sigma_k'(t)-\varepsilon, \sigma_k'(t)+\varepsilon]} |f(t,\sigma_k(t),\sigma_k'(t)) - f(t,\sigma_k(t),y)|, \quad k = 1,2$$

and

$$\begin{cases} \widetilde{J}_i(x) = x + J_i(\alpha(t_i,x)) - \alpha(t_i,x), \\ \widetilde{M}_i(y) = y + M_i(\beta(y)) - \beta(y), \qquad i = 1,2,\dots,m, \end{cases} \tag{4.6}$$

$$\widetilde{f}(t,x,y) = \begin{cases} f(t,\sigma_1(t),y) - \omega_1\big(t, \dfrac{\sigma_1(t)-x}{\sigma_1(t)-x+1}\big) - \dfrac{\sigma_1(t)-x}{\sigma_1(t)-x+1} \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } x < \sigma_1(t), \\ f(t,x,y) \qquad\qquad\qquad\qquad\quad \text{if } \sigma_1(t) \le x \le \sigma_2(t), \\ f(t,\sigma_2(t),y) + \omega_2\big(t, \dfrac{x-\sigma_2(t)}{x-\sigma_2(t)+1}\big) + \dfrac{x-\sigma_2(t)}{x-\sigma_2(t)+1} \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } x > \sigma_2(t). \end{cases} \tag{4.7}$$

We see that $\omega_k$ are $\mathbb{L}_1$-Carathéodory functions on $[0,T]\times[0,1]$ which are nonnegative and nondecreasing in the second variable and $\omega_k(0) = 0$, $k = 1,2$. Consequently, $\widetilde{f}$ is $\mathbb{L}_1$-Carathéodory on $[0,T] \times \mathbb{R}^2$. Furthermore, $\widetilde{J}_i$, $\widetilde{M}_i$ are continuous on $\mathbb{R}$, $i = 1,2,\dots,m$. Using (4.6) and (4.7), we get the auxiliary problem (2.1), (2.2), and

$$u(0) = u(T) = \alpha(0, u(0) + u'(0) - u'(T)). \tag{4.8}$$

• Step 2. *We prove that the problem* (2.1), (2.2), (4.8) *is solvable.*
Let $a : \mathbb{C}_\mathrm{D} \times \mathbb{R} \mapsto \mathbb{R}$ be given by (3.8), an operator $\widetilde{\mathcal{N}} : \mathbb{C}^1_\mathrm{D} \mapsto \mathbb{C}_\mathrm{D}$ by

$$(\widetilde{\mathcal{N}}(x))(t) = \int_0^t \widetilde{f}(s,x(s),x'(s))\,\mathrm{d}s + \sum_{i=1}^m \big[\phi\big(\widetilde{M}_i(x'(t_i))\big) - \phi\big(x'(t_i)\big)\big]\,\chi_{(t_i,T]}(t)$$

and an operator $\widetilde{\mathcal{J}} : \mathbb{C}^1_\mathrm{D} \mapsto \mathbb{C}^1_\mathrm{D}$ by

$$(\widetilde{\mathcal{J}}(x))(t) = \sum_{i=1}^m \big[\widetilde{J}_i(x(t_i)) - x(t_i)\big]\,\chi_{(t_i,T]}(t).$$

Finally, define an operator $\widetilde{\mathcal{F}} : \mathbb{C}^1_{\mathrm{D}} \mapsto \mathbb{C}^1_{\mathrm{D}}$ by

$$(4.9) \qquad (\widetilde{\mathcal{F}}(x))(t) = \int_0^t \phi^{-1}\Big(a\big(\widetilde{\mathcal{N}}(x), (\widetilde{\mathcal{J}}(x))(T)\big) + (\widetilde{\mathcal{N}}(x))(s)\Big)\, \mathrm{d}s \\ + \alpha(0, x(0) + x'(0) - x'(T)) + (\widetilde{\mathcal{J}}(x))(t).$$

As in the proof of Theorem 3.5 we get that $\widetilde{\mathcal{F}}$ is completely continuous and $u$ is a solution of (2.1), (2.2), (4.8) if and only if $u$ is a fixed point of $\widetilde{\mathcal{F}}$.

Denote by I the identity operator on $\mathbb{C}^1_{\mathrm{D}}$ and consider the parameter system of operator equations

$$(4.10) \qquad (\mathrm{I} - \lambda\widetilde{\mathcal{F}})u = 0, \quad \lambda \in [0,1].$$

For $R \in (0,\infty)$, define $\mathcal{B}(R) = \{u \in \mathbb{C}^1_{\mathrm{D}} : \|u\|_{\mathrm{D}} < R\}$. By (4.2) and (4.9), we can find $R_0 \in (0,\infty)$ such that $u \in \mathcal{B}(R_0)$ for each $\lambda \in [0,1]$ and each solution $u$ of (4.10). So, for each $R \geq R_0$ the operator $\mathrm{I} - \lambda\widetilde{\mathcal{F}}$ is a homotopy on $\mathrm{cl}(\mathcal{B}(R)) \times [0,1]$ and its Leray-Schauder degree $\deg(\mathrm{I} - \lambda\widetilde{\mathcal{F}}, \mathcal{B}(R))$ has the same value for each $\lambda \in [0,1]$. Since $\deg(\mathrm{I}, \mathcal{B}(R)) = 1$, we conclude that

$$(4.11) \qquad \deg(\mathrm{I} - \widetilde{\mathcal{F}}, \mathcal{B}(R)) = 1 \;\text{ for }\; R \in [R_0, \infty).$$

By (4.11), there is at least one fixed point of $\widetilde{\mathcal{F}}$ in $\mathcal{B}(R)$. Hence there exists a solution of the auxiliary problem (2.1), (2.2), (4.8).

• Step 3. *We find estimates for solutions of the auxiliary problem.*

Let $u$ be a solution of (2.1), (2.2), (4.8). We derive an estimate for $\|u\|_\infty$. By (4.6), (4.7) and (1.12), we obtain that $\widetilde{f}$, $\widetilde{J}_i$, $\widetilde{M}_i$, $i = 1, 2, \ldots, m$, satisfy (2.4)–(2.6). Moreover, in view of (4.3) we have

$$\sigma_1(0) \leq \alpha(0, u(0) + u'(0) - u'(T)) \leq \sigma_2(0).$$

Thus $u$ satisfies (2.8) by Lemma 2.4.

We find an estimate for $\|u'\|_\infty$. By the Mean Value Theorem and (2.8), there are $\xi_i \in (t_i, t_{i+1})$ such that

$$|u'(\xi_i)| \leq \frac{\|\sigma_1\|_\infty + \|\sigma_2\|_\infty}{\Delta}, \quad i = 1, 2, \ldots, m.$$

Having in mind notation of Step 1, we get

$$(4.12) \qquad |\phi(u'(\xi))| < c_0.$$

Moreover, by (2.8) and (4.7), $u$ satisfies (1.1) for a.e. $t \in [0,T]$. Therefore, integrating (1.1) and using (4.2), (4.5) and (4.12), we obtain

$$(4.13) \qquad \|u'\|_\infty < c_1 < c.$$

Hence, by (4.6) and (4.8), we see that $u$ fulfils (1.2) and $u(0) = u(T)$ (i.e. the first condition from (1.3) is satisfied).

• Step 4. *We verify that u fulfils the second condition in* (1.3)*.*

We must prove that $u'(0) = u'(T)$. By (4.8), this is equivalent to

$$\sigma_1(0) \le u(0) + u'(0) - u'(T) \le \sigma_2(0). \tag{4.14}$$

Suppose, on the contrary, that (4.14) is not satisfied. Let, for example,

$$u(0) + u'(0) - u'(T) > \sigma_2(0). \tag{4.15}$$

Then, by (4.3), we have $\alpha(0, u(0) + u'(0) - u'(T)) = \sigma_2(0)$. By (1.9) and (4.8), this yields

$$u(0) = u(T) = \sigma_2(0) = \sigma_2(T). \tag{4.16}$$

Inserting (4.16) into (4.15) we get

$$u'(0) > u'(T). \tag{4.17}$$

On the other hand, (4.16) together with (2.8) and (4.17) implies that

$$\sigma_2'(0) \ge u'(0) > u'(T) \ge \sigma_2'(T),$$

a contradiction to (1.9).

If we assume that $u(0) + u'(0) - u'(T) < \sigma_1(0)$, we can argue similarly and again derive a contradiction to (1.9).

So, we have proved that (4.14) is valid which means that $u'(0) = u'(T)$. Consequently, $u$ is a solution of (1.1)–(1.3) satisfying (2.8). □

*Proof of Theorem* 4.1. Put

$$c = r^* + \|\sigma_1'\|_\infty + \|\sigma_2'\|_\infty,$$

where $r^* \in (0, \infty)$ is given by Lemma 2.2 for $r = \|\sigma_1\|_\infty + \|\sigma_2\|_\infty$. For a.e. $t \in [0, T]$ and all $(x, y) \in \mathbb{R}^2$ define a function

$$g(t, x, y) = \begin{cases} f(t, x, y) & \text{if } |y| \le c, \\ (2 - \dfrac{|y|}{c})\, f(t, x, y) & \text{if } c < |y| < 2\,c, \\ 0 & \text{if } |y| \ge 2\,c. \end{cases} \tag{4.18}$$

Then $\sigma_1$ and $\sigma_2$ are respectively lower and upper functions of the auxiliary problem (1.2), (1.3), and

$$(\phi(u'))' = g(t, u, u'). \tag{4.19}$$

There is a function $m^* \in \mathbb{L}[0,T]$ such that $|f(t,x,y)| \le m^*(t)$ for a.e. $t \in [0,T]$ and all $(x,y) \in [\sigma_1(t), \sigma_2(t)] \times [-2\,c, 2\,c]$. Hence

$$|g(t,x,y)| \le m^*(t) \ \text{ for a.e. } \ t \in [0,T] \ \text{ and all } \ (x,y) \in [\sigma_1(t), \sigma_2(t)] \times \mathbb{R}.$$

Since $g$ is $\mathbb{L}_1$-Carathéodory on $[0,T] \times \mathbb{R}^2$, we can apply Proposition 4.2 on problem (4.19), (1.2), (1.3) and get that this problem has a solution $u$ fulfilling (2.8). Hence $\|u\|_\infty \le r$. Moreover, by (4.1), $u$ satisfies (2.21). Therefore, by Lemma 2.2, $\|u'\|_\infty \le r^* \le c$. This implies due to (4.18) that $u$ is a solution of 1.1)–(1.3). $\square$

The next simple existence criterion follows from Theorem 4.1 and Remark 1.3.

**4.3. Corollary.** *Assume that:*

(i) $M_i(0) = 0$ *and* $y\,M_i(y) \ge 0$ *for* $y \in \mathbb{R}$ *and* $i = 1, 2, \dots, m$;

(ii) *there are* $r_1$, $r_2 \in \mathbb{R}$ *such that* $r_1 < r_2$, $f(t, r_1, 0) \le 0 \le f(t, r_2, 0)$ *for a.e.* $t \in [0,T]$, $J_i(r_1) = r_1$, $J_i(x) \in [r_1, r_2]$ *if* $x \in [r_1, r_2]$, $J_i(r_2) = r_2$, $i = 1, 2, \dots, m$.

(iii) *there are* $h$ *and* $\omega$ *satisfying* (2.18) *and* (2.19) *with* $\sigma_1(t) \equiv r_1$ *and* $\sigma_2(t) \equiv r_2$ *and such that* (4.1) *holds.*

*Then the problem* (1.1)–(1.3) *has a solution* $u$ *fulfilling* $r_1 \le u \le r_2$ *on* $[0,T]$. $\square$

Let $r^*$ be given by Lemma 2.2 for $r = \|\sigma_1\|_\infty + \|\sigma_2\|_\infty$. Under the assumption

$$\sigma_1 < \sigma_2 \ \text{ on } \ [0,T] \quad \text{and} \quad \sigma_1(t_i+) < \sigma_2(t_i+) \quad \text{for} \ \ i = 1, 2, \dots, m \tag{4.20}$$

we can define an open set $\Omega$ by

$$\begin{aligned} \Omega = \{u \in \mathbb{C}^1_{\mathrm{D}} : \|u'\|_\infty < r^*,\ \sigma_1(t) < u(t) < \sigma_2(t) \ \text{ for } \ t \in [0,T], \\ \sigma_1(t_i+) < u(t_i+) < \sigma_2(t_i+) \ \text{ for } \ i = 1, 2, \dots, m\}. \end{aligned} \tag{4.21}$$

Next theorem gives the evaluation of the Leray-Schauder degree of the operator $\mathrm{I} - \mathcal{F}$ (corresponding to the problem (1.1)–(1.3)) on the set $\Omega$.

**4.4. Theorem.** *Let* (4.20) *and all the assumptions of Theorem* 4.1 *be satisfied. Further assume that* $\mathcal{F}$ *and* $\Omega$ *are respectively defined by* (3.12) *and* (4.21). *If* $\mathcal{F}(u) \ne u$ *for each* $u \in \partial\Omega$, *then*

$$\deg(\mathrm{I} - \mathcal{F}, \Omega) = 1.$$

*Proof.* Consider $c$ and $g$ from the proof of Theorem 4.1 and define $\widetilde{J}_i$, $\widetilde{M}_i$, $i = 1, 2, \dots, m$, and $\widetilde{f}$ by (4.6) and (4.7), where we insert $g$ instead of $f$. Suppose that $\mathcal{F}u \neq u$ for each $u \in \partial\Omega$, define $\widetilde{\mathcal{F}}$ by (4.9) and put $\Omega_1 = \{u \in \Omega : \sigma_1(0) < u(0) + u'(0) - u'(T) < \sigma_2(0)\}$. We have

$$\mathcal{F} = \widetilde{\mathcal{F}} \quad \text{on} \quad \mathrm{cl}(\Omega_1) \tag{4.22}$$

and

$$\Big(\mathcal{F}u = u \quad \text{and} \quad u \in \Omega\Big) \implies u \in \Omega_1. \tag{4.23}$$

By the proof of Proposition 4.2, each fixed point $u$ of $\widetilde{\mathcal{F}}$ satisfies (1.3), (2.8) and, consequently, $\|u\|_\infty \le r$. Hence, in view of (4.1), (4.7) and (4.18), we have

$$|(\phi(u'(t)))'| = |g(t, u(t), u'(t))| \le \omega(|\phi(u'(t))|)\,(|u'(t)| + h(t))$$

for a.e. $t \in [0, T]$ and for $|u'(t)| \ge 1$. Therefore Lemma 2.2 implies that $\|u'\|_\infty \le r^*$. So, $u \in \mathrm{cl}(\Omega)$ and, due to (1.3), $u \in \Omega_1$. Now, choose $R$ in (4.11) so that $\mathcal{B}(R) \supset \Omega$. Then, by (4.22), (4.23) and by the excision property of the degree, we get

$$\deg(\mathrm{I} - \mathcal{F}, \Omega) = \deg(\mathrm{I} - \widetilde{\mathcal{F}}, \Omega_1) = \deg(\mathrm{I} - \widetilde{\mathcal{F}}, \Omega_1)) = \deg(\mathrm{I} - \widetilde{\mathcal{F}}, \mathcal{B}(R)) = 1. \qquad \square$$

**4.5. Remark.** Following the ideas of [16], the evaluation of $\deg(\mathrm{I} - \mathcal{F}, \Omega)$ enables us to prove the existence of solutions to the problem (1.1)–(1.3) also for non-ordered lower/upper functions. This will be included in our next preprint [17].

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